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ON BESSEL DISTRIBUTIONS FOR QUASI-SPLIT GROUPS

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Abstract. We show that the Bessel distribution attached by Gelfand and Kazhdan and by Shalika to a generic representation of a quasi-split reductive group over a local field is given by a function when it is restricted to the open Bruhat cell. As in the case of the character distribution, this function is real analytic for archimedean fields and locally constant for non-archimidean fields.

1. Introduction

The purpose of this paper is to provide a first step towards a regularity theorem for Bessel distributions. Such a theorem can play an analog role to the Regularity Theorem of Harish-Chandra ([13] Theorem 2, [14] Theorem 1) which is a key ingredient in Harish-Chandra's Plancherel Theorem. Moreover, the Bessel distributions appear naturally in the relative trace formula ([16], [17], [4]) and play an analogous role to the character distributions role in the regular trace formula ([1]).

Similar results were obtained by Hakim [12] for certain spherical distributions and by Rader and Rallis [22] for general spherical distributions attached to p-adic symmetric spaces.

In this present paper we will be concerned with representations that appear as components of the representation

$$\psi^* = \mathrm{ind}_U^G \psi.$$

Here G is (the rational points of) a quasi-split reductive group over a local field k, U is the unipotent radical of a Borel subgroup of G and ψ is a non-degenerate character of U. These representations (in the case of a finite field) first appeared in the paper of Gelfand and Graev [10]. The Bessel distributions first appeared in the paper of Gelfand and Kazhdan [11] for the group $GL_n(k)$, where k is a non-archimedean local field, and were used to prove the uniqueness of the Whittaker model. This result was elegantly generalized by Shalika [23] to quasi-split groups over any locally compact field. We view this current paper as a natural appendix to Shalika's paper. We shall retain his notations and his style of proof by proving our result first for $G = GL_n$ and indicating how to generalize it later.

Our aim is to attach a function, which we shall call the Bessel function, to the Bessel distribution that we are considering. This function will give the distribution on the open Bruhat cell. Our method of proof uses Harish-Chandra's techniques for invariant distributions and does not use the uniqueness of the Whittaker model. It is possible ([9], [24], [2], [3], [7]), at least in the non-archimedean case, to use

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the uniqueness in order to define Bessel functions without referring to the Bessel distributions. It is a natural question to ask if these Bessel functions agree with the Bessel functions which we introduce here. In the case of Gl_n over a non-archimedean field the answer is yes [3]. Moreover, in [2] and [3] we proved that for GL_2 and GL_3 these Bessel functions are locally integrable and give the Bessel distribution on the whole group. Bessel functions for $GL_2(\mathbf{R})$ were defined in [6]. In [4] it is proved that they coincide with the Bessel functions defined here.

In section 2 we set up the notation and state our main result. In section 3 we prove the non-archimedean result for GL_n and in section 4 we prove the archimedean result for $GL_n(\mathbf{R})$. In section 5 we give some rank 1 examples and in section 6 we indicate how to generalize our result to every quasi-split group.

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2. Statement of the main result

We retain the notation of [23]. We record them here for the benefit of the reader.

Let k be a field. If $\mathbf{M}$ is an algebraic group defined over k, $\mathbf{M}_k$ will denote the group of rational points. Let $\mathbf{G} = GL_n$. Let $\mathbf{H}$ denote the subgroup of diagonal matrices in $\mathbf{G}$. Let $\mathbf{U}$ denote the subgroup of upper-triangular matrices and $\bar{\mathbf{U}}$ the subgroup of $\mathbf{G}$ which is opposite to $\mathbf{U}$ (that is, lower triangular). If

$$w_0 = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix},$$

then $\bar{\mathbf{U}} = w_0 \mathbf{U} w_0^{-1}$.

If k is a local field, the groups $G = \mathbf{G}_k$, $H = \mathbf{H}_k$, $U = \mathbf{U}_k$ and $\bar{U} = \bar{\mathbf{U}}_k$ may be regarded as locally compact groups. Let $N(H)$ be the normalizer of H and let ψ be a one-dimensional unitary representation (character) of U. We shall say that ψ is non-degenerate if, for $s \in N(H)$ and not in H, the restriction of ψ to each of the groups $U \cap s^{-1}\bar{U}s$ is non-trivial.

For k non-archimedean (resp. archimedean), let $C_c^\infty(G)$ denote the space of complex valued, locally constant (resp. smooth) functions on G with compact support.

For $f \in C_c^\infty(G)$, $x, y \in G$, let

$$(L_x f)(y) = f(x^{-1}y), \ \ (R_x f)(y) = f(yx).$$

Let G act on distributions (on the left and right) by duality.

If k is archimedean, we may consider G as a real Lie group. We let $\mathfrak{g}$ be the Lie Algebra of G and $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}^{\mathbf{C}}$. We let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra. Let π be an irreducible admissible representation of G acting on a Hilbert space $\mathfrak{H}$. Let $\mathcal{D}(\pi)$ denote the linear subspace of $\mathfrak{H}$ spanned by all elements of the form $\pi(f)v$ with $f \in C_c^\infty(G)$ and $v \in \mathfrak{H}$. G leaves $\mathcal{D}(\pi)$ stable. If k is archimedean, we topologize $\mathcal{D}(\pi)$ by the semi-norms $||v||_D = ||Dv||$, $D \in \mathcal{U}(\mathfrak{g}), v \in \mathcal{D}(\pi)$.

Let $\mathcal{D}'(\pi)$ denote the vector space of all continuous linear functionals on $\mathcal{D}(\pi)$ (the algebraic dual if k is non-archimedean). Let $\mathcal{D}'_\psi(\pi)$ denote the subspace of $\lambda \in \mathcal{D}'(\pi)$ such that

$$\lambda(\pi(u)v) = \psi(u)\lambda(v), \quad u \in U, v \in \mathcal{D}(\pi).$$

Let $(\pi^*, \mathfrak{H}^*)$ denote the representation contragredient to $(\pi, \mathfrak{H})$. Given $\lambda \in \mathcal{D}'_\psi(\pi)$ and $\lambda' \in \mathcal{D}'_{\psi^{-1}}(\pi^*)$ we can define the distribution $J = J_{\pi,\lambda,\lambda'}$ on $C_c^\infty(G)$ by

$$J(f) = \langle \lambda, \pi^*(f)(\lambda') \rangle, \quad f \in C_v^\infty(G). \tag{2.1}$$

Here $\langle\ ,\ \rangle$ is a non-degenerate G invariant bilinear form on $\mathcal{D}'(\pi) \times \mathcal{D}(\pi)$. If k is non-archimedean, then we can view $\pi^*(f)(\lambda') \in \mathcal{D}(\pi)$ through an isomorphism of π with $(\pi^*)^*$. The same is true for k archimedean and f (left) K finite. For the general case see [23], p. 183 and p. 184. Our distribution J coincides with the distribution T in [23], p. 184. It is easy to see that J satisfies $L_u(J) = \psi^{-1}(u)J$ and $R_u J = \psi(u)J$ for all $u \in U$. These properties allow descent from G to H.

Remark 2.1. It follows from [23], p. 184 that if k is archimedean, then J is an eigendistribution for the center of $\mathcal{U}(\mathfrak{g})$.

Remark 2.2. If ψ is non-degenerate then by [23], $\mathcal{D}'_\psi(\pi)$ and $\mathcal{D}'_{\psi^{-1}}(\pi^*)$ are either both one-dimensional or both zero. In the case that they are one-dimensional, then the distribution $J_{\pi,\lambda,\lambda'}$ is determined uniquely up to scalars by ψ and we denote it by $J_{\pi,\psi}$. In that case we call it the Bessel distribution associated to π.

We are now ready to state the main result of this paper.

Theorem 2.3. *Let $J = J_{\pi,\lambda,\lambda'}$ be the distribution defined above. Let $X = UHw_0U$ be the open Bruhat cell in G and dx be a Haar measure on G restricted to X. Then there exist a function $j = j_{\pi,\lambda,\lambda'} : X \to \mathbb{C}$ such that*

$$J(f) = \int_X j(x)f(x)dx, \quad f \in C_c^\infty(X).$$

Moreover, if k is non-archimedean, then j is locally constant and if k is archimedean, then j is real analytic.

Remark 2.4. The theorem above holds for every ψ, degenerate or non-degenerate. However, the nature and asymptotics of the function $j = j_{\pi,\lambda,\lambda'}$ changes when ψ is degenerate. When ψ is degenerate we no longer expect j to be locally integrable. See Remark 5.1 for an example.

It will be convenient to reformulate Theorem 2.3 as follows. Let $\tilde{\lambda} = \pi^*(w_0)\lambda$. Then $\tilde{\lambda}$ satisfies

$$\tilde{\lambda}(\pi(\bar{u})v) = \psi_1(\bar{u})\tilde{\lambda}(v), \quad \bar{u} \in \bar{U}, v \in \mathcal{D}(\pi),$$

where ψ_1 is the character of $\bar{U}$ defined by

$$\psi_1(\bar{u}) = \psi(w_0^{-1}\bar{u}w_0).$$

Define $J_{\pi,\tilde{\lambda},\lambda'}$ as in (2.1). Then

$$J_{\pi,\tilde{\lambda},\lambda'} = L_{w_0}(J_{\pi,\lambda,\lambda'}).$$

It follows that proving Theorem 2.3 is equivalent to proving that $J_{\pi,\tilde{\lambda},\lambda'}$ is given by a function on the open set $\bar{U}HU$. We shall formulate now a slightly more general theorem.

Let ψ_1 be a character of $\bar{U}$ and ψ_2 be a character of U. Let $\lambda_1 \in \mathcal{D}'(\pi)$ and $\lambda_2 \in \mathcal{D}'(\pi^*)$ be such that

$$\lambda_1(\pi(\bar{u})v) = \psi_1(\bar{u})\lambda_1(v), \quad \bar{u} \in \bar{U}, v \in \mathcal{D}(\pi), \tag{2.2}$$

$$\lambda_2(\pi(u)v^*) = \psi_2(u)\lambda_2(v^*), \quad u \in U, v^* \in \mathcal{D}(\pi^*). \tag{2.3}$$

Define

$$J_{\pi,\lambda_1,\lambda_2}(f) = \langle \lambda_1, \pi^*(f)(\lambda_2)\rangle, \quad f \in C_c^\infty(G). \tag{2.4}$$

Then $J = J_{\pi,\lambda_1,\lambda_2}$ satisfies

$$L_{\bar{u}}(J) = \psi_1^{-1}(\bar{u})J, \quad \bar{u} \in \bar{U},$$

and

$$R_u J = \psi_2(u)J, \quad u \in U.$$

If k is archimedean, then J is an eigendistribution for $\mathcal{Z}(\mathfrak{g})$.

Theorem 2.5. *Let $Y = \bar{U}HU$ and let dy be a Haar measure on G restricted to Y. Then there exist a function $j = j_{\pi,\lambda_1,\lambda_2} : Y \to \mathbb{C}$ such that*

$$J_{\pi,\lambda_1,\lambda_2}(f) = \int_Y j(y)f(y)dy, \quad f \in C_c^\infty(Y).$$

Moreover, if k is non-archimedean, then j is locally constant and if k is archimedean, then j is real analytic.

3. The p-adic case

3.1. **Admissible distributions on Abelian groups.** In this section we recall some definitions and results from [14] regarding admissible distributions. Let G be an *abelian* p-adic group (by which we mean a totally disconnected *abelian* topological group with a base of open compact subgroups). Let $x \in G$ and let K be an open compact subgroup of G. For every character χ of K we define a function χ_x on xK by letting

$$\chi_x(xk) = \chi(k), \quad k \in K.$$

We can also view χ_x as a function on every set containing xK by letting it vanish outside of xK.

By a distribution Θ on a set $X \subseteq G$ we mean a linear functional from $C_c^\infty(X)$ to $\mathbf{C}$.

Definition 3.1. Let Θ be a distribution on an open set X of G. Let $x \in X$. We say that Θ is admissible at x if there exists an open compact subgroup K of G such that $xK \subset X$ and such that for every non-trivial character χ of K we have

$$\Theta(\chi_x) = 0.$$

We say that θ is admissible on X if it is admissible at every point $x \in X$.

Lemma 3.2. *Let G be abelian and X an open subset of G. A distribution Θ on X is admissible if and only if there exist a locally constant function θ on X such that*

$$\Theta(f) = \int_X \theta(x)f(x)dx$$

for all $f \in C_c^\infty(X)$. Here dx is a fixed Haar measure on G.

Proof. Assume Θ is admissible. Let $x \in X$. It is enough to prove that the statement holds for a small neighborhood of x. Let K be as in Definition 3.1. By Fourier inversion, every $f \in C_c^\infty(xK)$ can be written as

$$f = \sum_\chi \lambda_\chi \chi_x$$

where all but a finite number of the scalars λ_χ vanish. Hence

$$\Theta(f) = \lambda_1 \Theta(1_x) = \Theta(1_x) \int_X f(y)dy$$

so our function θ is given on xK by $\Theta(1_x)1_x$, hence it is constant on xK and locally constant on X.

The other direction is also easy and is left for the reader. □

3.2. **Minimal K-types for congruence subgroups of GL_n.** We recall some results of Howe and Moy [15] on minimal K-types for congruence subgroups of GL_n. These results were later generalized by Moy and Prasad [21] for reductive groups. Our proof of Theorem 2.5 for the general case of a quasi-split group over a non-archimedean local field will use the same argument as in the GL_n case utilizing the general result of Moy and Prasad.

Let k be a non-archimedean local field. Let R be the ring of integers in k and P the maximal ideal in R. We let $G = GL_n(k)$ and $K = GL_n(R)$. Let H, U, $\bar{U}$ be as in section 2. Let $K_m, m \geq 1$, be the congruence filtration of K where $K_m = 1 + M_n(P^m)$. K_{m+1} is normal in K_m and K_m / K_{m+1} is isomorphic to $M_n(P^m)/M_n(P^{m+1})$ through the isomorphism $A \mapsto A - 1$. The character group of the Abelian group $M_n(P^m)/M_n(P^{m+1})$ is isomorphic to $M_n(P^{-(m+1)})/M_n(P^{-m})$ by the following isomorphism. Let η be a character of k such that η is trivial on R and non-trivial on P^{-1}. For every element of A of $M_n(P^{-(m+1)})$ we define a character η_A on $M_n(P^m)$ trivial on $M_n(P^{m+1})$ by

$$\eta_A(X) = \eta(\mathrm{trace}(AX)).$$

The map $A \mapsto \eta_A$ is our isomorphism. Thus every character χ of K_m trivial on K_{m+1} is associated to a class in $M_n(P^{-(m+1)})/M_n(P^{-m})$. The pair (K_m, χ) is called an unrefined minimal K type if the coset associated to χ in $M_n(P^{-(m+1)})$ does not contain a nilpotent element. Howe and Moy ([15], Theorem 1.1) and Bushnell [5] proved existence properties of unrefined minimal K types in irreducible admissible representation of $GL_n(k)$. Howe and Moy ([15], Theorem 4.1) also proved some uniqueness properties. Moy and Prasad [21] generalized these results to arbitrary reductive groups. To state the existence result we would need to introduce the Iwahori filtrations. However, for our purpose, we shall only need a weak form of the uniqueness result and only for congruence subgroups:

Theorem 3.3 ([15], Corollary 4.2)**.** *Let π be an irreducible admissible representation of $GL_n(k)$. Then there exists at most one integer m such that the restriction of π to K_m contains an unrefined minimal K type.*

Let $H_m = H \cap K_m$ and let χ be a non-trivial character of H_m which is trivial on H_{m+1}. We can extend χ to a character $\tilde{\chi}$ on K_m using the Iwahori decomposition of K_m,

$$K_m = (\bar{U} \cap K_m) H_m (U \cap K_m)$$

and by defining

$$\tilde{\chi}(\bar{u}_1 h u_2) = \chi(h), \quad \bar{u}_1 \in \bar{U} \cap K_m, h \in H_m, u_2 \in U \cap K_m.$$

$\tilde{\chi}$ is associated with a coset in $M_n(P^{-(m+1)}) / M_n(P^{-m})$ and it is easy to see that this coset contains a diagonal element d. Since χ is non-trivial, d is not in $M_n(P^{-m})$. Let i be an integer such that $|d_{i,i}|$, the absolute value of the (i,i) entry of d is maximal. Then $|d_{i,i}| = q^r$ for some $r > m$. Every element a in the coset of d is of the form $a = d + p$ for some $p \in M_n(P^{-m})$. It is easy to see that $|(a^k)_{i,i}| = q^{kr}$, hence a is not nilpotent, that is, the coset of d does not contain nilpotent elements. It follows that $(K_m, \tilde{\chi})$ is an unrefined minimal K type. The following corollary to Theorem 3.3 is immediate.

Corollary 3.4. *Let π be an irreducible admissible representation of G. There exist an integer M such that for every $m \geq M$ and for every non-trivial character χ on H_m which is trivial on H_{m+1}, the restriction of π to K_m does not contain $\tilde{\chi}$.*

We can extend every non-trivial character of H_m (not necessarily trivial on H_{m+1}) to a character on an open compact subgroup in the following way. Let $l > m$ be the smallest integer such that χ is trivial on H_l. Let

$$K_\chi = (\bar{U} \cap K_{l-1}) H_m (U \cap K_{l-1}).$$

It is easy to check that K_χ is a group and that χ extends to a character $\tilde{\chi}$ of K_χ as above. We now have

Corollary 3.5. *Let π be an irreducible admissible representation of G. Let M be as in Corollary 3.4 and let $m \geq M$. Let χ be a non-trivial character of H_m and $\tilde{\chi}$ be the extended character on K_χ. Then the restriction of π to K_χ does not contain $\tilde{\chi}$.*

Proof. Let $l = l_\chi$ be as in the proof of Corollary 3.4 and let μ denote the restriction of χ to H_{l-1}. Let $\tilde{\mu}$ be the extension of μ to K_{l-1}. By Corollary 3.4, the restriction of π to K_{l-1} does not contain $\tilde{\mu}$. Since $K_\chi \supseteq K_{l-1}$ and the restriction of $\tilde{\chi}$ to K_{l-1} is $\tilde{\mu}$, our result follows. $\square$

3.3. **Proof of Theorem 2.5 for the p-adic case.** As above we assume that k is a non-archimedean local field. Given a distribution J on $Y = \bar{U} H U$ satisfying

$$L_{\bar{u}_1} J = \psi_1^{-1}(\bar{u}_1) J, \quad \bar{u}_1 \in \bar{U},$$

and

$$R_{u_2} J = \psi_2(u_2) J, \quad u_2 \in U,$$

we can attach to it a distribution σ_J on H in the following way. For every $\alpha \in C_c^\infty(Y)$ we define $\beta_\alpha \in C_c^\infty(H)$ by

$$\beta_\alpha(h) = \int_{\bar{U} \times U} \alpha(\bar{u}_1 h u_2) \psi_1^{-1}(\bar{u}_1) \psi_2^{-1}(u_2) d\bar{u}_1 du_2. \tag{3.1}$$

By Proposition 1.12 in [23] the map $\alpha \mapsto \beta_\alpha$ is surjective and there exists a unique distribution σ_J on H such that

$$J(\alpha) = \sigma_J(\beta_\alpha). \tag{3.2}$$

The distribution σ_J determines the distribution J (on Y) completely. If σ_J is given by a locally constant function $j_H : H \to \mathbf{C}$, then J is given by the locally constant function $j : Y \to \mathbf{C}$ given by $j(\bar{u}_1 h u_2) = \psi_1(\bar{u}_1) \psi_2(u_2) j_H(h) \Delta^{-1}(h)$, where $\Delta(h)$

is a modular function (Jacobian) such that $dg = d\bar{u}_1(\Delta(h)dh)du_2$. Hence it follows from Lemma 3.2 that to prove that J is given by a function on Y, it is enough to prove that σ_J is admissible. To conclude the proof of Theorem 2.5 for the non-archimedean case we now prove that $\sigma_{J_{\pi,\lambda_1,\lambda_2}}$ is admissible.

Proof. Let π be an irreducible admissible representation of G. Let λ_1, λ_2 be linear functionals on the space of π and π^* respectively satisfying the properties (2.2) and (2.3) respectively. Let $J_{\pi,\lambda_1,\lambda_2}$ be the distribution defined in (2.4) and $\sigma_{J_{\pi,\lambda_1,\lambda_2}}$ the induced distribution on H. Let $h_0 \in H$. We would like to show that $\sigma_{J_{\pi,\lambda_1,\lambda_2}}$ is admissible at h_0. We have

$$h_0 K_m = (h_0(\bar{U} \cap K_m)h_0^{-1})(h_0 H_m)(U \cap K_m).$$

Let M be a positive integer as in Corollary 3.5 and let $m \geq M$ be such that ψ_1 is trivial on $h_0(\bar{U} \cap K_m)h_0^{-1}$ and such that ψ_2 is trivial on $(U \cap K_m)$. In order to show that $\sigma_{J_{\pi,\lambda_1,\lambda_2}}$ is admissible at h_0 we shall show that $\sigma_{J_{\pi,\lambda_1,\lambda_2}}(\chi_{h_0}) = 0$ for every non-trivial character χ of H_m. (Here H_m is an open compact subgroup of H and plays the role of the subgroup K in Definition 3.1).

Let χ be a non-trivial character of H_m and let $\tilde{\chi}$ be the character of K_χ as defined in Corollary 3.5. (Notice that here K_χ is an open compact subgroup of G which contains H_m. K_χ depends on χ but its Iwahori decomposition will always have H_m as the middle piece). Let $\tilde{\chi}_{h_0}$ be the function which is defined on $h_0 K_\chi$ by translating $\tilde{\chi}$ and which vanishes outside of $h_0 K_\chi$. It follows from Corollary 3.5 that $\pi^*(\tilde{\chi}_{h_0})\lambda_2 = 0$. Hence $J_{\pi,\lambda_1,\lambda_2}(\tilde{\chi}_{h_0}) = 0$. It is easy to see that $\beta_{\tilde{\chi}_{h_0}} = c\chi_{h_0}$ for some non-zero scalar c. Hence $\sigma_{J_{\pi,\lambda_1,\lambda_2}}(\chi_{h_0}) = 0$ and we have proved that $\sigma_{J_{\pi,\lambda_1,\lambda_2}}$ is admissible at h_0. ☐

4. The archimedean case

In this section we will prove Theorem 2.5 for the case where $G = GL_n(\mathbf{R})$. Our argument here works for every split real reductive group. For the general case of quasi-split real reductive groups (which includes the case of complex reductive groups) we shall need a slightly different approach which will be presented in section 5. The difference is that in the case of split real groups, the "radial component" of the Casimir element gives rise to an elliptic differential operator. This is not true in the general case; hence, we will have to show that there exist a member of the center of the universal enveloping algebra that gives rise to an elliptic differential operator. The argument is standard and follows Harish-Chandra's proof for the character case.

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{R})$ be the Lie algebra of G. A basis for $\mathfrak{g}$ is given by the elementary matrices $E_{i,j}$, $1 \leq i, j \leq n$, where $E_{i,j}$ has a 1 at the (i,j)th entry and zero elsewhere. The Casimir element Δ in the center of the universal enveloping algebra $\mathcal{Z}(\mathfrak{g})$ can be represented in the form

$$\begin{aligned}\Delta &= E_{1,1}^2 + E_{2,2}^2 + ... + E_{n,n}^2 \\ &\quad + (n-1)E_{1,1} + (n-3)E_{2,2} + ... + (3-n)E_{n-1,n-1} + (1-n)E_{n,n} \\ &\quad + \sum_{1 \leq j < i \leq n} 2E_{i,j}E_{j,i}.\end{aligned} \tag{4.1}$$

Let $\Delta_1 = E_{1,1}^2 + E_{2,2}^2 + ... + E_{n,n}^2 + (n-1)E_{1,1} + ... + (1-n)E_{n,n}$. We identify $\mathcal{U}(\mathfrak{g})$ with the algebra of left invariant differential operators on G. $\mathcal{U}(\mathfrak{g})$ acts on distributions J on G, so that

$$(-1)(DJ)(f) = J(Df), \quad D \in \mathfrak{g}, f \in C_c^\infty(G).$$

Using Remark 2.1, it is easy to see that the following theorem implies Theorem 2.5.

Theorem 4.1. *Let J be a distribution on G satisfying*

$$L_{\bar{u}} J = \psi_1^{-1}(\bar{u}) J, \qquad \bar{u} \in \bar{U}, \tag{4.2}$$

$$R_u J = \psi_2(u) J, \qquad u \in U, \tag{4.3}$$

$$\Delta J = \kappa J \ \ (\textit{for some } \kappa \in \mathbf{C}). \tag{4.4}$$

Then there exists a real analytic function j on $Y = \bar{U}HU$ so that

$$J(f) = \int_Y f(y) j(y) dy, \quad f \in C_c^\infty(Y).$$

Proof. We restrict J to Y and associate to it a distribution σ_J on H as in (3.2). Here we shall describe σ_J in another way. The map $p : \bar{U} \times H \times U \to Y$ given by

$$p(\bar{u}_1, h, u_2) = \bar{u}_1 h u_2$$

induces an isomorphism from $C_c^\infty(\bar{U} \times H \times U)$ to $C_c^\infty(Y)$. We would like to compute the effect of Δ on this map. In other words, if $f \in C_c^\infty(Y)$ and if under the above isomorphism

$$f \mapsto \phi$$

for some $\phi \in C_c^\infty(\bar{U} \times H \times U)$, then

$$\Delta(f) \mapsto \alpha(\Delta)\phi \tag{4.5}$$

for some variable coefficient differential operator $\alpha(\Delta)$ on $C_c^\infty(\bar{U} \times H \times U)$. We shall now compute $\alpha(\Delta)$. To do that we first compute the effect of the basis elements $E_{i,j}$. It is useful to notice that $\operatorname{Ad} u_2(\Delta) = \Delta$.

Assume

$$f(y) = f_{a \otimes b \otimes c}(\bar{u}_1 h u_2) = a(\bar{u}_1) b(h) c(u_2),$$

for $a \in C_c^\infty(\bar{U})$, $b \in C_c^\infty(H)$ and $c \in C_c^\infty(U)$. Then

$$\begin{aligned} \operatorname{Ad} u_2(E_{i,i})(f)(\bar{u}_1 h u_2) &= \frac{d}{dt} f(\bar{u}_1 h u_2 \operatorname{Ad} u_2(\exp(tE_{i,i})))|_{t=0} \\ &= \frac{d}{dt} f(\bar{u}_1 h(\exp(tE_{i,i}))u_2)|_{t=0} \\ &= a(\bar{u}_1)(E_{i,i}b)(h)c(u_2). \end{aligned} \tag{4.6}$$

Similarly, for $i > j$,

$$\begin{aligned} \operatorname{Ad} u_2(E_{i,j})(f)(\bar{u}_1 h u_2) &= \frac{d}{dt} f(\bar{u}_1 h(\exp(tE_{i,j}))u_2)|_{t=0} \\ &= \frac{d}{dt} f(\bar{u}_1 \operatorname{Ad} h(\exp(tE_{i,j}))hu_2)|_{t=0} \\ &= \frac{d}{dt} f(\bar{u}_1(\exp((th_{i,i}/h_{j,j})E_{i,j}))hu_2)|_{t=0} \\ &= [(R_{E_{i,j}}a)(\bar{u}_1)][(h_{i,i}/h_{j,j})b(h)][c(u_2)] \end{aligned} \tag{4.7}$$

Finally, for $i > j$,

$$\begin{aligned}\operatorname{Ad} u_2(E_{j,i})(f)(\bar{u}_1 h u_2) &= \frac{d}{dt} f(\bar{u}_1 h(\exp(tE_{j,i}))u_2)|_{t=0} \\ &= a(\bar{u}_1)b(h)(L_{E_{i,i}}c)(u_2).\end{aligned} \tag{4.8}$$

Hence the differential operator $\alpha(\Delta)$ is given by

$$\alpha(\Delta) = 1 \otimes \Delta_1 \otimes 1 + \sum_{i>j} 2R_{E_{i,j}} \otimes \alpha_{i,j} \otimes L_{E_{j,i}}$$

where $\alpha_{i,j}$ is the multiplication operator

$$(\alpha_{i,j}b)(h) = \frac{h_{i,i}}{h_{j,j}} b(h), \quad b \in C_c^\infty(H),$$

and $R_{E_{i,j}}, L_{E_{i,j}}$ are the "right by " and "left by " differential operators, respectively, attached to $E_{i,j}$. Given a distribution J on Y we can define a distribution $\tilde{J}$ on $\bar{U} \times H \times U$ by letting $\tilde{J}(a \otimes b \otimes c) = J(f_{a\otimes b\otimes c})$. Now assume that J satisfies (4.4), (4.2) and (4.3). By (4.4), we have that $\alpha(\Delta)\tilde{J} = \kappa\tilde{J}$. By (4.2) and (4.3), there exist a distribution σ_J on H such that

$$\tilde{J}(a \otimes b \otimes c) = \left(\int a(\bar{u})\psi_1(\bar{u})d\bar{u}\right)\left(\int c(u)\psi_2(u)du\right)\sigma_J(b). \tag{4.9}$$

Notice that this is the same distribution σ_J which was defined in (3.2). By (4.9) we have

$$\begin{aligned}\tilde{J}(\alpha(\Delta)(a \otimes b \otimes c)) &= \tilde{J}(1 \otimes \Delta_1 \otimes 1(a \otimes b \otimes c)) \\ &\quad + \sum_{i>j} \tilde{J}(R_{E_{i,j}} \otimes \alpha_{i,j} \otimes L_{E_{j,i}}(a \otimes b \otimes c)) \\ &= \left(\int a(\bar{u})\psi_1(\bar{u})d\bar{u}\right)\left(\int c(u)\psi_2(u)du\right)\sigma_J(\Delta_1 b) \\ &\quad + \sum_{i>j}\left(\int (R_{E_{i,j}}a)(\bar{u})\psi_1(\bar{u})d\bar{u}\right)\left(\int (L_{E_{j,i}}c)(u)\psi_2(u)du\right)\sigma_J(\alpha_{i,j}b).\end{aligned} \tag{4.10}$$

Let $\psi^1_{i,j}$ and $\psi^2_{i,j}$, $i > j$, be the purely imaginary numbers given by

$$\psi_1(\exp(tE_{i,j})) = \exp(\psi^1_{i,j}t), \quad \psi_2(\exp(tE_{j,i})) = \exp(\psi^2_{i,j}t).$$

From (4.10) we obtain the variable coefficient differential operator $\gamma(\Delta)$ on H which is given by

$$\gamma(\Delta) = \Delta_1 + \sum_{i=1}^{n-1} \psi^1_{i,i+1}\psi^2_{i,i+1}\alpha_{i,i+1}. \tag{4.11}$$

It follows from (4.4) that σ_J satisfies the differential equation

$$\gamma(\Delta)(\sigma_J) = \kappa\sigma_J.$$

Since $\gamma(\Delta) - \kappa$ is an elliptic differential operator, it follows from the regularity theorem for elliptic differential operators that there exists a real analytic function j_H on H such that

$$\sigma_J(\phi) = \int_H j_H(h)\phi(h)dh, \quad \phi \in C_c^\infty(H).$$

$\square$

5. Rank 1 examples

In this section we shall give some examples of Bessel functions of rank 1 groups. These functions are basically classical Bessel functions or their analog for the p-adic case. We remark that Bessel functions for $GL_2(\mathbf{R})$ have been defined and computed in a different way in [6]. It is proved in [4] that these Bessel functions are the same as the ones we defined here. Our Bessel functions here are solutions of a second order differential equation. We will restrict ourselves to writing this equation explicitly for $SL_2(\mathbf{R})$ and giving the two-dimensional general solution. To find the precise solution (up to scalar) one can restrict the Bessel function in [6] to the group $SL_2(\mathbf{R})$ for the appropriate representation. Bessel functions for principal series of GL_2 over a non-archimedean local field are computed in [4]. Here we shall give the result without proof.

5.1. **Bessel functions for $SL_2(\mathbf{R})$.** Let $G = SL_2(\mathbf{R})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$. Let $X, Y, T \in \mathfrak{g}$ be

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$\triangle = 1/2T^2 + T + 2YX.$$

Then $\triangle$ is the Casimir element in $\mathcal{Z}(\mathfrak{g})$. Let

$$U = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbf{R} \right\}, \quad \bar{U} = \left\{ \bar{n}(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \mid x \in \mathbf{R} \right\}.$$

Let

$$H = \left\{ h(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a \in \mathbf{R}^* \right\}.$$

and

$$w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

For $r \in \mathbf{R}$ we let $\psi = \psi_r$ be a character of U defined by

$$\psi_r(n(x)) = e^{irx}.$$

We shall restrict ourselves to the situation of Theorem 2.3. Let J be a distribution on G satisfying

$$L(n)J = \psi^{-1}(n)J, \quad R(n)J = \psi(n)J, \quad n \in U, \tag{5.1}$$

and that

$$\triangle J = \kappa J. \tag{5.2}$$

Let $X = UHwU$ be the open Bruhat cell in G. As in (3.1) we define a mapping from $C_c^\infty(X)$ to $C_c^\infty(H)$ by

$$\phi_\alpha(h) = \int_{U \times U} \alpha(n_1 h w n_2) \psi(n_1) \psi(n_2) dn_1 dn_2.$$

We define a distribution σ_J on H by $\sigma_J(\phi_\alpha) = J(\alpha)$. Since J satisfies (5.1), σ_J is well-defined. It is easy to check that

$$\phi_{\triangle\alpha} = (1/2T^2 + T + 2r^2 h^{-2})\phi_\alpha. \tag{5.3}$$

We shall confuse $\phi(h)$ with $\phi(a) = \phi(h(a))$, $a \in \mathbf{R}^*$, $h \in H$. Let $D = 1/2T^2 + T + 2r^2a^{-2} - \kappa$. It follows from (5.2) and from (5.3) that

$$\sigma_J(D\phi) = 0, \quad \phi \in C_c^\infty(H). \tag{5.4}$$

For a differential operator D on H we let D^t be the differential operator satisfying

$$\int D\phi(h(a))y(h(a))da = \int \phi(h(a))D^t y(h(a))da, \quad \phi, y \in C_c^\infty(H).$$

Here da is the standard Lebesgue measure on R (and not the Haar measure on H which is identified with the Haar measure on R^* which is $dh = da/|a|$). Since (5.4) is an elliptic equation, there exist a real analytic function $y(a)$ such that σ_J is given by

$$\sigma_J(\phi) = \int \phi(a)y(a)da, \quad \phi \in C_c^\infty(H).$$

By (5.4), $y(a)$ satisfies the differential equation $D^t y = 0$. Writing this equation explicitly we have

$$y'' + a^{-1}y + (4r^2 - (1+2\kappa)a^{-2})y = 0. \tag{5.5}$$

If $4r^2 = 1$, then the reader will immediately recognize that (5.5) is the classical Bessel equation for J_ν where $\nu^2 = 1+2\kappa$. For the general case we need to distinguish two cases. Set $\nu^2 = 1 + 2\kappa$.

If $r \neq 0$, then a general solution for (5.5) is of the form

$$y(a) = c_1 J_\nu(2ra) + c_2 Y_\nu(2ra), \quad c_1, c_2 \in \mathbf{C},$$

where J_ν and Y_ν are the classical Bessel functions defined in ([18], 5.4). We can replace Y_ν with $J_{-\nu}$ if ν is not an integer.

If $r = 0$ and $\nu \neq 0$, then a general solution for (5.5) is of the form

$$y(h) = c_1|h|^\nu + c_2|h|^{-\nu}, \quad c_1, c_2 \in \mathbf{C},$$

If $\nu = 0$ we replace $c_2|h|^{-\nu}$ with $c_2\mathrm{Log}(|h|)$.

Remark 5.1. The different set of solutions gives different asymptotics at ∞. The first set of solutions has order $|a|^{-1/2}$ for $a \to \infty$ while the second case is the degenerate case where ψ is trivial and the asymptotics depend on the eigenvalue κ of the Casimir element $\triangle$.

Remark 5.2. If J is a distribution as above, then by Theorem 4.1 there exist a real analytic function $j : UHwU \to \mathbf{C}$ such that

$$J(\alpha) = \int_{UHwU} \alpha(g)j(g)dg, \quad \alpha \in C_c^\infty(UHwU).$$

Here dg is a Haar measure on G. j is closely related to the function y above. To compute j we notice that $dg = dn_1|a|^2dhdn_2 = dn_1|a|dadn_2$ and that

$$J(\alpha) = \int_{U\times H\times U} \alpha(n_1h(a)wn_2)\psi(n_1)y(a)\psi(n_2)dn_1dadn_2.$$

Hence we have

$$j(n(x)h(a)wn(y)) = e^{irx}|a|y(a)e^{iry}.$$

Remark 5.3. For a given representation of $SL_2(\mathbf{R})$ or $GL_2(\mathbf{R})$ it is possible to compute the constants c_1 and c_2 appearing in the formulas for $y(a)$ above. (See [6] and [4] and the example below.)

5.2. $GL_2(F)$**, where** F **is non-archimedean.** Let F be a non-archimedean local field and let $G = GL_2(F)$. Let B be the upper triangular Borel subgroup of G and let $\chi = (\chi_1, \chi_2)$ be a quasi-character of B. Here χ_1, χ_2 are quasi-characters of F^* and χ can be viewed as a character of the diagonal subgroup. Let π_χ be the infinite dimensional component of $Ind_B^G \chi$. Let ψ be a character of F which is also viewed as a character of U by the formula $\psi(n(x)) = \psi(x)$. By the formulas in [4] we have

$$j_{\pi_\chi,\psi}(h(a)w) = \int^* \chi_1(-a/z)\chi_2(-a/h)|a/z|\psi(a^2/z - z)dz.$$

Here $\int^* f(x)dx = \lim_{m\to\infty} \int_{q^{-m}\leq|x|\leq q^m} f(x)dx$.

Remark 5.4. The same formula holds when $F = \mathbf{R}$ with a suitable interpretation for $\int^*$ (cf. [4]). Indeed, let $\chi_1(x) = |x|^{q_1}$, $\chi_2(x) = |x|^{q_2}$ and $\psi(x) = e^{ix}$. Then if we ignore the question of convergence, we get that the integral above represents the K-Bessel function $hK_\nu(2ih)$ (cf. [18], (5.10.25)), where $\nu = q_1 - q_2$. But the K-Bessel of an imaginary argument ih is a solution of the regular Bessel equation for J_ν which is consistent with our solutions in the $SL_2(\mathbf{R})$ case above.

6. The general case

In this section we shall outline the proof of Theorem 2.5 for the general case of quasi-split reductive groups. Most of the arguments presented in section 3 and section 4 for GL_n carry over to the general case and we shall indicate how to generalize them.

Let $\mathbf{G}$ be a quasi-split reductive group over a local field k. Let $\mathbf{B}$ be a k split Borel subgroup and let $\mathbf{S}$ be a maximal k split torus contained in $\mathbf{B}$. Let $\mathbf{H} = Z(\mathbf{S})$ be the centralizer of $\mathbf{S}$. Then $\mathbf{H}$ is a maximal torus in $\mathbf{G}$ and $\mathbf{B} = \mathbf{HU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let $\bar{\mathbf{U}}$ be the group opposite to $\mathbf{U}$.

6.1. **The non-archimedean case.** Let k be a non-archimedean local field and let $H = \mathbf{H}_k$ and H_i, $i = 1, 2, ...$, be open compact subgroups of H as defined in [21], 3.2, where they are denoted Z_n or in [19], where they are denoted Z_t^0. If G is split, then

$$H_i = \{h \in H : \omega(\chi(h) - 1) \geq i \text{ for all } k\text{-rational characters } \chi \text{ of } \mathbf{H}\}.$$

where ω is the non-archimedean valuation on k. Let Δ be a basis of the affine root system and let $\Theta \subset \Delta$. Let $P = P_\theta$ be the parahoric subgroup associated to Θ (see [20], 2.2) and let P_n, $n = 0, 1, ...$, be the canonical filtration of P. (We can choose Θ to be a basis of the spherical root system to get the congruence subgroups of section 3. See the example in [19], 3.3.) Let $s_n = s_{n,\Theta}$ be the integer defined in ([20], 2.2). Then P_n share many properties of the congruence subgroups K_n defined in section 3. In particular, P_n has an Iwahori decomposition $P_n = (\bar{U}\cap K_n)H_{s_n}(U\cap K_n)$, P_{n+1} is normal in P_n and P_n/P_{n+1} is isomorphic to a quotient of certain R-Lie algebras, where R is the ring of integers in k. Every non-trivial character of H_{s_n} which is trivial on $H_{s_{n+1}}$ can be extended to a character of P_n and forms an unrefined minimal K type as defined in [21]. From here the proof follows exactly as the proof in section 3.3. The uniqueness theorem ([15], Theorem 4.1, Corollary 4.2) of Howe and Moy for minimal K types of the congruence subgroups K_n that is used in 3.3 is replaced by the uniqueness theorem of Moy and Prasad ([21], Theorem 3.5) for unrefined minimal K types of the filtration subgroups P_n.

6.2. **The archimedean case—a Harish-Chandra map.** Here k is archimedean and $G = \mathbf{G}_k$ is considered as a real Lie group with Lie algebra $\mathfrak{g}$. The proof of Theorem 4.1 and in particular (4.6), (4.7), (4.8) and (4.10) yields a map from $\mathcal{U}(\mathfrak{g})$ to the algebra of variable coefficients differential operators on H which we will describe now. Our interest is the restriction of this map to $\mathcal{Z}(G)$, the center of $\mathcal{U}(\mathfrak{g})$.

Let $\mathfrak{n} = \mathrm{Lie}(U)$ and $\bar{\mathfrak{n}} = \mathrm{Lie}(\bar{U})$. Let $d\psi_1$ and $d\psi_2$ be the differentials of ψ_1 and ψ_2 respectively and we extend them to characters of $\mathcal{U}(\mathfrak{n})$ and $\mathcal{U}(\bar{\mathfrak{n}})$ in an obvious way. Let $\mathfrak{h} = \mathrm{Lie}(H)$. Let Φ be the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}^{\mathbf{c}}$, let $\Phi^+ = \{\alpha_1, ..., \alpha_m\}$ be the set of positive roots determined by $\mathfrak{n}$. For every $\alpha \in \Phi^+$ we fix non-zero $X_\alpha \in \mathfrak{n}$ and $X_{-\alpha} \in \bar{\mathfrak{n}}$ in the root space of α and $-\alpha$ respectively. Let $\{H_1, ..., H_l\}$ be a basis of $\mathfrak{h}$. Then the elements

$$u((q_i),(r_i),(p_i)) = X_{-\alpha_1}^{q_1} \cdots X_{-\alpha_m}^{q_m} H_1^{r_1} \cdots H_l^{r_l} X_{\alpha_1}^{p_1} \cdots X_{\alpha_m}^{p_m}$$

form a basis for the vector space of $\mathcal{U}(\mathfrak{g})$. We define a Harish-Chandra map γ from $\mathcal{U}(\mathfrak{g})$ to the algebra of differential operators on H by defining it on the basis elements $u((q_i),(m_i),(p_i))$ as follows:

$$\begin{aligned} \gamma(u((q_i),(r_i),(p_i))) &= d\psi_1(X_{-\alpha_1})^{q_1} \cdots d\psi_1(X_{-\alpha_m})^{q_m} \\ &\times d\psi_2(X_{\alpha_1})^{p_1} \cdots d\psi_2(X_{\alpha_m})^{p_m} \\ &\times (-q_1\alpha_1 - ... - q_m\alpha_m) H_1^{r_1} \cdots H_l^{r_l} \end{aligned} \tag{6.1}$$

and extend to $\mathcal{U}(\mathfrak{g})$ by linearity. It is clear that when we apply γ to the Casimir element Δ in the universal enveloping algebra of $\mathfrak{gl}_n$ we get the same differential operator $\gamma(\Delta)$ as defined in (4.11).

Now we replace (4.4) in Theorem 4.1 with the more general condition that J is an eigendistribution for $\mathcal{Z}(\mathfrak{g})$. That is, there exist a character κ of $\mathcal{Z}(\mathfrak{g})$ such that

$$zJ = \kappa(z)J, \quad z \in \mathcal{Z}(\mathfrak{g}). \tag{6.2}$$

Given a distribution J on G satisfying (4.2), (4.3) and (6.2), we restrict J to $\bar{U}HU$ and produce a distribution σ_J on H as in section 4. Since J is an eigendistribution with respect to $\mathcal{Z}(\mathfrak{g})$ it follows that σ_J is an eigendistribution with respect to $\gamma(\mathcal{Z}(\mathfrak{g}))$. In order to finish the proof we must show that there exist elliptic differential operators in $\gamma(\mathcal{Z}(\mathfrak{g}))$. (If the group is split over $\mathbf{R}$, then the image of the Casimir Δ will be elliptic.) To do this we compare our map to the usual Harish-Chandra map $\tilde{\gamma} : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h})$. $\tilde{\gamma}$ is a particular choice of γ for the case where $d\psi_1 = 0$ and $d\psi_2 = 0$.

By the proof of [8] Theorem 7.4.5, $\tilde{\gamma}$ preserves the grading on $\mathcal{Z}(\mathfrak{g})$ and $S(\mathfrak{h})$. It follows that in the representation of an element $z \in \mathcal{Z}(\mathfrak{g})$ as a linear combination of $u((q_i),(r_i),(p_i))$ there appears a highest order term $u((q_i),(r_i),(p_i))$ such that $p_i = 0, q_i = 0, i = 1, ..., m$ (i.e., $u((q_i),(r_i),(p_i)) \in \mathcal{U}(\mathfrak{h})$). It follows that the highest order terms in $\gamma(z)$ and $\tilde{\gamma}(z)$ are the same. Since $\tilde{\gamma}(\mathcal{Z}(\mathfrak{g}))$ contains elliptic operators, it follows that $\gamma(\mathcal{Z}(\mathfrak{g}))$ contains elliptic operators and our proof of Theorem 2.5 is now complete.

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