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BI-LIPSCHITZ HOMOGENEOUS CURVES IN $\mathbb{R}^2$ ARE QUASICIRCLES

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Abstract. We show that a bi-Lipschitz homogeneous curve in the plane must satisfy the bounded turning condition, and that this is false in higher dimensions. Combined with results of Herron and Mayer this gives several characterizations of such curves in the plane.

1. Introduction

A set $X \subset \mathbb{R}^d$ is called homogeneous with respect to a set of mappings $\mathcal{M}$ if for any two points $x, y \in X$ there is an $f \in \mathcal{M}$ such that $f(x) = y$. In this paper we are primarily concerned with the case when $\mathcal{M}$ is the set K-$BL(X)$ of K-bi-Lipschitz mappings of X to itself. K-bi-Lipschitz means that $K^{-1}|x-y| \le |f(x) - f(y)| \le K|x-y|$. In this case we say X is K-bi-Lipschitz homogeneous and for convenience write "X is K-BLH". We say X is bi-Lipschitz homogeneous if there is some $K < \infty$ for which this holds and write "X is BLH". If $X \subset \mathbb{R}^d$ is compact then Baire's theorem implies that if X is BLH, then it is K-BLH for some $K < \infty$ (e.g., the argument of Theorem 3.1 of [10], or Theorem 6.1 of [7]). However, D. Herron has constructed an example of a non-compact curve in the plane which is BLH, but not for a fixed K (personal communication). Our main result extends work of Ghamsari and Herron [5] and Herron and Mayer [6] by showing

Theorem 1.1. *If Γ is a BLH closed Jordan curve in $\mathbb{R}^2$, then it is a quasicircle.*

There are many equivalent definitions of quasicircles in the plane, but for our purposes it is most convenient to say Γ is a quasicircle iff it satisfies the bounded turning condition: there is a $C < \infty$ so that if $x, y \in \Gamma$, then the diameter of the smaller arc connecting x and y is less than $C|x-y|$. The class of curves with bounded turning will be denoted by BT. In [5] Ghamsari and Herron showed Theorem 1.1 was true if Γ was also rectifiable. In [6], Herron and Mayer proved it when Γ was homogeneous with respect to a group of uniformly bi-Lipschitz mappings. They also give an example of a curve in $\mathbb{R}^3$ which is BLH but not BT.

By combining Theorem 1.1 with the results of [6] we can deduce the following characterizations of BLH curves in the plane. The relevant definitions will be given following the statement of the theorem.

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Corollary 1.2. *Suppose* Γ *is a closed Jordan curve in* $\mathbb{R}^2$. *Then the following are equivalent*:

1. Γ *is BLH.*
2. *There is an* $h : S^1 \to \Gamma$ *which conjugates rotations to bi-Lipschitz self-maps of* Γ.
3. Γ *is homogeneous with respect to* $BL(\mathbb{R}^2)$.
4. Γ *is the orbit of a 1 parameter group in* $BL(\mathbb{R}^2)$.
5. Γ *has a very weak quasihomogeneous parameterization.*
6. Γ *has a weak quasihomogeneous parameterization.*
7. Γ *has a quasihomogeneous parameterization.*
8. *There is a* φ *so that* Γ *is* φ*-chord-arc.*
9. Γ *is pointwise bi-BLH.*
10. Γ *is BLH on arcs.*

In $\mathbb{R}^d$, $d \geq 3$, conditions (6)–(10) are equivalent and characterize BLH curves which are also BT. In $\mathbb{R}^3$, we will give examples to show that for the analogous conditions, (1) $\not\Rightarrow$ (2), (3) $\not\Rightarrow$ (4) and (5) $\not\Rightarrow$ (6). The remaining (non-trivial) implications are open so far as I know.

Now for the definitions. In what follows, the notation $a \simeq_C b$ means that $C^{-1} \leq \frac{a}{b} \leq C$, and we write $a \simeq b$ if this is true for some $C < \infty$. Similarly, for $a \lesssim_C b$, $a \lesssim b$, $a \gtrsim_C b$ and $a \gtrsim b$.

Suppose Γ is a closed Jordan curve in $\mathbb{R}^d$. We say h is *weak quasihomogeneous* (or WQH) if there is a $K < \infty$ so that for $x, y, u, v \in S^1$, $|x-y| \leq |u-v|$ implies $|h(x)-h(y)| \leq K|h(u)-h(v)|$. We say $h : S^1 \to \Gamma$ is *quasihomogeneous* (or QH) if there is a homeomorphism η of $[0,\infty)$ to itself such that

$$\frac{|h(x)-h(y)|}{|h(u)-h(v)|} \leq \eta(\frac{|x-y|}{|u-v|}),$$

for all distinct $x, y, u, v \in S^1$. These are homogeneous versions of the better known quasisymmetric conditions. A map $h : S^1 \to \Gamma$ is called *weakly quasisymmetric* (or WQS) if there is a $K < \infty$ so that

$$|y-x| \leq |z-x| \quad \text{implies} \quad |f(y)-f(x)| \leq K|f(z)-f(x)|,$$

for $x, y, z \in S^1$ and is called *quasisymmetric* (or QS) if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \eta(\frac{|y-x|}{|z-x|}),$$

for all triples in S^1. In the plane, both conditions are equivalent to Γ being a quasicircle. See [15]. Finally, we say that h is *very weakly quasihomogeneous* (or VWQH) if there is a $K < \infty$ so that for $x, y, u, v \in S^1$, $|x-y| = |u-v|$ implies $|h(x)-h(y)| \leq K|h(u)-h(v)|$.

Given an increasing function φ on $[0,\infty)$, the corresponding Hausdorff content is defined as

$$\mathcal{H}^{\varphi}_{\infty}(E) = \inf\{\sum_j \varphi(r_j) : E \subset \bigcup_j B(x_j, r_j)\}.$$

Hausdorff φ-measure is defined as

$$\mathcal{H}^{\varphi}(E) = \lim_{\delta \to 0} \inf\{\sum_j \varphi(r_j) : E \subset \bigcup_j B(x_j, r_j), r_j \leq \delta\}.$$

Both quantities have the same null sets, but only the latter is a measure. If $\varphi(t) = t^\alpha$, then $\mathcal{H}^\varphi$ is called α-dimensional measure and is denoted $\mathcal{H}^\alpha$. If $\alpha = 1$, then it is called linear measure. If $\alpha = d$ is the dimension of the space, then $\mathcal{H}^d$ and $\mathcal{H}^d_\infty$ are comparable.

Given an increasing function φ, a curve Γ is called *φ-chord-arc* if given $x, y \in \Gamma$ the smaller diameter subarc γ of Γ connecting these points has $\mathcal{H}^\varphi$ measure $\simeq \varphi(|x-y|)$. The usual notion of *chord-arc curve* corresponds to the special case $\varphi(t) = t$. The α-chord-arc condition with $\varphi(t) = t^\alpha$ has been carefully studied by Ghamsari and Herron in [4].

The curve Γ is *pointwise bi-BLH* if given any $C < \infty$, there is a $K < \infty$ so that for any $x, y, u, v \in \Gamma$ with $|x-y| \simeq_C |u-v|$ there is a K-bi-Lipschitz self-map f of Γ so that $f(x) = u$ and $f(y) = v$. Similarly, Γ is *BLH on arcs* if given any $C < \infty$ there is a $K < \infty$ so that for any two arcs $\gamma_1, \gamma_2 \subset \Gamma$ with $\mathrm{diam}(\gamma_1) \simeq_C \mathrm{diam}(\gamma_2)$, there is a K-bi-Lipschitz self-map g of Γ so that $f(\gamma_1) = \gamma_2$.

Curves homogeneous with respect to quasiconformal maps are considered in [2], [3] and [10]. Bi-homogeneous sets for quasiconformal mappings are considered in [9]. Also see [11] for further results on BLH curves in higher dimensions. I thank David Herron for his comments on a earlier version of this manuscript, and in particular for pointing out the results of [5] and [6].

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is by contradiction. We shall assume Γ is a BLH closed Jordan curve in $\mathbb{R}^d$ which fails the bounded turning condition and we shall deduce that for every point $x \in \Gamma$ and arbitrarily small $\epsilon > 0$ there is a radius $r > 0$ so that the ball $B(x, r)$ contains $\simeq \epsilon^{-2}$ points of Γ which are pairwise $\gtrsim \epsilon r$ apart. Using this we can rescale Γ and pass to a limit, obtaining a BLH closed set which has positive area, but which is not the whole plane. Finally, we prove that a closed BLH set must either be the whole plane or have zero area. The contradiction proves that Γ must have been a quasicircle after all.

In our definition of BLH, we did not assume that the self mappings of the curve Γ preserve orientation. We start by showing that we may assume that they do. The following is the same as Lemma 2.6 of [5], but we include a proof for completeness.

Lemma 2.1. *Suppose Γ is a Jordan curve which is K-bi-Lipschitz homogeneous. Then Γ is homogeneous with respect to K^2-bi-Lipschitz maps which preserve orientation.*

Proof. Fix $x, y \in \Gamma$. Let E_x be the set of $z \in \Gamma$ so that there is an orientation preserving K-bi-Lipschitz map sending x to z. Then $x \in E_x$ (because of the identity map) and E_x is closed (by equicontinuity and Ascoli's theorem we can take limits). Similarly, for E_y. If $E_x \cup E_y \neq \Gamma$, then choose $z \notin E_x \cup E_y$, and K-bi-Lipschitz, orientation reversing maps f, g which map x to z and y to z respectively. Then $g^{-1} \circ f$ is the desired map from x to y. If $E_x \cup E_y = \Gamma$, then since Γ is connected we must have $E_x \cap E_y \neq \emptyset$. Choose z in the intersection and again the corresponding composition is the desired map. $\square$

The following lemma contains the main idea in proving Theorem 1.1. It says that a BLH curve which is not BT must look 'thick' in a certain sense.

Lemma 2.2. *Suppose Γ is a closed Jordan curve in $\mathbb{R}^d$ which is K-bi-Lipschitz homogeneous with respect to orientation preserving self maps, but which does not* *satisfy the bounded turning condition. Then there is a $C < \infty$ so that for any $x \in \Gamma$ and for any $\frac{1}{10} > \epsilon > 0$ there is a $0 < \delta \le \epsilon$ and $r > 0$ so that the ball $B(x,r)$ contains more than $C^{-1}\delta^{-2}$ points of Γ, any two of which are at least distance $\delta r/C$ apart.*

Proof. Since Γ does not satisfy the bounded turning condition, we may assume, by dilating and translating if necessary, that $0 \in \Gamma$ and there is a point $x \in \Gamma$ with $|x| \le \epsilon$ and so that the smaller diameter arc $\gamma \subset \Gamma$ connecting 0 to x leaves the ball $B(0,1)$. Let γ_0 be the smallest closed subarc of γ containing 0 with both endpoints on the circle $|x| = K^{-2}/10$. Similarly, let γ_1 be the smallest open subarc of γ containing 0 and with both endpoints on $\{|x| = 1\}$. Let δ be the distance from γ_0 to $\Gamma \setminus \gamma_1$. Then δ is clearly positive since we are taking the distance between disjoint compact sets. Moreover, $\delta \le \epsilon$ since $0 \in \gamma_0$ and $x \in \Gamma \setminus \gamma_1$. Let $y_0 \in \gamma_0$ and $y_1 \in \Gamma \setminus \gamma_1$ be points where the minimum distance is obtained. Note that the shortest subarc of γ connecting γ_1 to y_1 has diameter at least $1/2$. Let γ_2 be the minimum closed subarc of Γ which contains γ_1 and has its endpoints on $\{|z| = 1/2\}$. Then clearly $y_1 \notin \gamma_2$.

Let $\mathcal{F}$ be the set of orientation preserving, K-bi-Lipschitz maps of Γ to itself. Choose $f_1 \in \mathcal{F}$ so that $f_1(y_0) = y_1$. Let $y_2 = f_1(y_1)$. In general, choose $f_n \in \mathcal{F}$ so that $f_n(y_0) = y_n$ and set $y_{n+1} = f_n(y_1)$. Since f_n is K-bi-Lipschitz,

$$|y_{n+1} - y_n| = |f_n(y_1) - f_n(y_0)| \le K|y_1 - y_0| \le K\delta.$$

Thus if $n \le K^{-2}\delta^{-1}$, all the points lie in the ball $B(0, 2K^{-1})$. Since each f_k is orientation preserving, we see that the points $\{y_k\}$ are distinct.

Choose a collection of $K^{-2}\delta^{-1}/20$ points $\{y_0^j\}$ on the arc γ_0 so that $y_0^1 = y_0$ and so that any two of them are at least distance δ apart. For example, we can choose y_0^j in $\Gamma \cap \{|z| = j\delta\}$ (at most two of these will have to be omitted because they are too close to y_0). For $n = 1, \dots, \delta^{-1}$ let $y_n^j = f_n(y_0^j)$. Note that y_0^{k+1} comes after y_0^k. This gives us a collection of $K^{-2}\delta^{-2}$ points and we claim that for any two of them the distance apart is at least δ/K

To prove this, consider y_n^j and y_m^k. If $n = m = 0$, then $|y_0^j - y_0^k| \ge \delta$ by construction. If $n = m \ne 0$, then

$$|y_n^j - y_n^k| = |f_n(y_0^j) - f_n(y_0^k)| \ge K^{-1}|y_0^j - y_0^k| \ge K^{-1}\delta$$

since f_n is K-bi-Lipschitz. Finally, suppose $n \ne m$. Without loss of generality, we may assume $n < m$. Let $w_0 = f_n^{-1}(y_m^0)$ and $w_1 = f_n^{-1}(y_m^k)$. Since our maps preserve orientation, $w_0 \notin \gamma_2$. Then $w_1 \in \tilde{\gamma} = f_n^{-1}(f_m(\gamma_0))$, which is an arc of diameter at most $K^2 \operatorname{diam}(\gamma_0) \le 1/10$. Since $\tilde{\gamma}$ contains w_0 (which is not in γ_2), and has small diameter, it cannot intersect γ_1. Thus $w_1 \notin \gamma_1$. Thus by our choice of δ,

$$|y_n^j - y_n^k| = |f_n(y_0^j) - f_n(w)| \ge K^{-1}|y_0^j - w_1| \ge K\delta.$$

To finish the proof we have to check that the collection of points described exists for any $z \in \Gamma$, instead of just the special point we have chosen. However, if we simply map 0 to z by an element of $\mathcal{F}$ the constructed points map to a collection with similar properties (but possibly larger constant C). This completes the proof of the lemma. $\square$

We now continue with the proof of Theorem 1.1. First recall the Hausdorff metric on closed sets on $\mathbb{R}^d$,

$$d_H(E,F) = \inf\{\epsilon : F \subset E_\epsilon \quad \text{and} \quad E \subset F_\epsilon\},$$

where $X_\epsilon = \{x : \mathrm{dist}(x,X) < \epsilon\}$. This makes the closed subsets of $\mathbb{R}^d$ into a complete metric space. We will say a sequence $\{E_n\}$ converges locally in the Hausdorff metric if $\{E_n \cap B(0,R)\}$ converges for every R. Every sequence of closed sets has a subsequence which converges locally.

Now suppose Γ is a BLH curve in $\mathbb{R}^d$. Choose a point $z \in \Gamma$ which is on the boundary of an open disk disjoint from Γ. Choose $\{\epsilon_n\} \to 0$ and use the previous lemma to find scales $\{r_n\}$ and collections $\mathcal{B}_n$ of disjoint balls as above. After rescaling to set $r_n = 1$, we have a sequence of curves $\{\Gamma_n\}$ and a sequence $\{\delta_n\} \to 0$ so that for each n there are at least $C\delta_n^{-2}$ points of $\Gamma_n \cap B(z,1)$ which are at least distance $C\delta_n$ apart. Let $\tilde{\Gamma}_n$ be the union of Γ_n and balls of radius $C\delta_n$ around each point. Clearly, $\tilde{\Gamma}_n$ has area which is bounded uniformly away from zero. By passing to a subsequence if necessary we may assume that $\{\Gamma_n\}$ converges locally in the Hausdorff metric to a closed set X and it is easy to check that X is also K-bi-Lipschitz homogeneous (use equicontinuity and Ascoli's theorem to take limits of mappings). The set X is not the entire plane because the point z was on the boundary of a disk disjoint from Γ. This easily implies X omits a half-plane. Moreover, X is also the local Hausdorff limit of a subsequence of $\{\tilde{\Gamma}_n\}$ and hence has positive area by the following easy fact.

Lemma 2.3. *If $\{X_n\}$ and X are compact sets in $\mathbb{R}^d$ and $X_n \to X$ in the Hausdorff metric. Then for any function φ,*

$$\mathcal{H}^\varphi_\infty(X) \geq \liminf_{n\to\infty} \mathcal{H}^\varphi_\infty(X_n).$$

In particular, if $d = 2$ and $\varphi(t) = t^2$, then $\mathrm{area}(X) \geq \limsup_n \mathrm{area}(X_n)$.

Proof. For any $\epsilon > 0$ choose a cover of X by open balls $\{B(x_j,r_j)\}$ so that $\sum \varphi(r_j) \leq \mathcal{H}^\varphi_\infty(X) + \epsilon$. Then $X_n \subset U$ for large enough n and hence

$$\mathcal{H}^\varphi_\infty(X) + \epsilon \geq \mathcal{H}^\varphi_\infty(U) \geq \limsup_n \mathcal{H}^\varphi_\infty(X_n).$$

Taking $\epsilon \to 0$ gives the desired conclusion. $\square$

We finish the proof of Theorem 1.1 by observing that the following result gives a contradiction. A set X is *porous at scales* $\leq R$ if there is a $M < \infty$ so that for every $x \in X$ and every $r \leq R$, there is a point $y \in \mathbb{R}^d$ so that $|x-y| \leq r$ and $\mathrm{dist}(y,X) \geq r/M$. This condition easily implies X has zero d-dimensional measure since this condition must fail for small r if x is a Lebesgue point of density of X (e.g. Proposition I.2, [14]). (In fact, if X is porous, then it has Hausdorff dimension strictly less than d.)

Lemma 2.4. *Suppose $X \subset \mathbb{R}^d$ is closed and BLH. Then X is porous at scales less than R, where R is the radius of any ball in the complement of X. In particular, either $X = \mathbb{R}^d$ or X has zero d-dimensional measure.*

Proof. If $X \neq \mathbb{R}^d$, then we can find a point $z \in X$ which is on the boundary of an open ball $B(x,R)$ which is disjoint from X. Now suppose X is not porous at scales $\leq R$. Then for any n there is a point $w_n \in X$ and a $0 < r_n < R$ so that every point of $B(w,r)$ is within distance r_n of a point of X. Define sets $Z_n = (X-z)/r_n$

and $W_n = (X - w_n)/r_n$ and a K-bi-Lipschitz map $f_n : Z_n \to W_n$ which maps 0 to 0. Passing to a subsequence, if necessary, we can pass to the limit to obtain closed sets Z and W and a bi-Lipschitz mapping $f : Z \to W$ taking 0 to 0. Since 0 is a boundary point of Z, but an interior point of W, this is clearly a contradiction, so we are done. $\square$

The argument of this section shows that if Γ is a BLH curve in $\mathbb{R}^d$, which is not BT, then it has rescalings which converge to a set of positive 2-dimensional measure which is not the whole space. The point is that this is only a contradiction when $d = 2$. We shall see an example in Section 4 of a BLH curve in $\mathbb{R}^3$ which has rescalings converging to a cylinder, $S^1 \times \mathbb{R}$.

3. Proof of Corollary 1.2

Most of this is immediate from Theorem A of [6], once we know that a BLH curve must be a quasicircle. To be more precise, their result says that $\Gamma \subset \mathbb{R}^d$ is BLH and BT iff it has a quasihomogeneous parameterization, i.e., condition (7) of Corollary 1.2. This trivially implies (6) and (5), and using Theorem C of [6], it implies (4) (and hence (2) and (3)). The equivalence of (1) and (8) is given by their Theorem A. The only conditions which they do not explicity mention are (9) and (10), so we include a short proof that these are equivalent BLH+BT in any dimension.

(6) $\Rightarrow$ (9), (10). Condition (6) implies Γ is BT, so it suffices to prove either (9) or (10) and the other is immediate. We will prove (9). Suppose γ_1 and γ_2 are subarcs of Γ so that $\mathrm{diam}(\gamma_1) \simeq \mathrm{diam}(\gamma_2)$. Let $I_j = h^{-1}(\gamma_j)$ for $j = 1, 2$. We claim I_1 and I_2 must have comparable lengths. Suppose not, e.g., suppose $|I_1| > N|I_2|$. Then by (6) , γ_1 contains N disjoint subarcs, all with diameter comparable to $\mathrm{diam}(\gamma_1)$. This contradicts the following:

Lemma 3.1. *If γ is an arc with the bounded turning property, then it can only contain a bounded number of disjoint subarcs with diameter* $\geq \mathrm{diam}(\gamma)/2$.

Proof. Suppose $\{\gamma_j\}$, $j = 1, \dots, M$, are disjoint subarcs with diameter $\geq \mathrm{diam}(\gamma)$ and let x_j denote the initial point of each arc. Since all the $\{x_j\}$ in a ball of radius $\simeq \mathrm{diam}(\gamma)$, there must be two of them which are at most $C\,\mathrm{diam}(\gamma)/M^{1/n}$ apart. The arc between these two is contained in γ and contains one of the arcs γ_j, hence has diameter $\geq \frac{1}{2}\,\mathrm{diam}(\gamma)$. The bounded turning condition now implies M is uniformly bounded. $\square$

Now back to the proof. We now have that $|I_1| \simeq |I_2|$. Thus there is a bi-Lipschitz mapping of the circle which takes I_1 to I_2. Clearly, h conjugates this to a bi-Lipschitz mapping of Γ taking γ_1 to γ_2 as desired.

(9) $\Rightarrow$ (1). The BLH condition is trivial, so we only have to show that (9) implies BT. Suppose γ is the smaller arc between x and y on Γ. Orient γ so x is the initial point and let z be the first point on γ where $|x - z| = |x - y|$. Let $\gamma_0 \subset \gamma$ be the arc from x to z. By (9) there is a bi-Lipschitz map of γ taking γ_0 to γ, hence $\mathrm{diam}(\gamma) \lesssim \mathrm{diam}(\gamma_0) \lesssim |x - y|$, as desired.

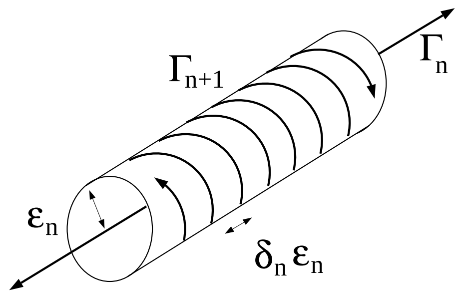

FIGURE 1.

(10) ⇒ (1). As above, we need only show that (10) implies BT. Suppose γ is the smaller arc between x and y on Γ. Choose a point $z \in \gamma$ so that $|x-z| \simeq |z-y| \simeq \operatorname{diam}(\gamma)$ and let γ_0 be the arc from x to z. Then by (10) there is a bi-Lipschitz map of Γ fixing x and taking z to y, so $\operatorname{diam}(\gamma) \simeq |x-z| \simeq |x-y|$, as desired.

4. BLH curves which are not BT

The following is essentially the same as Example 5.7 of [6]. We include it so that we can make a few further observations about the example.

Example 4.1. There is a BLH curve in $\mathbb{R}^3$ which is not BT.

Proof. We will define Γ as a limit of curves $\{\Gamma_n\}$. Let Γ_0 be a circle. Choose ϵ_0 small and let T_0 be $\{z \in \mathbb{R}^3 : \operatorname{dist}(z, \Gamma_0) = \epsilon_0\}$, i.e., T_0 is a long, thin torus with Γ_0 as an axis. Choose δ_0 small and let Γ_1 be a smooth spiral curve on T_0 with period $\delta_0\epsilon_0$. In general, given the smooth curve Γ_n, choose ϵ_n so small that $T_n = \{z \in \mathbb{R}^3 : \operatorname{dist}(z, \Gamma_n) = \epsilon_n\}$ locally looks like a flat cylinder (up to a very small distortion). Choose δ_n small and let Γ_{n+1} be a spiral on T_n with period $\delta_n\epsilon_n$. See Figure 1. It is clear that the normalized arclength parameterization satisfies the very weak homogeneity condition, and hence so does the limiting curve Γ. Thus Γ has a transitive 1-parameter bi-Lipschitz group, but is clearly not bounded turning if $\delta_n \to 0$. □

Example 4.2. Given $\epsilon > 0$, the curve Γ in Example 4.1 can be constructed so that every element of the circle group acting on Γ has a $(1+\epsilon)$-bi-Lipschitz extension to all of $\mathbb{R}^3$.

Proof. At each stage of the construction, we can choose ϵ_n so small that the group acting on Γ_n also preserves T_n and preserves distances up to a factor of $\epsilon 2^{-n}$ for points inside T_n that are at most ϵ_n apart. This group does not act on Γ_{n+1}, but for any $x, y \in \Gamma_{n+1}$ can be used to move x to a point that is at most "one twist" of Γ_{n+1} away from y. We can then compose with a rotation and translation of T_n along its axis that maps x to y. Use the previously constructed map at points more than $M\epsilon_n$ from T_n and in the region $\{\epsilon_n \le \operatorname{dist}(z, T_n) \le M\epsilon_n\}$ interpolate between the two maps by a twist. Passing to the limit shows that Γ is homogeneous with respect to $(1+\epsilon)$-bi-Lipschitz maps on all of $\mathbb{R}^3$. □

Example 4.3. The curve in the previous two examples can be constructed so that it is not the orbit of uniformly bi-Lipschitz group acting on $\mathbb{R}^3$.

Proof. Choose two points x and y that are $\delta_n\epsilon_n$ apart and joined by an arc of diameter $\epsilon_n/2$, i.e., x and y are nearby points on adjacent turnings of Γ at scale ϵ_n. Let γ be a path of length $\delta_n\epsilon_n$ connecting x and y and suppose there is a 1-parameter K-bi-Lipschitz group acting on $\mathbb{R}^3$ for which Γ is an orbit. Move x forward along Γ by distance ϵ_n, i.e., move it around the spiral $\epsilon_n\delta_{n+1}^{-1}$ times and consider what happens to the curve γ. The path cannot cross the middle half of the tube T_n. Therefore, the image of γ must wrap around the tube $\epsilon_n/(\epsilon_{n+1}\delta_{n+1})$ times and hence have length $\gtrsim \epsilon_n/\delta_{n+1} \simeq \operatorname{diam}(\gamma)/\delta_{n+1}$. Since $\delta_n \to 0$ we get a contradiction to the fact that γ and its image must have comparable length. Thus Γ cannot be the orbit of a continuous, uniformly bi-Lipschitz group of all of $\mathbb{R}^3$. $\square$

5. Further questions

We saw that in $\mathbb{R}^2$ that every BLH curve has a parameterization which conjugates rotations on the circle to bi-Lipschitz self-maps of Γ. What about higher dimensions?

Question 5.1. Is a curve in $\mathbb{R}^d$, $d > 2$ BLH iff it has a VWQH parameterization?

What happens if bi-Lipschitz maps are replaced by a larger class, e.g. quasisymmetric maps?

Question 5.2. Suppose Γ is a closed Jordan curve in $\mathbb{R}^2$ which is homogeneous with respect to QS maps. Must Γ be a quasicircle?

We can also ask how to characterize the BLH compact sets in $\mathbb{R}^d$, for each $d \geq 1$, although there may not be any simple answer in this generality. Note that there are non-locally connected examples even in $\mathbb{R}^2$, e.g., a Cantor set across a circle. Even in the case of compact, connected plane sets, this question may not be easy. Continua which are homogeneous with respect to self-homeomorphisms is a large and active area and even in the plane homogeneous continua have not been completely classified, although examples other than Jordan curves are known (see [13], [12] for a survey of some recent results). It is known, for example, that Jordan curves are the only homogeneous continua in the plane which contain arcs, or are locally connected, or are decomposable (i.e., can be written as a union of two proper sub-continua). Thus in Theorem 1.1 we could replace "Jordan curve" by just "curve" (a continuous image of S^1) or "locally connected continuum". It seems reasonable that none of the more exotic homogeneous planar continua can occur in the bi-Lipschitz case.

Question 5.3. If X is a K-BLH continua in $\mathbb{R}^2$, must it be a closed Jordan curve (and hence a quasicircle)? What if we assume K is close to 1?

This in not true in $\mathbb{R}^3$ because one can build a solenoid which is bi-Lipschitz homogeneous, but not locally connected (locally it looks like a Cantor set across an interval). Does a bi-Lipschitz homogeneous continuum in $\mathbb{R}^d$ always contain an arc?

The only known homogeneous continua in the plane are a point, a closed Jordan curve, the pseudo-arc and a circle of pseudo-arcs [12]. Thus a special case of the question above is

Question 5.4. Is there a pseudo-arc which is BLH?

See [1] for a definition of pseudo-arc and a proof that it is homogeneous. Briefly, a pseudo-arc is the intersection of sets D_n where each D_n is a union of a "chain" of open sets $\mathcal{D}_n = \{D_n^i\}$. Each element D_n^i of the chain has diameter $\leq 1/n$ and intersects D_n^j iff $|i-j| = 1$ and the chain $\mathcal{D}_n$ is "crooked" in the previous chain $\mathcal{D}_{n-1}$. This means that each link of $\mathcal{D}_n$ is contained in a link of $\mathcal{D}_{n-1}$ and that if $i < j$ and D_n^i intersects D_{n-1}^h and D_n^j intersects D_{n-1}^k with $|h-k| > 2$, then there are links D_n^r and D_n^s with $i < r < s < j$ so that D_n^r and D_n^s are subsets of links of $\mathcal{D}_{n-1}$ which are adjacent to D_{n-1}^k and D_{n-1}^h respectively.

If the answer to the previous question is no, then what is the best modulus of continuity that a transitive set of homeomorphisms of a pseudo-arc can have? If there is a BLH pseudo-arc (or other exotic example), what strengthening of bi-Lipschitz homogeneity implies implies Jordan curve? Note that in [8] it is proven that any homogeneous, compact metric space has a compatible metric with respect to which the space is BLH. Thus the fact, the continuum is in the plane must play an important role.

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