

DEGREE-ONE MAPS BETWEEN HYPERBOLIC 3-MANIFOLDS WITH THE SAME VOLUME LIMIT

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Dedicated to Professor Shin'ichi Suzuki on his sixtieth birthday

ABSTRACT. Suppose that $f_n : M_n \rightarrow N_n$ ($n \in \mathbf{N}$) are degree-one maps between closed hyperbolic 3-manifolds with

$$\lim_{n \rightarrow \infty} \text{Vol}(M_n) = \lim_{n \rightarrow \infty} \text{Vol}(N_n) < \infty.$$

Then, our main theorem, Theorem 2, shows that, for all but finitely many $n \in \mathbf{N}$, f_n is homotopic to an isometry. A special case of our argument gives a new proof of Gromov-Thurston's rigidity theorem for hyperbolic 3-manifolds without invoking any ergodic theory. An example in §3 implies that, if the degree of these maps is greater than 1, the assertion corresponding to our theorem does not hold.

In this paper, we will study degree-one maps $f : M \rightarrow N$ between closed, hyperbolic manifolds. According to a stronger version of Mostow's Rigidity Theorem, Gromov-Thurston's Rigidity Theorem in [20, Chapter 6], the volume $\text{Vol}(N)$ of N is not greater than $\text{Vol}(M)$. Moreover, in the case of dimension ≥ 3 , the equality $\text{Vol}(M) = \text{Vol}(N)$ holds if and only if f is homotopic to an isometry. In any dimension n other than three, H.C. Wang [23] proved that the number of closed, hyperbolic n -manifolds with volume less than a given constant is finite up to isometry if $n \geq 4$ and up to homeomorphism if $n = 2$. On the other hand, in dimension three, the Hyperbolic Dehn Surgery Theorem [20] implies that there are infinitely many, closed, hyperbolic 3-manifolds with bounded volume. So, we are peculiarly interested in degree-one maps between hyperbolic 3-manifolds. Though this stronger rigidity theorem is powerful, one cannot apply it directly to some of the subjects which are important in the study of degree-one maps between geometric 3-manifolds. We refer to [16], [1], [6], [7], [24] and their references for various results and related topics concerning such subjects.

Consider a C^1 -map g homotopic to a degree-one map $f : M \rightarrow N$ between closed, hyperbolic 3-manifolds. Let $Dg : T(M) \rightarrow T(N)$ be the derivative of g . We may regard that g is close to a local isometry if, for any unit tangent vector v to M , the norm $\|Dg(v)\|$ is close to 1. The *metrical distortion* $\delta(g)$ of g is defined by

$$\delta(g) = \max\{|\|Dg(v)\| - 1|; v \in T(M) \text{ with } \|v\| = 1\}.$$

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Theorem 1 shows that, even if $\text{Vol}(M)/\text{Vol}(N)$ is arbitrarily close to 1, in general, any C^1 -map homotopic to f is far away from being an isometry.

Theorem 1. *Let $\varepsilon > 0$ be any positive number. Then, there exist closed, connected, orientable, hyperbolic 3-manifolds M, N with $(1 - \varepsilon)\text{Vol}(M) \leq \text{Vol}(N) < \text{Vol}(M)$ and satisfying the following.*

(0.1) *For any $R > 0$, they admit a degree-one map $f_R : M \rightarrow N$ such that the metrical distortion $\delta(g)$ of any C^1 -map $g : M \rightarrow N$ homotopic to f_R is greater than R .*

The following is our main theorem, where we consider degree-one maps without fixing domain and target manifolds and present a result contrasting with Theorem 1.

Theorem 2. *Let $f_n : M_n \rightarrow N_n$ ($n \in \mathbf{N}$) be degree-one maps between closed, connected, orientable, hyperbolic 3-manifolds with $\lim_{n \rightarrow \infty} \text{Vol}(M_n) = \lim_{n \rightarrow \infty} \text{Vol}(N_n) < \infty$. Then, for all but finitely many $n \in \mathbf{N}$, f_n is homotopic to an isometry.*

The proof of Theorem 2 is done by showing that the restriction $f_n|_{M_{n,\text{thick}(\varepsilon)}} : M_{n,\text{thick}(\varepsilon)} \rightarrow N_n$ is homotopic to a homeomorphism $j_n : M_{n,\text{thick}(\varepsilon)} \rightarrow N_{n,\text{thick}(\varepsilon)} \subset N_n$ for all sufficiently large $n \in \mathbf{N}$. However, the fact that $M_{n,\text{thick}(\varepsilon)}$ and $N_{n,\text{thick}(\varepsilon)}$ have the same topological type seems not to be an immediate consequence from the condition on volume limit. Our proof is based on the argument similar to that in Soma [17], where a certain rigidity theorem for hyperbolic 3-manifolds of infinite volume was proved by using the notion of w -nets, which consist of regular ideal simplices in \mathbf{H}^3 (see [17, p. 1238]). In the present case, we will define simplicial honeycombs $\mathcal{H}_{z,t}^{(m)}$ and twins $\mathcal{T}_{z,t}$ consisting of (proper) straight 3-simplices in \mathbf{H}^3 well approximated by regular ideal simplices. By using these tools, one can show that not only $M_{n,\text{thick}(\varepsilon)}$ is homeomorphic to $N_{n,\text{thick}(\varepsilon)}$, but also the homeomorphism is realized by a map homotopic to $f_n|_{M_{n,\text{thick}(\varepsilon)}}$.

In the original proof of Gromov-Thurston's Rigidity Theorem, a degree-one map $f : M \rightarrow N$ between closed hyperbolic 3-manifolds is lifted to a map $\tilde{f} : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ between the universal coverings which is radially extended to a $\pi_1(M)$ -equivariant measurable map $\tilde{f}_\infty : \partial\mathbf{H}^3 \rightarrow \partial\mathbf{H}^3$. In order to show that the extension is possible, the assumption of the equality $\text{Vol}(M) = \text{Vol}(N)$ was crucial. In the general case without setting this assumption, one cannot invoke the argument in [20] to guarantee the existence of \tilde{f}_∞ even if $\text{Vol}(M)$ and $\text{Vol}(N)$ are arbitrarily close to each other. As a special case of our argument, we will have a new proof of their rigidity theorem without using radially extended maps and hence without relying on any ergodic theory, see Remark 1 in §3.

When the degree of maps is greater than 1, the assertion corresponding to Theorem 2 does not hold. For any $d_0 \in \mathbf{N}$ with $d_0 > 1$, there exist degree- d_0 maps $f_n : M_n \rightarrow N_n$ between closed, connected, orientable, hyperbolic 3-manifolds with $\lim_{n \rightarrow \infty} \text{Vol}(M_n) = d_0 \lim_{n \rightarrow \infty} \text{Vol}(N_n) < \infty$, but such that each f_n is not homotopic to a (locally isometric) covering, see Example 1 in §3. Let $\rho_n : \pi_1(M_n) \rightarrow \text{PSL}_2(\mathbf{C})$ be the composition of the homomorphism $(f_n)_* : \pi_1(M_n) \rightarrow \pi_1(N_n)$ induced from the f_n and the holonomy $\text{hol}_n : \pi_1(N_n) \rightarrow \text{PSL}_2(\mathbf{C})$ of N_n . Then, this example also implies that $\{\rho_n\}$ is a sequence of torsion-free, discrete representations to $\text{PSL}_2(\mathbf{C})$ with $\lim_{n \rightarrow \infty} \text{Vol}(\rho_n) = \lim_{n \rightarrow \infty} \text{Vol}(M_n) < \infty$, but each ρ_n is not faithful, where $\text{Vol}(\rho_n)$ is the volume of ρ_n in the sense of Goldman [5]. We compare this fact with Goldman's rigidity theorem which says that, for any closed

hyperbolic 3-manifold M , a representation $\rho : \pi_1(M) \longrightarrow \text{Isom}^+(\mathbf{H}^3)$ is discrete and faithful if and only if $\text{Vol}(\rho) = \text{Vol}(M)$, see Dunfield [2, Theorem 6.1] for a clear proof.

The following corollary is a stronger version of Gromov-Thurston’s Rigidity Theorem.

Corollary 1. *For any $V > 0$, there exists a constant $c = c(V) > 0$ depending only on V and satisfying the following.*

(0.2) *Let $f : M \longrightarrow N$ be a degree-one map between closed, connected, orientable, hyperbolic 3-manifolds with $\text{Vol}(M) \leq V$. Then, f is homotopic to an isometry if and only if $(1 - c)\text{Vol}(M) \leq \text{Vol}(N)$.*

Note that Theorem 1 implies that $\lim_{V \rightarrow \infty} c(V) = 0$ for any $c(V)$ satisfying the property (0.2). From this rigidity theorem, we know that there are no degree-one maps between distinct 3-manifolds obtained by most Dehn surgeries on the same hyperbolic knot in S^3 . Corollary 2 is another application of Corollary 1, which concerns one of the main subjects on degree-one maps between 3-manifolds, for example see [10, Problem 3.100], [18], [19] and [15].

Corollary 2. *For any $V > 0$, there exists a natural number $n_0 = n_0(V)$ depending only on V and satisfying the following.*

(0.3) *Consider any ascending sequence*

$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \dots \xleftarrow{f_n} M_n$$

of non-homotopy equivalence, degree-one maps between closed, connected, orientable, hyperbolic 3-manifolds. If $n \geq n_0$, then the volume of M_n is greater than V .

Since the set \mathbf{V} of volumes of closed hyperbolic 3-manifolds is a well ordered subset of \mathbf{R} (see [20, Corollary 6.6.1]), any strictly decreasing sequence in \mathbf{V} is finite. From this fact, we know that any descending sequence of non-homotopy equivalence, degree-one maps between closed hyperbolic 3-manifolds is finite. On the other hand, strictly increasing, infinite sequences in \mathbf{V} are not necessarily unbounded. However, Corollary 2 implies that such a sequence in \mathbf{V} defined from ascending sequence of degree-one maps is always unbounded.

We will finish this section by outlining how to prove our rigidity theorem on degree-one maps $f : M \longrightarrow N$ between closed, hyperbolic 3-manifolds with $\text{Vol}(M)/\text{Vol}(N)$ sufficiently close to 1. As will be seen in Lemmas 1 and 2 in §1, for “most” of the straight 3-simplices $\sigma : \Delta^3 \longrightarrow M$ in M well approximated by regular ideal simplices, the straightened 3-simplex $\text{straight}(f \circ \sigma)$ of $f \circ \sigma$ in N is also approximated by such an ideal simplex. By using this fact, we wish to show that a lift $\tilde{f} : \mathbf{H}^3 \longrightarrow \mathbf{H}^3$ of f to the universal coverings is close to a Möbius transformation in a domain near the boundary $\partial\mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$. Lemma 6 implies that there exists a certain subset W of $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ adjacent to $\partial\mathbf{H}^3$ such that $\tilde{f}|_W$ is close to the identity. In fact, by using a simplicial honeycomb $\mathcal{H}_{z,t}^{(m)}$ satisfying a certain effectiveness condition, one can show that $\tilde{f}|_{V_z^{(m)} \times \{t\}}$ is close to the identity if we normalize \tilde{f} by composing some isometric transformation on \mathbf{H}^3 (Lemma 3), where $V_z^{(m)}$ is a subset of \mathbf{C} consisting of vertices of 9^m regular subtriangles of the same size in a certain regular triangle \hat{T}_z . Then, a simplicial twin $\mathcal{T}_{z,t}$ is used to show that such a normalization of \tilde{f} is taken independently of parameters (z, t) (Lemma 4). The subset W as above is contained in the union of $V_z^{(m)} \times \{t\}$ ’s for “most”

(z, t) in the parameter space $B_r(2R) \times (0, s] \subset \mathbf{C} \times \mathbf{R}_+$ given in §2. Then, we will show that $f|M_{\text{thick}(\varepsilon)}$ is homotopic to a homeomorphism $j : M_{\text{thick}(\varepsilon)} \rightarrow N_{\text{thick}(\varepsilon)}$ by relying on the fact that $\tilde{f}|W$ is close to the identity. Since j maps a meridian of each component of $M_{\text{thin}(\varepsilon)}$ to a meridian of the corresponding component of $N_{\text{thin}(\varepsilon)}$, j is extended to a homeomorphism $\hat{j} : M \rightarrow N$, which is homotopic to an isometry.

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1. PRELIMINARIES

Throughout the paper, all manifolds are assumed to be orientable, and suitably oriented in all cases. First of all, we will review briefly the fundamental notation and definitions needed in later sections, and refer to Thurston [20] for details on hyperbolic geometry, and to Hempel [8], Jaco [9] for 3-manifold topology.

Let $\mathbf{C} \times \mathbf{R}_+ = \{(z, t) \in \mathbf{C} \times \mathbf{R}; t > 0\}$ be the 3-dimensional upper half space with the Euclidean metric $ds_E^2 = |dz|^2 + dt^2$. Hyperbolic 3-space \mathbf{H}^3 is the Riemannian 3-manifold with the underlying space $\mathbf{C} \times \mathbf{R}_+$ and the metric $ds_H^2 = ds_E^2/t^2$. The group $\text{Isom}^+(\mathbf{H}^3)$ of orientation-preserving, isometric transformations on \mathbf{H}^3 is naturally identified with the group $\text{PSL}_2(\mathbf{C})$ of Möbius transformations on the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. It is easily seen that the compact-open topology on $\text{Isom}^+(\mathbf{H}^3)$ coincides with the quotient topology on $\text{PSL}_2(\mathbf{C})$ induced from that on $\text{SL}_2(\mathbf{C}) \subset \mathbf{C}^4$. The quotient space $M = \mathbf{H}^3/\Gamma$ of \mathbf{H}^3 by a torsion-free, discrete subgroup Γ of $\text{Isom}^+(\mathbf{H}^3)$ is called a *hyperbolic 3-manifold*. Then, the quotient map $p : \mathbf{H}^3 \rightarrow M$ is the locally isometric, universal covering. For an $\varepsilon > 0$, the ε -thin part $M_{\text{thin}(\varepsilon)}$ of M is the set consisting of all points $x \in M$ such that there exists a non-contractible loop l in M with $l \ni x$ and of length $\leq \varepsilon$. The complement $M_{\text{thick}(\varepsilon)} = M - \text{int}M_{\text{thin}(\varepsilon)}$ is called the ε -thick part of M . A *Margulis tube* is an embedded, equidistant, tubular neighborhood of a short closed geodesic in M . A \mathbf{Z} or a $\mathbf{Z} \times \mathbf{Z}$ -cusp C is a subset of M such that each component of $p^{-1}(C)$ is a horoball the stabilizer of which in Γ is isomorphic to either \mathbf{Z} or $\mathbf{Z} \times \mathbf{Z}$. By the Margulis Lemma [20, Corollary 5.10.2], there exists an $\varepsilon_0 > 0$ independent of Γ such that, for any $\varepsilon > 0$ less than ε_0 , each component of $M_{\text{thick}(\varepsilon)}$ is either a Margulis tube or \mathbf{Z} or $\mathbf{Z} \times \mathbf{Z}$ -cusp. The M is called *geometrically finite* if M contains a convex core $C(M)$ such that $C(M) \cap M_{\text{thick}(\varepsilon)}$ is compact. If $\text{Vol}(M) < \infty$, then M has at most finitely many Margulis tubes. If necessary replacing ε by a sufficiently smaller positive number, we may assume that $M_{\text{thin}(\varepsilon)}$ has no Margulis tube components. Then, each component of $M_{\text{thin}(\varepsilon)}$ is a $\mathbf{Z} \times \mathbf{Z}$ -cusp.

Let Δ^k be a regular k -simplex of edge length 1 in the Euclidean k -space. For any smooth manifold N , let $C^1(\Delta^k, N)$ be the topological space of C^1 -maps $\Delta^k \rightarrow N$ with C^1 -topology. We denote by $\mathcal{C}_k(N)$ the \mathbf{R} -vector space consisting of Borel measures μ on $C^1(\Delta^k, N)$ with compact support and bounded total variation. Here, the *total variation* $\|\mu\|$ of μ is defined by

$$\|\mu\| = \int_{C^1(\Delta^k, N)} d\mu_+ + \int_{C^1(\Delta^k, N)} d\mu_-,$$

where μ_+, μ_- are the positive and negative parts of μ respectively, that is, $\mu_+ \geq 0, \mu_- \geq 0$ and $\mu = \mu_+ - \mu_-$. An element of $\mathcal{C}_k(N)$ is called a k -chain. The boundary

operator $\partial_k : \mathcal{C}_k(N) \rightarrow \mathcal{C}_{k-1}(N)$ is naturally defined. Then, the \mathbf{R} -vector space of k -cycles is denoted by $\mathcal{Z}_k(N)$. The homology determined by $(\mathcal{C}_*(N), \partial_*)$ is equal to the standard homology $H_*(N, \mathbf{R})$ for N , for example see Zastrow [25]. Here, the duality between the homology and cohomology is given by

$$(1.1) \quad \langle \mu, \omega \rangle = \int_{C^1(\Delta^k, N)} \left(\int_{\Delta^k} \sigma^*(\omega) \right) d\mu(\sigma),$$

where $\mu \in \mathcal{Z}_k(N)$ and ω is a closed k -form on N , see [20, Chapter 6] for details.

Here, we return to the case where M is a hyperbolic 3-manifold with the universal covering $p : \mathbf{H}^3 \rightarrow M$. For any continuous map $\tilde{\sigma} : \Delta^3 \rightarrow \mathbf{H}^3$, let $\text{straight}(\tilde{\sigma}) : \Delta^3 \rightarrow \mathbf{H}^3$ be the affine map with respect to the Euclidean structure on Δ^3 and the quadratic model on \mathbf{H}^3 such that $\text{straight}(\tilde{\sigma}(v_j)) = \tilde{\sigma}(v_j)$ for all vertices v_1, \dots, v_4 of Δ^3 . We note that the image $\text{straight}(\tilde{\sigma})(\Delta^3)$ is a (possibly degenerate) straight 3-simplex in \mathbf{H}^3 . For a continuous map $\sigma : \Delta^3 \rightarrow M$, $\text{straight}(\sigma) = p \circ \text{straight}(\tilde{\sigma}) : \Delta^3 \rightarrow M$ is called the 3-simplex obtained by *straightening* σ , where $\tilde{\sigma} : \Delta^3 \rightarrow \mathbf{H}^3$ is a lift of σ to \mathbf{H}^3 . If $z = \sum r_i \sigma_i$ ($r_i \in \mathbf{R}$) is a 3-cycle on M , then $\text{straight}(z) = \sum r_i \text{straight}(\sigma_i)$ is also a 3-cycle on M with $[\text{straight}(z)] = [z]$ in $H_3(M, \mathbf{R})$.

Now, we suppose that the M is closed. For any non-degenerate, straight 3-simplex $\sigma : \Delta^3 \rightarrow \mathbf{H}^3$, we will define a (Borel measure) 3-cycle $z_M(\sigma) \in \mathcal{Z}_3(M)$ associated to σ such that $\|z_M(\sigma)\| = \text{Vol}(M)$ and $[z_M(\sigma)] = \text{Vol}(\sigma)[M]$ in $H_3(M; \mathbf{R})$, where $[M]$ is the fundamental class of M . The 3-chain $\text{smear}_M(\sigma) \in \mathcal{C}_3(M)$ satisfying the following (1.2) and (1.3) is defined in [20, Chapter 6] by using a left-right invariant Haar measure μ_0 on $\text{Isom}^+(\mathbf{H}^3) = \text{PSL}_2(\mathbf{C})$. The measure μ_0 is normalized so that for any bounded Borel subset U of \mathbf{H}^3 ,

$$\mu_0(\{\alpha \in \text{Isom}^+(\mathbf{H}^3); \alpha(0, 1) \in U\}) = \text{Vol}(U),$$

where $(0, 1) \in \mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ is the base point of \mathbf{H}^3 . The μ_0 will play an important role in the proof of the key lemma, Lemma 6, for Theorem 2.

(1.2) The support $\text{supp}(\text{smear}_M(\sigma))$ consists of all C^1 -maps $\tau : \Delta^3 \rightarrow M$ admitting lifts $\tilde{\tau} : \Delta^3 \rightarrow \mathbf{H}^3$ with $\tilde{\tau} = \alpha \circ \tilde{\sigma}$ for some $\alpha \in \text{Isom}^+(\mathbf{H}^3)$.

(1.3) Consider any Borel subset \mathcal{A} of $\text{Isom}^+(\mathbf{H}^3)$ such that $\mathcal{A} \cap \gamma \mathcal{A} = \emptyset$ for any non-trivial covering transformations $\gamma \in \pi_1(M) - \{1\}$. Then, we have $\text{smear}_M(\sigma)(A) = \mu_0(\mathcal{A})$ for the subset $A = \{p \circ \alpha \circ \sigma; \alpha \in \mathcal{A}\} \subset \text{supp}(\text{smear}_M(\sigma))$.

Intuitively, the measure $\text{smear}_M(\sigma)$ “smears M evenly (1.3)” “with straight, singular 3-simplices in M isometric to σ (1.2)”. Then, we define $z_M(\sigma) \in \mathcal{Z}_3(M)$ by

$$z_M(\sigma) = \frac{1}{2}(\text{smear}_M(\sigma) - \text{smear}_M(\sigma_-)),$$

where $\sigma_- : \Delta^3 \rightarrow \mathbf{H}^3$ is a mirror image of σ . Since, by (1.3) and the normalization of μ_0 , $\|\text{smear}_M(\sigma)\| = \|\text{smear}_M(\sigma_-)\| = \text{Vol}(M)$, $\|z_M(\sigma)\|$ is also equal to $\text{Vol}(M)$. Moreover, by (1.1) and (1.2),

$$\langle z_M(\sigma), \Omega_M \rangle = \int_{C^1(\Delta^3, M)} \left(\int_{\Delta^3} \tau^*(\Omega_M) \right) d(z_M(\sigma))(\tau) = \text{Vol}(\sigma)\text{Vol}(M),$$

where Ω_M is the volume form on M . This implies that $[z_M(\sigma)] = \text{Vol}(\sigma)[M]$ in $H_3(M, \mathbf{R})$. Even in the case where M is of finite volume but not closed, $z_M(\sigma)$ is defined similarly. Then, $z_M(\sigma)$ is a measure on $C^1(\Delta^3, M)$ with locally compact support, and represents the element $\text{Vol}(\sigma)[M]$ in the locally finite homology $H_3^{\text{l.f.}}(M, \mathbf{R})$.

Let $\{N_n\}$ be a sequence of Riemannian manifolds (possibly with boundary) of the same dimension such that each N_n has a base point x_n . We say that $\{N_n\}$ converges smoothly to a Riemannian manifold N with a base point x_0 if there exist sequences $\{\varepsilon_n\}, \{R_n\}$ with $\varepsilon_n \searrow 0, R_n \nearrow \infty$ and diffeomorphisms $f_n : \mathcal{N}_{R_n}(x_n, N_n) \rightarrow \mathcal{N}_{R_n}(x_0, N)$ satisfying $1 - \varepsilon_n < \|Df_n(v)\| < 1 + \varepsilon_n$ for all unit tangent vectors $v \in T_x(N_n)$ with $x \in \mathcal{N}_{R_n}(x_n, N_n)$. Here, $\mathcal{N}_R(x, N)$ denotes the R -neighborhood of x in N . When $n \in \mathbf{N}$ is sufficiently large, N_n is said to be well approximated by N . Let $\{\Delta_n\}_{n=1}^\infty$ be a set of non-degenerate, straight 3-simplices in \mathbf{H}^3 such that the base point of each Δ_n is its internal center. According to [20, Chapter 7], $\{\Delta_n\}$ converges smoothly to a regular ideal 3-simplex Δ_{reg} in \mathbf{H}^3 if and only if $\lim_{n \rightarrow \infty} \text{Vol}(\Delta_n) = \text{Vol}(\Delta_{\text{reg}})$. Throughout the remainder of this paper, we set $\mathbf{v}_3 = \text{Vol}(\Delta_{\text{reg}})$.

Let X, N be hyperbolic 3-manifolds of finite volume, and $p : \mathbf{H}^3 = \tilde{X} \rightarrow X$ the universal covering of X . A continuous map $\varphi : X \rightarrow N$ can be lifted to a map $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{N}$ between the universal coverings. For any $\eta > 0$, a non-degenerate, straight 3-simplex $\sigma : \Delta^3 \rightarrow \tilde{X}$ is said to be η -effective (resp. η -ineffective) with respect to $\tilde{\varphi}$ if $\iota \text{Vol}(\text{straight}(\tilde{\varphi} \circ \sigma)) > \mathbf{v}_3 - \eta$ (resp. $\iota \text{Vol}(\text{straight}(\tilde{\varphi} \circ \sigma)) \leq \mathbf{v}_3 - \eta$), where $\iota = \text{Vol}(\sigma)/|\text{Vol}(\sigma)|$. Since $\tilde{\varphi}$ is continuous, the η -effectiveness is an open condition. That is, any straight 3-simplex $\tau : \Delta^3 \rightarrow X$ close to an η -effective 3-simplex σ in $C^1(\Delta^3, X)$ is also η -effective. In particular, there exists an open neighborhood \mathcal{U} of the identity in $\text{Isom}^+(\mathbf{H}^3)$ such that $\alpha \circ \sigma$ is η -effective for any $\alpha \in \mathcal{U}$. We say that a straight 3-simplex $\tau : \Delta^3 \rightarrow X$ is η -effective with respect to φ if its lift $\tilde{\tau} : \Delta^3 \rightarrow \tilde{X}$ is η -effective with respect to $\tilde{\varphi}$. For a straight 3-simplex $\sigma : \Delta^3 \rightarrow \mathbf{H}^3$, let $\mathcal{C}_{\text{effec}}^\eta$ be the subset of $\text{supp}(z_X(\sigma))$ consisting of η -effective 3-simplices with respect to φ , and set $\mathcal{C}_{\text{ineffec}}^\eta = \text{supp}(z_X(\sigma)) - \mathcal{C}_{\text{effec}}^\eta$. Then, we have

$$z_X(\sigma) = z_{\text{effec}}^\eta + z_{\text{ineffec}}^\eta \quad \text{and} \quad \|z_X(\sigma)\| = \text{Vol}(X) = \|z_{\text{effec}}^\eta\| + \|z_{\text{ineffec}}^\eta\|,$$

where $z_{\text{effec}}^\eta = z_X(\sigma)|_{\mathcal{C}_{\text{effec}}^\eta}$ and $z_{\text{ineffec}}^\eta = z_X(\sigma)|_{\mathcal{C}_{\text{ineffec}}^\eta}$.

Here, let us suppose that (i) $X_{\text{thin}(\delta)}$ consists only of $\mathbf{Z} \times \mathbf{Z}$ -cusps for some $\delta > 0$, (ii) N is closed, and (iii) $\varphi(P)$ is either a closed geodesic or a point in N for each component P of $X_{\text{thin}(\delta)}$. In the former case of the condition (iii), we assume moreover that $\tilde{\varphi}(\tilde{P})$ is a geodesic line in \tilde{N} for any component \tilde{P} of $p^{-1}(P)$. The last condition (iii) is used to show that, for any 3-cycle z in $(X, X_{\text{thin}(\delta)})$, $\text{straight}(\varphi_*(z))$ is a 3-cycle in $(N, \varphi(X_{\text{thin}(\delta)}))$. In fact, for any singular 2-simplex $\tau : \Delta^2 \rightarrow X_{\text{thin}(\delta)}$, a lift $\tilde{\varphi} \circ \tau : \Delta^2 \rightarrow \mathbf{H}^3$ of $\varphi \circ \tau$ is contained in a geodesic (or a point) $\tilde{\varphi}(\tilde{P})$, so $\text{straight}(\tilde{\varphi} \circ \tau)$ is also in $\tilde{\varphi}(\tilde{P})$. Thus, $\partial \text{straight}(\varphi_*(z)) = \text{straight}(\varphi_*(\partial z))$ is a 2-cycle in $\varphi(X_{\text{thin}(\delta)})$. The fundamental classes of $(X, X_{\text{thin}(\delta)})$ and $(N, \varphi(X_{\text{thin}(\delta)}))$ are denoted respectively by $[X, X_{\text{thin}(\delta)}] \in H_3(X, X_{\text{thin}(\delta)}; \mathbf{Z}) \cong \mathbf{Z}$ and $[N, \varphi(X_{\text{thin}(\delta)})] \in H_3(N, \varphi(X_{\text{thin}(\delta)}); \mathbf{Z}) \cong H_3(N; \mathbf{Z}) \cong \mathbf{Z}$. Then, we have the following lemma.

Lemma 1. *If $\varphi_*([X, X_{\text{thin}(\delta)}]) = [N, \varphi(X_{\text{thin}(\delta)})]$ and $\text{Vol}(\sigma) > \mathbf{v}_3 - \varepsilon$ for some $\varepsilon > 0$, then*

$$\|z_{\text{ineffec}}^\eta\| < \frac{\mathbf{v}_3}{\eta} (\text{Vol}(X) - \text{Vol}(N)) + \frac{\varepsilon}{\eta} \text{Vol}(N).$$

Proof. Since $[z_X(\sigma)] = \text{Vol}(\sigma)[X]$ in $H_3^{\text{l.f.}}(X; \mathbf{R})$, $\varphi_*(z_X(\sigma))$ represents $\text{Vol}(\sigma) \times [N, \varphi(X_{\text{thin}(\delta)})]$ in $H_3(N, \varphi(X_{\text{thin}(\delta)}); \mathbf{R})$. This shows that

$$\begin{aligned} \text{Vol}(\sigma)\text{Vol}(N) &= \langle \varphi_*(z_X(\sigma)), \Omega_N \rangle \\ &= \langle \text{straight}(\varphi_*(z_X(\sigma))), \Omega_N \rangle \\ &= \langle \text{straight}(\varphi_*(z_{\text{effec}}^\eta)), \Omega_N \rangle + \langle \text{straight}(\varphi_*(z_{\text{ineffec}}^\eta)), \Omega_N \rangle \\ &\leq \mathbf{v}_3 \|z_{\text{effec}}^\eta\| + (\mathbf{v}_3 - \eta) \|z_{\text{ineffec}}^\eta\|, \end{aligned}$$

where Ω_N is the volume form on N . Thus, our desired inequality is obtained from the facts $\|z_{\text{effec}}^\eta\| = \text{Vol}(X) - \|z_{\text{ineffec}}^\eta\|$ and $\text{Vol}(\sigma) > \mathbf{v}_3 - \varepsilon$. \square

Let \mathcal{F} be a measurable subset of $\text{Isom}^+(\mathbf{H}^3)$ such that $\mathcal{F} \cap \gamma\mathcal{F} = \emptyset$ for any non-trivial covering transformation $\gamma \in \pi_1(X) - \{1\}$ of \mathbf{H}^3 and such that $\alpha \circ \sigma : \Delta^3 \rightarrow \mathbf{H}^3$ is η -ineffective with respect to $\tilde{\varphi}$ for any $\alpha \in \mathcal{F}$. Since the correspondence $\alpha \mapsto p \circ \alpha \circ \sigma$ maps \mathcal{F} injectively into $\mathcal{C}_{\text{ineffec}}^\eta$, by (1.3), $\mu_0(\mathcal{F}) \leq \text{smear}_X(\sigma)(\mathcal{C}_{\text{ineffec}}^\eta)$. Since $2\|z_{\text{ineffec}}^\eta\| = \text{smear}_X(\sigma)(\mathcal{C}_{\text{ineffec}}^\eta) + \text{smear}_X(\sigma_-(\mathcal{C}_{\text{ineffec}}^\eta))$, Lemma 1 implies the following.

Lemma 2. *With the same assumptions as in Lemma 1,*

$$\mu_0(\mathcal{F}) < \frac{2\mathbf{v}_3}{\eta}(\text{Vol}(X) - \text{Vol}(N)) + \frac{2\varepsilon}{\eta}\text{Vol}(N).$$

Lemma 2 means that, for any subset \mathcal{F} as above, $\mu_0(\mathcal{F})$ is small enough if both $\varepsilon > 0$ and $\text{Vol}(X) - \text{Vol}(N)$ are sufficiently small compared with η .

2. SIMPLICIAL HONEYCOMBS AND TWINS

For an $a > 0$ and $z \in \mathbf{C}$, we denote by $B_a(z)$ the disk in \mathbf{C} centered at z of radius a . Let R, r, s be positive numbers with $r < R$ and $s < 1$. For a $z \in B_r(2R)$, consider the regular triangle \hat{T}_z in \mathbf{C} centered at the origin $0 \in \mathbf{C}$ and such that z is a vertex of \hat{T}_z . Divide \hat{T}_z into 9^m regular triangles $T_{z,1}, T_{z,2}, \dots, T_{z,9^m}$ of the same size for $m \in \mathbf{N}$. These subtriangles are numbered so that the first six triangles

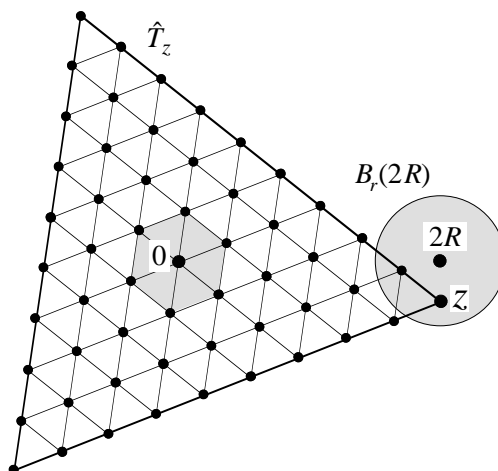


FIGURE 1. The case of $m = 2$. The shaded hexagon represents the union $T_{z,1} \cup \dots \cup T_{z,6}$.

$T_{z,1}, \dots, T_{z,6}$ have the origin of \mathbf{C} as a common vertex, see Fig. 1. Let $\{v_{i0}, v_{i1}, v_{i2}\}$ be the set of vertices of $T_{z,i}$, and let $v_{i0} = 0$ if $i = 1, 2, \dots, 6$. Note that the vertex union $V_z^{(m)} = \bigcup_{i=1}^{9^m} \{v_{i0}, v_{i1}, v_{i2}\} - \{0\}$ consists of $3^{m+1}(3^{m-1} + 1)/2$ points evenly distributed in $\widehat{T}_z - \{0\}$. For any $i = 1, 2, \dots, 6$ and $t \in (0, s]$, consider the straight 3-simplex $\Delta_{z,i,t}$ in \mathbf{H}^3 spanned by the four points $(0, 1/s)$, $(0, s)$, (v_{i1}, t) and $(v_{i2}, t) \in \mathbf{C} \times \mathbf{R}_+ = \mathbf{H}^3$. For $i = 7, 8, \dots, 9^m$, let $\Delta_{z,i,t}$ be the straight 3-simplex spanned by $(0, 1/s)$, (v_{i0}, t) , (v_{i1}, t) and (v_{i2}, t) . We say that the set $\mathcal{H}_{z,t}^{(m)} = \{\Delta_{z,i,t}; i = 1, 2, \dots, 9^m\}$ is the *simplicial honeycomb* in \mathbf{H}^3 of type (z, m, t) (for short (z, m, t) -honeycomb), see Fig. 2.

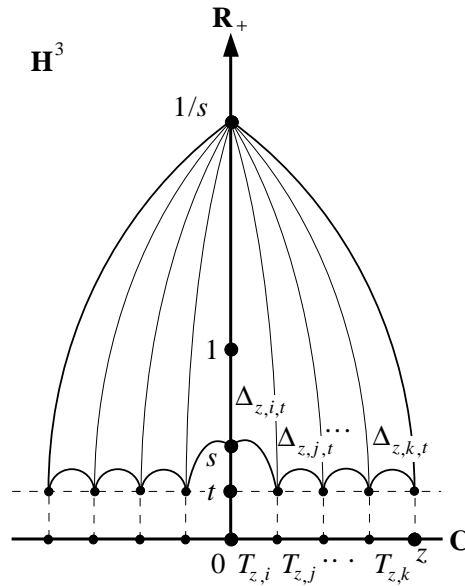


FIGURE 2.

Consider an open condition \mathbf{P} for straight 3-simplices in \mathbf{H}^3 . Fix an open neighborhood \mathcal{A} of the identity in $\text{Isom}^+(\mathbf{H}^3)$. Let $\mathcal{E}^{(m,s)} \subset \mathcal{A} \times B_r(2R) \times (0, s]$ be the (α, z, t) such that $\alpha\Delta_{z,i,t}$ satisfies \mathbf{P} for all $i = 1, 2, \dots, 9^m$. The set $\mathcal{E}^{(m,s)}$ can be decomposed as follows. Let $\mathcal{F}_{z,i,t}^{(m)}$ be the closed subset of \mathcal{A} consisting of those α for which $\alpha\Delta_{z,i,t}$ does not satisfy \mathbf{P} . Then,

$$\mathcal{E}^{(m,s)} = \{ (\alpha, z, t); (z, t) \in B_r(2R) \times (0, s], \alpha \in \mathcal{A} - \bigcup_{i=1}^{9^m} \mathcal{F}_{z,i,t}^{(m)} \}.$$

Since \mathbf{P} is an open condition, $\mathcal{E}^{(m,s)}$ is open in $\mathcal{A} \times B_r(2R) \times (0, s]$.

Let S_0 be the regular triangle in \mathbf{C} spanned by $(0, w_1, w_2)$, where $w_1 = 2Re^{\pi/6}/\sqrt{3}$ and $w_2 = 2Re^{-\pi/6}/\sqrt{3}$. Note that \widehat{T}_{2R} can be divided into nine triangles isometric to S_0 . For any $z \in B_r(2R)$, consider the triangle S_z in \mathbf{C} spanned by (w_1, w_2, z) . The triangle S_z is regular if and only if $z = 2R$. Let ∇_0 be the straight 3-simplex in $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ spanned by $(0, 1/s)$, $(0, s)$, (w_1, s) , (w_2, s) , and let $\nabla_{z,t}$ be that spanned by $(0, 1/s)$, (w_1, s) , (w_2, s) , (z, t) for $t \in (0, s]$. We say that $\mathcal{T}_{z,t} = \{\nabla_0, \nabla_{z,t}\}$ is the *simplicial twin* in \mathbf{H}^3 of type (z, t) (for short (z, t) -twin), see Fig. 3. For an open condition \mathbf{P}' for straight 3-simplices in \mathbf{H}^3 , let

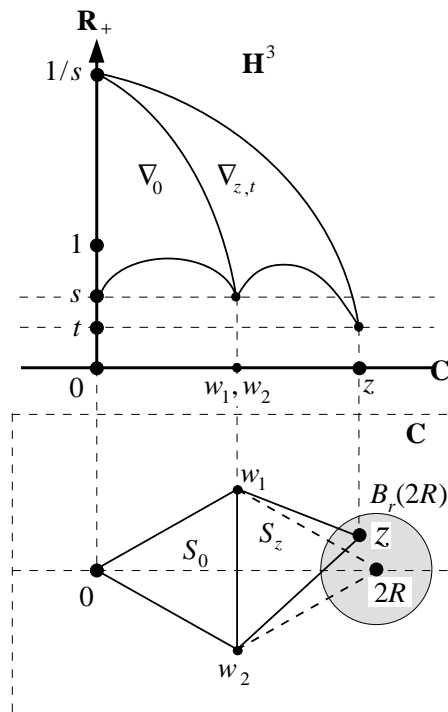


FIGURE 3.

$\mathcal{E}' \subset \mathcal{A} \times B_r(2R) \times (0, s]$ be the (α, z, t) such that both $\alpha \nabla_0$ and $\alpha \nabla_{z,t}$ satisfy \mathbf{P}' . The set \mathcal{E}' can be decomposed as follows. Let \mathcal{G}_0 (resp. $\mathcal{G}_{z,t}$) be the closed subset of \mathcal{A} consisting of those α for which $\alpha \nabla_0$ (resp. $\alpha \nabla_{z,t}$) does not satisfy \mathbf{P}' . Then,

$$\mathcal{E}' = \{ (\alpha, z, t); (z, t) \in B_r(2R) \times (0, s], \alpha \in \mathcal{A} - \mathcal{G}_0 \cup \mathcal{G}_{z,t} \}$$

Since \mathbf{P}' is an open condition, \mathcal{E}' is open in $\mathcal{A} \times B_r(2R) \times (0, s]$.

These $\mathcal{E}^{(m,s)}$ and \mathcal{E}' for certain open conditions will be used in the proof of Lemma 6. For $\Delta \in \mathcal{H}_{z,t}^{(m)} \cup \mathcal{T}_{z,t}$, any vertex of Δ other than $(0, 1/s)$ is called a *lower vertex*. We say that $\mathcal{H}_{z,t}^{(m)}$ (resp. $\mathcal{T}_{z,t}$) satisfies \mathbf{P} if each Δ in $\mathcal{H}_{z,t}^{(m)}$ (resp. in $\mathcal{T}_{z,t}$) satisfies \mathbf{P} .

For a given continuous map $\tilde{\varphi} : \mathbf{H}^3 \rightarrow \mathbf{H}^3$, we will study the η -effectiveness condition $\mathbf{P}_\eta(\tilde{\varphi})$ with respect to $\tilde{\varphi}$. That is, a straight 3-simplex Δ in \mathbf{H}^3 of $\text{Vol}(\Delta) > 0$ satisfying $\mathbf{P}_\eta(\tilde{\varphi})$ means that $\text{Vol}(\text{straight}(\tilde{\varphi}(\Delta))) > \mathbf{v}_3 - \eta$.

Now, let us consider honeycombs and twins under the fixed constant R . For any $m \in \mathbf{N}$, there exists a constant $\varepsilon = \varepsilon_m(s) > 0$ with $\lim_{s \rightarrow 0} \varepsilon_m(s) = 0$ and such that each $\Delta_{z,i,t}$ in $\mathcal{H}_{z,t}^{(m)}$ has the volume greater than $\mathbf{v}_3 - \varepsilon$ if $(z, t) \in B_r(2R) \times (0, s]$. For a $(z, t) \in B_r(2R) \times (0, s]$, if necessary deforming $\tilde{\varphi}$ slightly by homotopy, one can suppose that $\tilde{\varphi}(0, 1/s)$, $\tilde{\varphi}(0, s)$, $\tilde{\varphi}(z, t)$ are not on any single geodesic line in \mathbf{H}^3 . Then, the composition $\tilde{\varphi}^{(z,t)} = \beta \circ \tilde{\varphi} : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ is called the (z, t) -normalization of φ if we choose $\beta \in \text{Isom}^+(\mathbf{H}^3)$ so that $\tilde{\varphi}^{(z,t)}(0, 1/s)_{\mathbf{C}} = \tilde{\varphi}^{(z,t)}(0, s)_{\mathbf{C}} = 0$, $\tilde{\varphi}^{(z,t)}(0, s)_{\mathbf{R}} < \tilde{\varphi}^{(z,t)}(0, 1/s)_{\mathbf{R}}$ and $\tilde{\varphi}^{(z,t)}(z, t)_{\mathbf{C}} = z$. Here, we set $x_{\mathbf{C}} = z$ and $x_{\mathbf{R}} = t$ for a point $x = (z, t) \in \mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$.

we have

$$(2.3) \quad |x_{\mathbf{R}} - \tilde{\varphi}^\circ(x)_{\mathbf{R}}| < \frac{\delta}{2}.$$

The inequalities (2.2) and (2.3) imply

$$\text{dist}_{\mathbf{C} \times \mathbf{R}_+}(x, \tilde{\varphi}^\circ(x)) < \delta.$$

This completes the proof. □

There exists a constant $\varepsilon' = \varepsilon'(s, r) > 0$ with $\lim_{(s,r) \rightarrow (0,0)} \varepsilon' = 0$ and such that both $\nabla_0, \nabla_{z,t}$ in $\mathcal{T}_{z,t}$ have the volumes greater than $\mathbf{v}_3 - \varepsilon'$ if $(z, t) \in B_r(2R) \times (0, s]$.

Lemma 4. *For any $\delta > 0, 0 < s < 1$ and $0 < r < \delta/16$, there exists a constant $\eta_2 = \eta_2(\delta) > 0$ such that, if both $\mathcal{T}_{z,t}$ and $\mathcal{T}_{z',t'}$ satisfy $\mathbf{P}_{\eta_2}(\tilde{\varphi})$ for $(z, t), (z', t') \in B_r(2R) \times (0, s]$, then $\tilde{\varphi}^\circ(z, t)_{\mathbf{C}}$ is contained in $B_{\delta/8}(2R)$, where $\tilde{\varphi}^\circ$ is the (z', t') -normalization of $\tilde{\varphi}$.*

Proof. The proof is done by an argument similar to that in Lemma 3. Since $\mathcal{T}_{t,z}$ and $\mathcal{T}_{z',t'}$ contain the same 3-simplex ∇_0 , $\text{straight}(\tilde{\varphi}(\nabla_{z,t}))$ and $\text{straight}(\tilde{\varphi}(\nabla_{z',t'}))$ have the three common vertices $\tilde{\varphi}(0, 1/s), \tilde{\varphi}(w_1, s), \tilde{\varphi}(w_2, s)$. If one can choose $\eta_2 = \eta_2(\delta) > 0$ sufficiently small, then both $\text{straight}(\tilde{\varphi}(\nabla_{z,t}))$ and $\text{straight}(\tilde{\varphi}(\nabla_{z',t'}))$ are well approximated by a common, regular ideal 3-simplex. Thus, $\eta_2 > 0$ can be taken so that the Euclidean distance between $\tilde{\varphi}^\circ(z, t)$ and $\tilde{\varphi}^\circ(z', t')$ in $\mathbf{C} \times \mathbf{R}_+$ is less than $\delta/16$. Since $\tilde{\varphi}^\circ(z', t')_{\mathbf{C}} = z'$,

$$\begin{aligned} \text{dist}_{\mathbf{C}}(2R, \tilde{\varphi}^\circ(z, t)_{\mathbf{C}}) &\leq \text{dist}_{\mathbf{C}}(2R, z') + \text{dist}_{\mathbf{C}}(\tilde{\varphi}^\circ(z', t')_{\mathbf{C}}, \tilde{\varphi}^\circ(z, t)_{\mathbf{C}}) \\ &\leq r + \text{dist}_{\mathbf{C} \times \mathbf{R}_+}(\tilde{\varphi}^\circ(z', t'), \tilde{\varphi}^\circ(z, t)) \\ &< \frac{\delta}{8}. \end{aligned}$$

This completes the proof. □

Recall that $V_z^{(m)}$ is the set of vertices other than the origin $0 \in \mathbf{C}$ of the 9^m regular triangles $T_{z,1}, \dots, T_{z,9^m}$ in \hat{T}_z . We denote by $\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\cdot)$ the standard Euclidean measure on $\mathbf{C} \times \mathbf{R}_+$.

The following lemma is essentially implied by Sublemma 4.6 in [17], so we only give an outline of the proof.

Lemma 5. *Suppose that δ, r, s are any positive numbers with $\delta < 1, r < R$ and $s < 1$. Then, there exists $m = m(\delta, r) \in \mathbf{N}$ independent of s and satisfying*

$$\frac{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\hat{T}_{2R} \times (0, s] \cap (\bigcup_{(z,t) \in L} V_z^{(m)} \times \{t\}))}{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\hat{T}_{2R} \times (0, s])} > 1 - \delta,$$

where L is any open subset of $B_r(2R) \times (0, s]$ with

$$\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(L) > \left(1 - \frac{\delta}{2}\right) \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(B_r(2R) \times (0, s]).$$

Outline of the Proof. From the Fubini Theorem, it suffices to show the existence of such an $m(\delta, r)$ for any level of height $t \in (0, s]$. Let $f : B_r(2R) \times \mathbf{C} \rightarrow \mathbf{C}$ be a map defined by $f(z, w) = zw/2R$. For any $v \in V_{2R}^{(m)}$, $f(z, v)$ is an element of $V_z^{(m)}$ corresponding to v , and $f(B_r(2R) \times \{v\})$ is a disk in \mathbf{C} centered at v of radius $r|v|/2R$, which is independent of m . Thus, we have

$$\lim_{m \rightarrow \infty} \text{meas}_{\mathbf{C}}(\hat{T}_{2R} \cap f(B_r(2R) \times V_{2R}^{(m)})) = \text{meas}_{\mathbf{C}}(\hat{T}_{2R}).$$

Moreover, since $f(\cdot, v) : B_r(2R) \rightarrow \mathbf{C}$, $v \in V_{2R}^{(m)}$, is a similar map, for any Borel subset λ of $B_r(2R)$ with

$$\begin{aligned} \text{meas}_{\mathbf{C}}(\lambda) &= (1 - \delta')\text{meas}_{\mathbf{C}}(B_r(2R)), \\ \text{meas}_{\mathbf{C}}(f(\lambda \times \{v\})) &= (1 - \delta')\text{meas}_{\mathbf{C}}(f(B_r(2R) \times \{v\})). \end{aligned}$$

From these facts, one can have a constant $m(\delta, r) \in \mathbf{N}$ satisfying the conclusion of Lemma 5. □

3. PROOFS OF THE MAIN THEOREM AND COROLLARIES

For a constant $K \geq 1$, a homeomorphism $f : (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is *K-quasi-isometric* if the map satisfies

$$K^{-1}d_X(x, x') \leq d_Y(f(x), f(x')) \leq Kd_X(x, x')$$

for any $x, x' \in X$. One can suppose that a *K*-quasi-isometry is close to an isometry if and only if *K* is close to 1.

Our main theorem, Theorem 2, is proved by reduction to absurdity. So, we suppose that there exist degree-one maps $f_n : M_n \rightarrow N_n$ ($n = 1, 2, 3, \dots$) such that infinitely many f_n 's of them are not homotopic to isometries, where M_n, N_n are closed, connected, hyperbolic 3-manifolds with $\lim_{n \rightarrow \infty} \text{Vol}(M_n) = \lim_{n \rightarrow \infty} \text{Vol}(N_n) < \infty$. By passing to a subsequence if necessary, we may assume that all f_n 's are not homotopic to isometries, and hence they are not homotopy equivalences by Mostow's Rigidity Theorem [13]. By Jørgensen's Theorem (see [20, Chapter 6]), we may also assume that there exist connected, hyperbolic 3-manifolds X, Y such that each M_n (resp. N_n) is obtained by hyperbolic Dehn surgery on X (resp. on Y). Moreover, we have sequences $\{\varepsilon_n\}, \{K_n\}$ with $\varepsilon_n \searrow 0, K_n \searrow 1$ so that there exist K_n -quasi-isometries $g_n : X_{\text{thick}(\varepsilon_n)} \rightarrow M_{n, \text{thick}(\varepsilon_n)}$ and $h_n : N_{n, \text{thick}(\varepsilon_n)} \rightarrow Y_{\text{thick}(\varepsilon_n)}$. Our assumption on volume limit implies

$$(3.1) \quad \text{Vol}(X) = \text{Vol}(Y) = \lim_{n \rightarrow \infty} \text{Vol}(N_n).$$

Take an $\varepsilon_0 > 0$ less than the Margulis constant such that each component of $X_{\text{thin}(\varepsilon_0)}$ or $Y_{\text{thin}(\varepsilon_0)}$ is a $\mathbf{Z} \times \mathbf{Z}$ -cusp. Let $p : \mathbf{H}^3 = \tilde{X} \rightarrow X$ be the universal covering. Fix a point x_0 in $X_{\text{thick}(\varepsilon_0)}$, and choose a point \tilde{x}_0 in \tilde{X} with $p(\tilde{x}_0) = x_0$. The coordinate on \mathbf{H}^3 is rearranged so that $\tilde{x}_0 = (0, 1)$ in $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$.

First of all, we will fix a positive number R and a connected, open neighborhood \mathcal{A} of the identity in $\text{Isom}^+(\mathbf{H}^3)$, which are not changed throughout the proof. Let $\gamma_1, \dots, \gamma_k$ be a set of generators of $\pi_1(X, x_0)$ consisting of non-parabolic elements. We choose $R > 0$ such that at least one of the two fixed point of each γ_i is contained in $B_{R/3}(0)$. If necessary replacing γ_i by γ_i^{-1} , we may assume that $B_{R/3}(0)$ contains the attracting fixed point of γ_i . We choose \mathcal{A} such that (i) $\mathcal{A} \cap \gamma\mathcal{A} = \emptyset$ for any $\gamma \in \pi_1(X, x_0) - \{1\}$, and (ii) $\text{dist}_{\mathbf{H}^3}(\tilde{x}_0, \alpha\tilde{x}_0) < \varepsilon_0$ and $\alpha^{-1}(B_{R/3}(0)) \subset B_{R/2}(0)$ for any $\alpha \in \mathcal{A}$.

Let us denote a unique closed geodesic of a Margulis tube V in M_n by c_V . If necessary deforming f_n by homotopy, we may assume that, for each component V of $M_{n, \text{thin}(\varepsilon_n)}$, $f_n(V)$ is a closed geodesic l in N_n if $f_n|_{c_V}$ is homotopic to l in N_n , and $f_n(V)$ is a point if $f_n|_{c_V}$ is contractible in N_n . For a deformation retract $r_n : X \rightarrow X_{\text{thick}(\varepsilon_n)}$, we set $\varphi_n = f_n \circ g_n \circ r_n : X \rightarrow N_n$. Let $\tilde{\varphi}_n : \mathbf{H}^3 = \tilde{X} \rightarrow \mathbf{H}^3 = \tilde{N}_n$ be a lift of φ_n to the universal covering. Note that $\tilde{\varphi}_n \circ \gamma = (\varphi_n)_*(\gamma) \circ \tilde{\varphi}_n$

for any $\gamma \in \pi_1(X, x_0)$. Here, $\gamma : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ is a covering transformation such that, for an oriented arc \tilde{c} in \mathbf{H}^3 connecting \tilde{x}_0 with $\gamma(\tilde{x}_0)$, $p \circ \tilde{c}$ represents $\gamma \in \pi_1(X, x_0)$.

The key to the proof of Theorem 2 is the following lemma.

Lemma 6. *Let $\delta > 0$ be any positive number. Then, we have $n_0 = n_0(\delta) \in \mathbf{N}$ and $0 < s_0 < 1$ such that, for any integer $n \geq n_0$, there are $\alpha_n \in \mathcal{A}$ and a Borel subset W_n of $\widehat{T}_{2R} \times (0, s_0]$ satisfying the following (3.2) and (3.3).*

$$(3.2) \quad \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(W_n) > (1 - \delta)\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\widehat{T}_{2R} \times (0, s_0]).$$

$$(3.3) \quad \text{dist}_{\mathbf{C} \times \mathbf{R}_+}(\tilde{\varphi}_n^\circ(x), x) < \delta \quad \text{for any } x \in W_n,$$

where $\tilde{\varphi}_n^\circ$ is the (z', t') -normalization of $\tilde{\varphi}_n \circ \alpha_n$ for some $(z', t') \in B_{\delta/16}(2R) \times (0, s_0]$.

Proof. Take $\eta_2 = \eta_2(\delta) > 0$ satisfying Lemma 4. Since $\lim_{(s,r) \rightarrow (0,0)} \varepsilon'(s, r) = 0$, there exist $0 < s_2 < 1$ and $0 < r < \delta/16$ such that $2\varepsilon'\eta_2^{-1}\text{Vol}(N_n) < \delta\mu_0(\mathcal{A})/16$ for any $0 < s \leq s_2$ and $n \in \mathbf{N}$, where μ_0 is the normalized Haar measure on $\text{Isom}_+(\mathbf{H}^3)$ given in §1. By (3.1), we have $n_1 \in \mathbf{N}$ such that, for any integer $n \geq n_1$, $2\mathbf{v}_3\eta_2^{-1}(\text{Vol}(X) - \text{Vol}(N_n)) < \delta\mu_0(\mathcal{A})/16$. Then, by Lemma 2,

$$(3.4) \quad \mu_0(\mathcal{G}_0 \cup \mathcal{G}_{z,t}) \leq \mu_0(\mathcal{G}_0) + \mu_0(\mathcal{G}_{z,t}) < \frac{\delta\mu_0(\mathcal{A})}{4},$$

where $\mathcal{G}_0, \mathcal{G}_{z,t}$ are the closed subsets of \mathcal{A} given in §2 for $\mathbf{P} = \mathbf{P}_{\eta_2}(\tilde{\varphi}_n)$.

Let $m \in \mathbf{N}$ be the integer given in Lemma 5 for $\delta > 0, r > 0$ as above, and let $\eta_1 = \eta_1(\delta, m) > 0$ be a constant satisfying Lemma 3. Since $\lim_{s \rightarrow 0} \varepsilon_m(s) = 0$, there exists $0 < s_3 \leq s_1(\delta, m)$ such that $2\varepsilon_m(s)\eta_1^{-1}\text{Vol}(N_n) < 9^{-m}\delta\mu_0(\mathcal{A})/8$ for any $0 < s \leq s_3$ and $n \in \mathbf{N}$. There exists $n_2 \in \mathbf{N}$ such that, for any integer $n \geq n_2$, $2\mathbf{v}_3\eta_1^{-1}(\text{Vol}(X) - \text{Vol}(N_n)) < 9^{-m}\delta\mu_0(\mathcal{A})/8$. Again by Lemma 2, we have

$$(3.5) \quad \sum_{i=1}^{9^m} \mu_0(\mathcal{F}_{z,i,t}^{(m)}) < \frac{\delta\mu_0(\mathcal{A})}{4},$$

where $\mathcal{F}_{z,i,t}^{(m)}$ is the closed subset of \mathcal{A} given in §2 for $\mathbf{P} = \mathbf{P}_{\eta_1}(\tilde{\varphi}_n)$. We set $n_0 = \max\{n_1, n_2\}$ and $s_0 = \min\{s_2, s_3\}$. For any integer $n \geq n_0$, let $\mathcal{E}^{(m,s_0)}$ and \mathcal{E}' be the open subsets of $\mathcal{A} \times B_r(2R) \times (0, s_0]$ given as in §2. By (3.4) and (3.5),

$$\frac{\mu_0 \times \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\mathcal{E}^{(m,s_0)} \cap \mathcal{E}')}{\mu_0 \times \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\mathcal{A} \times B_r(2R) \times (0, s_0])} > 1 - \frac{\delta}{2}.$$

Thus, we have $\alpha_n \in \mathcal{A}$ satisfying

$$\frac{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(L_n)}{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(B_r(2R) \times (0, s_0])} > 1 - \frac{\delta}{2},$$

where $L_n \subset B_r(2R) \times (0, s_0]$ is the α_n -section of $\mathcal{E}^{(m,s_0)} \cap \mathcal{E}'$, that is,

$$\{\alpha_n\} \times L_n = (\{\alpha_n\} \times \mathbf{C} \times \mathbf{R}_+) \cap \mathcal{E}^{(m,s_0)} \cap \mathcal{E}'.$$

By Lemma 5,

$$W_n = \widehat{T}_{2R} \times (0, s_0] \cap \left(\bigcup_{(z,t) \in L_n} V_z^{(m)} \times \{t\} \right)$$

is a Borel subset of $\widehat{T}_{2R} \times (0, s_0]$ satisfying (3.2). Note that, for any $(z', t') \in L_n$, $\mathcal{H}_{z',t'}^{(m)}$ and $\mathcal{T}_{z',t'}$ satisfy respectively $\mathbf{P}_{\eta_1}(\tilde{\varphi}_n \circ \alpha_n)$ and $\mathbf{P}_{\eta_2}(\tilde{\varphi}_n \circ \alpha_n)$. Let $\tilde{\varphi}_n^\circ =$

$\beta_n \circ \tilde{\varphi}_n \circ \alpha_n$ be the (z', t') -normalization of $\tilde{\varphi}_n \circ \alpha_n$ for a fixed $(z', t') \in L_n$. For any $x \in W_n$, there exists $(z, t) \in L_n$ so that x is a lower vertex of some $\Delta_{z,t} \in \mathcal{H}_{z,t}^{(m)}$. By Lemma 4, $\tilde{\varphi}_n^\circ(z, t)_{\mathbf{C}}$ is contained in $B_{\delta/8}(2R)$. Hence, by Lemma 3, we have (3.3). This completes the proof. \square

Proof of Theorem 2. For any $n \in \mathbf{N}$, consider the new coordinate on $\tilde{X} = \mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ with $x^{\text{new}} = \alpha_n^{-1}(x^{\text{old}})$, and so $(\alpha_n \tilde{x}_0)^{\text{new}} = (0, 1)$. The covering transformation γ_{ni} of $p : \mathbf{H}^3 \rightarrow X$ corresponding to $\gamma_i \in \pi_1(X)$ ($i = 1, \dots, k$) with respect to the new coordinate is $\alpha_n^{-1} \circ \gamma_i \circ \alpha_n$. Since $\text{dist}_{\mathbf{H}^3}(\tilde{x}_0, \alpha_n \tilde{x}_0) < \varepsilon_0$, if necessary passing to a subsequence, we may assume that $\{\alpha_n\}$ converges to α_0 in $\text{Isom}^+(\mathbf{H}^3)$. Then, $\{\gamma_{ni}\}$ converges to $\gamma_{0i} = \alpha_0^{-1} \circ \gamma_i \circ \alpha_0$. Note that the attracting fixed point of γ_{ni} is contained in $B_{R/2}(0)$. Similarly, the new coordinate on $\tilde{N} = \mathbf{C} \times \mathbf{R}_+$ is defined with the isometry $\beta_n : \mathbf{H}^3 \rightarrow \mathbf{H}^3$ so that the covering transformation $(\varphi_n)_*(\gamma_i)^{\text{new}}$ corresponding to $(\varphi_n)_*(\gamma_i) \in \pi_1(N)$ is $\beta_n \circ (\varphi_n)_*(\gamma_i)^{\text{old}} \circ \beta_n^{-1}$. The equality $(\varphi_n)_*(\gamma_i)^{\text{old}} \circ \tilde{\varphi}_n = \tilde{\varphi}_n \circ \gamma_i$ implies $(\varphi_n)_*(\gamma_i)^{\text{new}} \circ \tilde{\varphi}_n^\circ = \tilde{\varphi}_n^\circ \circ \gamma_{ni}$. For simplicity, we set $(\varphi_n)_*(\gamma_i)^{\text{new}} = (\varphi_n)_*(\gamma_i)$.

For any $\delta > 0$, let $n_0(\delta)$ be a positive integer given in Lemma 6. There exists a sequence $\{\delta_n\}_{n=\nu}^\infty$ with $\delta_n > 0$, $\lim_{n \rightarrow \infty} \delta_n = 0$ and $n \geq n_0(\delta_n)$, where $\nu = n_0(1)$. Here, we will show that there exists a constant $c_0 > 0$ such that, for the Borel subset $O_{ni} = W_n \cap \gamma_{ni}^{-1}(W_n)$ of W_n ,

$$(3.6) \quad \liminf_{n \rightarrow \infty} \frac{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(O_{ni})}{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(W_n)} > c_0.$$

Since $0 < |d\gamma_{0i}(z_0)/dz| < 1$ at the attracting fixed point $z_0 \in B_{R/2}(0)$ of γ_{0i} , there are real numbers c_1, c_2 with $0 < c_1 < c_2 < 1$ and a small round disk D in \widehat{T}_{2R} with $\gamma_{0i}(D) \subset \text{int}D$ and $c_1 < |d\gamma_{0i}(z)/dz| < c_2$ for all $z \in D$. Since $\{\gamma_{ni}\} \rightarrow \gamma_{0i}$ in $\text{Isom}^+(\mathbf{H}^3)$, if we take $s_0 > 0$ in Lemma 6 small enough, then $c_1 < \|d\gamma_{ni}(\mathbf{v})\| < c_2$ for all sufficiently large $n \in \mathbf{N}$ and for all unit vectors \mathbf{v} tangent to $D \times (0, s_0]$, where $\|\cdot\|$ denotes the norm on the Euclidean half space $\mathbf{C} \times \mathbf{R}_+$. Since $c_2 < 1$ and $\gamma_{ni}^{-1}(D) \supset \text{int}D$, one can suppose $\gamma_{ni}^{-1}(D \times (0, s_0]) \supset D \times (0, s_0]$. We set $U_n = D \times (0, s_0] \cap W_n$ and $c_3 = \text{Area}(D)/\text{Area}(\widehat{T}_{2R})$. By (3.2) of Lemma 6,

$$\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(U_n) \geq \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(D \times (0, s_0]) - \delta_n \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\widehat{T}_{2R} \times (0, s_0]),$$

and

$$\begin{aligned} \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\gamma_{ni}^{-1}(U_n)) &\geq \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\gamma_{ni}^{-1}(D \times (0, s_0])) \\ &\quad - \delta_n c_1^{-3} \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\widehat{T}_{2R} \times (0, s_0]). \end{aligned}$$

These inequalities imply that

$$\begin{aligned} \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(U_n \cap \gamma_{ni}^{-1}(U_n)) &\geq \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(D \times (0, s_0]) \\ &\quad - \delta_n (1 + c_1^{-3}) \text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\widehat{T}_{2R} \times (0, s_0]). \end{aligned}$$

The situation is illustrated in Fig. 5. Since

$$\frac{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(O_{ni})}{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(W_n)} \geq \frac{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(U_n \cap \gamma_{ni}^{-1}(U_n))}{\text{meas}_{\mathbf{C} \times \mathbf{R}_+}(\widehat{T}_{2R} \times (0, s_0])},$$

the left hand side of (3.6) is greater than $c_3 - \delta_n(1 + c_1^{-3})$ if $0 < \delta_n < \delta_0$ for a fixed small number $\delta_0 > 0$. This shows (3.6).

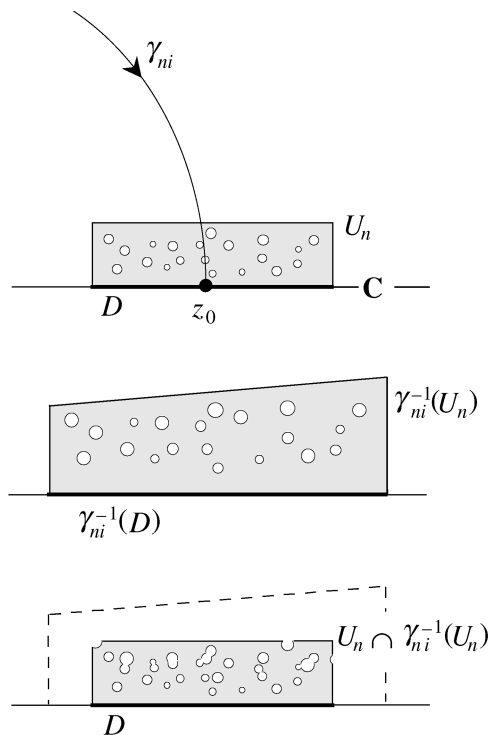


FIGURE 5.

Let $q_n : \mathbf{H}^3 \rightarrow N_n$ be the universal covering, and let c_{ni} be an oriented loop in N_n based at $y_n = q_n(0, 1)$ and representing $(\varphi_n)_*(\gamma_i)$. By invoking the two facts that (i) $(\varphi_n)_*(\gamma_i) \circ \tilde{\varphi}_n^\circ = \tilde{\varphi}_n^\circ \circ \gamma_{ni}$ and (ii) $\tilde{\varphi}_n^\circ$ satisfies (3.3) in $O_{ni} \cup \gamma_{ni}(O_{ni})$, for each $i \in \{1, \dots, k\}$, we will show the following.

(3.7) The sequence $\{(\varphi_n)_*(\gamma_i)\}$ converges to γ_{0i} in $\text{Isom}^+(\mathbf{H}^3)$.

By (3.6), we have a constant $d_0 > 0$ independent of n and three points a_{nt} ($t = 1, 2, 3$) in O_{ni} satisfying $\text{dist}_{\mathbf{C} \times \mathbf{R}_+}(a_{nt}, a_{nu}), \text{dist}_{\mathbf{C} \times \mathbf{R}_+}(\gamma_{ni}(a_{nt}), \gamma_{ni}(a_{nu})) > d_0$ for $1 \leq t, u \leq 3$ with $t \neq u$. By taking a subsequence if necessary, one can suppose that $\{a_{nt}\}$ converges in $\overline{\mathbf{C}} \times \overline{\mathbf{R}}_+$ to a point $a_{0t} \in \mathbf{C}$ for $t = 1, 2, 3$. Then, $\{\gamma_{ni}(a_{nt})\}$ converges to $\gamma_{0i}(a_{0t})$. Since $a_{nt} \in W_n$, by (3.3) with $\{\delta_n\} \rightarrow 0$, we have $\text{dist}(\tilde{\varphi}_n^\circ(a_{nt}), a_{nt}) \rightarrow 0$. Then, the convergence $\{\tilde{\varphi}_n^\circ(a_{nt})\} \rightarrow a_{0t}$ follows from $\text{dist}(a_{nt}, a_{0t}) \rightarrow 0$. For the proof of (3.7), we will show the following in advance.

(3.8) $\{(\varphi_n)_*(\gamma_i) \circ \tilde{\varphi}_n^\circ(a_{nt})\} = \{\tilde{\varphi}_n^\circ \circ \gamma_{ni}(a_{nt})\} \rightarrow \gamma_{0i}(a_{0t})$.

Since $\gamma_{ni}(a_{nt}) \in W_n$, by (3.3), $\text{dist}(\tilde{\varphi}_n^\circ \circ \gamma_{ni}(a_{nt}), \gamma_{ni}(a_{nt})) \rightarrow 0$. Set $K = \hat{T}_{2R} \times [0, 1] \cap \gamma_{0i}^{-1}(\hat{T}_{2R} \times [0, 1])$. Since γ_{0i} maps the compact set K into the compact set $\hat{T}_{2R} \times [0, 1]$ and since $\{\gamma_{ni}\} \rightarrow \gamma_{0i}$ in $\text{Isom}^+(\mathbf{H}^3)$, $\{\gamma_{ni}\}$ converges uniformly to γ_{0i} in a small compact neighborhood $N(K)$ of K . Since $a_{n1}, a_{n2}, a_{n3} \in N(K)$ for all sufficiently large $n \in \mathbf{N}$, $\text{dist}(\gamma_{ni}(a_{nt}), \gamma_{0i}(a_{nt})) \rightarrow 0$. Since $\{a_{nt}\} \rightarrow a_{0t}$, $\text{dist}(\gamma_{0i}(a_{nt}), \gamma_{0i}(a_{0t})) \rightarrow 0$. Thus, we have $\text{dist}(\tilde{\varphi}_n^\circ \circ \gamma_{ni}(a_{nt}), \gamma_{0i}(a_{0t})) \rightarrow 0$. This shows (3.8).

By (3.8), $\{\gamma_{0i}^{-1} \circ (\varphi_n)_*(\gamma_i) \circ \tilde{\varphi}_n^\circ(a_{nt})\} \rightarrow a_{0t}$. For simplicity, we set $\psi_n = \gamma_{0i}^{-1} \circ (\varphi_n)_*(\gamma_i)$. We have a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that both

$\tilde{\varphi}_n^\circ(a_{nt})$ and $\psi_n \circ \tilde{\varphi}_n^\circ(a_{nt})$ are contained in the closed ε_n -neighborhood $N_{\varepsilon_n}(a_{0t})$ of a_{0t} in $\overline{\mathbf{C}} \times \overline{\mathbf{R}}_+$. Note that $N_{\varepsilon_n}(a_{0t})$ is a half ball, and $\text{dist}(a_{0t}, a_{0u}) \geq d_0 \gg \varepsilon_n$ for $t \neq u$. The subset $A(\varepsilon_n)$ of $\text{Isom}^+(\mathbf{H}^3)$ consisting of Möbius transformations η with $\eta(N_{\varepsilon_n}(a_{0t})) \cap N_{\varepsilon_n}(a_{0t}) \neq \emptyset$ for $t = 1, 2, 3$ is compact, and $\bigcap_{n=1}^\infty A(\varepsilon_n) = \{\text{Id}\}$. Then, $\psi_n \in A(\varepsilon_n)$ converges to the identity in $\text{Isom}^+(\mathbf{H}^3)$, or equivalently $(\varphi_n)_*(\gamma_i)$ converges to γ_{0i} in $\text{Isom}^+(\mathbf{H}^3)$. This shows (3.7).

By (3.7), one can choose c_{ni} 's ($n \in \mathbf{N}$) which are equicontinuous, and with

$$L = \sup\{\text{length}_{N_n}(c_{ni}); i = 1, \dots, k, n \in \mathbf{N}\} < \infty.$$

For all $n \in \mathbf{N}$ with $\varepsilon_n < \varepsilon_0$ and $\text{dist}_{N_n}(\partial N_{n,\text{thin}(\varepsilon_0)}, \partial N_{n,\text{thin}(\varepsilon_n)}) > 2L$, $c_{n1} \cup \dots \cup c_{nk}$ is disjoint from $N_{n,\text{thin}(\varepsilon_n)}$. Otherwise, $c_{n1} \cup \dots \cup c_{nk}$ would be contained in $N_{n,\text{thin}(\varepsilon_0)}$, and hence $(\varphi_n)_*(\gamma_{ni})$ ($i = 1, \dots, k$) would generate a cyclic group. This contradicts that the sequences $\{(\varphi_n)_*(\gamma_{ni})\}$ converge to γ_{0i} 's which generate non-elementary group $\pi_1(X)$. Since $h_n : N_{n,\text{thick}(\varepsilon_n)} \rightarrow Y_{\text{thick}(\varepsilon_n)}$ is a K_n -quasi-isometry with $K_n \searrow 1$ for $i = 1, \dots, k$, again by passing to a subsequence, we may assume that each $\{h_n \circ c_{ni}\}_{n=1}^\infty$ converges uniformly to a curve $c_{\infty i}$ in Y . This shows that $h_n \circ c_{ni}$ is homotopic to $c_{\infty i}$ in Y for all sufficiently large $n \in \mathbf{N}$. Let $\gamma_{\infty i}$ be the element of $\pi_1(Y, y_\infty)$ represented by $c_{\infty i}$, and Γ_∞ the subgroup of $\pi_1(Y, y_\infty)$ generated by $\gamma_{\infty 1}, \dots, \gamma_{\infty k}$, where $y_\infty = \lim_{n \rightarrow \infty} h_n(y_n)$. Consider the universal covering $q_\infty : \mathbf{H}^3 \rightarrow Y$ of Y . Again by the K_n -quasi-isometricity of h_n with $K_n \searrow 1$, each $\{(\varphi_n)_*(\gamma_{ni})\}$ converges to $\gamma_{\infty i}$ in $\text{Isom}^+(\mathbf{H}^3)$, where $\gamma_{\infty i} \in \pi_1(Y, y_\infty)$ is regarded as a covering transformation on \mathbf{H}^3 with respect to a suitable coordinate on \mathbf{H}^3 with $q_\infty(0, 1) = y_\infty$. This shows that γ_{0i} is equal to $\gamma_{\infty i}$ as elements of $\text{Isom}^+(\mathbf{H}^3)$ for $i = 1, 2, \dots, k$, and hence in particular $\pi_1(X, x'_0) = \Gamma_\infty$ for $x'_0 = p(\alpha_0 \tilde{x}_0)$. Since $\text{Vol}(X) = \text{Vol}(Y)$, Γ_∞ is equal to $\pi_1(Y, y_\infty)$. It follows that there exists an isometry $\psi : X \rightarrow Y$ with $\psi_*(\gamma_{0i}) = \gamma_{\infty i}$ for $i = 1, 2, \dots, k$. By our construction of ψ , the homeomorphism

$$j_n = h_n^{-1} \circ (\psi|_{X_{\text{thick}(\varepsilon_n)}}) \circ g_n^{-1} : M_{n,\text{thick}(\varepsilon_n)} \rightarrow N_{n,\text{thick}(\varepsilon_n)} \subset N_n$$

is homotopic to $f_n|_{M_{n,\text{thick}(\varepsilon_n)}} : M_{n,\text{thick}(\varepsilon_n)} \rightarrow N_n$. Let m_V be a meridian of a component V of $M_{n,\text{thin}(\varepsilon_n)}$. Since $j_n(m_V) \simeq f_n(m_V)$ is contractible in N_n , $j_n(m_V)$ is a meridian of the component of $N_{n,\text{thin}(\varepsilon_n)}$ bounded by $j_n(\partial V)$. Then, j_n would be extended to a homeomorphism $\hat{j}_n : M_n \rightarrow N_n$ homotopic to f_n , a contradiction. This completes our reduction to absurdity and hence the proof of Theorem 2. □

Remark 1. In the special case of $M_n = X$, $N_n = Y$ and $f_n = f$ for all $n \in \mathbf{N}$, the map $\psi : X \rightarrow Y$ above is an isometry homotopic to $f : X \rightarrow Y$. Thus, we have given a new proof of Gromov-Thurston's Rigidity Theorem without using the radially extended map $\tilde{f}_\infty : S_\infty^2 \rightarrow S_\infty^2$.

We will now present the example which shows that Theorem 2 is false for maps of higher degree.

Example 1. Let K be a figure-eight knot in S^3 , and let $p : M \rightarrow S^3$ be a d_0 -fold, cyclic branched covering of S^3 branched over K for $d_0 \in \mathbf{N}$ with $d_0 > 1$. The complement $S^3 - K$ has a complete hyperbolic structure with $\text{Vol}(S^3 - K) = 2\mathbf{v}_3$. Consider a regular neighborhood V of K in S^3 , and the preimage $\tilde{V} = p^{-1}(V)$. Let $\mu, \lambda \subset \partial V$ be a meridian and a longitude for (S^3, V) respectively. Then, $\tilde{\mu} = p^{-1}(\mu)$ is a meridian of the solid torus \tilde{V} . Take a component $\tilde{\lambda}$ of $p^{-1}(\lambda)$.

The homomorphism $(p|\partial\tilde{V})_* : H_1(\partial\tilde{V}; \mathbf{Z}) \rightarrow H_1(\partial V; \mathbf{Z})$ maps $[\tilde{\mu}]$ to $d_0[\mu]$, and $[\tilde{\lambda}]$ to $[\lambda]$. For any $n \in \mathbf{N}$, let s_n be a simple loop in ∂V representing $[\mu] + n[\lambda]$ in $H_1(\partial V; \mathbf{Z})$. Then, $\tilde{s}_n = p^{-1}(s_n)$ is also a simple loop in $\partial\tilde{V}$ representing $[\tilde{\mu}] + d_0n[\tilde{\lambda}]$ in $H_1(\partial\tilde{V}; \mathbf{Z})$. Let M_n be the 3-manifold obtained by attaching a solid torus \tilde{V}_0 to $M - \text{int}\tilde{V}$ by a homeomorphism $\partial\tilde{V}_0 \rightarrow \partial\tilde{V}$ which maps a meridian of $\partial\tilde{V}_0$ onto \tilde{s}_n . Let N_n be the 3-manifold obtained by attaching a solid torus V_0 to $S^3 - \text{int}V$ by a homeomorphism $\partial V_0 \rightarrow \partial V$ which maps a meridian of ∂V_0 onto s_n . Then, the restriction $p|(M - \text{int}\tilde{V}) : M - \text{int}\tilde{V} \rightarrow S^3 - \text{int}V$ is extended to a d_0 -fold, cyclic branched covering $f_n : M_n \rightarrow N_n$ branched over the core Σ_n of V_0 . By the Hyperbolic Dehn Surgery Theorem [20, Theorem 5.9], for all sufficiently large $n \in \mathbf{N}$, both M_n and N_n have hyperbolic structures, and $\lim_{n \rightarrow \infty} \text{Vol}(M_n) = d_0 \lim_{n \rightarrow \infty} \text{Vol}(N_n) = 2d_0\mathbf{v}_3$. For any point $x_0 \in \Sigma_n$, the preimage $f_n^{-1}(x_0)$ consists of a single point \tilde{x}_0 in M_n . Then, any simple closed loop l in N_n with $l \cap \Sigma_n = \{x_0\}$ is covered homeomorphically by a loop \tilde{l} in M_n with $\tilde{l} \cap f_n^{-1}(\Sigma_n) = \{\tilde{x}_0\}$. This shows that $(f_n)_* : \pi_1(M_n) \rightarrow \pi_1(N_n)$ is surjective, and hence f_n is not homotopic to an (unbranched) d_0 -fold covering.

Proof of Corollary 1. We suppose that there are no constants $c > 0$ satisfying the property (0.2) in Corollary 1. Then, for any $n \in \mathbf{N}$, there would exist a degree-one, but non-homotopy equivalence map $f_n : M_n \rightarrow N_n$ between closed, connected, hyperbolic 3-manifolds with $\text{Vol}(M_n) \leq V$ and $(1 - 1/n)\text{Vol}(M_n) \leq \text{Vol}(N_n)$. Since $\text{Vol}(N_n) \leq \text{Vol}(M_n)$, there exist subsequences $\{M_{n_i}\}, \{N_{n_i}\}$ with

$$\lim_{n_i \rightarrow \infty} \text{Vol}(M_{n_i}) = \lim_{n_i \rightarrow \infty} \text{Vol}(N_{n_i}) \leq V.$$

This contradicts Theorem 2, and hence we have a constant $c > 0$ satisfying (0.2). \square

Proof of Corollary 2. Consider any ascending sequence

$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \dots \xleftarrow{f_n} M_n$$

of non-homotopy equivalence, degree-one maps between closed, hyperbolic 3-manifolds. If $\text{Vol}(M_n) \leq V$, then Corollary 1 implies

$$(1 - c(V))^n \text{Vol}(M_n) > \text{Vol}(M_0) \geq \mathbf{v}_0,$$

where $\mathbf{v}_0 > 0$ is the smallest volume of hyperbolic 3-manifolds, see [20, Chapter 6] for the existence of \mathbf{v}_0 . It follows that

$$n < \frac{\log \mathbf{v}_0 - \log V}{\log(1 - c(V))}.$$

Thus, the smallest integer n_0 not less than $(\log \mathbf{v}_0 - \log V)/\log(1 - c(V))$ satisfies the property (0.3) of Corollary 2. \square

4. DIVERGENT SEQUENCES OF DEGREE-ONE MAPS

In this section, we will prove Theorem 1 by using the argument similar to that in Fujii-Soma [4].

Let A be a compact, connected, hyperbolic 3-manifold such that the boundary ∂A consists of two totally geodesic components Σ, Σ' . Moreover, we suppose that A admits an orientation-reversing, isometric involution $\tau : A \rightarrow A$ with $\tau(\Sigma) = \Sigma'$, and contains a two-sided, closed surface F such that $A - F$ is connected, for example see [4, Lemma 1] for the existence of such a 3-manifold A . A connected sum $A' = A \# S^1 \times S^2$ is obtained from $A - \text{int}B_0$ and $S^1 \times S^2 - \text{int}B_1$ by attaching ∂B_1 to ∂B_0

by an orientation-reversing homeomorphism, where B_0, B_1 are 3-balls embedded in $\text{int}A$ and $S^1 \times S^2$ respectively. One can choose B_1 so that $S^1 \times \{x\} \cap B_1 = \emptyset$ for some $x \in S^2$. For any $m \in \mathbf{N}$, let $\varphi_m : A' \rightarrow A$ be a proper, degree-one map such that $\varphi_m|(A - \text{int}B_0)$ is the identity and the algebraic intersection number $[(\varphi_m)_*(S^1 \times \{x\})] \cdot [F] = m$. By Theorem 6.1 in Myers [14] together with Thurston’s Uniformization Theorem [21] and Mostow’s Rigidity Theorem, there exists a null-homotopic knot K in A' with $K \cap S^1 \times \{x\} = \emptyset$ such that $A' - K$ admits a complete, hyperbolic structure with totally geodesic boundary. By the Hyperbolic Dehn Surgery Theorem, one can obtain a compact hyperbolic 3-manifold \widehat{A} with totally geodesic boundary by Dehn surgery on A' along K . By Proposition 3.2 in Boileau-Wang [1], there exists a degree-one map $\widehat{\varphi} : \widehat{A} \rightarrow A'$ extending the identity of the complement of a small neighborhood of K in A' . Boileau and Wang considered only the case where 3-manifolds containing null-homotopic knots are irreducible. However, their proof holds even in the non-irreducible case without any alterations. Thus, we have a degree-one map $\psi_m = \varphi_m \circ \widehat{\varphi} : \widehat{A} \rightarrow A$ such that $\psi_m|\partial\widehat{A} : \partial\widehat{A} \rightarrow \partial A$ is the identity of $\Sigma \cup \Sigma'$. Let A_1, \dots, A_{2n} be $2n$ copies of A , and let $\eta_i : A \rightarrow A_i$ be the identity for $i = 1, 2, \dots, 2n$. We set $\eta_i(\Sigma) = \Sigma_i$ and $\eta_i(\Sigma') = \Sigma'_i$. Construct the “long” hyperbolic 3-manifold C_n from A_1, A_2, \dots, A_{2n} by connecting A_i with A_{i+1} by the orientation-reversing isometry $(\eta_{i+1}|\Sigma) \circ (\tau|\Sigma') \circ (\eta_i^{-1}|\Sigma'_i) : \Sigma'_i \rightarrow \Sigma_{i+1}$ for $i = 1, 2, \dots, 2n - 1$. Let M_n (resp. N_n) be a closed 3-manifold obtained from \widehat{A} (resp. A) and C_n by identifying $\Sigma \cup \Sigma'$ and $\partial C_n = \Sigma_1 \cup \Sigma'_{2n}$ by the orientation-reversing homeomorphisms $(\eta_1|\Sigma) \circ (\tau|\Sigma') : \Sigma' \rightarrow \Sigma_1$ and $(\eta_{2n}|\Sigma') \circ (\tau|\Sigma) : \Sigma \rightarrow \Sigma'_{2n}$. Note that N_n admits the hyperbolic structure extending those on A and C_n . Though M_n also admits a hyperbolic structure by Thurston’s Uniformization Theorem, the structure is not in general the extension of those on \widehat{A} and C_n .

Now, we will show that, for a sufficiently large $n \in \mathbf{N}$, $M = M_n$ and $N = N_n$ satisfy the property (0.1) of Theorem 1.

Proof of Theorem 1. According to Freedman-Hass-Scott [3], we may assume that each $\Sigma'_i = \Sigma_{i+1}$ is a least area surface in M_n among its homotopy class. Then, M_n is divided by these least area surfaces into the pieces $\widehat{A}^{(n)}, A_1^{(n)}, \dots, A_{2n}^{(n)}$ corresponding to $\widehat{A}, A_1, \dots, A_{2n}$ respectively. Then, one can show the following (4.1) and (4.2) by the argument quite similar to that in [4], which is based on McMullen’s work in [11] and [12].

$$(4.1) \quad \sup_{n \in \mathbf{N}} \{ \max\{ \text{Vol}(\widehat{A}^{(n)}), \text{Vol}(A_1^{(n)}), \dots, \text{Vol}(A_{2n}^{(n)}) \} \} = L < \infty.$$

(4.2) Suppose that $0 < \eta < 1$ is arbitrarily given. Then, there exists $n_0(\eta) \in \mathbf{N}$ such that

$$1 - \eta < \frac{\text{Vol}(A_j^{(n)})}{\text{Vol}(A)} < 1 + \eta$$

for any $n \in \mathbf{N}$ with $n > n_0$ and any $j = n_0 + 1, n_0 + 2, \dots, 2n - n_0 - 1, 2n - n_0$.

Intuitively, the (4.1) implies that the induced metrics on $\widehat{A}^{(n)}$ and $A_i^{(n)}$ from that on M_n remain in “bounded regions” independent of n . In fact, (4.1) can be proved also by using the two facts that (i) the topological type of the union B of $\widehat{A}^{(n)}$ or $A_i^{(n)}$ and its adjacent pieces (e.g. $B = A_1^{(n)} \cup \widehat{A}^{(n)} \cup A_{2n}^{(n)}$) is independent of n , and (ii) the closure of the set of all complete, geometrically finite, hyperbolic

structures on $\text{int}B$ is compact in the algebraic deformation space by Thurston [22]. (4.2) is derived from the fact that, except for the first and last n_0 pieces, all other $A_i^{(n)}$'s are well and uniformly approximated by A , see [4, §3] for details. By (4.1) and (4.2), for any $n \in \mathbf{N}$ with $n > n_0$, we have

$$2(n - n_0)(1 - \eta)\text{Vol}(A) < \text{Vol}(M_n) < (2n_0 + 1)L + 2(n - n_0)(1 + \eta)\text{Vol}(A).$$

Since $\text{Vol}(N_n) = (2n + 1)\text{Vol}(A)$, we have $\lim_{n \rightarrow \infty} \text{Vol}(M_n)/\text{Vol}(N_n) = 1$. For any $\varepsilon > 0$, there exists $n_1 \in \mathbf{N}$ with $(1 - \varepsilon)\text{Vol}(M_{n_1}) \leq \text{Vol}(N_{n_1})$. Set $M = M_{n_1}$, $N = N_{n_1}$, and let $f_m : M \rightarrow N$ be the degree-one map such that $f_m|_{\widehat{A}} = \psi_m$ and $f_m|_{A_i} : A_i \rightarrow A_i$ ($i = 1, 2, \dots, 2n_1$) is the identity of A . For a simple loop l in M with $\widehat{\varphi}(l) = S^1 \times \{x\}$, we have

$$[(f_m)_*(l)] \cdot [F] = [(\varphi_m)_*(S^1 \times \{x\})] \cdot [F] = m.$$

It follows that

$$\lim_{m \rightarrow \infty} \inf \{ \delta(g); g \text{ is a } C^1\text{-map homotopic to } f_m \} = \infty.$$

This completes the proof. \square

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