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COHOMOLOGY OF PROJECTIVE SPACE SEEN BY RESIDUAL COMPLEX

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Abstract. A subcomplex of a residual complex on projective space is constructed for computing the cohomology modules of locally free sheaves. A constructive new proof of the Bott formula is given by explicitly exhibiting bases for the cohomology modules.

1. Introduction

This paper is about the cohomology structure of locally free sheaves on a projective space. It is a part of the program for rendering concrete realizations of Grothendieck duality and finding applications of such realizations. In the paper, we propose a new method to organize cohomology data using residual complexes.

Let κ be a field. The cohomology modules of quasi-coherent sheaves on the n-dimensional projective space $\mathbb{P}^n$ over κ are usually calculated using Čech cohomology. The need for Čech cohomology is supported by the common belief that injective resolutions of a given sheaf of modules are inaccessible from the computational point of view. Our philosophy about this belief is that, for sheaves of modules related to duality theory, some injective complexes can be explicitly constructed and give rise directly to cohomology information. For instance, the Gorenstein property of the ring $\kappa[[X^{t_1}, \cdots, X^{t_n}]]$ is characterized by a value semigroup via a complex of injective modules

$$\cdots \to 0 \to \kappa((X)) \to \operatorname{Hom}^c_\kappa(\kappa[[X^{t_1}, \cdots, X^{t_n}]], \kappa) \to 0 \to \cdots,$$

see [7]. In [9], some classical theorems on the projective plane, such as the Newton theorem and the Cayley-Bacharach theorem, are obtained from a complex of injective sheaves of modules. The complexes used in the above-cited works are called residual complexes, and they become subtle in higher dimensional projective spaces. In this paper, our philosophy on injective resolutions is emphasized by illustrating how the cohomology modules of locally free sheaves on $\mathbb{P}^n$ are determined by the structure of a residual complex on $\mathbb{P}^n$.

We recall the definition of residual complexes. Let $\mathcal{X}$ be a locally Noetherian scheme. A residual complex $\mathcal{M}^\bullet$ on $\mathcal{X}$ is a complex of quasi-coherent injective $\mathcal{O}_\mathcal{X}$-modules, bounded below, with coherent cohomology sheaves, and such that there

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is an isomorphism

$$\bigoplus_{i\in\mathbb{Z}}\mathcal{M}^i \simeq \bigoplus_{\mathfrak{p}\in\mathcal{X}} J(\mathfrak{p}),$$

where $J(\mathfrak{p})$ is the quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module which is the constant sheaf $M(\mathfrak{p})$, a given injective hull of the residue field $\kappa(\mathfrak{p})$ over the local ring $\mathcal{O}_{\mathcal{X},\mathfrak{p}}$, on $\{\mathfrak{p}\}^-$, and 0 elsewhere [3, page 304]. The importance of residual complexes is shown in [3], which lays the foundation of Grothendieck duality using the language of derived categories and derived functors. However, the subtlety of the coboundary maps

$$\mathcal{M}^i \to \mathcal{M}^{i+1} \qquad (i\in\mathbb{Z})$$

of residual complexes, already mentioned by Grothendieck [2, page 113], is not fully treated in [3]. Fortunately, residual complexes and their properties are now better understood in concrete terms, thanks to [5], [6], [8], [10], [11], [12], [13], [14], [15], [16] and others. With success in concrete realizations of Grothendieck duality, it is desirable to see what information on schemes one gets from these concrete aspects of duality.

This paper focuses on cohomology modules of locally free sheaves of modules on $\mathbb{P}^n$. The next section sketches a residual complex $J^\bullet$ on $\mathbb{P}^n$ which is a simplified version of the residual complex constructed in [5]. As $J^\bullet$ is an injective resolution of $\mathcal{O}_{\mathbb{P}^n}(-n-1)$, the complex $J^\bullet\otimes\mathcal{F}(n+1)$ provides an injective resolution for any locally free sheaf of modules $\mathcal{F}$ on $\mathbb{P}^n$. In section 3, we investigate the structure of $J^\bullet\otimes\mathcal{F}(n+1)$. We construct a subcomplex $\mathcal{F}^{(\bullet)}$ of $J^\bullet\otimes\mathcal{F}(n+1)$ which gives rise to the cohomology modules of $\mathcal{F}$. In the last section, we explicitly give a basis for $\mathrm{H}^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n/\kappa}(m))$ by working on the complex $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(\bullet)}$ to illustrate that $\mathcal{F}^{(\bullet)}$ is indeed suitable for computing the cohomology modules of $\mathcal{F}$. More precisely, we give a basis $\{\omega^{j_1\cdots j_p}_{t_0\cdots\underline{t_{n-q}}\cdots t_n}\}$ for $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(q)}$ and explicitly describe coboundary maps

$$d^{(q)}\colon \Omega^p_{\mathbb{P}^n/\kappa}(m)^{(q)} \to \Omega^p_{\mathbb{P}^n/\kappa}(m)^{(q+1)}$$

in terms of operators $\frac{X_\ell}{X_{n-q}}$ on $\omega^{j_1\cdots j_p}_{t_0\cdots\underline{t_{n-q}}\cdots t_n}$. We will prove the following facts:

1. $\mathrm{H}^0(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n/\kappa}(m))$ has a basis consisting of

$$\sum_{\ell=0}^{p}(-1)^{\ell+1}\tfrac{X_{j_\ell}}{X_n}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots t_{n-i}0\cdots\underline{0}}$$

   satisfying
   - $0\le i\le n-p$,
   - $0\le j_0<\cdots<j_p=n-i$,
   - $t_0,\cdots,t_{n-i}\ge 0$,
   - $t_0+\cdots+t_{n-i}=m-p-1$.

   (Note that, to satisfy these conditions, p has to be less than m.)
2. For $0<q<n$, $\mathrm{H}^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n/\kappa}(m))$ has a basis consisting of

$$\omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}\cdots(-1)}.$$

   (For this element to be well-defined, p has to equal q and m has to vanish.)
3. $\mathrm{H}^n(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n/\kappa}(m))$ has a basis consisting of

$$\omega^{12\cdots ij_{i+1}\cdots j_p}_{\underline{(-1)}\cdots(-1)t_i\cdots t_n}$$

satisfying

- $0 \leq i \leq p$,
- $i < j_{i+1} < \cdots < j_p \leq n$,
- $t_i < -1$,
- $t_{i+1}, \cdots, t_n < 0$,
- $t_i + \cdots + t_n = m - p + i - 1$.

(Note that, to satisfy these conditions, m has to be less than $p - n$.)

Counting the cardinality of these bases (see Corollary 4.4 and Corollary 4.9), the Bott formula is recovered.

2. Residual Complex on Projective Space

As we work on a projective space $\mathbb{P}^n = \mathrm{Proj}(\kappa[X_0, X_1, \cdots, X_n])$ over κ, the approach in [5] to residual complexes can be simplified. In particular, a canonical injective hull of the residue field of a point on $\mathbb{P}^n$ can be constructed without the identifying process in [5, section 2].

For each point $\mathfrak{p} \in \mathbb{P}^n$, the $\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}$-module

$$M_{\mathbb{P}^n}(\mathfrak{p}) := \mathrm{H}^d_{\mathfrak{m}_\mathfrak{p}}(\Omega^n_{\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}/\kappa}),$$

where d is the height of $\mathfrak{p}$ and $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}$, is an injective hull of the residue field of

$$\kappa\left[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}\right]_\mathfrak{p}.$$

Let $J_{\mathbb{P}^n}(\mathfrak{p})$ be the constant sheaf which is $M_{\mathbb{P}^n}(\mathfrak{p})$ on $\{\mathfrak{p}\}^-$, and 0 elsewhere. We write $M_{\mathbb{P}^n}(\mathfrak{p})$ and $J_{\mathbb{P}^n}(\mathfrak{p})$ simply as $M(\mathfrak{p})$ and $J(\mathfrak{p})$, if it is clear from the context that we are working on $\mathbb{P}^n$. Cover $\mathbb{P}^n$ by affine open subsets

$$V_i = \mathrm{Spec}\,\kappa\left[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}\right].$$

If $\mathfrak{p}$ is contained in some V_i, then there is a canonical isomorphism

$$M(\mathfrak{p}) \simeq \mathrm{H}^d_{\mathfrak{m}_\mathfrak{p}}(\Omega^n_{\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}]_\mathfrak{p}/\kappa}).$$

By identifying the above two modules, elements in $M(\mathfrak{p})$ can be described by generalized fractions

$$\begin{bmatrix} \omega \\ f_1^{t_1}, \cdots, f_d^{t_d} \end{bmatrix},$$

where

$$\omega \in \Omega^n_{\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}]_\mathfrak{p}/\kappa}, \qquad t_1, \cdots, t_d \geq 1,$$

and $f_1, \cdots, f_d$ is a system of parameters of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}]_\mathfrak{p}$. The reader is referred to [6, Chapter 2] for the definition and properties of generalized fractions. In this paper, we allow t_ℓ to be any integer: By convention,

$$\begin{bmatrix} \omega \\ f_1^{t_1}, \cdots, f_d^{t_d} \end{bmatrix} = 0,$$

if some $t_\ell \leq 0$.

Example 1. For each $1 \leq i \leq n$, let $\mathfrak{p}$ be the generic point of the closed subscheme $\mathbb{P}^i := \mathrm{Proj}(\kappa[X_0, \cdots, X_i])$ of $\mathbb{P}^n$. The elements of $M(\mathfrak{p})$ are sums of the elements of the form

$$\begin{bmatrix} \frac{h}{g} d\frac{X_1}{X_0} \wedge \cdots \wedge d\frac{X_n}{X_0} \\ (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \cdots, (\frac{X_n}{X_0})^{t_n} \end{bmatrix},$$

where $g, h \in \kappa[\frac{X_1}{X_0}, \cdots, \frac{X_i}{X_0}]$ and $t_{i+1}, \cdots, t_n \geq 1$.

Example 2. Let $\mathfrak{p}$ be a point in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ for some $1 \leq i \leq n$. As a prime ideal of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]$, $\mathfrak{p}$ is generated by a prime ideal $\mathfrak{p}'$ of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]$ and $\frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}$. The elements of $M(\mathfrak{p})$ are sums of the elements of the form

$$\begin{bmatrix} \frac{h}{g} d\frac{X_0}{X_i} \wedge \cdots \wedge d\frac{X_{i-1}}{X_i} \wedge d\frac{X_{i+1}}{X_i} \wedge \cdots \wedge d\frac{X_n}{X_i} \\ f_1, \cdots, f_\ell, (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n} \end{bmatrix},$$

where

$$h \in \kappa\left[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}\right], g \in \kappa\left[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}\right] \setminus \mathfrak{p}', f_1, \cdots, f_\ell \in \kappa\left[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}\right]$$

form a system of parameters of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]_{\mathfrak{p}'}$, and $t_{i+1}, \cdots, t_n \geq 1$.

Let

$$J^j_{\mathbb{P}^n} := \bigoplus_{\mathrm{ht}\,\mathfrak{p}=j} J(\mathfrak{p}).$$

To define coboundary maps in order to make a residual complex

$$\cdots \to 0 \to J^0_{\mathbb{P}^n} \to J^1_{\mathbb{P}^n} \to \cdots \to J^n_{\mathbb{P}^n} \to 0 \to \cdots$$

on $\mathbb{P}^n$, it suffices to define

$$d_{\mathbb{P}^n;\mathfrak{p},\mathfrak{q}} \colon M(\mathfrak{p}) \to M(\mathfrak{q})$$

for each pair of points $\mathfrak{p}$ and $\mathfrak{q}$ with $\mathrm{ht}\,\mathfrak{q} = 1 + \mathrm{ht}\,\mathfrak{p}$. We write $d_{\mathbb{P}^n;\mathfrak{p},\mathfrak{q}}$ simply as $d_{\mathfrak{p},\mathfrak{q}}$, if it is clear from the context that we are working on $\mathbb{P}^n$. A description of the maps $d_{\mathfrak{p},\mathfrak{q}}$ can be found in [5]. Here we spell out two special cases of $d_{\mathfrak{p},\mathfrak{q}}$ that we use is this paper: Let

$$\omega_i^{j_1 \cdots j_p} = d\frac{X_{j_1}}{X_i} \wedge \cdots \wedge d\frac{X_{j_p}}{X_i}$$

and

$$\omega_i = \omega_i^{0 \cdots \hat{i} \cdots n},$$

where $0 \leq i, j_1, \cdots, j_p \leq n$ and $0 \cdots \hat{i} \cdots n$ is the sequence obtained by removing i from the sequence $0 \cdots n$. The symbols $\omega_i^{j_1 \cdots j_p}$ and ω_i are just shorthand for elements in some modules depending on the context.

Formula 1. *For each $1 \leq i \leq n$, let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the generic point of $\mathbb{P}^i$ (resp. $\mathbb{P}^{i-1}$). The map $d_{\mathfrak{p},\mathfrak{q}}$ is given by*

$$d_{\mathfrak{p},\mathfrak{q}}\left(\begin{bmatrix} (\frac{X_i}{X_0})^{-t_i} \frac{h}{g}\omega_0 \\ (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \cdots, (\frac{X_n}{X_0})^{t_n} \end{bmatrix}\right) = \begin{bmatrix} \frac{h}{g}\omega_0 \\ (\frac{X_i}{X_0})^{t_i}, (\frac{X_{i+1}}{X_0})^{t_{i+1}}, \cdots, (\frac{X_n}{X_0})^{t_n} \end{bmatrix},$$

where $t_{i+1}, \cdots, t_n \geq 1$, $t_i \in \mathbb{Z}$, and $g, h \in \kappa[\frac{X_1}{X_0}, \cdots, \frac{X_i}{X_0}]$ have no factors $\frac{X_i}{X_0}$.

Formula 2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ with $\operatorname{ht}\mathfrak{q} = 1 + \operatorname{ht}\mathfrak{p}$ for some $1 \leq i \leq n$. As a prime ideal of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]$, $\mathfrak{p}$ (resp. $\mathfrak{q}$) is generated by a prime ideal $\mathfrak{p}'$ (resp. $\mathfrak{q}'$) of $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]$ and $\frac{X_{i+1}}{X_i}, \cdots, \frac{X_n}{X_i}$. The map $d_{\mathbb{P}^n;\mathfrak{p},\mathfrak{q}}$ is given by*

$$d_{\mathbb{P}^n;\mathfrak{p},\mathfrak{q}}\left(\begin{bmatrix} \frac{h}{g}\omega_i \\ f_1, \cdots, f_\ell, (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n} \end{bmatrix}\right) = \begin{bmatrix} \frac{h'}{g'}\omega_i \\ f'_1, \cdots, f'_{\ell+1}, (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n} \end{bmatrix},$$

where

- $t_{i+1}, \cdots, t_n \geq 1$,
- $h, h', f_1, \cdots, f_\ell, f'_1, \cdots, f'_{\ell+1} \in \kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]$,
- $g \in \kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}] \setminus \mathfrak{p}'$,
- $g' \in \kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}] \setminus \mathfrak{q}'$,

satisfy the following conditions:

- $f_1, \cdots, f_\ell$ *form a system of parameters of* $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]_{\mathfrak{p}'}$,
- $f'_1, \cdots, f'_{\ell+1}$ *form a system of parameters of* $\kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]_{\mathfrak{q}'}$,
- $$d_{\mathbb{P}^i;\mathfrak{p}',\mathfrak{q}'}\left(\begin{bmatrix} \frac{h}{g}\omega_i^{0\cdots(i-1)} \\ f_1, \cdots, f_\ell \end{bmatrix}\right) = \begin{bmatrix} \frac{h'}{g'}\omega_i^{0\cdots(i-1)} \\ f'_1, \cdots, f'_{\ell+1} \end{bmatrix}.$$

We remark that it is a subtle problem to give an explicit formula for h', g', and $f'_1, \cdots, f'_{\ell+1}$, see [9, Example 3].

The complex $J^\bullet_{\mathbb{P}^n}$ gives an injective resolution of $\omega_{\mathbb{P}^n} := \Omega^n_{\mathbb{P}^n/\kappa}$. To describe a short exact sequence

$$0 \to \omega_{\mathbb{P}^n} \to J^0_{\mathbb{P}^n} \to J^1_{\mathbb{P}^n},$$

it suffices to describe a map

$$\Gamma(V_i, \omega_{\mathbb{P}^n}) \to \Gamma(V_i, J^0_{\mathbb{P}^n})$$

for each $0 \leq i \leq n$. The map we need is trivial in formalism: it has the form

$$h\omega_i \mapsto \frac{h}{1}\omega_i,$$

where $h \in \kappa[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]$.

3. Complexes Giving Rise to Cohomology

The residual complex $J^\bullet_{\mathbb{P}^n}$ can be organized to reveal information about the cohomology modules of locally free sheaves of $\mathbb{P}^n$: Let $\mathbb{P}^{-1} = \emptyset$. For any locally free sheaf $\mathcal{F}$ of $\mathbb{P}^n$ and $0 \leq i, j \leq n$, we define

$$\mathcal{F}^{i,j} := \bigoplus M(\mathfrak{p}) \otimes_{\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}} \mathcal{F}(n+1)_\mathfrak{p},$$

where $\mathfrak{p}$ ranges over all points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ with height j. Then $\mathcal{F}^{i,j} = 0$ for $j < n-i$, and

$$\Gamma(\mathbb{P}^n, J^j \otimes \mathcal{F}(n+1)) \simeq \bigoplus_{i=n-j}^{n} \mathcal{F}^{i,j}.$$

As $d_{\mathfrak{p},\mathfrak{q}}$ vanishes unless $\mathfrak{q}$ is an immediate specialization of $\mathfrak{p}$ and the immediate specializations of the points in $\mathbb{P}^i \setminus \mathbb{P}^{i-1}$ lies in $\mathbb{P}^i$, the non-trivial coboundary maps of the complex $\Gamma(\mathbb{P}^n, J^\bullet_{\mathbb{P}^n} \otimes \mathcal{F}(n+1))$ consist of at most those $\mathcal{F}^{i,j} \to \mathcal{F}^{k,j+1}$ with $i \geq k$. We indicate these maps in the following diagram:

$$\begin{tikzcd}
\mathcal{F}^{n,0} \arrow[r] \arrow[dr] & \mathcal{F}^{n,1} \arrow[r] \arrow[dr] \arrow[ddr] & \cdots \arrow[r] & \mathcal{F}^{n,n-1} \arrow[r] \arrow[dr] \arrow[dddr] \arrow[ddddr] & \mathcal{F}^{n,n} \\
 & \mathcal{F}^{n-1,1} \arrow[r] & \cdots \arrow[r] & \mathcal{F}^{n-1,n-1} \arrow[r] \arrow[ddr] \arrow[dddr] & \mathcal{F}^{n-1,n} \\
 & & \ddots & \vdots & \vdots \\
 & & & \mathcal{F}^{1,n-1} \arrow[r] \arrow[dr] & \mathcal{F}^{1,n} \\
 & & & & \mathcal{F}^{0,n}
\end{tikzcd}$$

Let $\mathcal{F}^{(n)} = \mathcal{F}^{0,n}$, and let $\mathcal{F}^{(i)}$ be the kernel of the map $\mathcal{F}^{n-i,i} \to \mathcal{F}^{n-i,i+1}$ for $0 \leq i \leq n-1$.

Lemma 3.1. *For each $1 \leq i \leq n$, the subcomplex*

$$\cdots \to 0 \to \omega^{n-i,i}_{\mathbb{P}^n} \to \omega^{n-i,i+1}_{\mathbb{P}^n} \to \cdots \to \omega^{n-i,n}_{\mathbb{P}^n} \to 0 \to \cdots$$

of $\Gamma(\mathbb{P}^n, J^\bullet_{\mathbb{P}^n})$ has only the i-th cohomology non-trivial, which as a κ-vector space has a basis consisting of the elements of the form

$$\begin{bmatrix} (\frac{X_0}{X_i})^{t_0} \cdots (\frac{X_{i-1}}{X_i})^{t_{i-1}} \omega_i \\ (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n} \end{bmatrix},$$

where $t_0, \cdots, t_{i-1} \geq 0$ and $t_{i+1}, \cdots, t_n \geq 1$.

Proof. From the construction, $J^\bullet_{\mathbb{P}^i}$ is a resolution of $\omega_{\mathbb{P}^i}$. Thanks to Formula 2, the complex $\omega^{n-i,\bullet}_{\mathbb{P}^n}$ as a κ-vector space is a direct sum of $\mathbb{N}^{n-i}$ copies (indexed by $(\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n}$, where $t_{i+1}, \cdots, t_n \geq 1$) of $\Gamma(\mathbb{P}^i \setminus \mathbb{P}^{i-1}, J^\bullet_{\mathbb{P}^i})$. Hence $\omega^{n-i,\bullet}_{\mathbb{P}^n}$ has only the $(n-i)$-th cohomology non-trivial. Since the kernel of $\omega^{i,0}_{\mathbb{P}^i} \to \omega^{i,1}_{\mathbb{P}^i}$ consists of elements of the form

$$hd\frac{X_0}{X_i} \wedge \cdots \wedge d\frac{X_{i-1}}{X_i},$$

where $h \in \kappa[\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}]$, the $(n-i)$-th cohomology of $\omega^{n-i,\bullet}_{\mathbb{P}^n}$ is generated by the elements of the form

$$\begin{bmatrix} (\frac{X_0}{X_i})^{t_0} \cdots (\frac{X_{i-1}}{X_i})^{t_{i-1}} \omega_i \\ (\frac{X_{i+1}}{X_i})^{t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{t_n} \end{bmatrix},$$

where $t_0, \cdots, t_{i-1} \geq 0$ and $t_{i+1}, \cdots, t_n \geq 1$. These elements are linearly independent. $\square$

Theorem 3.2. *For any locally free sheaf of modules $\mathcal{F}$ on $\mathbb{P}^n$, $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F})$ is isomorphic to the i-th cohomology module of the subcomplex*

$$\cdots \to 0 \to \mathcal{F}^{(0)} \to \mathcal{F}^{(1)} \to \cdots \to \mathcal{F}^{(n-1)} \to \mathcal{F}^{(n)} \to 0 \to \cdots$$

of the complex $\Gamma(\mathbb{P}^n, J^\bullet_{\mathbb{P}^n} \otimes \mathcal{F}(n+1))$.

Proof. First we compare the complexes $\mathcal{F}^{i,\bullet}$ and $\omega^{i,\bullet}_{\mathbb{P}^n}$ for $1 \leq i \leq n$. As modules, $\omega^{i,\bullet}_{\mathbb{P}^n}$ is the direct sum of those injective hulls $M(\mathfrak{p})$ with $\mathfrak{p} \in V_i \cap \mathbb{P}^i$, and $\mathcal{F}^{i,\bullet}$ is the direct sum of $\operatorname{rank} \mathcal{F}$ copies of $\omega^{i,\bullet}_{\mathbb{P}^n}$. Since the coboundary maps of $\mathcal{F}^{i,\bullet}$ and $\omega^{i,\bullet}_{\mathbb{P}^n}$ are given by the coboundary maps of $\Gamma(V_i, J^\bullet_{\mathbb{P}^n} \otimes \mathcal{F}(n+1))$ and $\Gamma(V_i, J^\bullet_{\mathbb{P}^n})$, respectively, as complexes, $\mathcal{F}^{i,\bullet}$ is also isomorphic to the direct sum of $\operatorname{rank} \mathcal{F}$ copies of $\omega^{i,\bullet}_{\mathbb{P}^n}$. Therefore $\mathcal{F}^{i,\bullet}$ has only one non-vanishing cohomology $H^{n-i}(\mathcal{F}^{i,\bullet})$, which is $\mathcal{F}^{(n-i)}$. With this fact, it is a standard trick to derive the theorem by diagram chasing. The details are left to the reader. $\square$

Note that $\mathcal{F}^{(\bullet)}$ is also a subcomplex of

$$\cdots \to 0 \to \mathcal{F}^{n,0} \to \mathcal{F}^{n-1,1} \to \cdots \to \mathcal{F}^{1,n-1} \to \mathcal{F}^{0,n} \to 0 \to \cdots,$$

which is isomorphic to the sum of $\operatorname{rank} \mathcal{F}$ copies of

$$\cdots \to 0 \to \omega^{n,0}_{\mathbb{P}^n} \to \omega^{n-1,1}_{\mathbb{P}^n} \to \cdots \to \omega^{1,n-1}_{\mathbb{P}^n} \to \omega^{0,n}_{\mathbb{P}^n} \to 0 \to \cdots. \tag{1}$$

As the coboundary map of the complex (1) is explicitly described in Formula 1, the dimension $h^i(\mathbb{P}^n, \mathcal{F})$ of the κ-vector space $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F})$ can be calculated directly.

4. Bott Formula

The following formula is well-known.

Bott Formula ([1]).

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) = \begin{cases} \binom{m-1}{p}\binom{m+n-p}{m}, & \textit{for } q=0,\ 0 \leq p \leq n,\ p < m; \\ 1, & \textit{for } m=0,\ 0 \leq p = q \leq n; \\ \binom{-m-1}{n-p}\binom{-m+p}{-m}, & \textit{for } q=n,\ 0 \leq p \leq n,\ m < p-n; \\ 0, & \textit{otherwise.} \end{cases}$$

It is interesting to know what the combinatorial numbers that occur in this formula mean. In this section, we interpret these numbers by exhibiting a basis for the κ-vector space $\mathrm{H}^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m))$.

For $i, j_1, \cdots, j_p, t_0, \cdots, t_n \in \mathbb{Z}$ satisfying

- $0 \leq i, j_1, \cdots, j_p \leq n$,
- $t_0 + \cdots + t_n = m - p - 1$,

we define an element $\omega^{j_1 \cdots j_p}_{t_0 \cdots \underline{t_i} \cdots t_n}$ in $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{n-i,i}$: If $p > 0$,

$$\omega^{j_1 \cdots j_p}_{t_0 \cdots \underline{t_i} \cdots t_n} := \begin{bmatrix} (\frac{X_0}{X_i})^{t_0} \cdots (\frac{X_{i-1}}{X_i})^{t_{i-1}}\, \omega_i \\ (\frac{X_{i+1}}{X_i})^{-t_{i+1}}, \cdots, (\frac{X_n}{X_i})^{-t_n} \end{bmatrix} \otimes \omega_i^{j_1 \cdots j_p} \otimes X_i^{m+n+1}.$$

If $p=0$, the symbol $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ is understood as $\omega_{t_0\cdots \underline{t_i}\cdots t_n}$ and defined by

$$\omega_{t_0\cdots \underline{t_i}\cdots t_n} := \begin{bmatrix} (\frac{X_0}{X_i})^{t_0}\cdots(\frac{X_{i-1}}{X_i})^{t_{i-1}}\,\omega_i \\ (\frac{X_{i+1}}{X_i})^{-t_{i+1}},\cdots,(\frac{X_n}{X_i})^{-t_n} \end{bmatrix} \otimes X_i^{m+n+1}.$$

These elements satisfy the following properties:

1. $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}=0$ if and only if one of the following conditions holds:
   - $j_\ell = i$ for some ℓ.
   - $j_k = j_\ell$ for some $k\neq\ell$.
   - $t_\ell \geq 0$ for some $\ell > i$.
2. If $j_k\neq j_\ell$ for $k\neq \ell$ and σ is a permutation of $j_1\cdots j_p$, then
$$\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n} = \epsilon(\sigma)\omega^{\sigma(j_1)\cdots\sigma(j_p)}_{t_0\cdots \underline{t_i}\cdots t_n},$$
   where $\epsilon(\sigma)$ is the sign of the permutation σ.
3. For $0\leq k\leq n$ not equal i,
$$\frac{X_k}{X_i}\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n} = \omega^{j_1\cdots j_p}_{t'_0\cdots \underline{t'_i}\cdots t'_n},$$
   where $t'_i = t_i - 1$ and $t'_\ell = t_\ell + \delta_{k\ell}$ if $\ell \neq i$.

Note that, for $0\leq i\leq n$, $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n-i)}$ as a κ-vector space has a basis consisting of those $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ satisfying

- $t_0,\cdots,t_{i-1}\geq 0$,
- $t_{i+1},\cdots,t_n < 0$,
- $0\leq j_1 < \cdots < j_p \leq n$,
- $j_\ell \neq i$ for all ℓ.

Given an element

$$\Psi = \sum a^{j_1\cdots j_p}_{t_0\cdots t_n}\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n} \qquad (a^{j_1\cdots j_p}_{t_0\cdots t_n}\in\kappa)$$

in $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n-i)}$ satisfying the above conditions, we call $a^{j_1\cdots j_p}_{t_0\cdots t_n}$ the coefficient of Ψ at $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$.

Proposition 4.1. *For $1\leq i\leq n$, the map $\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n-i)}\to\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n-i+1)}$, denoted by $d^{(n-i)}$, is given by*

$$\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n} \mapsto \sum_{\ell=0}^{p}(-1)^{\ell+1}\frac{X_{j_\ell}}{X_{i-1}}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots \underline{t_{i-1}}\cdots t_n},$$

where $j_0 = i$.

Proof. Assume first that $p=0$. Write $\omega_{t_0\cdots \underline{t_i}\cdots t_n}$ in terms of $\frac{X_1}{X_0},\cdots,\frac{X_n}{X_0}$ and X_0:

$$\omega_{t_0\cdots \underline{t_i}\cdots t_n} = (-1)^i\begin{bmatrix} (\frac{X_1}{X_0})^{t_1}\cdots(\frac{X_{i-1}}{X_0})^{t_{i-1}}(\frac{X_i}{X_0})^{t_i+1}\omega_0 \\ (\frac{X_{i+1}}{X_0})^{-t_{i+1}},\cdots,(\frac{X_n}{X_0})^{-t_n} \end{bmatrix}\otimes X_0^{m+n+1}$$

By Formula 1, $\omega_{t_0\cdots \underline{t_i}\cdots t_n}$ maps to

$$(-1)^i\begin{bmatrix} (\frac{X_1}{X_0})^{t_1}\cdots(\frac{X_{i-1}}{X_0})^{t_{i-1}}\omega_0 \\ (\frac{X_i}{X_0})^{-t_i-1},(\frac{X_{i+1}}{X_0})^{-t_{i+1}},\cdots,(\frac{X_n}{X_0})^{-t_n} \end{bmatrix}\otimes X_0^{m+n+1},$$

which equals $-\frac{X_i}{X_{i-1}}\omega_{t_0\cdots \underline{t_{i-1}}\cdots t_n}$.

Now assume $p>0$. If we switch j_k and j_{k+1} for some $1\le k<p$, then it is easy to see that both $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ and $\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_{i-1}}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots\underline{t_{i-1}}\cdots t_n}$ change signs. If $j_k=j_{k+1}$ for some $1\le k<p$, then it is easy to see that both $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ and $\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_{i-1}}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots\underline{t_{i-1}}\cdots t_n}$ vanish. Therefore we may assume that $0\le j_1<\cdots<j_p\le n$. If $j_k=i$ for some $1\le k\le p$, it is also easy to see that both $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ and $\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_{i-1}}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots\underline{t_{i-1}}\cdots t_n}$ vanish. Therefore we may assume furthermore that $j_k\ne i$ for $1\le k\le p$.

Write $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ in terms of $\frac{X_1}{X_0},\cdots,\frac{X_n}{X_0}$ and X_0:

$$\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}=\begin{bmatrix}(\frac{X_1}{X_0})^{t_1}\cdots(\frac{X_i}{X_0})^{t_i}\omega_0\\ (\frac{X_{i+1}}{X_0})^{-t_{i+1}},\cdots,(\frac{X_n}{X_0})^{-t_n}\end{bmatrix}\otimes\sum_{\ell=0}^p(-1)^{i+\ell}\frac{X_{j_\ell}}{X_0}\omega_0^{j_0\cdots\hat{j_\ell}\cdots j_p}\otimes X_0^{m+n+1}$$

By Formula 1, $\omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_i}\cdots t_n}$ maps to

$$\begin{bmatrix}(\frac{X_1}{X_0})^{t_1}\cdots(\frac{X_{i-1}}{X_0})^{t_{i-1}}\omega_0\\ (\frac{X_{i}}{X_0})^{-t_{i}},\cdots,(\frac{X_n}{X_0})^{-t_n}\end{bmatrix}\otimes\sum_{\ell=0}^p(-1)^{i+\ell}\frac{X_{j_\ell}}{X_0}\omega_0^{j_0\cdots\hat{j_\ell}\cdots j_p}\otimes X_0^{m+n+1},$$

which equals

$$\begin{aligned}&\begin{bmatrix}(\frac{X_0}{X_{i-1}})^{t_0}\cdots(\frac{X_{i-2}}{X_{i-1}})^{t_{i-2}}\omega_{i-1}\\ (\frac{X_{i}}{X_{i-1}})^{-t_{i}},\cdots,(\frac{X_n}{X_{i-1}})^{-t_n}\end{bmatrix}\otimes\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_0}\\ &\cdot\left(\frac{X_0}{X_{i-1}}\omega_{i-1}^{j_0\cdots\hat{j_\ell}\cdots j_p}-\frac{X_{j_0}}{X_{i-1}}\omega_{i-1}^{0j_1\cdots\hat{j_\ell}\cdots j_p}+\frac{X_{j_1}}{X_{i-1}}\omega_{i-1}^{0j_0j_2\cdots\hat{j_\ell}\cdots j_p}-\cdots\right)\otimes X_{i-1}^{m+n+1}.\end{aligned}$$

If $j_1>0$, the element $\omega_{i-1}^{0j_0\cdots\hat{j_k}\cdots\hat{j_\ell}\cdots j_p}$ appears twice in the above summation with opposite signs for each $0\le k<\ell\le p$. If $j_1=0$, the above summation has only one non-trivial term, namely the term $\ell=1$. In either cases, the above element equals

$$\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_{i-1}}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots\underline{t_{i-1}}\cdots t_n}.$$

□

Definition 4.2. For $0\le i\le n-p$, $V^i_{p,m}$ is defined to be the vector space generated by those

$$\sum_{\ell=0}^p(-1)^{\ell+1}\frac{X_{j_\ell}}{X_n}\omega^{j_0\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots t_{n-i}0\cdots\underline{0}}$$

satisfying the conditions

- $t_0,\cdots,t_{n-i}\ge 0$,
- $j_0<\cdots<j_p=n-i$.

Theorem 4.3.

$$\text{Kernel}(d^{(0)})=V^0_{p,m}\oplus\cdots\oplus V^{n-p}_{p,m}\tag{2}$$

Proof. If $p=0$, then

$$\begin{aligned}
\mathrm{Kernel}(d^{(0)}) &= \langle \omega_{t_0\cdots \underline{t_n}} \mid t_0,\cdots,t_{n-1}\geq 0 \text{ and } t_n\geq -1\ \rangle,\\
V^0_{p,m} &= \langle \omega_{t_0\cdots \underline{t_n}} \mid t_0,\cdots,t_n\geq 0\ \rangle,\\
V^n_{p,m} &= \langle \omega_{m0\cdots 0\underline{(-1)}}\rangle\\
V^i_{p,m} &= \langle \omega_{t_0\cdots t_{n-i}0\cdots 0\underline{(-1)}} \mid t_0,\cdots,t_{n-i-1}\geq 0 \text{ and } t_{n-i}\geq 1\rangle
\end{aligned}$$

for $0<i<n$. It is easy to see (2) in such case.

Now we assume that $p>0$. It is straightforward to check that the vector spaces $V^0_{p,m},\cdots,V^{n-p}_{p,m}$ are contained in the kernel of the map $d^{(0)}$. For $-1\leq i\leq n-p$, we define

$$W_{n-i} := \mathrm{Kernel}(d^{(0)})\cap \langle \omega^{j_1\cdots j_p}_{t_0\cdots t_{n-i-1}0\cdots 0\underline{(-1)}} \mid j_1<\cdots<j_p<n-i\ \rangle.$$

Note that $W_{n+1}=\mathrm{Kernel}(d^{(0)})$ and $W_p=0$. We will show that

$$W_{n-i+1}=W_{n-i}\oplus V^i_{p,m}$$

for $0\leq i\leq n-p$, from which the theorem follows easily.

We first show that $W_{n-i+1}=W_{n-i}+V^i_{p,m}$. Let

$$\Psi=\sum a^{j_1\cdots j_p}_{t_0\cdots t_{n-i}}\omega^{j_1\cdots j_p}_{t_0\cdots t_{n-i}0\cdots 0\underline{(-1)}} \qquad (a^{j_1\cdots j_p}_{t_0\cdots t_{n-i}}\in\kappa) \tag{3}$$

be an element in W_{n-i+1}, where the summation runs over those $t_0,\cdots,t_{n-i}$ and $j_1,\cdots,j_p$ satisfying the conditions

- $t_0,\cdots,t_{n-i}\geq 0$,
- $j_1<\cdots<j_p<n-i+1$.

By subtracting from Ψ the element

$$\sum a^{j_1\cdots j_p}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)}\Bigg(\omega^{j_1\cdots j_p}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)0\cdots 0\underline{(-1)}} + \sum_{\ell=1}^{p}(-1)^{\ell+p+1}\frac{X_{j_\ell}}{X_n}\omega^{j_1\cdots\hat{j_\ell}\cdots j_p(n-i)}_{t_0\cdots t_{n-i}0\cdots\underline{0}}\Bigg)$$

in $V^i_{p,m}$, where the summation runs over those $t_0,\cdots,t_{n-i},j_1,\cdots,j_p$ satisfying the conditions

- $t_0,\cdots,t_{n-i}\geq 0$,
- $j_1<\cdots<j_p<n-i$,

we may assume that the representation (3) of Ψ satisfies the additional condition

- $j_p<n-i$, $t_{n-i}=0$

or the additional condition

- $j_p=n-i$.

In other words, we may assume that

$$\Psi=\Psi_1+\Psi_2,$$

where

$$\Psi_1=\sum a^{j_1\cdots j_p}_{t_0\cdots t_{n-i-1}0}\omega^{j_1\cdots j_p}_{t_0\cdots t_{n-i-1}0\cdots 0\underline{(-1)}}$$

with indices running over those $t_0,\cdots,t_{n-i-1},j_1,\cdots,j_p$ satisfying the conditions

- $t_0,\cdots,t_{n-i-1}\geq 0$,

- $j_1 < \cdots < j_p < n-i$;

and

$$\Psi_2 = \sum a^{j_1\cdots j_{p-1}(n-i)}_{t_0\cdots t_{n-i}} \omega^{j_1\cdots j_{p-1}(n-i)}_{t_0\cdots t_{n-i}0\cdots 0\underline{(-1)}}$$

with indices running over those $t_0, \cdots, t_{n-i}, j_1, \cdots, j_{p-1}$ satisfying the conditions

- $t_0, \cdots, t_{n-i} \geq 0$,
- $j_1 < \cdots < j_{p-1} < n-i$.

The coefficient of $d^{(0)}(\Psi)$ at $\omega^{nj_1\cdots j_{p-1}}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)0\cdots 0\underline{(-1)}(-1)}$ is

$$(-1)^{p+1} a^{j_1\cdots j_{p-1}(n-i)}_{t_0\cdots t_{n-i}},$$

which has to be zero. Hence $\Psi_2 = 0$, and it follows that $W_{n-i+1} = W_{n-i} + V^i_{p,m}$.

Now we show that $W_{n-i} \cap V^i_{p,m} = 0$. For $i = 0$, this is trivial. So we assume that $i > 0$. Let

$$\Psi = \sum b^{j_0\cdots j_p}_{t_0\cdots t_{n-i}} \left(\sum_{\ell=0}^{p} (-1)^{\ell+1} \frac{X_{j_\ell}}{X_n} \omega^{j_0\cdots \hat{j_\ell}\cdots j_p}_{t_0\cdots t_{n-i}0\cdots \underline{0}} \right)$$

be an element in $W_{n-i} \cap V^i_{p,m}$, where the (first) summation runs over those $t_0, \cdots,$ t_{n-i} and $j_0, \cdots, j_p$ satisfying the conditions

- $t_0, \cdots, t_{n-i} \geq 0$,
- $j_0 < \cdots < j_p = n-i$.

As an element in $V^i_{p,m}$, the coefficient of Ψ at $\omega^{j_0\cdots j_{p-1}}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)0\cdots 0\underline{(-1)}}$ is

$$(-1)^{p+1} b^{j_0\cdots j_p}_{t_0\cdots t_{n-i}}.$$

However, as an element in W_{n-i}, the coefficient of Ψ at $\omega^{j_0\cdots j_{p-1}}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)0\cdots 0\underline{(-1)}}$ is zero. Therefore all $b^{j_0\cdots j_p}_{t_0\cdots t_{n-i}}$ vanish, and hence $\Psi = 0$. □

Corollary 4.4.

$$h^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) = \binom{m-1}{p}\binom{m+n-p}{m}.$$

Proof. Modulo the subspace generated by the elements of the form $\omega^{j'_1\cdots j'_{p-1}(n-i)}_{t'_0\cdots \underline{t'_n}}$, the elements

$$\sum_{\ell=0}^{p} (-1)^{\ell+1} \tfrac{X_{j_\ell}}{X_n} \omega^{j_0\cdots \hat{j_\ell}\cdots j_p}_{t_0\cdots t_{n-i}0\cdots \underline{0}} \quad \text{and} \quad (-1)^{p+1} \omega^{j_0\cdots j_{p-1}}_{t_0\cdots t_{n-i-1}(t_{n-i}+1)0\cdots 0\underline{(-1)}}$$

$(j_0 < \cdots < j_p = n-i)$ are the same. Hence, for a fixed i, the elements

$$\sum_{\ell=0}^{p} (-1)^{\ell+1} \tfrac{X_{j_\ell}}{X_n} \omega^{j_0\cdots \hat{j_\ell}\cdots j_p}_{t_0\cdots t_{n-i}0\cdots \underline{0}}$$

are linearly independent. Therefore

$$\begin{aligned}
h^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) &= \sum_{i=0}^{n-p} \dim V^i_{p,m} \\
&= \sum_{i=0}^{n-p} \binom{n-i}{p}\binom{m+n-p-i-1}{n-i} \\
&= \binom{m-1}{p} \sum_{i=0}^{n-p} \binom{m+n-p-i-1}{m-1} \\
&= \binom{m-1}{p}\binom{m+n-p}{m}.
\end{aligned}$$

□

Theorem 4.5. *For $0 < q < n$, $\mathrm{H}^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m))$ vanishes unless $m = 0$ and $p = q$; $\mathrm{H}^q(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n/\kappa})$ is non-trivial and generated (after identifying with the q-th cohomology of $\Omega^{q\,(\bullet)}_{\mathbb{P}^n/\kappa}$) by the equivalence class containing $\omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}\cdots(-1)}$.*

Proof. If $p = 0$, then

$$\mathrm{Kernel}(d^{(q)}) = \langle \omega_{t_0\cdots \underline{t_{n-q}}\cdots t_n} \mid t_{n-q} \geq -1 \rangle = \mathrm{Image}(d^{(q-1)}).$$

This implies $\mathrm{H}^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$. So, to prove the theorem, we may assume that $p > 0$.

Let Ψ be an element in the kernel of the map

$$d^{(q)} \colon \Omega^p_{\mathbb{P}^n/\kappa}(m)^{(q)} \to \Omega^p_{\mathbb{P}^n/\kappa}(m)^{(q+1)}.$$

We claim that, by adding a suitable element in the image of $d^{(q-1)}$, Ψ becomes

$$\begin{cases} \omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}\cdots(-1)}, & \text{if } m = 0 \text{ and } p = q; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that Ψ can be written as

$$\Psi = \sum a^{j_1\cdots j_p}_{t_0\cdots t_n} \omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_{n-q}}\cdots t_n}, \tag{4}$$

where $a^{j_1\cdots j_p}_{t_0\cdots t_n} \in \kappa$ and the summation runs over those $t_0, \cdots, t_n, j_1, \cdots, j_q \in \mathbb{Z}$ satisfying

- $t_0, \cdots, t_{n-q-1} \geq 0$,
- $t_{n-q+1}, \cdots, t_n < 0$,
- $t_0 + \cdots + t_n = m - p - 1$,
- $0 \leq j_1 < \cdots < j_p \leq n$,
- $j_\ell \neq n - q$ for all ℓ.

Step 1. We may assume $t_{n-q} = -1$.

If $t_{n-q} \leq -2$ and $j_\ell \neq n-q-1$ for all ℓ, then the coefficient of $d^{(q)}(\Psi)$ at $\frac{X_{n-q}}{X_{n-q-1}} \omega^{j_1\cdots j_p}_{t_0\cdots \underline{t_{n-q-1}}\cdots t_n}$ is $-a^{j_1\cdots j_p}_{t_0\cdots t_n}$, which has to be zero. Using this fact, one sees that if $t_{n-q} \leq -2$ and $j_\ell = n-q-1$ for some ℓ, then the coefficient of $d^{(q)}(\Psi)$ at $\omega^{(n-q)j_1\cdots \hat{j_\ell}\cdots j_p}_{t_0\cdots \underline{t_{n-q-1}}\cdots t_n}$ is $(-1)^{\ell+1} a^{j_1\cdots j_p}_{t_0\cdots t_n}$, which is thus also zero. Therefore $a^{j_1\cdots j_p}_{t_0\cdots t_n} = 0$ for $t_{n-q} \leq -2$.

For $t_{n-q} \geq 0$, if $j_\ell \neq n-q+1$ for all ℓ, then

$$\omega^{j_1\cdots j_p}_{t_0\cdots\underline{t_{n-q}}\cdots t_n} + d^{(q-1)}\left(\frac{X_{n-q}}{X_{n-q+1}}\omega^{j_1\cdots j_p}_{t_0\cdots\underline{t_{n-q+1}}\cdots t_n}\right)$$

is contained in the κ-vector space generated by those $\omega^{(n-q+1)j'_2\cdots j'_p}_{t'_0\cdots\underline{t'_{n-q}}\cdots t'_n}$ satisfying $t'_{n-q} \geq 0$. So we may assume that $j_\ell = n-q+1$ for some ℓ. In this case,

$$\omega^{j_1\cdots j_p}_{t_0\cdots\underline{t_{n-q}}\cdots t_n} = d^{(q-1)}\left((-1)^{\ell-1}\omega^{(n-q)j_1\cdots\hat{j_\ell}\cdots j_p}_{t_0\cdots\underline{t_{n-q+1}}\cdots t_n}\right).$$

Therefore, to prove the claim, we may assume that the representation (4) of Ψ satisfies the additional condition:

- $t_{n-q} = -1$.

Step 2. We may assume $j_{\ell_0} = n-q+1$ for some ℓ_0 and $t_{n-q+1} = -1$.

If $j_\ell \neq n-q+1$ for all ℓ, then

$$\omega^{j_1\cdots j_p}_{t_0\cdots t_{n-q-1}\underline{(-1)}t_{n-q+1}\cdots t_n} + d^{(q-1)}\left(\omega^{j_1\cdots j_p}_{t_0\cdots t_{n-q-1}0\underline{(t_{n-q+1}-1)}t_{n-q+2}\cdots t_n}\right)$$

is contained in the κ-vector space generated by those $\omega^{(n-q+1)j'_2\cdots j'_p}_{t'_0\cdots t'_{n-q-1}\underline{(-1)}t'_{n-q+1}\cdots t'_n}$. Therefore, by adding an element of the image of $d^{(q-1)}$ to Ψ, we may assume furthermore that the representation (4) of Ψ satisfies one more condition:

- $j_{\ell_0} = n-q+1$ for some ℓ_0.

For $t_{n-q+1} \leq -2$, the coefficient of $d^{(q)}(\Psi)$ at $\frac{X_{n-q+1}}{X_{n-q-1}}\omega^{(n-q)j_1\cdots\widehat{j_{\ell_0}}\cdots j_p}_{t_0\cdots\underline{t_{n-q-1}}(-1)t_{n-q+1}\cdots t_n}$ is $(-1)^{\ell_0+1}a^{j_1\cdots j_p}_{t_0\cdots t_{n-q-1}(-1)t_{n-q+1}\cdots t_n}$, which has to be zero. Therefore the representation (4) of Ψ also satisfies the condition

- $t_{n-q+1} = -1$.

Step 3. We may assume $q \leq p$, $t_{n-q} = \cdots = t_n = -1$, and $j_{p-q+i} = n-q+i$ for $1 \leq i \leq q$.

If $q > p$, repeating the process in Step 2, we may assume that the representation (4) is of the form

$$\Phi = \sum a^{(n-q+1)\cdots(n-q+p)}_{t_0\cdots t_{n-q-1}(-1)\cdots(-1)t_{n-q+p+1}\cdots t_n}\omega^{(n-q+1)\cdots(n-q+p)}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)t_{n-q+p+1}\cdots t_n}.$$

Since

$$\begin{aligned} d^{(q-1)}&(\omega^{(n-q+2)\cdots(n-q+p+1)}_{t_0\cdots t_{n-q-1}0\underline{(-1)}\cdots(-1)(t_{n-q+p+1}-1)t_{n-q+p+2}\cdots t_n}) \\ &= (-1)^{p+1}\omega^{(n-q+1)\cdots(n-q+p)}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)t_{n-q+p+1}\cdots t_n}, \end{aligned}$$

by adding to Ψ an element of the image of $d^{(q-1)}$, Ψ becomes zero.

For $q \leq p$, repeating the process in Step 2, we may write

$$\Phi = \sum a^{j_1\cdots j_{p-q}(n-q+1)\cdots n}_{t_0\cdots t_{n-q-1}(-1)\cdots(-1)}\omega^{j_1\cdots j_{p-q}(n-q+1)\cdots n}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)}. \tag{5}$$

Step 4. We may assume $q = p$.

Assume $p > q$. If $t_0 > 0$ and $j_1 > 0$, then

$$\omega^{j_1\cdots j_{p-q}(n-q+1)(n-q+2)\cdots n}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)} - d^{(q-1)}\left((-1)^{p-q}\omega^{0j_1\cdots j_{p-q}(n-q+2)(n-q+3)\cdots n}_{(t_0-1)t_1\cdots t_{n-q-1}0\underline{(-1)}\cdots(-1)}\right)$$

is contained in the κ-vector space generated by those $\omega^{0j'_2\cdots j'_{p-q}(n-q+1)(n-q+2)\cdots n}_{t'_0\cdots t'_{n-q-1}\underline{(-1)}\cdots(-1)}$. Therefore, in the case $p > q$, we may assume that the representation (5) also satisfies the condition

- $t_0 = 0$ or $j_1 = 0$.

If $j_1 = 0$, the coefficient of $d^{(q)}(\Psi)$ at $\omega^{(n-q)j_2\cdots j_{p-q}(n-q+1)(n-q+2)\cdots n}_{(t_0+1)t_1\cdots t_{n-q-2}\underline{(t_{n-q-1}-1)}(-1)\cdots(-1)}$ is

$$a^{0j_2\cdots j_{p-q}(n-q+1)(n-q+2)\cdots n}_{t_0\cdots t_{n-q-1}(-1)\cdots(-1)},$$

which has to be zero. Therefore we may assume that the representation (5) satisfies the condition

- $t_0 = 0$ and $j_1 > 0$.

Repeating this process, finally we may assume that Ψ is of the form

$$\Phi = \sum a^{(n-p)\cdots(n-q-1)(n-q+1)\cdots n}_{0\cdots 0t_{n-p}\cdots t_{n-q-1}(-1)\cdots(-1)}\omega^{(n-p)\cdots(n-q-1)(n-q+1)\cdots n}_{0\cdots 0t_{n-p}\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)}$$

For such a representation, the coefficient of $d^{(q)}(\Psi)$ at $\omega^{(n-p)\cdots(n-q-2)(n-q)\cdots n}_{0\cdots 0t_{n-p}\cdots \underline{t_{n-q-1}}(-1)\cdots(-1)}$ is $a^{(n-p)\cdots(n-q-1)(n-q+1)\cdots n}_{0\cdots 0t_{n-p}\cdots t_{n-q-1}(-1)\cdots(-1)}$, which has to be zero. Therefore, if $p > q$, adding to Ψ a suitable element in the image of $d^{(q-1)}$, Ψ becomes zero. After these steps, we may write

$$\Phi = \sum a^{(n-q+1)\cdots n}_{t_0\cdots t_{n-q-1}(-1)\cdots(-1)}\omega^{(n-q+1)\cdots n}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)}. \tag{6}$$

Step 5. We may assume $m = 0$ and $t_0 = \cdots = t_{n-q-1} = 0$.

If $m < 0$, it is impossible that the conditions

- $t_0, \cdots, t_{n-q-1} \geq 0$,
- $t_0 + \cdots + t_{n-q-1} - q - 1 = m - q - 1$

of the representation (6) are satisfied. Hence $\Psi = 0$ if $m < 0$. If $m > 0$, that is, $t_\ell > 0$ for some $0 \leq \ell \leq n-q-1$, then

$$d^{(q-1)}\left(\omega^{\ell(n-q+2)(n-q+3)\cdots n}_{t_0\cdots t_{\ell-1}(t_\ell-1)t_{\ell+1}\cdots t_{n-q-1}0\underline{(-1)}\cdots(-1)}\right) = \omega^{(n-q+1)(n-q+2)\cdots n}_{t_0\cdots t_{n-q-1}\underline{(-1)}\cdots(-1)}.$$

Hence we may assume that the representation (6) of Ψ satisfies the conditions

- $m = 0$,
- $t_0 = \cdots = t_{n-q-1} = 0$,

that is, we may assume that Ψ is generated by $\omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}\cdots(-1)}$. This final step verifies the claim.

It is easy to check that $\omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}(-1)\cdots(-1)}$ is in the kernel of $d^{(q)}$. To prove the theorem, it remains to show that $\omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}(-1)\cdots(-1)}$ is not in the image of $d^{(q-1)}$. Assume the contrary; then there exist $j_1, \cdots, j_q, t_0, \cdots, t_n, \ell_0 \in \mathbb{Z}$ satisfying

- $t_0, \cdots, t_{n-q} \geq 0$,
- $t_{n-q+2}, \cdots, t_n < 0$,
- $t_0 + \cdots + t_n = -q - 1$,
- $0 \leq j_1 < \cdots < j_q \leq n$,
- $j_\ell \neq n - q + 1$ for all ℓ

such that

$$\frac{X_{j_{\ell_0}}}{X_{n-q}}\omega^{(n-q+1)j_1\cdots\widehat{j_{\ell_0}}\cdots j_q}_{t_0\cdots\underline{t_{n-q}}\cdots t_n} = \omega^{(n-q+1)(n-q+2)\cdots n}_{0\cdots 0\underline{(-1)}(-1)\cdots(-1)}.$$

$\{n-q+1, j_1, \cdots, \widehat{j_{\ell_0}}, \cdots, j_q\} = \{n-q+1, \cdots, n\}$ implies that $j_{\ell_0} \le n-q$. This contradicts $t_{j_{\ell_0}} \ge 0$. $\square$

Corollary 4.6. *For* $0 < q < n$*,*

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) = \begin{cases} 1, & \text{if } m = 0 \text{ and } p = q; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.7. For $0 \le i \le p$, $\check{V}^{n-i}_{p,m}$ is defined to be the vector space generated by those $\omega^{12\cdots i j_{i+1}\cdots j_p}_{\underline{(-1)}\cdots(-1)t_i\cdots t_n}$ satisfying

- $i < j_{i+1} < \cdots < j_p \le n$,
- $t_i < -1$,
- $t_{i+1}, \cdots, t_n < 0$,
- $t_i + \cdots + t_n = m - p + i - 1$.

Note that the elements $\omega^{12\cdots i j_{i+1}\cdots j_p}_{\underline{(-1)}\cdots(-1)t_i\cdots t_n}$ satisfying the above conditions are linearly independent.

Theorem 4.8.

$$\Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n)} = \check{V}^{n-p}_{p,m} \oplus \cdots \oplus \check{V}^n_{p,m} \oplus \operatorname{Image}(d^{(n-1)})$$

Proof. If $p = 0$, then

$$\operatorname{Image}(d^{(n-1)}) = \langle d^{(n-1)}(\omega_{t_0\underline{t_1}\cdots t_n}) |\ t_0 \ge 0\ \rangle = \langle \omega_{\underline{t_0}\cdots t_n} |\ t_0 \ge -1\ \rangle.$$

As

$$\check{V}^n_{p,m} = \langle \omega_{\underline{t_0}\cdots t_n} |\ t_0 < -1\ \rangle,$$

it is easy to verify the theorem in this case. If $p = n$, then

$$\begin{aligned} \operatorname{Image}(d^{(n-1)}) &= \langle \omega^{1\cdots n}_{\underline{t_0}\cdots t_n} |\ t_0 \ge 0\ \rangle, \\ \check{V}^{n-i}_{p,m} &= \langle \omega^{1\cdots n}_{\underline{(-1)}\cdots(-1)t_i\cdots t_n} |\ t_i < -1\ \rangle \end{aligned}$$

for $0 \le i \le n$. It is also easy to verify the theorem in this case. So we assume that $0 < p < n$ in the rest of the proof.

In the proof, we assume that the indices $j_1, \cdots, j_p$ satisfy the condition $j_1 < \cdots < j_p$. We define

$$\begin{aligned} W_0' &:= \langle d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}) |\ j_1 = 0\ \rangle, \\ W_0'' &:= \langle d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}) |\ j_1 > 0 \text{ and } t_0 > 0\ \rangle, \\ W_0 &:= W_0' + W_0'', \\ W_1 &:= \langle d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}) |\ j_1 > 0,\ t_0 = 0, \text{ and } t_1 < -1\ \rangle. \end{aligned}$$

Note that, if $j_1 > 0$, $t_0 = 0$, and $t_1 > -1$, then

$$d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}) = \sum_{\ell=1}^{p} (-1)^{\ell+1} \omega^{1 j_1\cdots\hat{j_\ell}\cdots j_p}_{\underline{(-1)}t_1\cdots(t_{j_\ell}+1)\cdots t_n} = 0.$$

For $1 < i \le p+1$, we define

$$\begin{aligned} W_i := \langle d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}\underline{t_2}\cdots t_n}) |\ & j_1 = 2, \cdots, j_{i-1} = i, \\ & t_0 = 0,\ t_1 = \cdots = t_{i-1} = -1, \text{ and } t_i < -1\ \rangle. \end{aligned}$$

Note that

$$d^{(n-1)}(\omega^{23\cdots(p+1)}_{0\underline{(-1)}(-1)\cdots(-1)t_{p+2}\cdots t_n}) = 0.$$

Hence

$$\begin{aligned} W_{p+1} \quad := \quad & \langle d^{(n-1)}(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}) |\ j_1 = 2, \cdots, j_p = p+1, \\ & \qquad\qquad t_0 = 0,\, t_1 = \cdots = t_p = -1\ \rangle. \end{aligned}$$

If $1 < i \le p$ and $j_i > i+1$, then

$$\begin{aligned} & d^{(n-1)}(\omega^{23\cdots ij_i\cdots j_p}_{0\underline{(-1)}(-1)\cdots(-1)t_{i+1}\cdots t_n}) \\ & \quad = \sum_{\ell=i}^{p}(-1)^{i+\ell} d^{(n-1)}(\omega^{23\cdots i(i+1)j_i\cdots\hat{j}_\ell\cdots j_p}_{0\underline{(-1)}\cdots(-1)(t_{i+1}-1)t_{i+2}\cdots t_{j_{\ell-1}}(t_{j_\ell}+1)t_{j_{\ell+1}}\cdots t_n}) \in W_{i+1}. \end{aligned}$$

Therefore

$$\mathrm{Image}(d^{(n-1)}) = W_0 + \cdots + W_{p+1}.$$

We define

$$\begin{aligned} \Omega^{(-1)!} \quad & := \quad \Omega^p_{\mathbb{P}^n/\kappa}(m)^{(n)}, \\ \Omega^{i!} \quad & := \quad \langle \omega^{12\cdots ij_{i+1}\cdots j_p}_{\underline{(-1)}(-1)\cdots(-1)t_{i+1}\cdots t_n} \rangle \end{aligned}$$

for $0 \le i \le p$. Note that $\Omega^{p!} = W_{p+1}$. We will show that

$$\Omega^{(i-1)!} = \check{V}^{n-i}_{p,m} \oplus W_i \oplus \Omega^{i!}$$

for $0 \le i \le p$, from which the theorem follows easily.

Case 1 $(i = 0)$. If $j_1 = 0$, then

$$d^{(n-1)}\left(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}\right) = \omega^{1j_2\cdots j_p}_{\underline{t_0}t_1t_2\cdots t_n}.$$

If $j_1 > 0$ and $t_0 > 0$, then

$$d^{(n-1)}\left(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}\right) = -\omega^{j_1\cdots j_p}_{\underline{(t_0-1)}(t_1+1)t_2\cdots t_n} + \sum_{\ell=1}^{p}(-1)^{\ell+1}\omega^{1j_1\cdots\hat{j}_\ell\cdots j_p}_{\underline{(t_0-1)}t_1\cdots(t_{j_\ell}+1)\cdots t_n}.$$

The summation in the above equation is contained in W_0'. Hence

$$W_0 = \langle \omega^{j_1\cdots j_p}_{\underline{t_0}t_1\cdots t_n} |\ t_0 \ge 0\ \rangle.$$

Recall that

$$\begin{aligned} \check{V}^n_{p,m} \quad & = \quad \langle \omega^{j_1\cdots j_p}_{\underline{t_0}t_1\cdots t_n} |\ t_0 < -1\ \rangle, \\ \Omega^{0!} \quad & = \quad \langle \omega^{j_1\cdots j_p}_{\underline{t_0}t_1\cdots t_n} |\ t_0 = -1\ \rangle. \end{aligned}$$

Therefore

$$\Omega^{(-1)!} = \check{V}^n_{p,m} \oplus W_0 \oplus \Omega^{0!}.$$

Case 2 $(i = 1)$. If $j_1 > 0$, $t_0 = 0$, and $t_1 < -1$, then

$$d^{(n-1)}\left(\omega^{j_1\cdots j_p}_{t_0\underline{t_1}t_2\cdots t_n}\right) = -\omega^{j_1\cdots j_p}_{\underline{(-1)}(t_1+1)t_2\cdots t_n} + \sum_{\ell=1}^{p}(-1)^{\ell+1}\omega^{1j_1\cdots\hat{j}_\ell\cdots j_p}_{\underline{(-1)}t_1\cdots(t_{j_\ell}+1)\cdots t_n}.$$

The summation in the above equation is contained in $\check{V}^{n-1}_{p,m}$. Therefore

$$\Omega^{0!} = \check{V}^{n-1}_{p,m} \oplus W_1 \oplus \Omega^{1!}.$$

Case 3 $(1 < i \leq p)$. If $t_i < -1$, then

$$d^{(n-1)}\left(\omega^{2\cdots ij_i\cdots j_p}_{0\underline{(-1)}\cdots(-1)t_i\cdots t_n}\right) = (-1)^i\omega^{12\cdots(i-1)j_i\cdots j_p}_{\underline{(-1)}\cdots(-1)(t_i+1)t_{i+1}\cdots t_n} + \sum_{\ell=i}^{p}(-1)^{\ell+1}\omega^{12\cdots ij_i\cdots\hat{j_\ell}\cdots j_p}_{\underline{(-1)}\cdots(-1)t_i\cdots(t_{j_\ell}+1)\cdots t_n}.$$

The summation in the above equation is contained in $\check{V}^{n-i}_{p,m}$; therefore

$$\Omega^{(i-1)!} = \check{V}^{n-i}_{p,m} \oplus W_i \oplus \Omega^{i!}.$$

□

Corollary 4.9.

$$h^n(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) = \binom{-m-1}{n-p}\binom{-m+p}{-m}.$$

Proof. Indeed,

$$\begin{aligned} h^n(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n/\kappa}(m)) &= \sum_{i=0}^{p}\dim \check{V}^{n-i}_{p,m} \\ &= \sum_{i=0}^{p}\binom{n-i}{p-i}\binom{-m+p-i-1}{n-i} \\ &= \binom{-m-1}{n-p}\sum_{i=0}^{p}\binom{-m+p-i-1}{-m-1} \\ &= \binom{-m-1}{n-p}\binom{-m+p}{-m}. \end{aligned}$$

□

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