

## $SL_n$ -CHARACTER VARIETIES AS SPACES OF GRAPHS

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ABSTRACT. An  $SL_n$ -character of a group  $G$  is the trace of an  $SL_n$ -representation of  $G$ . We show that all algebraic relations between  $SL_n$ -characters of  $G$  can be visualized as relations between graphs (resembling Feynman diagrams) in any topological space  $X$ , with  $\pi_1(X) = G$ . We also show that all such relations are implied by a single local relation between graphs. In this way, we provide a topological approach to the study of  $SL_n$ -representations of groups.

The motivation for this paper was our work with J. Przytycki on invariants of links in 3-manifolds which are based on the Kauffman bracket skein relation. These invariants lead to a notion of a skein module of  $M$  which, by a theorem of Bullock, Przytycki, and the author, is a deformation of the  $SL_2$ -character variety of  $\pi_1(M)$ . This paper provides a generalization of this result to all  $SL_n$ -character varieties.

### 1. INTRODUCTION

In this paper we introduce a new method in the study of representations of groups into affine algebraic groups. Although we consider only  $SL_n$ -representations, the results of this paper can be generalized to other affine algebraic groups; see [Si].

For any group  $G$  and any commutative ring  $R$  with 1 there is a commutative  $R$ -algebra  $Rep_n^R(G)$  and the universal  $SL_n$ -representation

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$$

such that any representation of  $G$  into  $SL_n(A)$ , where  $A$  is an  $R$ -algebra, factors through  $j_{G,n}$  in a unique way. This universal property uniquely determines  $Rep_n^R(G)$  and  $j_{G,n}$  up to an isomorphism.

$GL_n(R)$  acts on  $Rep_n^R(G)$  (see Section 2), and the subring of  $Rep_n^R(G)$  composed of the elements fixed by the action,  $Rep_n^R(G)^{GL_n(R)}$ , is called the *the universal  $SL_n$ -character ring of  $G$* . This ring contains essential information about  $SL_n$ -representations of  $G$ . In particular, if  $R$  is an algebraically closed field of characteristic 0, then there are natural bijections between the following three sets:

- the set of all  $R$ -algebra homomorphisms  $Rep_n^R(G)^{GL_n(R)} \rightarrow R$ ,
- the set of all semisimple  $SL_n(R)$ -representations of  $G$  up to conjugation, and
- the set of  $SL_n(R)$ -characters of  $G$ .

It is convenient to think about  $Rep_n^R(G)^{GL_n(R)}$  as the coordinate ring of a scheme,  $\mathfrak{X}_n(G) = Spec(Rep_n^R(G)^{GL_n(R)})$ , called the  $SL_n$ -character variety of  $G$ .

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As explained in Section 6, the algebra  $\text{Rep}_n^R(G)^{GL_n(R)}$  encodes all algebraic relations between the  $SL_n$ -characters of  $G$ . Unfortunately, it is very difficult to give a finite presentation of  $\text{Rep}_n^R(G)^{GL_n(R)}$  and, hence, to describe  $\mathfrak{X}_n(G)$ , even for groups  $G$  with relatively simple presentations.

In this paper, we present a topological approach to the study of  $SL_n$ -character varieties. We prove that  $R[\mathfrak{X}_n(G)] = \text{Rep}_n^R(G)^{GL_n(R)}$  is spanned by a special class of graphs (resembling Feynman diagrams) in  $X$ , where  $X$  is any topological space with  $\pi_1(X) = G$ ; see Theorem 3.7. Moreover, all relations between the elements of this spanning set are induced by specific local relations between the graphs, called *skein relations*.

We postpone a detailed study of applications of our graphical calculus to the theory of  $SL_n$ -representations of groups to future papers. In this paper, we content ourselves with an example, in which we apply our method to a study of  $SL_3$ -representations of the free group on two generators. In this algebraically non-trivial example a huge reduction of computational difficulties can be achieved by the application of our geometric method.

This work is related to several areas of mathematics and physics:

**Knot theory (and skein modules).** Skein relations between links were used to define the famous polynomial invariants of links, like the Conway, Jones, and Homfly polynomials, [Co, Jo, FYHLMO, P-T, Ka]. In this paper we apply skein relations to the representations of groups.

The motivation for this work was our earlier work on skein modules, [PS-2]. The main theorem of this paper generalizes the Bullock-Przytycki-Sikora theorem relating the Kauffman bracket skein module of a manifold  $M$  to the  $SL_2(\mathbb{C})$ -character variety of  $\pi_1(M)$ ; see [B-2, PS-2].

**Quantum invariants of 3-manifolds.** We hope that this work will help understand the connections between quantum invariants of 3-manifolds and representations of their fundamental groups. It follows from the work of Yokota [Yo] that for any 3-manifold  $M$ , the  $SU_n$ -quantum invariants of  $M$  can be defined by using our graphs considered up to relations which are  $q$ -deformations of our skein relations.

**Spin networks and gauge theory.** The graphs considered in this paper have an interpretation as spin networks; see [Si]. They are also very similar to graphs used by physicists in non-abelian gauge theory (QCD); see [Cv].

**Number theory.** After a preliminary version of this paper was made available, M. Kapranov pointed out to us that our work is related to the work of Wiles and others on “pseudo-representations.” In his work (related to Fermat’s Last Theorem), Wiles gave necessary and sufficient conditions under which a complex-valued function on  $G$  is a  $GL_2(\mathbb{C})$ -character of  $G$ . His ideas were developed further and generalized to all  $GL_n$ -characters by Taylor, [Ta]. See also [Ny, Ro]. These results provide a description of the coordinate ring of  $GL_n$ -character varieties quotiented by nilpotent elements. Our results are similar in spirit, but they are concerned with  $SL_n$ -representations and they are stronger, since they describe  $R[\mathfrak{X}_n(G)]$  (i.e.  $\text{Rep}_n^R(G)^{GL_n(R)}$ ) exactly (with possible nilpotent elements).

The plan of this paper is as follows. In Section 2 we introduce some basic notions and facts concerning representations of groups. In Section 3 we define the algebra  $\mathbb{A}_n(X)$  in terms of graphs in  $X$  and formulate (Theorems 3.6 and 3.7) the main results of the paper asserting that  $\mathbb{A}_n(X)$  is isomorphic to  $\text{Rep}_n^R(G)^{GL_n(R)}$ , where

$G = \pi_1(X)$ . The proof requires introducing another algebra,  $\mathbb{A}_n(X, x_0)$ , associated with any pointed topological space  $(X, x_0)$ . The algebra  $\mathbb{A}_n(X, x_0)$  is an interesting object by itself, and for  $n = 2$  it already appeared in the theory of skein modules as a relative skein algebra. Sections 4 and 5 are devoted to the proof of the results of Section 3. In the final section we consider trace identities and use our results to describe the  $SL_3$ -character variety of the free group on two generators.

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2. BACKGROUND FROM REPRESENTATION THEORY

In this section we introduce the basic elements of the theory of  $SL_n$ -representations of groups. We follow the approach of Brumfiel and Hilden, ([B-H], Chapter 8), which although formally restricted to  $SL_2$ -representations, has a straightforward generalization to  $SL_n$ -representations for any  $n$ . Compare also [L-M], [Pro-2].

Let  $G$  be a group and let  $R$  be a commutative ring with unity. There is a commutative  $R$ -algebra  $Rep_n^R(G)$ , called *the universal representation algebra*, and *the universal representation*

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G)),$$

with the following property: For any commutative  $R$ -algebra  $A$  and any representation  $\rho : G \rightarrow SL_n(A)$  there is a unique homomorphism of  $R$ -algebras  $h_\rho : Rep_n^R(G) \rightarrow A$  which induces a homomorphism of groups

$$SL_n(h_\rho) : SL_n(Rep_n^R(G)) \rightarrow SL_n(A)$$

such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\ & \searrow \rho & \downarrow SL_n(h_\rho) \\ & & SL_n(A) \end{array}$$

This universal property uniquely determines  $Rep_n^R(G)$  up to an isomorphism of  $R$ -algebras.

The universal representation algebra of  $G$  may also be constructed explicitly in the following way. Let  $\langle g_i, i \in I | r_j, j \in J \rangle$  be a presentation of  $G$  such that all relations  $r_j$  are monomials in non-negative powers of generators,  $g_i$ . Such a presentation exists for every group  $G$ . Since we work with groups which are not necessarily finitely presented,  $I$  and  $J$  may be infinite. Let  $P_n(I)$  be the ring of polynomials over  $R$  in variables  $x_{jk}^i$ , where  $i \in I$  and  $j, k \in \{1, 2, \dots, n\}$ . Let  $A_i$ , for  $i \in I$ , be the matrix  $(x_{jk}^i) \in M_n(P_n(I))$ . For any word  $r_j = g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_k}^{n_k}$  consider the corresponding matrix  $M_j = A_{i_1}^{n_1} A_{i_2}^{n_2} \dots A_{i_k}^{n_k} \in M_n(P_n(I))$ . Let  $\mathcal{I}$  be the two-sided ideal in  $P_n(I)$  generated by polynomials  $Det(A_i) - 1$ , for  $i \in I$ , and by all entries of matrices  $M_j - Id$ , for  $j \in J$ , where  $Id$  is the identity matrix. We denote the quotient  $P_n(I)/\mathcal{I}$  by  $Rep_n^R(G)$  and the quotient map  $P_n(I) \rightarrow Rep_n^R(G)$  by  $\eta$ . Let  $\bar{x}_{jk}^i = \eta(x_{jk}^i)$  and  $\bar{A}_i = (\bar{x}_{jk}^i) \in M_n(Rep_n^R(G))$ .

Note that we divided  $P_n(I)$  by all relations necessary for the existence of a representation

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$$

such that  $j_{G,n}(g_i) = \bar{A}_i$ . The algebra  $Rep_n^R(G)$  and the representation  $j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$  are exactly the universal representation algebra and the universal  $SL_n$ -representation of  $G$ .

Let  $A \in GL_n(R)$ . By the definition of the universal representation ring,  $Rep_n^R(G)$ , there is a unique homomorphism  $f_A : Rep_n^R(G) \rightarrow Rep_n^R(G)$  such that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\
 A^{-1}j_{G,n}A \searrow & & \downarrow SL_n(f_A) \\
 & & SL_n(Rep_n^R(G))
 \end{array}$$

One easily observes that  $f_A$  is an automorphism of  $Rep_n^R(G)$  and that the assignment  $A \rightarrow f_A$  defines a left action of  $GL_n(R)$  on  $Rep_n^R(G)$ . We denote this action by  $A*$ , i.e.  $f_A(r) = A * r$ , for any  $r \in Rep_n^R(G)$  and  $A \in GL_n(R)$ . We call the ring  $Rep_n^R(G)^{GL_n(R)}$  consisting of elements of  $Rep_n^R(G)$  fixed by the action of  $GL_n(R)$  the *universal character ring of  $G$* . This term indicates a connection between the ring  $Rep_n^R(G)^{GL_n(R)}$  and  $SL_n$ -characters of  $G$ , i.e. traces of  $SL_n$ -representations of  $G$ . In the simplest case, when  $R$  is an algebraically closed field of characteristic 0, and  $G$  is finitely generated, this connection can be described as follows. Every representation  $\rho : G \rightarrow SL_n(R)$  induces a homomorphism  $h_\rho : Rep_n^R(G) \rightarrow R$ , whose restriction to  $Rep_n^R(G)^{GL_n(R)}$  we denote by  $h'_\rho$ . The next proposition follows from geometric invariant theory and from the results of [L-M].

**Proposition 2.1.** *Under the above assumptions, the following sets are in a natural correspondence given by bijections  $\rho \rightarrow h'_\rho$ , and  $\rho \rightarrow \chi = tr \circ \rho$  :*

- the set of all semisimple  $SL_n$ -representations of  $G$ ;
- the set of all  $R$ -homomorphisms  $Rep_n^R(G)^{GL_n(R)} \rightarrow R$ ;
- the set of all  $SL_n(R)$ -characters of  $G$ .

By the proposition, we can identify the above sets and denote them by  $X_n(G)$ . By the second definition,  $X_n(G)$  is the affine algebraic set composed of the closed points of the  $SL_n$ -character variety,  $\mathfrak{X}_n(G)$ , defined in the introduction. In other words,

$$R[X_n(G)] = R[\mathfrak{X}_n(G)]/\sqrt{0} = Rep_n^R(G)^{GL_n(R)}/\sqrt{0}.$$

As shown in [L-M, KM],  $Rep_n^R(G)^{GL_n(R)}$  may contain nilpotent elements and, therefore,  $\mathfrak{X}_n(G)$  contains more subtle information about  $SL_n$ -representations of  $G$  than  $X_n(G)$ . The ring  $Rep_n^R(G)^{GL_n(R)}$  will be given a topological description in Section 3.

The  $GL_n(R)$  action on  $Rep_n^R(G)$  induces an action of  $GL_n(R)$  on the ring of  $n \times n$  matrices over  $Rep_n^R(G)$ . If  $M = (m_{ij}) \in M_n(Rep_n^R(G))$  and  $A \in GL_n(R)$ , then

$$(2.1) \quad A * M = A \begin{pmatrix} A * m_{11} & A * m_{12} & \dots & A * m_{1n} \\ \vdots & \vdots & \dots & \vdots \\ A * m_{n1} & A * m_{n2} & \dots & A * m_{nn} \end{pmatrix} A^{-1}.$$

There is an equivalent definition of the action of  $GL_n(R)$  on  $Rep_n^R(G)$  and on  $M_n(Rep_n^R(G))$ . In order to introduce it we will first define  $GL_n(R)$ -actions on  $P_n(I)$  and  $M_n(P_n(I))$ . We can consider  $P_n(I)$  as a ring of polynomial functions defined

on the product of  $I$  copies of  $M_n(R)$ ,  $M_n(R)^I \rightarrow R$ , by identifying  $x_{jk}^{i_0} \in P_n(I)$  with a map assigning to  $(M_i)_{i \in I} \in M_n(R)^I$  the  $(j, k)$ -entry of  $M_{i_0}$ . Therefore,

$$P_n(I) = \text{Poly}(M_n(R)^I, R).$$

With this identification any entry in a matrix  $M$  in  $M_n(P_n(I))$  is a polynomial function on  $M_n(R)^I$ . Therefore we can think of elements of  $M_n(P_n(I))$  as coordinate-wise polynomial functions  $M_n(R)^I \rightarrow M_n(R)$ ,

$$M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)).$$

If  $X, Y$  are sets with a left  $G$ -action, then the set of all functions  $\text{Fun}(X, Y)$  has a natural left  $G$ -action defined for any  $f : X \rightarrow Y$  and  $g \in G$  by  $g * f(x) = gf(g^{-1}x)$ , for  $x \in X$ .  $GL_n$  acts on  $M_n(R)$  and on  $M_n(R)^I$  by conjugation, and it acts trivially on  $R$ . These actions induce  $GL_n(R)$ -actions on  $\text{Fun}(M_n(R)^I, R)$  and on  $\text{Fun}(M_n(R)^I, M_n(R))$ , which restrict to

$$P_n(I) = \text{Poly}(M_n(R)^I, R)$$

and

$$M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)).$$

The following statement is a consequence of the above definitions.

**Lemma 2.2.** (1) *The natural embedding of  $P_n(I)$  into  $M_n(P_n(I))$  as scalar matrices is  $GL_n(R)$ -equivariant.*

(2)  *$A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$  is invariant under the action of  $GL_n(R)$ , for any  $i_0 \in I$ .*

Now, we are going to show that  $\eta : P_n(I) \rightarrow \text{Rep}_n^R(G)$  is  $GL_n(R)$ -equivariant, and hence the action of  $GL_n(R)$  on  $P_n(I)$  induces a  $GL_n(R)$ -action on  $\text{Rep}_n^R(G)$  which coincides with the  $GL_n(R)$ -action on  $\text{Rep}_n^R(G)$  defined previously.

**Proposition 2.3.** *The following diagram commutes:*

$$(2.2) \quad \begin{array}{ccc} M_n(P_n(I)) & \xrightarrow{M_n(\eta)} & M_n(\text{Rep}_n^R(G)) \\ \downarrow \text{Tr} & & \downarrow \text{Tr} \\ P_n(I) & \xrightarrow{\eta} & \text{Rep}_n^R(G) \end{array}$$

and all maps appearing in it intertwine with the  $GL_n(R)$ -action.

*Proof.* Since the commutativity of the above diagram is obvious, we will prove only that the trace functions and homomorphisms  $\eta$ ,  $M_n(\eta)$  are  $GL_n(R)$ -equivariant.

- The trace map  $\text{Tr} : M_n(R) \rightarrow R$  is  $GL_n(R)$ -equivariant. Therefore the induced map

$$\text{Tr} : M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)) \rightarrow \text{Poly}(M_n(R)^I, R) = P_n(I)$$

is also  $GL_n(R)$ -equivariant.

- If  $M = (m_{ij}) \in M_n(\text{Rep}_n^R(G))$  and  $A \in GL_n(R)$ , then  $A * M$  is given by matrix (2.1), whose trace is  $\text{Tr}(A * M) = \sum_{i=1}^n A * m_{ii} = A * \text{Tr}(M)$ . Therefore

$$\text{Tr} : M_n(\text{Rep}_n^R(G)) \rightarrow \text{Rep}_n^R(G)$$

is  $GL_n(R)$ -equivariant.

- Recall that  $P_n(I)$  is generated by elements  $x_{jk}^i$ . Therefore in order to prove that  $\eta$  is  $GL_n(R)$ -equivariant it is enough to show that  $\eta(A * x_{jk}^i) = A * \bar{x}_{jk}^i$ , for any  $i \in I, j, k \in \{1, 2, \dots, n\}$ , where  $\bar{x}_{jk}^i = \eta(x_{jk}^i) \in \text{Rep}_n^R(G)$ .

For any  $i_0 \in I, j_{G,n}(g_{i_0}) = \bar{A}_{i_0} \in SL_n(\text{Rep}_n^R(G))$ . By the definition of the  $GL_n(R)$ -action on  $\text{Rep}_n^R(G)$ ,  $A^{-1}j_{G,n}(g_{i_0})A$  is the matrix obtained from  $j_{G,n}(g_{i_0})$  by acting on all its entries by  $A$ . Therefore

$$(2.3) \quad A * \bar{x}_{jk}^{i_0} = (j, k)\text{-entry of } A^{-1}\bar{A}_{i_0}A.$$

Having described  $A * \bar{x}_{jk}^{i_0}$ , we need to give an explicit description of  $A * x_{jk}^{i_0} \in P_n(I)$ . Recall that we identified  $x_{jk}^{i_0}$  with the map  $M_n(R)^I \rightarrow R$  assigning to  $\{M_i\}_{i \in I}$  the  $(j, k)$ -entry of  $M_{i_0}$ . The definition of the  $GL_n(R)$ -action on maps between  $GL_n(R)$ -sets implies that

$$(A * x_{jk}^{i_0})(\{M_i\}_{i \in I}) = A * \left( x_{jk}^{i_0}(A^{-1} * \{M_i\}_{i \in I}) \right).$$

Since  $GL_n(R)$  acts by simultaneous conjugation on  $M_n(R)^I$  and it acts trivially on  $R$ , the right side of the above equation is equal to the  $(j, k)$ -entry of  $A^{-1}M_{i_0}A$ . But the entries of  $M_{i_0}$  are given by the values of functions  $x_{jk}^{i_0}$  evaluated on  $\{M_i\}_{i \in I}$ . Therefore

$$(2.4) \quad A * x_{jk}^{i_0} = (j, k)\text{-entry of } A^{-1}(x_{jk}^{i_0})A.$$

Finally, (2.3) and (2.4) imply that

$$\begin{aligned} \eta(A * x_{jk}^{i_0}) &= \eta(\text{the } (j, k)\text{-entry of } A^{-1}A_{i_0}A) \\ &= \text{the } (j, k)\text{-entry of } A^{-1}\bar{A}_{i_0}A = A * \bar{x}_{jk}^{i_0}. \end{aligned}$$

- We prove that  $M_n(\eta)$  is equivariant. Let  $M = (m_{jk}) \in M_n(P_n(I))$ . Notice that the definition of  $GL_n(R)$ -action on  $M_n(P_n(I))$  implies that  $A * M = A(A * m_{jk})A^{-1}$ . Therefore

$$M_n(\eta)(A * M) = M_n(\eta)(A(A * m_{jk})A^{-1}) = A(\eta(A * m_{jk}))A^{-1}.$$

Since  $\eta$  is  $GL_n(R)$ -equivariant, the matrix on the right side of the above equation is  $A(A * \eta(m_{jk}))A^{-1} = A * (\eta(m_{jk}))$ . Therefore  $M_n(\eta)$  is also  $GL_n(R)$ -equivariant.  $\square$

The above proposition implies that there exists a function

$$\text{Tr} : M_n(\text{Rep}_n^R(G))^{GL_n(R)} \rightarrow \text{Rep}_n^R(G)^{GL_n(R)}.$$

This function will be given a simple topological interpretation in the next section.

**Proposition 2.4.** *The image of the universal representation*

$$j_{G,n} : G \rightarrow M_n(\text{Rep}_n^R(G))$$

*is invariant under the action of  $GL_n(R)$ .*

*Proof.* Since the elements  $g_i$  generate  $G$ , it is sufficient to show that  $j_{G,n}(g_i) \in M_n(\text{Rep}_n^R(G))^{GL_n(R)}$ . By Lemma 2.2(2),  $A_i \in M_n(P_n(I))^{GL_n(R)}$ . The map  $M_n(\eta)$  is equivariant. Therefore it takes the invariant  $A_i$  to the invariant  $M_n(\eta)(A_i) = j_{G,n}(g_i)$ .  $\square$

3. SKEIN ALGEBRAS

In this section we assign to each path-connected topological space  $X$  a commutative  $R$ -algebra  $\mathbb{A}_n(X)$  and to each pointed path-connected topological space  $(X, x_0)$  an  $R$ -algebra  $\mathbb{A}_n(X, x_0)$ . These algebras encode the most important information about the  $SL_n$ -representations of  $\pi_1(X, x_0)$ . We will show that if  $R$  is a field of characteristic 0 (but not necessarily algebraically closed), then  $\mathbb{A}_n(X)$  is isomorphic to the universal character ring  $Rep_n^R(G)^{GL_n(R)}$ , where  $G = \pi_1(X, x_0)$ , and  $\mathbb{A}_n(X, x_0)$  is isomorphic to  $M_n(Rep_n^R(G))^{GL_n(R)}$ .

We start with a definition of a graph which is the most suitable for our purposes. A graph  $D = (\mathcal{V}, \mathcal{E}, \mathcal{L})$  consists of a vertex-set  $\mathcal{V}$ , a set of oriented edges  $\mathcal{E}$ , and a set of oriented loops  $\mathcal{L}$ . Each edge  $E \in \mathcal{E}$  has a beginning  $b(E) \in \mathcal{V}$  and an end  $e(E) \in \mathcal{V}$ . Loops have neither beginnings nor ends. If  $b(E) = v$  or  $e(E) = v$ , then  $E$  is *incident* with  $v$ . For any  $v \in \mathcal{V}$ , all edges incident to  $v$  are ordered by consecutive integers  $1, 2, \dots$ . Therefore the beginning and the end of each edge is assigned a number.

The sets  $\mathcal{V}, \mathcal{E}, \mathcal{L}$  are finite. We topologize each graph as a CW-complex. The topology of a graph coincides with the topology of its edges  $E \simeq [0, 1], E \in \mathcal{E}$ , and its loops  $L \simeq S^1, L \in \mathcal{L}$ . There is a natural notion of isomorphism of graphs.

Let  $\mathcal{G}$  be a set of representatives of all isomorphism classes of graphs defined above. We say that a vertex  $v$  is an  $n$ -valent source of a graph  $D$  if  $n$  distinct edges of  $D$  begin at  $v$  and no edge ends at  $v$ . Similarly, we say that  $v$  is an  $n$ -valent sink of  $D$  if  $n$  distinct edges end at  $v$  and no edge begins at  $v$ . Let  $\mathcal{G}_n$  denote the set of all graphs in  $\mathcal{G}$ , all of whose vertices are either  $n$ -valent sources or  $n$ -valent sinks. We assume that the empty graph  $\emptyset$  is also an element of  $\mathcal{G}_n$ . We denote the single loop in  $\mathcal{G}_n$ , i.e. the connected graph without any vertices, by  $S^1$ . Let  $\mathcal{G}'_n$  denote the set of all graphs  $D \in \mathcal{G}$  such that  $D$  has one 1-valent source and one 1-valent sink, and all other vertices of  $D$  are  $n$ -valent sources or  $n$ -valent sinks. We denote the single edge in  $\mathcal{G}'_n$ , i.e. the connected graph without any  $n$ -valent vertices, by  $[0, 1]$ .

Let  $X$  be a path-connected topological space. We will call any continuous map  $f : D \rightarrow X$ , where  $D \in \mathcal{G}_n$ , a graph in  $X$ . We identify two maps  $f_1, f_2 : D \rightarrow X$  if they are homotopic. Let us denote the set of all graphs in  $X$  by  $\mathcal{G}_n(X)$ . Similarly, we define  $\mathcal{G}_n(X, x_0)$  to be the set of all maps  $f : D \rightarrow X \times [0, 1]$ , where  $D \in \mathcal{G}'_n$  and  $f$  maps the 1-valent sink of  $D$  to  $(x_0, 1)$  and the 1-valent source of  $D$  to  $(x_0, 0)$ . We identify maps which are homotopic relative to  $(x_0, 0)$  and  $(x_0, 1)$ . We will call elements of  $\mathcal{G}_n(X, x_0)$  *relative graphs in  $X \times [0, 1]$* .

We introduce a few classes of graphs in  $\mathcal{G}_n(X)$  and  $\mathcal{G}_n(X, x_0)$  which will often be used later on in the paper. Let  $L_\gamma : S^1 \rightarrow X$  be a graph in  $X$  which represents the conjugacy class of  $\gamma \in \pi_1(X, x_0)$ . We denote by  $E_\gamma$  a relative graph  $E_\gamma : [0, 1] \rightarrow X \times [0, 1]$ ,  $E_\gamma(0) = (x_0, 0)$ ,  $E_\gamma(1) = (x_0, 1)$ , whose projection into  $X$ ,

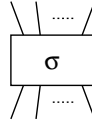
$$[0, 1] \xrightarrow{E_\gamma} X \times [0, 1] \rightarrow X,$$

represents  $\gamma \in \pi_1(X, x_0)$ . Let  $EL_\gamma : [0, 1] \cup S^1 \rightarrow X \times [0, 1]$  be a relative graph such that  $EL_\gamma(t) = (x_0, t)$  for  $t \in [0, 1]$ , and  $EL_\gamma|_{S^1} : S^1 \rightarrow X \times [0, 1] \rightarrow X$  represents the conjugacy class of  $\gamma \in \pi_1(X, x_0)$ .

For any two graphs  $f_1 : D_1 \rightarrow X$  and  $f_2 : D_2 \rightarrow X$ ,  $f_1, f_2 \in \mathcal{G}_n(X)$ , we define a product of them to be  $f_1 \cup f_2 : D_1 \cup D_2 \rightarrow X$ ,  $f_1 \cup f_2 \in \mathcal{G}_n(X)$ , where  $D_1 \cup D_2$  denotes the disjoint union of  $D_1$  and  $D_2$ . Therefore the free  $R$ -module  $R\mathcal{G}_n(X)$  on

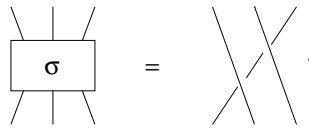
$\mathcal{G}_n(X)$  can be considered as a commutative  $R$ -algebra. The empty graph  $\emptyset : \emptyset \rightarrow X$  is an identity in  $R\mathcal{G}_n(X)$ .

In the next definition we will represent fragments of diagrams by coupons, as depicted below:

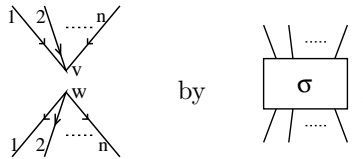


This coupon means a braid corresponding to a permutation  $\sigma \in S_n$ .<sup>1</sup>

**Example 3.1.** If  $\sigma = (1, 2, 3) \in S_3$ , then



Suppose that  $f : D \rightarrow X$  is a graph in  $X$  and  $f$  maps a source  $w$  and a sink  $v$  of  $D$  to the same point  $x_1 \in X$ . Let  $D_\sigma$  be a graph obtained from  $D$  by replacing



In the diagram we display  $v$  and  $w$  as separate points to accentuate the fact that they are distinct in the domain of the mapping  $f$ . There is an obvious way to modify  $f : D \rightarrow X$  to a map  $f_\sigma : D_\sigma \rightarrow X$ . We call  $(f, \{f_\sigma\}_{\sigma \in S_n})$  a family of *skein related graphs at  $x_1$* .

**Definition 3.2.** Let  $X$  be a path-connected topological space, and let  $I$  be the ideal in  $R\mathcal{G}_n(X)$  generated by two kinds of expressions:

- (1)  $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$ , where  $\epsilon(\sigma)$  denotes the sign of  $\sigma$  and  $(f, \{f_\sigma\}_{\sigma \in S_n})$  is a family of skein related graphs at some point  $x_1 \in X$ .
- (2)  $L_e - n$ , where  $e$  is the identity element in  $\pi_1(X, x_0)$  (i.e.  $L_e$  is a homotopically trivial loop).

Then the  $R$ -algebra  $\mathbb{A}_n(X) = R\mathcal{G}_n(X)/I$  is called the  $n$ -th skein algebra of  $X$ .

Similarly we define  $\mathbb{A}_n(X, x_0)$ . Let  $f_1 : D_1 \rightarrow X \times [0, 1], f_2 : D_2 \rightarrow X \times [0, 1]$  be elements of  $\mathcal{G}_n(X, x_0)$ . We define the product of them to be a map  $f_1 \cdot f_2 : D_1 \cup D_2 \rightarrow X \times [0, 1]$ , such that

$$(f_1 \cdot f_2)(d) = \begin{cases} (x, \frac{1}{2}t) & \text{if } d \in D_1 \text{ and } f_1(d) = (x, t), \\ (x, \frac{1}{2}t + \frac{1}{2}) & \text{if } d \in D_2 \text{ and } f_2(d) = (x, t). \end{cases}$$

This product extends to an associative (but generally non-commutative) product in  $R\mathcal{G}_n(X, x_0)$ . The identity in  $R\mathcal{G}_n(X, x_0)$  is a map  $f : E \rightarrow X \times [0, 1]$ , where  $E$  is a single edge and  $f$  maps  $E$  onto  $\{x_0\} \times [0, 1]$ .

<sup>1</sup> Since we consider graphs up to homotopy equivalence, it does not matter which braid corresponding to  $\sigma$  we take.



If  $f : D \rightarrow X \times [0, 1]$  is an element of  $\mathcal{G}_n(X, x_0)$  such that an  $n$ -valent source  $w$  and an  $n$ -valent sink  $v$  of  $D$  are mapped to a point  $x_1 \in X \times [0, 1]$ , then one can define  $D_\sigma$  and  $f_\sigma : D_\sigma \rightarrow X \times [0, 1]$ ,  $f_\sigma \in \mathcal{G}_n(X, x_0)$ , in exactly the same way as it was done for graphs in  $\mathcal{G}_n(X)$  in the paragraph preceding Definition 3.2. We say, as before, that  $(f, \{f_\sigma\}_{\sigma \in S_n})$  are graphs skein related at  $x_1$ .

**Definition 3.3.** Let  $X$  be a path-connected topological space with a specified point  $x_0 \in X$ , and let  $I'$  be the ideal in  $R\mathcal{G}_n(X, x_0)$  generated by expressions

- (1)  $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$ , where  $(f, \{f_\sigma\}_{\sigma \in S_n})$  is a family of skein related graphs at some point  $x_1 \in X \times [0, 1]$ .
- (2)  $EL_e - n$ , where  $e$  is the identity in  $\pi_1(X, x_0)$ .

Then the  $R$ -algebra  $\mathbb{A}_n(X, x_0) = R\mathcal{G}_n(X, x_0)/I'$  is called the  $n$ -th relative skein algebra of  $(X, x_0)$ .

Note that different choices of  $x_0 \in X$  give isomorphic algebras  $\mathbb{A}_n(X, x_0)$ .

Let  $f \in \mathcal{G}_n(X)$ ,  $f : D \rightarrow X$ . Let  $D' = D \cup E$ , where  $E$  is an edge disjoint from  $D$ . Then  $D'$  has one 1-valent sink  $e_0$  and one 1-valent source  $e_1$ ,  $\{e_0, e_1\} = \partial E$ , and  $D' \in \mathcal{G}'_n$ . We extend  $f$  to  $f' : D' \rightarrow X \times [0, 1]$  in such a way that  $f'(d) = (f(d), \frac{1}{2})$  for  $d \in D$ , and  $f'(t) = (x_0, t)$  for  $t \in [0, 1] \simeq E$ , where  $[0, 1] \simeq E$  is an orientation-preserving parameterization of  $E$ . This operation defines an embedding  $\iota : \mathcal{G}_n(X) \rightarrow \mathcal{G}_n(X, x_0)$ ,  $\iota(f) = f'$ . Notice that  $\iota$  induces a homomorphism  $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$ . Therefore we can consider  $\mathbb{A}_n(X, x_0)$  as an  $\mathbb{A}_n(X)$ -algebra.

Let  $f : D \rightarrow X \times [0, 1]$  be a map,  $f \in \mathcal{G}_n(X, x_0)$ . Let  $\overline{D} \in \mathcal{G}_n$  be a graph obtained by identification of the 1-valent sink with the 1-valent source in  $D$ . Let us compose  $f : D \rightarrow X \times [0, 1]$  with a projection  $X \times [0, 1] \rightarrow X$ . This composition gives a map  $\overline{f} : \overline{D} \rightarrow X$ ,  $\overline{f} \in \mathcal{G}_n(X)$ . Therefore, we have a function  $\overline{\cdot} : \mathcal{G}_n(X, x_0) \rightarrow \mathcal{G}_n(X)$ . This function can be extended to an  $R$ -linear homomorphism  $\mathbb{T} : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(X)$ . Notice that for any graph  $D \in \mathcal{G}_n(X)$ ,  $\mathbb{T}(\iota_*(D))$  is equal to a union of  $f : D \rightarrow X$  with a contractible loop in  $X$ . Hence by Definition 3.2(2)  $\mathbb{T}(\iota_*(D)) = n \cdot D$ . Since graphs in  $X$  span  $\mathbb{A}_n(X)$ , the composition of  $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$  with  $\mathbb{T} : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(X)$  is equal to  $n$  times the identity on  $\mathbb{A}_n(X)$ . This implies the following fact.

**Fact 3.4.** *If  $\frac{1}{n} \in R$ , then  $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$  is a monomorphism of rings.*

The next proposition summarizes basic properties of  $\mathbb{A}_n(X)$  and  $\mathbb{A}_n(X, x_0)$ .

**Proposition 3.5.** (1) *The assignment  $X \rightarrow \mathbb{A}_n(X)$  (respectively,  $(X, x_0) \rightarrow \mathbb{A}_n(X, x_0)$ ) defines a functor from the category of path-connected topological spaces (respectively, category of path-connected pointed spaces) to the category of commutative  $R$ -algebras (respectively, the category of  $R$ -algebras).*

- (2) *If  $f : X \rightarrow Y, f(x_0) = y_0$ , induces a surjection  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ , then the corresponding homomorphisms  $\mathbb{A}_n(f) : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(Y, y_0)$  and  $\mathbb{A}_n(f) : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(Y)$  are epimorphisms of  $R$ -algebras.*
- (3) *The algebra  $\mathbb{A}_n(X)$  is generated by loops in  $X$ , i.e. by graphs  $L_\gamma$ , for  $\gamma \in \pi_1(X, x_0)$ .*
- (4) *The algebra  $\mathbb{A}_n(X, x_0)$  is generated by graphs  $E_{g_i^{\pm 1}}$  and  $EL_\gamma$ , where  $\{g_i\}_{i \in I}$  is a set of generators of  $\pi_1(X, x_0)$  and  $\gamma \in \pi_1(X, x_0)$ .*

*Proof.* Since statements (1) and (2) of Proposition 3.5 are obvious, we give a proof of (3) and (4) only.

From the definition of a graph  $D \in \mathcal{G}_n$  or  $D \in \mathcal{G}'_n$  we see that it has an equal number of  $n$ -valent sinks and sources. Relation (1) of Definition 3.2 and of Definition 3.3 implies that each pair of vertices of  $f : D \rightarrow X$ ,  $f \in \mathcal{G}_n(X)$  (respectively, of  $f : D \rightarrow X \times [0, 1]$ ,  $f \in \mathcal{G}_n(X, x_0)$ ) composed of a sink and a source can be resolved and  $f$  can be replaced by a linear combination of graphs with a smaller number of sinks and sources. Therefore, after a finite number of steps each graph in  $\mathcal{G}_n(X)$  (respectively,  $\mathcal{G}_n(X, x_0)$ ) can be expressed as a linear combination of graphs without  $n$ -valent vertices.

(3) If  $f : D \rightarrow X$ ,  $f \in \mathcal{G}_n(X)$ , and  $D$  has no vertices, then  $D$  is a union of loops,  $D = S^1 \cup S^1 \cup \dots \cup S^1$ , and therefore  $f = L_{\gamma_1} \cdot L_{\gamma_2} \cdot \dots \cdot L_{\gamma_k} \in \mathbb{A}_n(X)$ , for some  $\gamma_1, \gamma_2, \dots, \gamma_k \in \pi_1(X, x_0)$ .

(4) If  $f : D \rightarrow X \times [0, 1]$ ,  $f \in \mathcal{G}_n(X, x_0)$ , and  $D$  has no  $n$ -valent vertices, then  $D = [0, 1] \cup S^1 \cup S^1 \cup \dots \cup S^1$ . Suppose that  $[0, 1] \xrightarrow{f} X \times [0, 1] \rightarrow X$  represents  $\gamma_0 \in \pi_1(X, x_0)$ , and  $f$  restricted to the  $j$ -th circle represents the conjugacy class of a  $\gamma_j \in \pi_1(X, x_0)$ ,  $j = 1, 2, \dots, k$ . Then  $f = E_{\gamma_0} \cdot EL_{\gamma_1} \cdot EL_{\gamma_2} \cdot \dots \cdot EL_{\gamma_k} \in \mathbb{A}_n(X, x_0)$ . Therefore  $\mathbb{A}_n(X, x_0)$  is generated by the elements  $EL_{\gamma}$  and  $E_{\gamma'}$ ,  $\gamma, \gamma' \in \pi_1(X, x_0)$ . But each  $E_{\gamma'}$  is a product of elements  $E_{g_i^{\pm 1}}$ , where  $\{g_i\}$  is a set of generators of  $\pi_1(X, x_0)$ . □

We will see later that  $\mathbb{A}_n(X)$  and  $\mathbb{A}_n(X, x_0)$  depend only on  $\pi_1(X, x_0)$ . Moreover, if  $\pi_1(X, x_0)$  is a finitely generated group, then the algebras  $\mathbb{A}_n(X)$  and  $\mathbb{A}_n(X, x_0)$  are also finitely generated.

Now we are ready to formulate the most important results of this paper.

**Theorem 3.6.** *Let  $X$  be any (path-connected) topological space, and let  $G = \pi_1(X, x_0), x_0 \in X$ . There are  $R$ -algebra homomorphisms*

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\text{Rep}_n^R(G))^{GL_n(R)}, \quad \theta : \mathbb{A}_n(X) \rightarrow \text{Rep}_n^R(G)^{GL_n(R)},$$

uniquely determined by the following conditions:

- (1)  $\Theta(E_{\gamma}) = j_{G,n}(\gamma)$  and  $\Theta(EL_{\gamma'}) = \text{Tr}(j_{G,n}(\gamma'))$ , for any  $\gamma, \gamma' \in \pi_1(X, x_0)$ .
- (2)  $\theta(L_{\gamma}) = \text{Tr}(j_{G,n}(\gamma))$ , for any  $\gamma \in \pi_1(X, x_0)$ .

Moreover, the following diagram commutes:

$$(3.1) \quad \begin{array}{ccc} \mathbb{A}_n(X, x_0) & \xrightarrow{\Theta} & M_n(\text{Rep}_n^R(G))^{GL_n(R)} \\ \downarrow \mathbb{T} & & \downarrow \text{Tr} \\ \mathbb{A}_n(X) & \xrightarrow{\theta} & \text{Rep}_n^R(G)^{GL_n(R)} \end{array}$$

**Theorem 3.7.** *If  $R$  is a field of characteristic 0, then*

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\text{Rep}_n^R(G))^{GL_n(R)}, \quad \theta : \mathbb{A}_n(X) \rightarrow \text{Rep}_n^R(G)^{GL_n(R)}$$

are isomorphisms of  $R$ -algebras.

Let  $R$  be a field of characteristic 0. It can be shown that if  $X$  is a 3-manifold, then  $\mathbb{A}_2(X)$  is isomorphic to the Kauffman bracket skein module of  $X$ ,  $\mathcal{S}_{2,\infty}(X, R, \pm 1)$ . Moreover, if  $X$  is a surface, then  $\mathbb{A}_2(X, x_0)$  is isomorphic to the relative Kauffman bracket skein module of  $X$ ,  $\mathcal{S}_{2,\infty}^{rel}(X, R, \pm 1)$ . See [PS-2], [H-P], for appropriate definitions and the notational conventions. The main results of [B-1], [B-2], [PS-1]

and [PS-2] relate the Kauffman bracket skein modules of 3-manifolds with the  $SL_2$ -representation theory of their fundamental groups. Theorem 3.7 generalizes these results to groups  $SL_n$ , for any  $n$ .

Moreover, it can be shown that in the case when  $X$  is any path-connected topological space,  $\mathbb{A}_2(X, x_0)$  and  $\mathbb{A}_2(X)$  can be given the following simple algebraic description: Let  $G = \pi_1(X)$  and let  $I$  be the ideal in the group ring  $RG$  generated by elements  $h(g + g^{-1}) - (g + g^{-1})h$ , where  $g, h \in G$ . There is an involution  $\tau$  on  $H(G) = RG/I$  sending  $g$  to  $g^{-1}$ . One can show that  $\mathbb{A}_2(X, x_0)$  is isomorphic to  $H(G)$  and  $\mathbb{A}_2(X)$  is isomorphic to  $H^+(G)$ , where  $H^+(G)$  is the subring of  $H(G)$  invariant under  $\tau$ . The algebras  $H(G), H^+(G)$  are introduced and thoroughly investigated in [B-H]. One of the main results of [B-H] is that  $H(G) = M_n(\text{Rep}_n^R(G))^{GL_n(R)}$  and  $H^+(G) = \text{Rep}_n^R(G)^{GL_n(R)}$ , for  $n = 2$ . (Compare also [Sa-1], [Sa-2].) Theorem 3.7 can be considered as a generalization of this result to all values of  $n$ .

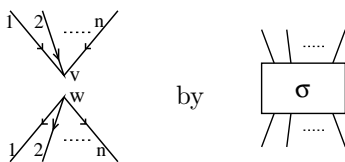
#### 4. PROOF OF THEOREM 3.6

Before we prove Theorem 3.6 we give new definitions of  $\mathbb{A}_n(X)$  and  $\mathbb{A}_n(X, x_0)$  which only use  $G = \pi_1(X, x_0)$ .

Let  $X$  be a path-connected topological space and  $x_0 \in X$ . For any graph in  $\mathcal{G}_n(X)$ , i.e. a map  $f : D \rightarrow X$  for some  $D \in \mathcal{G}_n$ , there is a map  $f' : D \rightarrow X$  homotopic to  $f$ , which maps all vertices of  $D$  to  $x_0$ . Therefore the homotopy class of  $f$  can be described by the graph  $D$  with each edge  $E$  labeled by an element of  $\pi_1(X, x_0)$  corresponding to the map  $f'_E : E \rightarrow X$  and each loop  $L$  labeled by the conjugacy class in  $\pi_1(X, x_0)$  corresponding to the map  $f'_{|L} : L \simeq S^1 \xrightarrow{f} X$ . This description does not need to be unique.

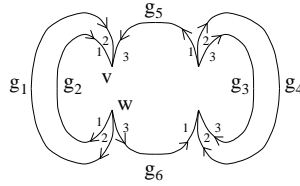
We denote the set of graphs in  $\mathcal{G}_n$  all of whose edges are labeled by elements of  $G$  and all loops are labeled by conjugacy classes in  $G$  by  $\mathcal{G}_n(G)$ . There is a natural multiplication operation on  $\mathcal{G}_n(G)$ . The product of  $D_1, D_2 \in \mathcal{G}_n(G)$  is the disjoint union of  $D_1$  and  $D_2$ . Therefore  $R\mathcal{G}_n(G)$  is a commutative  $R$ -algebra with  $\emptyset$  as the identity.

Let  $D$  be a graph in  $\mathcal{G}_n(G)$ . We have noticed already that  $D$  corresponds to a map  $f : D \rightarrow X$  which maps all vertices of  $D$  to  $x_0$  and restricted to edges and loops of  $D$  agrees with their labeling. Such  $f$  is unique up to a homotopy which fixes the vertices of  $D$ . Let  $w$  be a source and  $v$  be a sink in  $D$ . Since  $f$  maps  $v$  and  $w$  to the same point in  $X$ , there exists a map  $f_\sigma : D_\sigma \rightarrow X$  defined for any  $\sigma \in S_n$  as in the paragraph preceding Definition 3.2. Notice that  $f_\sigma$  maps all vertices of  $D_\sigma$  to  $x_0 \in X$ . Therefore, we can label all edges of  $D_\sigma$  by appropriate elements of  $G$  and all loops of  $D_\sigma$  by appropriate conjugacy classes in  $G$ , and hence consider  $D_\sigma$  as an element of  $\mathcal{G}_n(G)$ . Hence, we have showed that one can replace any source  $w$  and any sink  $v$  in an arbitrary graph  $D \in \mathcal{G}_n(G)$

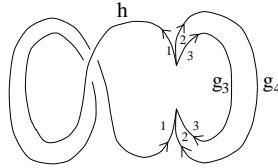


and obtain a well-defined graph  $D_\sigma \in \mathcal{G}_n(G)$ .

As an example consider the graph  $D$  presented below:



Replacing the vertices  $v, w$  by a coupon decorated by  $\sigma = (123) \in S_3$  gives a diagram  $D_\sigma$ :



where  $h = g_6 g_1 g_2 g_5$ .

Now we are ready to define  $\mathbb{A}_n(X)$  in terms of graphs in  $\mathcal{G}_n(G)$ . Namely, this algebra is isomorphic to  $R\mathcal{G}_n(\pi_1(X, x_0))/I$ , where  $I \triangleleft R\mathcal{G}_n(\pi_1(X, x_0))$  is an ideal generated by relations analogous to relations (1) and (2) of Definition 3.2 and by relations following from the fact that the assignment  $\mathcal{G}_n(G) \rightarrow \mathcal{G}_n(X)$  described above is onto but not 1-1. The problem comes from the fact that one can take a graph  $D \in \mathcal{G}_n$  whose edges and loops are labeled in two different ways such that the corresponding maps  $f, f' : D \rightarrow X$  sending the vertices of  $D$  to  $x_0$  are homotopic but not by a homotopy relative to the vertices of  $D$ . In order to resolve this problem we need to allow an operation which moves vertices of  $D$  around paths in  $X$  beginning and ending at  $x_0$ . Notice however that it suffices to move one vertex at a time. The following fact summarizes our observations.

**Fact 4.1.** *Let  $X$  be a path-connected topological space with a specified point  $x_0 \in X$ , and let  $G = \pi_1(X, x_0)$ . Let  $I$  be the ideal in  $R\mathcal{G}_n(G)$  generated by expressions of the following form:*

$$(4.1) \quad \begin{array}{c} i \quad 2 \quad \dots \quad n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ 1 \quad 2 \quad \dots \quad n \end{array} - \sum_{\sigma \in S_n} \epsilon(\sigma) \begin{array}{c} \dots \\ \swarrow \quad \downarrow \quad \searrow \\ \sigma \\ \swarrow \quad \downarrow \quad \searrow \\ \dots \end{array},$$

$$(4.2) \quad \begin{array}{c} \circlearrowleft \\ e \end{array} - n,$$

$$(4.3) \quad \begin{array}{c} hg_1 \quad hg_2 \quad \dots \quad hg_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array} - \begin{array}{c} g_1 \quad g_2 \quad \dots \quad g_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array},$$

$$(4.4) \quad \begin{array}{c} g_1 h \quad g_2 h \quad \dots \quad g_n h \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array} - \begin{array}{c} g_1 \quad g_2 \quad \dots \quad g_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 h \quad g_2 h \quad \dots \quad g_n h \end{array},$$

for any  $g_1, g_2, \dots, g_n, h \in G$ .

Then there is an isomorphism between the  $R$ -algebras  $\mathbb{A}_n(X)$  and  $RG_n(G)/I$  assigning to each graph  $f : D \rightarrow X, f \in \mathcal{G}_n(X)$ , with all vertices at  $x_0$  the graph  $D$  with every edge  $E$  of  $D$  decorated by the element of  $\pi_1(X, x_0)$  corresponding to  $f|_E : E \rightarrow X$ , and every loop  $L$  of  $D$  decorated by the conjugacy class in  $\pi_1(X, x_0)$  represented by  $f|_L : L \rightarrow X$ .

Now we will state a similar fact for  $\mathbb{A}_n(X, x_0)$ . Let  $\mathcal{G}'_n(G)$  be a set of graphs in  $\mathcal{G}'_n$  all of whose edges are labeled by elements of  $G$  and all of whose loops are labeled by conjugacy classes in  $G$ .

There is a multiplication operation defined on  $\mathcal{G}'_n(G)$  in the following way. Let  $D_1, D_2 \in \mathcal{G}'_n(G)$ , let  $v_i$  be the 1-valent source of  $D_i, i \in \{1, 2\}$ , and let  $w_i$  be the 1-valent sink of  $D_i$ . Let  $g_i$  be the label of the edge of  $D_i$  joining  $v_i$  with  $w_i$ . The graph  $D_1 \cdot D_2$  is obtained from the disjoint union of  $D_1$  and  $D_2$  by identifying  $v_1$  with  $w_2$ . The edge of  $D_1 \cdot D_2$  joining  $v_2$  with  $w_1$  is labeled by  $g_1 \cdot g_2$ . All other edges and loops of  $D_1 \cdot D_2$  inherit labels from  $D_1$  and  $D_2$ . A single edge labeled by  $e \in G$  is the identity in  $\mathcal{G}'_n(G)$ .

This multiplication extends to an associative (but not commutative) multiplication in  $RG'_n(G)$ .

**Fact 4.2.** *Let  $X$  be a path-connected topological space with a specified point  $x_0 \in X$ , and let  $G = \pi_1(X, x_0)$ . Let  $I'$  be the ideal in  $RG'_n(G)$  generated by the expressions (4.1),(4.3), (4.4) and*

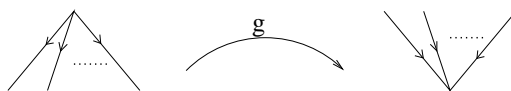
$$\left. \begin{array}{c} \downarrow e \\ \text{---} \circlearrowleft \text{---} \\ \uparrow e \end{array} \right|_{-n} e .$$

Then  $\mathbb{A}_n(X, x_0) \simeq RG'_n(G)/I'$ .

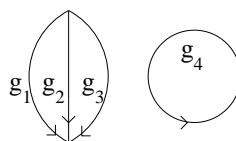
Facts 4.1 and 4.2 show that the algebras  $\mathbb{A}_n(X, x_0)$  and  $\mathbb{A}_n(X)$  depend only on  $\pi_1(X, x_0)$ . In fact 4.1 and 4.2 give us models for  $\mathbb{A}_n(X, x_0)$  and  $\mathbb{A}_n(X)$  built from  $\mathcal{G}_n(G)$  and  $\mathcal{G}'_n(G)$ . In the rest of this section we will use these models.

Let us fix a commutative ring  $R$  and a positive integer  $n$  and a topological space  $X$  with  $x_0 \in X, \pi_1(X, x_0) = G$ . Let  $\mathcal{R} = Rep_n^R(G)$  and let  $V = \mathcal{R}^n$  be a free  $n$ -dimensional module over  $\mathcal{R}$  with the standard basis,  $\{e_1, e_2, \dots, e_n\}, e_i = (0, 0, \dots, 1, \dots, 0)$ . The dual space  $V^*$  has the dual basis  $e^1, e^2, \dots, e^n, e^i(e_j) = \delta_{i,j}$ . We will always use the standard bases and therefore identify  $V^* \otimes V \simeq End_{\mathcal{R}}(V) \simeq M_n(\mathcal{R})$ .

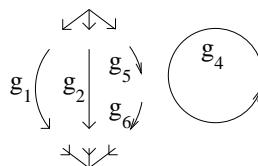
Let  $D$  be an element of  $\mathcal{G}_n(G)$  or  $\mathcal{G}'_n(G)$ . We can decompose  $D$  into arcs, sources and sinks:



**Example 4.3.**



can be decomposed to



where  $g_1, g_2, g_3, g_4, g_5, g_6 \in G$ ,  $g_3 = g_6g_5$ .

Notice that the decomposition of a graph is not unique, since we can cut each edge or loop into many pieces.

Let us assign to each  $n$ -valent source the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \dots \otimes e_{\sigma(n)} \in V^{\otimes n},$$

and to each  $n$ -valent sink the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e^{\sigma(1)} \otimes e^{\sigma(2)} \otimes \dots \otimes e^{\sigma(n)} \in (V^*)^{\otimes n}.$$

To each edge labeled by  $g$  we assign a tensor in  $V^* \otimes V \simeq \text{End}_{\mathcal{R}}(V)$  corresponding to  $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset M_n(\mathcal{R})$ .

Let  $D_0$  denote a graph  $D$  decomposed into pieces. We assign to  $D_0$  the tensor product of tensors corresponding to them. We denote this tensor by  $T(D_0)$ . Notice that  $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$ , where  $N =$  the number of 1-valent sources in  $D_0 =$  the number of 1-valent sinks in  $D_0$ .

Now we glue all components of  $D_0$  together to get the graph  $D$  back. Whenever we glue an end of one piece to a beginning of another piece in  $D_0$ , we make the corresponding contraction on  $T(D_0)$ . More specifically, suppose that the two free ends glued together correspond to the two underlined components:

$$T(D_0) \in V \otimes \dots \otimes \underline{V} \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes \underline{V^*} \otimes \dots \otimes V^*.$$

By applying to this tensor space the contraction map  $\underline{V} \otimes \underline{V^*} \rightarrow \mathcal{R}$  (which is the evaluation map  $(v, f) \rightarrow f(v)$ ), we send  $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$  to an element of  $V^{\otimes N-1} \otimes (V^*)^{\otimes N-1}$ . By repeating this process until we get the graph  $D$  back, we obtain an element of  $\mathcal{R}$ , if  $D \in \mathcal{G}_n(G)$ , or an element of  $M_n(\mathcal{R})$ , if  $D \in \mathcal{G}'_n(G)$ . Notice that the above construction does not depend on the particular decomposition of  $D$  into pieces. Therefore, we have defined functions

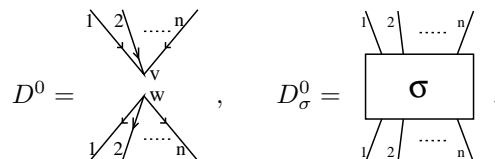
$$(4.5) \quad \Theta : \mathcal{G}'_n(G) \rightarrow M_n(\mathcal{R}), \quad \theta : \mathcal{G}_n(G) \rightarrow \mathcal{R}.$$

**Lemma 4.4.** *Let  $D$  be a graph in  $\mathcal{G}_n(G)$  or in  $\mathcal{G}'_n(G)$ . Let  $w$  be an  $n$ -valent source of  $D$  and  $v$  an  $n$ -valent sink of  $D$ . Let  $D_\sigma$ , for  $\sigma \in S_n$ , be defined as at the beginning of Section 4. If  $D \in \mathcal{G}_n(G)$ , then  $\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma) \theta(D_\sigma)$ . If  $D \in \mathcal{G}'_n(G)$ , then  $\Theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma) \Theta(D_\sigma)$ .*

*Proof.* We will prove Lemma 4.4 only for  $D \in \mathcal{G}_n(G)$ . For  $D \in \mathcal{G}'_n(G)$  the proof is identical.

Decompose  $D$  and  $D_\sigma$  into sources, sinks, and edges. We denote the fragment of the decomposition of  $D$  composed of the source  $w$  and the sink  $v$  by  $D^0$ . We

may assume that the decomposition of  $D_\sigma$  is identical to that of  $D$ , except that it contains a coupon  $D_\sigma^0$  instead of  $D^0$ :



We order the 1-valent sources and sinks of  $D_\sigma^0$  consistently with the ordering of the 1-valent vertices of  $D^0$ .

Let  $T(D^0)$  (respectively,  $T(D_\sigma^0)$ ) be the tensor associated to  $D^0$  (respectively,  $D_\sigma^0$ ). We assume that the  $i$ -th coordinate of  $T(D^0) \in V^{\otimes n} \otimes V^{*\otimes n}$  corresponds to the  $i$ -th source of  $D^0$ , if  $1 \leq i \leq n$ , or to the  $(i - n)$ -th sink of  $D^0$ , if  $n < i \leq 2n$ .

Recall that  $\theta(D), \theta(D_\sigma) \in \mathcal{R}$  are results of contractions of tensors associated with elements of decompositions of  $D$  and  $D_\sigma$ . Since the decompositions of  $D$  and  $D_\sigma$  chosen by us differ only by elements  $D^0, D_\sigma^0$ , the proof of Lemma 4.4 can be reduced to a local computation on tensors. Namely, it is enough to prove that

$$(4.6) \quad T(D^0) = \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0).$$

Notice that each edge of  $D_\sigma^0$  is labeled by the identity map in  $End_R(V)$ . This map is represented by  $\sum_{i=1}^n e_i \otimes e^i$  in  $V \otimes V^* \simeq End_R(V)$ . Therefore, if  $\sigma = id \in S_n$ , then

$$T(D_\sigma^0) = \sum_{i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_n}.$$

Similarly, for any  $\sigma \in S_n$ , we have

$$T(D_\sigma^0) = \sum_{i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes \dots \otimes e^{i_{\sigma(n)}}.$$

Therefore,

$$(4.7) \quad \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) = \sum_{\substack{\sigma \in S_n \\ i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}}} \epsilon(\sigma) e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes \dots \otimes e^{i_{\sigma(n)}}.$$

Note that we can assume that the numbers  $i_1, i_2, \dots, i_n$  appearing on the right side of (4.7) are all different. Indeed, if  $i_j = i_k, j \neq k$ , then there is an equal number of even and odd permutations contributing the term

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{j_1} \otimes e^{j_2} \otimes \dots \otimes e^{j_n}$$

to the sum on the right side of (4.7), for any  $j_1, j_2, \dots, j_n$ .

Therefore, we can assume that the numbers  $(i_1, i_2, \dots, i_n)$  appearing in each term of the sum on the right side of (4.7) form a permutation  $\tau$  of  $(1, 2, \dots, n)$ . Hence we

have

$$\begin{aligned} & \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) \\ &= \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \otimes e^{\tau(\sigma(1))} \otimes e^{\tau(\sigma(2))} \otimes \dots \otimes e^{\tau(\sigma(n))}. \end{aligned}$$

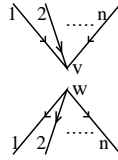
Substitute  $\tau'$  for  $\tau \circ \sigma$ . Then  $\epsilon(\sigma) = \epsilon(\tau)\epsilon(\tau')$ , and we get

$$\begin{aligned} & \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) \\ &= \sum_{\tau, \tau' \in S_n} \epsilon(\tau)\epsilon(\tau') e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \otimes e^{\tau'(1)} \otimes e^{\tau'(2)} \otimes \dots \otimes e^{\tau'(n)}. \end{aligned}$$

Notice that the right side of the above equation is equal to

$$\left( \sum_{\tau \in S_n} \epsilon(\tau) e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \right) \otimes \left( \sum_{\tau' \in S_n} \epsilon(\tau') e^{\tau'(1)} \otimes e^{\tau'(2)} \otimes \dots \otimes e^{\tau'(n)} \right).$$

But the expression above is exactly the tensor assigned to:



Therefore we have proved (4.6) and completed the proof of Lemma 4.4. □

The study of  $SL_n$ -actions on linear spaces was one of the main objectives of classical invariant theory. In particular, Weyl ([We]) determined all invariants of the action of  $SL(V)$  on  $V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$  and gave a full description of relations between them. The set of “typical” invariants consists of brackets  $[v_1, \dots, v_n] = \text{Det}(v_1, \dots, v_n)$ ,  $[\phi_1, \dots, \phi_n]^* = \text{Det}(\phi_1, \dots, \phi_n)$ , where  $v_1, \dots, v_n \in V$ ,  $\phi_1, \dots, \phi_n \in V^*$ , and contractions  $\phi_j(v_i)$ . The identity

$$(4.8) \quad [v_1, \dots, v_n][\phi_1, \dots, \phi_n]^* = \text{Det}(\phi_j(v_i))_{i,j=1}^n$$

is one of the fundamental identities relating the typical invariants. The bracket  $[\cdot, \cdot, \dots, \cdot]$  is a skew symmetric linear functional on  $V \otimes V \otimes \dots \otimes V$  and hence an element of  $\wedge^n V^*$ . Similarly,  $[\cdot, \cdot, \dots, \cdot]^* \in \wedge^n V$ . Note that the sources and sinks of graphs considered by us are labeled exactly by the tensors  $[\cdot, \cdot, \dots, \cdot]^*$  and  $[\cdot, \cdot, \dots, \cdot]$ . (However,  $V$  is in our case a free module over  $\mathcal{R} = \text{Rep}_n^R(G)$ .)

It follows from the proof of Lemma 4.4 that

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \left[ \begin{array}{c} \diagdown \quad \dots \quad \diagup \\ \boxed{\sigma} \\ \diagup \quad \dots \quad \diagdown \end{array} \right]$$

represents the tensor in  $\text{Hom}(V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*, \mathcal{R}) = V^* \otimes \dots \otimes V^* \otimes V \otimes \dots \otimes V$  assigning to  $(v_1, v_2, \dots, v_n, \phi_1, \phi_2, \dots, \phi_n)$  the value  $\text{Det}(\phi_j(v_i))_{i,j=1}^n$ . Therefore, the



identity

$$\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma)\theta(D_\sigma)$$

is essentially equivalent to (4.8).

**Lemma 4.5.** *Let  $L_g, E_g, EL_g$  be graphs defined as in Section 3 but considered as elements of  $\mathcal{G}_n(G)$  and  $\mathcal{G}'_n(G)$ , i.e.*

- (1)  $L_g \in \mathcal{G}_n(G)$  is a single loop labeled by the conjugacy class of  $g \in G$ ,
- (2)  $E_g \in \mathcal{G}'_n(G)$  is a single edge labeled by  $g \in G$ , and
- (3)  $EL_g \in \mathcal{G}'_n(G)$  is a graph composed of an edge labeled by the identity in  $G$  and of a loop labeled by the conjugacy class of  $g \in G$ .

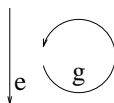
Under the above assumptions the functions  $\Theta$  and  $\theta$  satisfy conditions (1) and (2) of Theorem 3.6.

*Proof.* (1)  $L_g$  can be decomposed into a single arc



which has associated the tensor  $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset V^* \otimes V$ . The contraction of this tensor gives  $\theta(L_g) = Tr(j_{G,n}(g))$ .

- (2)  $E_g$  is a single arc. Therefore  $\Theta(E_g) = T(E_g) = j_{G,n}(g)$ .
- (3)  $EL_g$  can be decomposed into:



The tensor associated with this decomposition is  $id \otimes j_{G,n}(g) \in End(V) \otimes End(V)$ . After making a contraction corresponding to the identification of the ends of the arc, we get

$$\Theta(EL_g) = id \cdot Tr(j_{G,n}(g)) \in End(V).$$

□

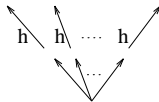
**Lemma 4.6.** *Let  $D, D' \in \mathcal{G}_n(G)$  be two graphs which are identical as unlabeled graphs and which have the same labeling of edges and loops except the labeling of edges incident to a vertex  $v$ . Moreover, suppose that*

- (1) *if  $v$  is a source, then the edges in  $D$  incident to  $v$  are labeled by  $g_1, g_2, \dots, g_n$  and the edges in  $D'$  incident to  $v$  are labeled by  $g_1h, g_2h, \dots, g_nh$  for some  $g_1, g_2, \dots, g_n, h \in G$ ;*
- (2) *if  $v$  is a sink, then the edges in  $D$  incident to  $v$  are labeled by  $g_1, g_2, \dots, g_n$  and the edges in  $D'$  incident to  $v$  are labeled by  $hg_1, hg_2, \dots, hg_n$  for some  $g_1, g_2, \dots, g_n, h \in G$ .*

Under the above assumptions  $\theta(D) = \theta(D')$ . An analogous fact is true for graphs in  $\mathcal{G}'_n(G)$ .

*Proof.* We prove part (1) only. The proof of part (2) is analogous.

Let  $v$  be a source. Notice that  $D$  and  $D'$  have identical decompositions into sinks, sources, and arcs except that  $D_0 = \begin{matrix} & \swarrow & \dots & \searrow \\ & & & \end{matrix}$  is an element of the decomposition of

$D$  and the diagram  $D'_0 =$   is a fragment of a decomposition of  $D'$ .

Therefore we need to show that the tensors assigned to the above diagrams are identical. Notice that the tensor associated to  $D_0$ ,  $T(D_0)$ , is an element of the one-dimensional  $\mathcal{R}$ -linear space of skew-symmetric tensors  $\bigwedge^n V \subset V^n$ . Let  $A : V \rightarrow V$  be an endomorphism given in the standard coordinates of  $V$  by  $j_{G,n}(h) \in SL_n(\mathcal{R})$ .  $A$  induces an endomorphism  $\wedge A : \bigwedge^n V \rightarrow \bigwedge^n V$  with the property that  $\wedge A(T(D_0)) = \text{Det}(A)T(D_0) \in \bigwedge^n V$ . Notice that  $\wedge A(T(D_0))$  is exactly the tensor associated to  $D'_0$ . Since  $\text{Det}(A) = 1$ , the tensors associated to  $D_0$  and  $D'_0$  are equal.  $\square$

*Proof of Theorem 3.6.* Let us extend the functions  $\theta$  and  $\Theta$  to all  $R$ -linear combinations of graphs in  $\mathcal{G}_n(G)$  and  $\mathcal{G}'_n(G)$  respectively. Facts 4.1 and 4.2 and Lemmas 4.4, 4.5, and 4.6 imply that these functions descend to  $R$ -linear homomorphisms

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\mathcal{R}), \quad \theta : \mathbb{A}_n(X) \rightarrow \mathcal{R}.$$

By Lemma 4.5,  $\Theta$  and  $\theta$  satisfy conditions (1) and (2) of Theorem 3.6.

We have showed in Proposition 3.5(4) that  $\mathbb{A}_n(X, x_0)$  is generated by elements  $E_\gamma$  and  $EL_{\gamma'}$ , for  $\gamma, \gamma' \in G = \pi_1(X, x_0)$ . By Proposition 2.4 and the paragraph preceding it,  $\Theta(E_\gamma) = j_{G,n}(\gamma)$  and  $\Theta(EL_{\gamma'}) = \text{Tr}(j_{G,n}(\gamma'))$  belong to

$$M_n(\text{Rep}_n^R(G))^{GL_n(R)}.$$

Therefore the image of  $\Theta$  lies in  $M_n(\text{Rep}_n^R(G))^{GL_n(R)}$ . We show analogously that the image of  $\theta$  lies in  $\text{Rep}_n^R(G)^{GL_n(R)}$ . Therefore the proof will be completed if we show that the diagram of Theorem 3.6 commutes.

Let  $D \in \mathcal{G}'_n(G)$ ,  $G = \pi_1(X, x_0)$ , represent an element of  $\mathbb{A}_n(X, x_0)$ . Then  $\Theta(D) \in M_n(\text{Rep}_n^R(G))^{GL_n(R)}$  is the result of a contraction of tensors associated with elements of some decomposition of  $D$ . Notice that  $\mathbb{T}(D)$  is an element of  $\mathbb{A}_n(X)$  represented by the diagram  $D$  with its 1-valent vertices identified.<sup>2</sup> Hence  $\theta(\mathbb{T}(D))$  is a contraction of  $\Theta(D)$ , i.e.  $\theta(\mathbb{T}(D)) = \text{Tr}(\Theta(D))$ . Since the elements  $D \in \mathcal{G}'_n(G)$  span  $\mathbb{A}_n(X, x_0)$ , the diagram of Theorem 3.6 commutes.  $\square$

### 5. PROOF OF THEOREM 3.7

Now we assume that  $R$  is a field of characteristic 0.

We start by stating the first and second fundamental theorems of invariant theory, following the approach of Procesi, [Pro-1] (compare also [Ra]).

Let  $I$  be an infinite set. Let  $P_n(I)$  and  $A_i$  be as before,

$$P_n(I) = R[x_{jk}^i, j, k \in \{1, 2, \dots, n\}, i \in I], \quad A_i = (x_{jk}^i) \in M_n(P_n(I)).$$

We are going to present Procesi's description of the ring  $M_n(P_n(I))^{GL_n(R)}$ .

Let  $T$  be a commutative  $R$ -algebra freely generated by the symbols

$$\text{Tr}(X_{i_1} X_{i_2} \dots X_{i_k}),$$

where  $i_1, i_2, \dots, i_k \in I$ . We adopt the convention that  $\text{Tr}(M) = \text{Tr}(N)$  if and only if the monomial  $N$  is obtained from  $M$  by a cyclic permutation. Let  $T\{X_i\}_{i \in I}$  be a non-commutative  $T$ -algebra freely generated by variables  $X_i$ ,  $i \in I$ . We

<sup>2</sup>Recall that  $\mathbb{T}$  was defined in the paragraph preceding Fact 3.4.

have a natural  $T$ -linear homomorphism  $Tr : T\{X_i\}_{i \in I} \rightarrow T$  which assigns to  $X_{i_1}X_{i_2}\dots X_{i_k} \in T\{X_i\}_{i \in I}$  an element  $Tr(X_{i_1}X_{i_2}\dots X_{i_k}) \in T$ .

There is a homomorphism  $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))$  uniquely determined by the conditions:

- $\pi(X_i) = A_i$ ,
- $\pi(Tr(X_{i_1}X_{i_2}\dots X_{i_k})) = Tr(A_{i_1}A_{i_2}\dots A_{i_k}) \in P_n(I) \subset M_n(P_n(I))$ .<sup>3</sup>

Proposition 2.3 and Lemma 2.2 imply that the image of  $\pi$  is fixed by the  $GL_n(R)$ -action on  $M_n(P_n(I))$ , i.e.  $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$ .

Notice that the following diagram commutes:

$$\begin{array}{ccc} T\{X_i\}_{i \in I} & \xrightarrow{\pi} & M_n(P_n(I))^{GL_n(R)} \\ \downarrow Tr & & \downarrow Tr \\ T & \xrightarrow{\pi|_T} & P_n(I)^{GL_n(R)} \end{array}$$

The following version of *The First Fundamental Theorem* of invariant theory of  $n \times n$  matrices is due to Procesi, [Pro-1].

**Theorem 5.1.**  $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$  is an epimorphism.

Before we state the second fundamental theorem of invariant theory of  $n \times n$  matrices, we need some preparations.

Suppose that  $\{1, 2, \dots, m\} \subset I$  and specify  $i_0 \in \{1, 2, \dots, m\}$ . We can present any  $\sigma \in S_m$  as a product of cycles in such a way that  $i_0$  is the first element of the first cycle,  $\sigma = (i_0, i_1, \dots, i_s)(j_0, j_1, \dots, j_t)\dots(k_0, k_1, \dots, k_v)$ . We define  $\Phi_{\sigma, i_0}(X_1, X_2, \dots, X_m)$  to be equal to

$$X_{i_0}X_{i_1}\dots X_{i_s}Tr(X_{j_0}X_{j_1}\dots X_{j_t})\dots Tr(X_{k_0}X_{k_1}\dots X_{k_v}) \in T\{X_i\}_{i \in I}.$$

We also define another expression which does not depend on  $i_0$ :

$$\Phi_\sigma(X_1, X_2, \dots, X_m) = Tr(X_{i_0}X_{i_1}\dots X_{i_s})Tr(X_{j_0}X_{j_1}\dots X_{j_t})\dots Tr(X_{k_0}X_{k_1}\dots X_{k_v}) \in T.$$

Let

$$F(X_1, X_2, \dots, X_m) = \sum_{\sigma \in S_m} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, \dots, X_m) \in T.$$

$F(X_1, X_2, \dots, X_{n+1})$  is called the fundamental trace identity of  $n \times n$  matrices.

Procesi argues that there exists a unique element  $G(X_1, X_2, \dots, X_n) \in T\{X_i\}_{i \in I}$ , involving only the variables  $X_1, \dots, X_n$  and the traces of monomials in these variables, such that

$$F(X_1, X_2, \dots, X_{n+1}) = Tr(G(X_1, X_2, \dots, X_n)X_{n+1}) \in T\{X_i\}_{i \in I}.$$

Procesi gives an explicit formula for  $G(X_1, X_2, \dots, X_n)$ , but we want to give a different formula, which will be more suitable for our purposes.

**Lemma 5.2.**

$$\begin{aligned} G(X_1, X_2, \dots, X_n) &= \sum_{\sigma \in S_n} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, \dots, X_n) - \sum_{\substack{i \in \{1, 2, \dots, n\} \\ \sigma \in S_n}} \epsilon(\sigma)\Phi_{\sigma, i}(X_1, X_2, \dots, X_n). \end{aligned}$$

<sup>3</sup>We identify  $P_n(I)$  with the scalar matrices in  $M_n(P_n(I))$ .

*Proof.* It follows from the remarks preceding Lemma 5.2 that it is enough to show that if we multiply the right side of the equation of Lemma 5.2 by  $X_{n+1}$ , then the trace of it will be equal to  $F(X_1, \dots, X_{n+1})$ , i.e. we have to prove that

$$(5.1) \quad \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(X_1, X_2, \dots, X_n) Tr(X_{n+1}) - \sum_{\substack{i \in \{1, 2, \dots, n\} \\ \sigma \in S_n}} \epsilon(\sigma) Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1}) = F(X_1, X_2, \dots, X_{n+1}).$$

Notice that  $\epsilon(\sigma) \Phi_\sigma(X_1, X_2, \dots, X_n) Tr(X_{n+1}) = \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1})$ , where  $\sigma' \in S_{n+1}$ ,  $\sigma'(i) = \sigma(i)$ , for  $i \in \{1, 2, \dots, n\}$ , and  $\sigma'(n+1) = n+1$ .

Similarly we can simplify  $Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1})$ . Suppose that

$$\sigma = (i_0, i_1, \dots, i_s)(j_0, j_1, \dots, j_t) \dots (k_0, k_1, \dots, k_v) \in S_n,$$

where  $i_0 = i$ . Then

$$Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1}) = \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}),$$

for  $\sigma' = (i_0, i_1, \dots, i_s, n+1)(j_0, j_1, \dots, j_t) \dots (k_0, k_1, \dots, k_v) \in S_{n+1}$ . Notice that  $\epsilon(\sigma') = -\epsilon(\sigma)$ . Therefore the left side of the equation (5.1) is equal to

$$\sum_{\substack{\sigma' \in S_{n+1}, \\ \sigma'(n+1)=n+1}} \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}) + \sum_{\substack{i \in \{1, 2, \dots, n\}, \sigma' \in S_{n+1} \\ \text{such that } \sigma'(n+1)=i}} \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}).$$

The above expression is obviously equal to  $F(X_1, X_2, \dots, X_{n+1})$ . □

Now we are ready to state *The Second Fundamental Theorem* of invariant theory of  $n \times n$  matrices, [Pro-1].

**Theorem 5.3.** *The kernel of  $\pi$  is generated by elements  $G(M_1, M_2, \dots, M_n)$  and  $F(N_1, N_2, \dots, N_{n+1})$ , where  $M_1, M_2, \dots, M_n, N_1, N_2, \dots, N_{n+1}$  are all possible monomials in the variables  $X_i, i \in I$ .*

Let  $X$  be a path-connected topological space. We choose a presentation  $\langle g_i, i \in I | r_j, j \in J \rangle$  of  $G = \pi_1(X, x_0)$  such that

- $I$  is an infinite set,
- the inverse of every generator is also a generator, and
- the defining relations  $r_j$  are products of non-negative powers of generators.

Note that such a presentation always exists (even if  $G$  is finitely generated).

Let  $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$  be an  $R$ -homomorphism such that  $\psi(X_i) = E_{g_i}$  and  $\psi(Tr(X_{i_1} X_{i_2} \dots X_{i_k})) = EL_{g_{i_1} g_{i_2} \dots g_{i_k}}$ . Recall that, by Proposition 3.4,  $\mathbb{A}_n(X)$  can be considered as a subalgebra of  $\mathbb{A}_n(X, x_0)$  in such a way that  $L_\gamma \in \mathbb{A}_n(X)$  is identified with  $EL_\gamma \in \mathbb{A}_n(X, x_0)$ . Hence  $\psi(Tr(X_{i_1} X_{i_2} \dots X_{i_k})) \in \mathbb{A}_n(X)$  and  $\psi$  restricts to  $\psi : T \rightarrow \mathbb{A}_n(X)$ . Moreover, the following diagram commutes:

$$(5.2) \quad \begin{array}{ccc} T\{X_i\}_{i \in I} & \xrightarrow{\psi} & \mathbb{A}_n(X, x_0) \\ \downarrow Tr & & \downarrow Tr \\ T & \xrightarrow{\psi} & \mathbb{A}_n(X) \end{array}$$

We are going to show that the kernel of  $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$  contains the kernel of  $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$  and therefore  $\psi$  descends to a homomorphism  $M_n(P_n(I))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$ .

We will need the following fact, due to Formanek (Proposition 45 [For]).

**Proposition 5.4.** *For any matrix  $A \in M_n(R)$*

$$Det(A) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) Tr(A^{c_1}) Tr(A^{c_2}) \dots Tr(A^{c_k}),$$

where  $c_1, c_2, \dots, c_k$  denote the lengths of all cycles in  $\sigma$ .

For completeness we sketch a proof of Proposition 5.4. A multilinearization of the determinant,  $Det : M_n(R) \rightarrow R$ , gives a function on  $n$ -tuples of  $n \times n$  matrices,

$$\mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} Det(X_\sigma),$$

where  $X_\sigma$  is a matrix whose  $i$ -th row is the  $i$ -th row of  $X_{\sigma(i)}$ . Note that  $\mathcal{M}(A, \dots, A) = n! Det(A)$ , and therefore the identity of Proposition 5.4 is a special case of the following identity:

$$\mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(X_1, \dots, X_n),$$

where  $\Phi_\sigma$  was defined in the second paragraph after Theorem 5.1. Formanek gives the following proof of the above identity. Assume that  $1, 2, \dots, n \in I$ . Since  $A_1, \dots, A_n \in M_n(P_n(I))$  represent generic matrices, in order to prove the above identity it is enough to show it for  $X_1 = A_1, \dots, X_n = A_n$ . Since  $\mathcal{M}(A_1, \dots, A_n)$  is an invariant polynomial function on  $n$ -tuples of matrices, the First Fundamental Theorem implies that  $\mathcal{M}(A_1, \dots, A_n)$  can be expressed in terms of traces of monomials in  $A_1, \dots, A_n$ . Since  $\mathcal{M}(A_1, \dots, A_n)$  is linear with respect to  $A_1, \dots, A_n$ , it is a linear combination of terms  $Tr(A_{i_1} \dots A_{i_s}) \dots Tr(A_{j_1} \dots A_{j_t})$ , where  $i_1, \dots, i_s, \dots, j_1, \dots, j_t$  form a permutation of  $1, 2, \dots, n$ . Therefore

$$\mathcal{M}(A_1, \dots, A_n) = \sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(A_1, \dots, A_n),$$

and hence

$$(5.3) \quad \mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(X_1, \dots, X_n),$$

for any  $n \times n$  matrices  $X_1, \dots, X_n$ . We need to prove that  $\alpha_\sigma = \epsilon(\sigma)$ . If we restrict the above equation to matrices  $A_1, \dots, A_n \in M_{n-1}(P_{n-1}(I))$  embedded into  $M_n(P_{n-1}(I))$  in the standard, non-unit-preserving way, we will get the following polynomial identity on  $(n - 1) \times (n - 1)$  matrices:

$$\sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(A_1, \dots, A_n) = 0.$$

It is not difficult to see that the Second Fundamental Theorem implies that  $F(A_1, \dots, A_n)$  is the only (up to scalar)  $n$ -linear trace identity of degree  $n$  on matrices  $A_1, \dots, A_n \in M_{n-1}(P_{n-1}(I))$ . Therefore  $\alpha_\sigma = \epsilon(\sigma)c$ , for some fixed  $c$ . Substituting the matrix  $(x_{ij})$  with a single non-zero entry  $x_{ii} = 1$  for  $X_i$  in (5.3), we get  $c = 1$ . Thus the proof of Proposition 5.4 is finished.

The specialization  $A = Id \in M_n(R)$  in Proposition 5.4 yields the following corollary.

**Corollary 5.5.** *For any positive integer  $n$ ,*

$$\sum_{\sigma \in S_n} \epsilon(\sigma) n^{c(\sigma)} = n!,$$

where  $c(\sigma)$  is the number of cycles in the cycle decomposition of  $\sigma$ .

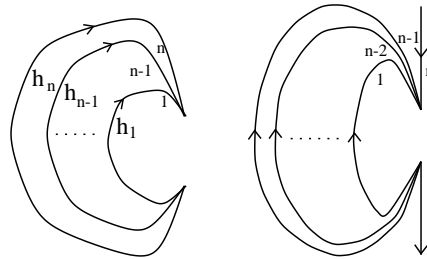
From the definition of  $T\{X_i\}_{i \in I}$  it immediately follows that for any family of matrices  $\{M_i\}_{i \in I} \subset M_n(R)$  there is a well-defined substitution

$$X_i \rightarrow M_i, \quad Tr(X_{i_1} X_{i_2} \dots X_{i_k}) \rightarrow Tr(M_{i_1} M_{i_2} \dots M_{i_k}) \in R \subset M_n(R),$$

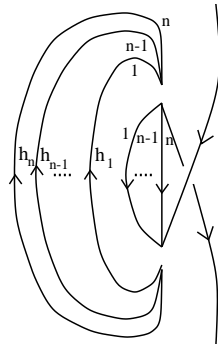
which can be extended to the whole ring  $T\{X_i\}_{i \in I}$ . Therefore, if  $H(X_{i_1}, \dots, X_{i_k})$  is an element of  $T\{X_i\}_{i \in I}$  involving variables  $X_{i_1}, \dots, X_{i_k}$ , then  $H(M_{i_1}, \dots, M_{i_k})$  is a well-defined matrix in  $M_n(R)$ .

**Lemma 5.6.** *If  $N_1, N_2, \dots, N_n$  are any monomials in the variables  $X_i$ ,  $i \in I$ , then  $\psi(G(N_1, N_2, \dots, N_n)) = 0$ .*

*Proof.* By the definition of  $\psi$  (given in the second paragraph after Theorem 5.3),  $\psi(N_i) = E_{h_i}$ , for some  $h_1, h_2, \dots, h_n \in G$ . Consider the following graph  $D$  in  $\mathcal{G}'_n(G)$ :



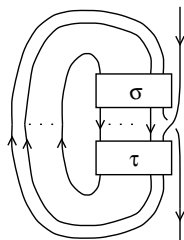
in which we omitted labels of edges labeled by the identity in  $G$ . Notice that  $D$  can also be presented in the following way:



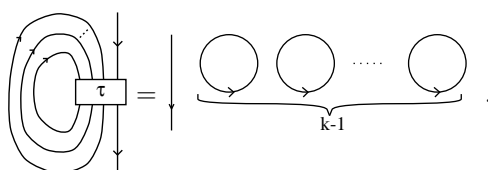
Since the vertices of  $D$  can be resolved in two possible ways (corresponding to the two diagrams above), we obtain the following equation:

$$(5.4) \quad \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) \epsilon(\tau) \left( \text{Diagram } \sigma \right) \left( \text{Diagram } \tau \right) = \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) \epsilon(\tau) D_{\sigma, \tau},$$

where  $D_{\sigma,\tau}$  is a graph of the form:



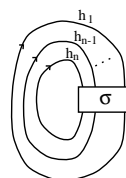
If  $\tau \in S_n$  decomposes into  $k = c(\tau)$  cycles, then



Therefore, by Corollary 5.5,

$$\sum_{\tau \in S_n} \epsilon(\tau) = \left( \text{graph with } \tau \text{ box and loops} \right) = \sum_{\tau \in S_n} \epsilon(\tau) n^{c(\tau)-1} = (n-1)!$$

Notice moreover that



is equal to  $\psi(\Phi_\sigma(N_1, N_2, \dots, N_n))$ . Therefore the left side of (5.4) is equal to the value of  $\psi$  on

$$(n-1)! \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(N_1, N_2, \dots, N_n).$$

Now we are going to consider the right side of (5.4). Notice that the single arc in  $D_{\sigma,\tau}$  is labeled by an element  $h_{i_s} \dots h_{i_1} h_{i_0} \in G$ , where  $i_0 = \tau(n)$ ,  $i_1 = \tau\sigma(i_0)$ ,  $\dots$ ,  $i_s = \tau\sigma(i_{s-1})$ , and  $\sigma(i_s) = n$ . Since  $\tau(\sigma(i_s)) = \tau(n) = i_0$ ,  $(i_s, i_{s-1}, \dots, i_1, i_0)$  is a cycle of the permutation  $(\tau\sigma)^{-1} \in S_n$ .

Note that every loop in  $D_{\sigma,\tau}$  is labeled by the conjugacy class of  $h_{j_t} h_{j_{t-1}} \dots h_{j_1} h_{j_0}$ , where  $(j_t, j_{t-1}, \dots, j_1, j_0)$  is a cycle of  $(\tau\sigma)^{-1} \in S_n$  disjoint from  $(i_s, i_{s-1}, \dots, i_1, i_0)$ . Therefore  $D_{\sigma,\tau}$  is the value of  $\psi$  on

$$N_{i_s} \dots N_{i_1} N_{i_0} Tr(N_{j_t} \dots N_{j_1} N_{j_0}) \dots Tr(N_{k_v} \dots N_{k_1} N_{k_0}),$$

where

$$(i_s, \dots, i_1, i_0)(j_t, \dots, j_1, j_0) \dots (k_v, \dots, k_1, k_0)$$

is the cycle decomposition of  $(\tau\sigma)^{-1}$ . The above expression is equal to

$$\Phi_{(\tau\sigma)^{-1}, i_s}(N_1, N_2, \dots, N_n) = \Phi_{(\tau\sigma)^{-1}, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n).$$

Therefore, the right side of (5.4) is the value of  $\psi$  on

$$\sum_{\sigma, \tau \in S_n} \epsilon(\sigma)\epsilon(\tau)\Phi_{(\tau\sigma)^{-1}, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n) \in T\{X_i\}_{i \in I}.$$

Let us replace  $(\tau\sigma)^{-1}$  by  $\kappa$  in the expression above. Then we get

$$\sum_{\sigma, \kappa \in S_n} \epsilon(\kappa)\Phi_{\kappa, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n) = (n-1)! \sum_{\kappa \in S_n, i \in \{1, 2, \dots, n\}} \epsilon(\kappa)\Phi_{\kappa, i}(N_1, N_2, \dots, N_n).$$

After comparing the above algebraic descriptions of the two sides of (5.4) we see that for any monomials  $N_1, N_2, \dots, N_n$  the following element of  $T\{X_i\}_{i \in I}$  belongs to  $\text{Ker } \psi$ :

$$\sum_{\sigma \in S_n} \epsilon(\sigma)\Phi_{\sigma}(N_1, N_2, \dots, N_n) - \sum_{\kappa \in S_n, i \in \{1, 2, \dots, n\}} \epsilon(\kappa)\Phi_{\kappa, i}(N_1, N_2, \dots, N_n).$$

By Lemma 5.2 the above expression is equal to  $G(N_1, N_2, \dots, N_n)$ . Therefore  $\psi(G(N_1, N_2, \dots, N_n)) = 0$ . □

**Lemma 5.7.** *Let  $N_1, N_2, \dots, N_{n+1}$  be any monomials in the variables  $X_i, i \in I$ . Then  $\psi(F(N_1, N_2, \dots, N_{n+1})) = 0$ .*

*Proof.* By definition,  $F(N_1, N_2, \dots, N_{n+1}) = \text{Tr}(G(N_1, N_2, \dots, N_n)N_{n+1})$ . By (5.2),  $\psi$  commutes with the trace function. Therefore

$$\begin{aligned} \psi(F(N_1, N_2, \dots, N_{n+1})) &= \psi(\text{Tr}(G(N_1, N_2, \dots, N_n)N_{n+1})) \\ &= \text{Tr}(\psi(G(N_1, N_2, \dots, N_n))\psi(N_{n+1})) = 0. \end{aligned}$$

□

Lemmas 5.6 and 5.7 and the Second Fundamental Theorem imply that the kernel of  $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$  contains the kernel of  $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$ . Therefore we have the following corollary.

**Corollary 5.8.** *There exists an  $R$ -algebra homomorphism  $\psi' : M_n(P_n(I))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$  such that  $\psi'(A_i) = E_{g_i}$  and  $\psi'(\text{Tr}(A_{i_1}A_{i_2}\dots A_{i_k})) = EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$ , for any  $i_1, i_2, \dots, i_k \in I$ .*

The epimorphism  $\eta : P_n(I) \rightarrow \text{Rep}_n^R(G)$  introduced in Section 2 induces an epimorphism  $M_n(\eta) : M_n(P_n(I)) \rightarrow M_n(\text{Rep}_n^R(G))$  and, therefore, by restriction, a homomorphism  $M_n(\eta)^{GL_n(R)} : M_n(P_n(I))^{GL_n(R)} \rightarrow M_n(\text{Rep}_n^R(G))^{GL_n(R)}$ . Our goal is to show that  $\psi'$  descends to

$$\psi'' : M_n(\text{Rep}_n^R(G))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$$

such that the following diagram commutes:

$$(5.5) \quad \begin{array}{ccc} M_n(P_n(I))^{GL_n(R)} & & \\ \downarrow M_n(\eta)^{GL_n(R)} & \searrow \psi' & \\ M_n(\text{Rep}_n^R(G))^{GL_n(R)} & \xrightarrow{\psi''} & \mathbb{A}_n(X, x_0) \end{array}$$



In order to prove this fact we need to show that  $Ker M_n(\eta)^{GL_n(R)} \subset Ker \psi'$ . We will use the following lemma.

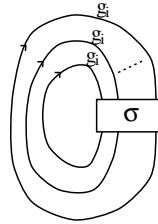
**Lemma 5.9.** (1)  $Det(A_i) \in P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$ .  
 (2)  $\psi'(Det(A_i)) = 1$ , for any  $i \in I$ .

*Proof.* (1) By Proposition 5.4,  $Det(A_i)$  can be expressed as a linear combination of traces of powers of  $A_i$ . By Lemma 2.2(2),  $A_i^k \in M_n(P_n(I))^{GL_n(R)}$ , and hence, by Proposition 2.3,  $Tr(A_i^k) \in P_n(I)^{GL_n(R)}$ . Finally, by Lemma 2.2(1) there is a natural embedding  $P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$ .

(2) If  $c_1, c_2, \dots, c_k$  are the lengths of all cycles of  $\sigma \in S_n$ , then  $\psi'$  maps

$$Tr(A_i^{c_1})Tr(A_i^{c_2})\dots Tr(A_i^{c_k})$$

to a graph



Therefore, by Proposition 5.4 and Fact 4.1,

$$\psi'(Det(A_i)) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \left[ \text{Diagram with } \sigma \text{ box and } c_i \text{ loops} \right] = \frac{1}{n!} \left[ \text{Diagram with } g_i \text{ loops} \right].$$

Analogously,

$$1 = \psi'(Det(\mathbf{1})) = \frac{1}{n!} \left[ \text{Diagram with } e \text{ loops} \right],$$

where  $e$  is the identity in  $G$ . But, by (4.3) (or, equivalently, (4.4)),

$$\left[ \text{Diagram with } g_i \text{ loops} \right] = \left[ \text{Diagram with } e \text{ loops} \right].$$

Therefore  $\psi'(Det(A_i)) = 1$ . □

The next proposition is due to Procesi. Since the proof of this proposition is hidden in the proof of Theorem 2.6 in [Pro-2], we will recall it here for completeness of this paper.<sup>4</sup>

**Proposition 5.10.** *Let  $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$  be a two-sided ideal and let  $\mathcal{J}'$  be the ideal in  $P_n(I)$  generated by the entries of elements of  $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \triangleleft M_n(P_n(I))$ . Then:*

- (1)  $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) = M_n(\mathcal{J}') \triangleleft M_n(P_n(I))$ .
- (2) *There is a unique  $GL_n(R)$ -action on  $M_n(P_n(I)/\mathcal{J}')$  such that the natural projection  $i : M_n(P_n(I)) \rightarrow M_n(P_n(I)/\mathcal{J}')$  is  $GL_n(R)$ -equivariant.*
- (3)  *$i$  induces a homomorphism*

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}$$

*which is an isomorphism of  $R$ -algebras.*

*Proof.* (1) This follows from the basic algebraic fact that every ideal  $\mathcal{I}$  in  $M_n(R)$ , for any ring  $R$  with 1, is of the form  $M_n(\mathcal{I}')$ , where  $\mathcal{I}'$  is the ideal in  $R$  generated by the entries of a generating set of the ideal  $\mathcal{I}$ .

(2) Let  $B \in GL_n(R)$ , and let  $B*$  denote the action of  $B$  on  $M_n(P_n(I))$ .  $B$  leaves  $M_n(\mathcal{J}')$  invariant. Indeed, any element  $C \in M_n(\mathcal{J}')$  is of the form  $\sum_i M_i C_i N_i$ , where  $M_i, N_i \in M_n(P_n(I))$ ,  $C_i \in \mathcal{J}$ , and therefore

$$B * C = \sum_i (B * M_i)(B * C_i)(B * N_i) \in M_n(\mathcal{J}').$$

This implies that the action of  $GL_n(R)$  on  $M_n(P_n(I)/\mathcal{J}')$  is well defined. All other statements of (2) are obvious consequences of this fact.

(3) For any rational action of  $GL_n(R)$  on any  $R$ -vector space  $N$  there exists a linear projection  $\nabla : N \rightarrow N^{GL_n(R)}$ , called the Reynolds operator, with the following properties:

- (i)  $\nabla(x) = x$  for  $x \in N^{GL_n(R)}$ , and, therefore,  $\nabla$  is an epimorphism.
- (ii)  $\nabla$  is natural with respect to  $GL_n(R)$ -equivariant maps  $N \rightarrow N'$ .
- (iii) If  $N$  is an algebra, then  $\nabla(xy) = x\nabla(y)$  and  $\nabla(yx) = \nabla(y)x$  for  $x \in N^{GL_n(R)}$  and  $y \in N$ .

For more information about this operator, see [MFK] or a more elementary text [Fog].

The homomorphism  $i$  restricted to  $M_n(P_n(I))^{GL_n(R)}$  induces a homomorphism

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}.$$

By Property (i) of  $\nabla$ ,  $j$  is an epimorphism. It remains to prove that  $j$  is injective.

Choose  $i_0 \in I$ . For any monomial  $m$  in  $P_n(I) = R[x_{jk}^i, i \in I, j, k = 1, 2, \dots, n]$  we define the degree of  $m$  to be the number of appearances of the variables  $x_{jk}^{i_0}$ ,  $j, k \in \{1, 2, \dots, n\}$ , in  $m$ . This induces a grading on  $P_n(I)$ . We can extend this grading on  $M_n(P_n(I))$  as follows. For any matrix  $A = (a_{jk}) \in M_n(P_n(I))$  with a single non-zero entry  $a_{st}$ ,  $deg(A) = deg(a_{st})$ . Note that the degree of the matrix  $A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$  considered in Section 2 is 1.

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<sup>4</sup>Compare also Proposition 9.5 in [B-H].

Let  $B \in GL_n(R)$ . By the definition of the  $GL_n(R)$ -action on  $M_n(P_n(I))$  and by Lemma 2.2(2),

$$B \begin{pmatrix} B * x_{11}^{i_0} & B * x_{12}^{i_0} & \dots & B * x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ B * x_{n1}^{i_0} & B * x_{n2}^{i_0} & \dots & B * x_{nn}^{i_0} \end{pmatrix} B^{-1} = \begin{pmatrix} x_{11}^{i_0} & x_{12}^{i_0} & \dots & x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ x_{n1}^{i_0} & x_{n2}^{i_0} & \dots & x_{nn}^{i_0} \end{pmatrix}.$$

Therefore  $B * x_{jk}^{i_0}$  is a linear combination of the variables  $x_{j'k'}^{i_0}$ ,  $j', k' = 1, 2, \dots, n$ , and hence the action of  $GL_n(R)$  preserves the grading of  $P_n(I)$ . For any  $M \in M_n(P_n(I))$ ,  $B * M$  is a matrix obtained by applying the action of  $B$  to all entries of  $M$  and then conjugating the resulting matrix by  $B$ . Therefore the action of  $GL_n(R)$  also preserves the grading of  $M_n(P_n(I))$ . The naturality of the Reynolds operators  $\nabla : P_n(I) \rightarrow P_n(I)^{GL_n(R)}$  and  $\nabla : M_n(P_n(I)) \rightarrow M_n(P_n(I))^{GL_n(R)}$  implies that they also preserve the gradings. This fact will be an important element of the proof of Proposition 5.10(3).

We need to show that

$$\mathcal{J} = M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)}.$$

However, it is sufficient to show that

$$\mathcal{J} \supset M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)},$$

since the opposite inclusion is obvious.

Let  $c = \sum_i a_i c_i b_i \in M_n(P_n(I))^{GL_n(R)}$ , where  $a_i, b_i \in M_n(P_n(I))$ ,  $c_i \in \mathcal{J}$ . We will show that  $c \in \mathcal{J}$ . Since  $c$  involves only finitely many variables  $x_{jk}^i$  and  $I$  is infinite, we can choose  $i_0 \in I$  such that  $x_{jk}^{i_0}$ ,  $j, k = 1, 2, \dots, n$ , do not appear in  $a_i, b_i, c_i$ . Thus  $\text{deg } a_i = \text{deg } b_i = \text{deg } c_i = 0$ .

Consider  $\text{Tr}(cA_{i_0})$ . By our assumptions about  $c$  and by Lemma 2.2(2),  $cA_{i_0} \in M_n(P_n(I))^{GL_n(R)}$ . Proposition 2.3 states that  $\text{Tr} : M_n(P_n(I)) \rightarrow P_n(I)$  is  $GL_n(R)$ -equivariant, and therefore  $\text{Tr}(cA_{i_0}) \in M_n(P_n(I))^{GL_n(R)}$ . Thus

$$\begin{aligned} \text{Tr}(cA_{i_0}) &= \text{Tr}(\nabla(cA_{i_0})) = \text{Tr}\left(\sum_i \nabla(a_i c_i b_i A_{i_0})\right) \\ &= \text{Tr}\left(\sum_i \nabla(b_i A_{i_0} a_i c_i)\right) = \text{Tr}\left(\sum_i \nabla(b_i A_{i_0} a_i) c_i\right). \end{aligned}$$

Note that  $b_i A_{i_0} a_i$  has degree 1 and, since  $\nabla$  preserves the grading,  $\nabla(b_i A_{i_0} a_i)$  is also of degree 1. By The First Fundamental Theorem of Invariant Theory (Theorem 5.1),  $M_n(P_n(I))^{GL_n(R)}$  is generated by the elements  $A_i$  and  $\text{Tr}(M)$ , where  $M$  varies over the set of monomials composed of non-negative powers of matrices  $A_i, i \in I$ . By our definition of degree,

$$\text{deg}(A_i) = \begin{cases} 1 & \text{if } i = i_0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\text{deg}(\text{Tr}(M)) = \text{number of appearances of } A_{i_0} \text{ in } M.$$

Therefore,  $\nabla(b_i A_{i_0} a_i)$  can be presented as

$$\sum_j p_{ij} A_{i_0} q_{ij} + \sum_k Tr(s_{ik} A_{i_0}) t_{ik},$$

for some elements  $p_{ij}, q_{ij}, s_{ik}, t_{ik} \in M_n(P_n(I))^{GL_n(R)}$  of degree 0. Thus

$$\begin{aligned} Tr(cA_{i_0}) &= Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i + \sum_i \sum_k Tr(s_{ik} A_{i_0}) t_{ik} c_i\right) \\ &= Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i\right) + \sum_i \sum_k Tr(s_{ik} A_{i_0}) Tr(t_{ik} c_i) \\ &= Tr\left(\left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right) A_{i_0}\right). \end{aligned}$$

Therefore

$$Tr\left(\left[c - \left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right)\right] A_{i_0}\right) = 0$$

in  $M_n(P_n(I))$ . The expression in brackets above has degree 0. Note that if  $deg d = 0$ ,  $d \in M_n(P_n(I))$ , then  $Tr(dA_{i_0}) = 0$  if and only if  $d = 0$ . Therefore

$$c = \sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik},$$

and hence  $c \in \mathcal{J}$ . This completes the proof of Proposition 5.10. □

Let  $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$  be the ideal generated by elements  $Det(A_i) - 1$ ,  $i \in I$ , and elements  $A_{i_1} A_{i_2} \dots A_{i_k} - 1$  corresponding to all defining relations  $r_j = g_{i_1} g_{i_2} \dots g_{i_k}$  of  $G$ . By Lemma 5.9(1) and Lemma 2.2(2),  $Det(A_i) - 1$  and  $A_{i_1} A_{i_2} \dots A_{i_k} - 1$  are indeed elements of  $M_n(P_n(I))^{GL_n(R)}$  and therefore  $\mathcal{J}$  is well defined. By Proposition 5.10(1) the ideal  $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \triangleleft M_n(P_n(I))$  is equal to  $M_n(\mathcal{J}')$ , where  $\mathcal{J}' \triangleleft P_n(I)$  is the ideal generated by coefficients of matrices belonging to  $\mathcal{J}$ . Notice that  $\mathcal{J}'$  is exactly the kernel of the epimorphism  $\eta : P_n(I) \rightarrow Rep_n^R(G)$  introduced in Section 2. Therefore by Proposition 5.10 the homomorphism

$$M_n(\eta)^{GL_n(R)} : M_n(P_n(I))^{GL_n(R)} \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$$

considered in diagram (5.5) descends to an isomorphism

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}.$$

**Proposition 5.11.**  $M_n(Rep_n^R(G))^{GL_n(R)}$  is generated by the elements  $j_{G,n}(g_i)$  and  $Tr(j_{G,n}(g_{i_1} g_{i_2} \dots g_{i_k}))$ , where  $i, i_1, i_2, \dots, i_k \in I$ .

*Proof.* From the paragraph preceding Proposition 5.11 it follows that  $M_n(\eta)^{GL_n(R)}$  is an epimorphism. By Theorem 5.1,  $M_n(P_n(I))^{GL_n(R)}$  is generated by the elements  $A_i$  and  $Tr(A_{i_1} A_{i_2} \dots A_{i_k})$ , where  $i, i_1, i_2, \dots, i_k \in I$ . The homomorphism  $M_n(\eta)^{GL_n(R)}$  carries these elements to  $j_{G,n}(g_i)$  and  $Tr(j_{G,n}(g_{i_1} g_{i_2} \dots g_{i_k}))$ , respectively. □

This proposition and Theorem 3.6 imply that  $\Theta$  is an epimorphism. We will show that it is also a monomorphism.

We have shown in Lemma 5.9 that  $Det(A_i) - 1 \in Ker \psi'$ .<sup>5</sup> Moreover, by the definition of  $\psi'$ ,  $A_{i_1}A_{i_2}\dots A_{i_k} - 1 \in Ker \psi'$ , for any  $i_1, i_2, \dots, i_k$  such that  $g_{i_1}g_{i_2}\dots g_{i_k} = e$  in  $G$ . Therefore  $\mathcal{J} \subset Ker \psi'$ , and we can factor  $\psi'$  to

$$\psi'' : M_n(Rep_n^R(G))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0),$$

such that diagram (5.5) commutes and, by Corollary 5.8,

- $\psi''(j_{G,n}(g_i)) = E_{g_i}$ ,
- $\psi''(Tr(j_{G,n}(g_{i_1}g_{i_2}\dots g_{i_k}))) = EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$ , for any  $i_1, i_2, \dots, i_k \in I$ .

Recall that our assumptions about the presentation of  $G$  (stated in the paragraph following Theorem 5.3) say that the inverse of any generator of  $G$  is also a generator and that every element of  $G$  is a product of non-negative powers of generators. Thus, by Proposition 3.5(4),  $\mathbb{A}_n(X, x_0)$  is generated by the elements  $E_{g_i}$  and  $EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$ , for  $i, i_1, i_2, \dots, i_k \in I$ . Since  $\psi'' \circ \Theta$  is the identity on the generators of  $\mathbb{A}_n(X, x_0)$ , it also is the identity on  $\mathbb{A}_n(X, x_0)$ . Therefore  $\Theta$  is a monomorphism.

In order to complete the proof of Theorem 3.7, we need to show that  $\theta$  is also an isomorphism.

Fact 3.4 implies that we have an embedding  $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$ ,  $\iota_*(L_g) = EL_g$ , for  $g \in G$ . Therefore we can consider  $\mathbb{A}_n(X)$  as a subring of  $\mathbb{A}_n(X, x_0)$ . Moreover, by Theorem 3.6,  $\theta$  is just the restriction of

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$$

to  $\mathbb{A}_n(X)$ . Therefore  $\theta$  is a monomorphism.

In order to show that  $\theta$  is an epimorphism we use once again an argument from invariant theory. By the naturality of the Reynolds operators  $\nabla : M_n(Rep_n^R(G)) \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$  and  $\nabla' : Rep_n^R(G) \rightarrow Rep_n^R(G)^{GL_n(R)}$ , the following diagram commutes:

$$\begin{array}{ccc} M_n(Rep_n^R(G)) & \xrightarrow{Tr} & Rep_n^R(G) \\ \downarrow \nabla & & \downarrow \nabla' \\ M_n(Rep_n^R(G))^{GL_n(R)} & \xrightarrow{Tr} & Rep_n^R(G)^{GL_n(R)} \end{array}$$

Since  $Tr : M_n(Rep_n^R(G)) \rightarrow Rep_n^R(G)$  and all Reynolds operators are epimorphic,  $Tr : M_n(Rep_n^R(G))^{GL_n(R)} \rightarrow Rep_n^R(G)^{GL_n(R)}$  is also epimorphic. But now commutativity of (3.1) implies that  $\theta$  is an epimorphism as well.

Therefore we have shown that  $\theta$  is an isomorphism. This completes the proof of Theorem 3.7.

### 6. SL<sub>n</sub>-CHARACTER VARIETIES

In this section we present one of the possible applications of Theorem 3.7 to a study of SL<sub>n</sub>-character varieties.

Let  $X$  be a path-connected topological space whose fundamental group  $G = \pi_1(X)$  is finitely generated. Let  $K$  be an algebraically closed field of characteristic 0. Recall that we noticed in Section 2 that the set of all SL<sub>n</sub>(K)-characters of  $G$ , denoted by  $X_n(G)$ , is an algebraic set whose coordinate ring is  $Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$ .

Let  $\chi_g = Tr(j_{G,n}(g)) \in Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$ , for any  $g \in G$ . It is not difficult to check that  $\chi_g$ , considered as an element of  $K[X_n(G)]$ , is a function which

<sup>5</sup>Recall that the map  $\psi'$  was defined in Corollary 5.8.

assigns to a character  $\chi$  the value  $\chi(g)$ . By Proposition 3.5(3) and Theorem 3.7,  $Rep_n^R(G)^{GL_n(K)}$  is generated by the elements  $Tr(j_{G,n}(g))$ . Therefore the functions  $\chi_g$  generate  $K[X_n(G)]$ .

By an  $SL_n$ -trace identity for  $G$  we mean a polynomial function in variables  $\chi_g, g \in G$ , which is identically equal to 0 on  $X_n(G)$ . For example,

$$\chi_g \chi_h = \chi_{gh} + \chi_{gh^{-1}}$$

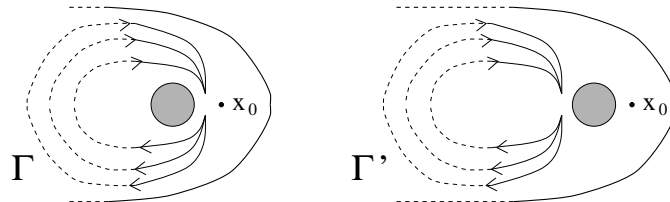
is the famous Fricke-Klein  $SL_2$ -trace identity, valid for any group  $G$  and any  $g, h \in G$ . From the above discussion it follows that the coordinate ring of  $X_n(G)$  can be considered as the quotient of the ring of polynomials in formal variables  $\chi_g, g \in G$ , by the ideal of all  $SL_n$ -trace identities for  $G$ . Therefore Theorem 3.7 implies the following corollary.

**Corollary 6.1.** *There is an isomorphism  $\Lambda : \mathbb{A}_n(X)/\sqrt{0} \rightarrow K[X_n(G)]$  such that  $\Lambda(L_g) = \chi_g$ . Under this isomorphism the identities on graphs in  $X$  induced by skein relations correspond to  $SL_n$ -trace identities for  $G$ .*

The above corollary is very useful in the study of trace identities, since it makes it possible to interpret them geometrically. Consider for example the following  $SL_3$ -trace identity, which holds for any  $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in G$  and any  $\chi \in X_3(G)$ , where  $G$  is an arbitrary group:

$$\begin{aligned} & \chi(\gamma_1)\chi(\gamma_2)\chi(\gamma_3) - \chi(\gamma_1)\chi(\gamma_2\gamma_3) - \chi(\gamma_2)\chi(\gamma_1\gamma_3) - \chi(\gamma_3)\chi(\gamma_1\gamma_2) \\ & + \chi(\gamma_1\gamma_2\gamma_3) + \chi(\gamma_1\gamma_3\gamma_2) - \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0)\chi(\gamma_3\gamma_0) \\ (6.1) \quad & + \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0\gamma_3\gamma_0) + \chi(\gamma_2\gamma_0)\chi(\gamma_1\gamma_0\gamma_3\gamma_0) + \chi(\gamma_3\gamma_0)\chi(\gamma_1\gamma_0\gamma_2\gamma_0) \\ & - \chi(\gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0) - \chi(\gamma_1\gamma_0\gamma_3\gamma_0\gamma_2\gamma_0) = 0. \end{aligned}$$

Our theory provides the following interpretation of this identity. Let  $x_0 \in X$  and  $G = \pi_1(X, x_0)$ . Let  $\gamma_0$  be a path in  $X$  representing a non-trivial element of  $\pi_1(X, x_0)$ . We assume that  $\gamma_0$  goes along a “hole” in  $X$  presented in the picture below. Consider the following two, obviously equivalent, graphs  $\Gamma$  and  $\Gamma'$  in  $X$ :



The graph  $\Gamma'$  is obtained from  $\Gamma$  by pulling its vertices along the “hole” in  $X$ . The obvious resolution of vertices in  $\Gamma$  and  $\Gamma'$  gives an equation involving closed loops in  $X$ . This equation corresponds to the trace identity (6.1).

There is a large body of literature about  $SL_2$ -character varieties and their applications. However, very little is known about  $SL_n$ -character varieties for  $n > 2$ . The reason for this is that the  $SL_n$ -trace identities, like (6.1), are intractable by classical (algebraic) methods. Since our theory often gives a simple geometric interpretation to complicated trace identities, it can be applied to a more detailed study of character varieties. This idea was already used in [PS-2] and [PS-3] to study  $SL_n$ -character varieties for  $n = 2$ . A generalization of these results for  $n > 2$ , which is based on our skein method, will appear in future papers. In this work we test our method on the simplest non-trivial example – we study  $SL_3$ -character

variety of the free group on two generators,  $F_2 = \langle g_1, g_2 \rangle$ . The basic problem is to determine the minimal dimension of the affine space in which  $X_3(F_2)$  is embedded, or equivalently, the minimal number of generators of  $K[X_3(F_2)]$ . A result of Procesi (Theorem 3.4(a) [Pro-1]) implies that  $K[X_3(F_2)]$  is generated by the elements  $\chi_{g_{i_1 g_{i_2} \dots g_{i_j}}}$ , where  $j \leq 7$  and  $i_1, i_2, \dots, i_j \in \{1, 2\}$ . A direct calculation shows that, after identifying words in  $g_1, g_2$  which are related by cyclic permutations, we get a set of 57 generators of  $K[X_3(F_2)]$ . It is difficult to obtain any further reduction of this set in any simple algebraic manner. However, our geometric method allows us to reduce this problem to the study of 3-valent graphs in the twice-punctured disc. By playing with pictures of such graphs one can reduce the number of generators of  $K[X_3(F_2)]$  to nine! These are

$$\chi_{g_1}, \chi_{g_2}, \chi_{g_1^2}, \chi_{g_2^2}, \chi_{g_1 g_2}, \chi_{g_1^2 g_2}, \chi_{g_1 g_2^2}, \chi_{g_1^2 g_2^2}, \chi_{g_1^2 g_2^2 g_1 g_2}.$$

Moreover, it is possible to show that this is the minimal number of generators, and  $X_3(F_2) \subset K^9$  is a solution set of one irreducible polynomial.

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