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THETA LIFTING OF HOLOMORPHIC DISCRETE SERIES: THE CASE OF $U(n,n) \times U(p,q)$

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Abstract. Let $(G, G') = (U(n,n), U(p,q))$ $(p+q \leq n)$ be a reductive dual pair in the stable range. We investigate theta lifts to G of unitary characters and holomorphic discrete series representations of G', in relation to the geometry of nilpotent orbits. We give explicit formulas for their K-type decompositions. In particular, for the theta lifts of unitary characters, or holomorphic discrete series with a scalar extreme K'-type, we show that the K structure of the resulting representations of G is almost identical to the $K_{\mathbb{C}}$-module structure of the regular function rings on the closure of the associated nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition. As a consequence, their associated cycles are multiplicity free.

1. Introduction

Let $\mathbb{G} = Sp(2N, \mathbb{R})$ be a real symplectic group of rank N. A pair of subgroups G and G' is called a *reductive dual pair* if G and G' are both reductive, and G' is the full centralizer of G in $\mathbb{G}$ and vice versa [2]. In this paper, we will be mainly concerned with the reductive dual pair

$$(G, G') = (U(n,n), U(p,q)) \subset \mathbb{G} = Sp(2N, \mathbb{R}), \tag{1.1}$$

where $N = 2n(p+q)$.

Let us consider the non-trivial double cover $\widetilde{\mathbb{G}} = Mp(2N, \mathbb{R})$ of $\mathbb{G}$, called the metaplectic group. For a subgroup L of $\mathbb{G}$, we denote the pullback of L in $\widetilde{\mathbb{G}}$ by $\widetilde{L}$. The metaplectic group $\widetilde{\mathbb{G}}$ has a distinguished unitary representation Ω, which has many different names in the literature; it is called the *oscillator representation*, or sometimes the metaplectic representation, the Weil (or Segal-Shale-Weil) representation, etc. Ω is very small, and in fact, its two irreducible constituents are among the four minimal representations of $\widetilde{\mathbb{G}}$ attached to the minimal (non-trivial) nilpotent orbit.

Using the oscillator representation Ω, Howe associates a given irreducible admissible representation π' of $\widetilde{G'}$ with an irreducible admissible representation π of $\widetilde{G}$, called the *theta lift* of π' ([5]). We shall denote this as $\pi = \theta(\pi')$ in this paper. Roughly speaking, π is the theta lift of π' if and only if there is a non-trivial morphism

$$\Omega \longrightarrow \pi \otimes \pi' \quad \text{as a } (\mathfrak{g}, \widetilde{K}) \times (\mathfrak{g}' \times \widetilde{K'})\text{-module}, \tag{1.2}$$

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where $\mathfrak{g}$ (respectively $\mathfrak{g}'$) is the complexification of the Lie algebra of G (respectively G') and K (respectively K') is a maximal compact subgroup of G (respectively G'). The morphism in (1.2) should be interpreted in the sense of Harish-Chandra modules. For a precise definition of $\theta(\pi')$, see §4.1.

Some of the most interesting representations arising from this framework are the theta lifts of unitary characters. The structure of these representations has been thoroughly investigated for various pairs (see, for example, [10], [11], [25], [20]). In particular, they are $\widetilde{K}$-multiplicity-free, and are of relatively small (and explicitly specified) Gelfand-Kirillov dimensions. Furthermore, they often appear as constituents of certain degenerate principal series representations.

Now assume that the pair (G, G') is in the stable range with G' the smaller member (see [1, §5] for its definition). For $(G, G') = (U(n, n), U(p, q))$, this assumption amounts to $p + q \leq n$. Then the theta lift $\theta(\chi)$ of a unitary character χ of G' is "unipotent" [6]. It is likely that this set of representations should play an important role in the classification theory of unitary representations of classical groups over $\mathbb{R}$.

In this paper, we first investigate the structure of $\theta(\chi)$ in relation to the geometry of a certain nilpotent orbit (for the pair $(G, G') = (U(n, n), U(p, q))$, $p + q \leq n$). Our geometric approach has several advantages to the other methods. One of the advantages is that we can determine the associated cycle of $\theta(\chi)$ almost immediately.

Here, we briefly recall the definition of associated cycle of an irreducible representation π of G (see [23] for details). The filtered algebra $U(\mathfrak{g})$ acts on the Harish-Chandra module X_π of π. We take a K-stable good filtration of X_π, and denote the associated graded module by $\operatorname{gr} X_\pi$, which is a $\operatorname{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g})$ module. Let $\mathcal{N}(\mathfrak{g}^*)$ be the cone of nilpotent elements in $\mathfrak{g}^*$. Then the annihilator ideal $\operatorname{Ann}(\operatorname{gr} X_\pi)$ defines an algebraic variety $\mathcal{AV}(\pi) \subseteq \mathcal{N}(\mathfrak{g}^*)$, called the associated variety of π. An irreducible component of $\mathcal{AV}(\pi)$ is the closure of a nilpotent $K_{\mathbb{C}}$ orbit. We can therefore write $\mathcal{AV}(\pi) = \bigcup_{i=1}^{l} \overline{\mathcal{O}_i}$, where $\mathcal{O}_i$ is a certain nilpotent $K_{\mathbb{C}}$ orbit. Let $P_i \subset S(\mathfrak{g})$ be a prime ideal which defines the irreducible component $\overline{\mathcal{O}_i}$. Then, the localization $(\operatorname{gr} X_\pi)_{P_i}$ at P_i is an Artinian module of the local ring $S(\mathfrak{g})_{P_i}$. We denote the length of $(\operatorname{gr} X_\pi)_{P_i}$ as an $S(\mathfrak{g})_{P_i}$ module by m_i, and define the associated cycle of π as a formal linear combination

$$\mathcal{AC}(\pi) = \sum_{i=1}^{l} m_i [\overline{\mathcal{O}_i}] \qquad (m_i = \operatorname{length}_{S(\mathfrak{g})_{P_i}} (\operatorname{gr} X_\pi)_{P_i}).$$

Now assume that the associated variety is irreducible, and write it as $\mathcal{AV}(\pi) = \overline{\mathcal{O}}$. If we know the K-type decomposition of π is (almost) the same as that of the ring of regular functions on $\overline{\mathcal{O}}$, we may immediately conclude that $\mathcal{AC}(\pi) = [\overline{\mathcal{O}}]$ without multiplicity. This happens to be the case for $\pi = \theta(\chi)$.

Here is another advantage of our method; namely that it works equally well for theta lifts of some irreducible admissible representations other than characters. As a typical example, we examine the theta lift of a holomorphic discrete series representation with a scalar extreme $\widetilde{K'}$-type.

Let π'_{hol} be a holomorphic discrete series representation of $\widetilde{G'} = U(p, q)^{\sim}$. Although π'_{hol} itself is fairly well understood, it is not so for its theta lift $\theta(\pi'_{\mathrm{hol}})$. By the general arguments of Adams [1, §5], most of $\theta(\pi'_{\mathrm{hol}})$ is realized as a derived functor module $A_{\mathfrak{q}}(\lambda)$, and consequently, its associated variety can be described in principle. Furthermore, the Blattner type formula for multiplicity of K-types of $A_{\mathfrak{q}}(\lambda)$ will then give the decomposition of $\theta(\pi'_{\mathrm{hol}})\big|_{\widetilde{K}}$. However, it is well-known that these general formulas are not very practical; for example, the Blattner type formula for K-types gives the multiplicity as a summation over a certain Weyl group, and it is often difficult to extract precise value from it.

In contrast, our method gives $\theta(\pi'_{\mathrm{hol}})\big|_{\widetilde{K}}$ completely in terms of the branching coefficients of finite dimensional representations of general linear groups, called Littlewood-Richardson coefficients, and there are known algorithms to calculate them effectively. Moreover, our method implies in a straightforward way that the associated cycle of $\theta(\pi'_{\mathrm{hol}})$ is multiplicity free if π'_{hol} has a scalar extreme $\widetilde{K'}$-type.

We shall be more precise in the following.

Let $\mathfrak{g} = \mathfrak{gl}_{2n}(\mathbb{C})$ be the complexified Lie algebra of $G = U(n,n)$. As is customary, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via an invariant form (the Killing form). Take a maximal compact subgroup $K = U(n) \times U(n)$ of G. Then it determines a (complexified) Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. We denote by $\mathcal{N}(\mathfrak{s})$ the cone of nilpotent elements in $\mathfrak{s}$. Then the complexification $K_{\mathbb{C}} = GL_n \times GL_n$ of K acts on $\mathcal{N}(\mathfrak{s})$ with finitely many orbits. We use similar notations for $G' = U(p,q)$.

Assume that $(G, G') = (U(n,n), U(p,q))$ is in the stable range with G' the small member, i.e., $p + q \leq n$, and take a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathcal{N}(\mathfrak{s}')$. Then we can define the theta lift $\mathcal{O} = \theta(\mathcal{O}')$ of $\mathcal{O}'$ in terms of certain geometric quotient maps with respect to the action of $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$ (see §3). It turns out that the associated variety of $\theta(\chi)$ ($\chi = \det^k$ for some $k \in \mathbb{Z}$) is the theta lift of the trivial orbit $\{0\} \subset \mathcal{N}(\mathfrak{s}')$, which we denote by $\mathcal{O}^{\mathbf{1}}_{p,q} = \theta(\{0\})$. Its Jordan type is $2^{p+q} \cdot 1^{2(n-(p+q))}$.

We introduce some notations. Denote by Λ_n^+ the set of dominant integral weights for $U(n)$ or GL_n:

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n, \lambda_1 \geq \cdots \geq \lambda_n\}.$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n^+$, τ_λ denotes an irreducible finite dimensional representation of GL_n with highest weight λ, and $\lambda^* = (-\lambda_n, \dots, -\lambda_1)$ denotes the highest weight of the contragredient representation τ_λ^*. We denote the set of all partitions of length k by $\mathcal{P}_k$. If $k \leq n$, then $\mathcal{P}_k$ may be considered as a subset of Λ_n^+ by adding $n-k$ zeros in the tail. Denote $\mathbb{I}_k = (1, \dots, 1) \in \mathcal{P}_k$. For $\alpha \in \mathcal{P}_p$ and $\beta \in \mathcal{P}_q$, we put

$$\alpha \odot \beta = (\alpha, 0, \dots, 0, \beta^*) \in \Lambda_n^+. \tag{1.3}$$

Note that the regular function ring $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ of the closure of $\mathcal{O}^{\mathbf{1}}_{p,q}$ inherits naturally a $K_{\mathbb{C}}$-action.

Theorem 1.1. *The $K_{\mathbb{C}}$-type decomposition of $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ and the $\widetilde{K}$-type decomposition of $\theta(\det^k)$ are multiplicity-free and are described as follows:*

$$\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}] \simeq \sum_{\alpha \in \mathcal{P}_p, \beta \in \mathcal{P}_q}^{\oplus} (\tau_{\alpha \odot \beta})^* \boxtimes \tau_{\alpha \odot \beta}, \qquad \textit{and}$$

$$\theta(\det^k)\big|_{\widetilde{K}} \simeq \begin{cases} \displaystyle\sum_{\substack{\alpha \in \mathcal{P}_p \\ \beta \in \mathcal{P}_q}}^{\oplus} \left(\tau_{(\alpha + k\mathbb{I}_p) \odot (\beta + k\mathbb{I}_q)} \otimes \chi_{p,q}\right)^* \boxtimes \left(\tau_{\alpha \odot \beta} \otimes \chi_{p,q}\right), & k \geq 0, \\ \displaystyle\sum_{\substack{\alpha \in \mathcal{P}_p \\ \beta \in \mathcal{P}_q}}^{\oplus} \left(\tau_{\alpha \odot \beta} \otimes \chi_{p,q}\right)^* \boxtimes \left(\tau_{(\alpha - k\mathbb{I}_p) \odot (\beta - k\mathbb{I}_q)} \otimes \chi_{p,q}\right), & k < 0, \end{cases}$$

where $\chi_{p,q} = \det^{\frac{p-q}{2}}$ *is a character of* $U(n)^{\sim}$. *Furthermore, the associated cycle of* $\theta(\det^k)$ *is given by*

$$\mathcal{AC}\,\theta(\det^k) = [\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}] \quad (\textit{multiplicity-free}).$$

As a corollary of the above theorem, we obtain

$$\begin{aligned}\operatorname{Dim}\theta(\det^k) &= \dim \mathcal{O}^{\mathbf{1}}_{p,q} = (p+q)(2n-(p+q)) \quad \text{and}\\ \operatorname{Deg}\theta(\det^k) &= \deg \overline{\mathcal{O}^{\mathbf{1}}_{p,q}},\end{aligned}$$

where $\operatorname{Dim}\pi$ denotes the Gelfand-Kirillov dimension of π and $\operatorname{Deg}\pi$ is the Bernstein degree [22].

Next, let us consider a holomorphic discrete series representation π'_{hol} of $G' = U(p,q)$. Let $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_-$ be a direct sum decomposition of $\mathfrak{s}'$ by $\operatorname{Ad} K'_{\mathbb{C}}$-invariant spaces. Then the associated variety of π'_{hol} is $\mathfrak{s}'_- = \overline{\mathcal{O}'_{\mathrm{hol}}}$ for an appropriate choice of the complex structure. Here $\mathcal{O}'_{\mathrm{hol}}$ is the open dense $K'_{\mathbb{C}}$-orbit in $\mathfrak{s}'_-$. Put $\mathcal{O}^{\mathrm{hol}}_{p,q} = \theta(\mathcal{O}'_{\mathrm{hol}})$, the theta lift of $\mathcal{O}'_{\mathrm{hol}}$. Then $\mathcal{O}^{\mathrm{hol}}_{p,q}$ is a 3-step nilpotent orbit with Jordan type $3^p \cdot 2^{q-p} \cdot 1^{2n-p-2q}$ (for $q \geq p$).

For $\mu, \nu, \lambda \in \Lambda_n^+$, define the Littlewood-Richardson coefficient $c_{\mu,\,\nu}^{\ \lambda}$ by the branching rule:

$$\tau_\mu \otimes \tau_\nu \simeq \sum_{\lambda \in \Lambda_n^+}{}^{\oplus} c_{\mu,\,\nu}^{\ \lambda}\ \tau_\lambda. \tag{1.4}$$

For $m, l \in \mathbb{Z}^+$, set

$$\begin{cases} a(m) = m + \frac{n}{2}, \quad b(l) = l + \frac{n}{2}, & n \text{ even},\\ a(m) = m + \frac{n-1}{2}, \quad b(l) = l + \frac{n+1}{2}, & n \text{ odd}.\end{cases} \tag{1.5}$$

Theorem 1.2. *Let* π'_{hol} *be a holomorphic discrete series of* $U(p,q)$ *with the following scalar extreme* K'*-type:*

$$\chi(m,l) = \det^{a(m)} \boxtimes \det^{-b(l)}, \qquad m, l \in \mathbb{Z}^+.$$

Then the $K_{\mathbb{C}} = GL_n \times GL_n$*-module structure of* $\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}]$ *and the* $\widetilde{K}$*-type decomposition of* $\theta(\pi'_{\mathrm{hol}})$ *are described as follows:*

$$\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}] \simeq \sum_{\substack{\alpha,\gamma \in \mathcal{P}_p\\ \beta,\delta \in \mathcal{P}_q}}{}^{\oplus} c_{\alpha,\,\beta^*}^{\ \gamma\odot\delta}\ (\tau_{\alpha\odot\beta})^* \boxtimes \tau_{\gamma\odot\delta}, \qquad \textit{and}$$

$$\theta(\pi'_{\mathrm{hol}})\big|_{\widetilde{K}} \simeq \sum_{\substack{\alpha,\gamma \in \mathcal{P}_p\\ \beta,\delta \in \mathcal{P}_q}}{}^{\oplus} c_{\alpha,\,\beta^*}^{\ \gamma\odot\delta}\ \big(\tau_{(\alpha+a(m)\mathbb{I}_p)\odot(\beta+b(l)\mathbb{I}_q)} \otimes \chi_{p,q}\big)^* \boxtimes \big(\tau_{\gamma\odot\delta} \otimes \chi_{p,q}\big).$$

Furthermore, the associated cycle of $\theta(\pi'_{\mathrm{hol}})$ *is given by*

$$\mathcal{AC}\,\theta(\pi'_{\mathrm{hol}}) = [\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}] \quad (\textit{multiplicity-free}).$$

Consequently, we have

$$\begin{aligned}\operatorname{Dim}\theta(\pi'_{\mathrm{hol}}) &= \dim \mathcal{O}^{\mathrm{hol}}_{p,q} = (p+q)(2n-(p+q)) + pq, \qquad \textit{and}\\ \operatorname{Deg}\theta(\pi'_{\mathrm{hol}}) &= \deg \overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}.\end{aligned}$$

The above results are also valid for the following reductive dual pairs in the stable range

$$(G, G') = \begin{cases} (O(p,q), Sp(2n,\mathbb{R})), & 2n < \min(p,q), \\ (U(p,q), U(r,s)), & r+s \leq \min(p,q), \\ (Sp(p,q), O^*(2n)), & n \leq \min(p,q) \end{cases}$$

with appropriate modifications. We shall leave this to the interested reader.

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2. Moment maps and the null cone

Unless otherwise stated, we always consider a reductive dual pair

$$(G, G') = (U(n,n), U(p,q))$$

in the *stable range*, where G' is the smaller member, i.e., we assume that $p+q \leq n$ throughout this paper except for §§4.1 and 4.2.

Let $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ be the complexified Lie algebra of G and fix a maximal compact subgroup $K = U(n) \times U(n)$ in G. It naturally determines a complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, which is realized explicitly as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} = \begin{pmatrix} \mathfrak{gl}_n & 0 \\ 0 & \mathfrak{gl}_n \end{pmatrix} \oplus \begin{pmatrix} 0 & M_n \\ M_n & 0 \end{pmatrix}, \tag{2.1}$$

where $M_n = M_n(\mathbb{C})$ denotes the space of all $n \times n$ matrices over $\mathbb{C}$. Therefore we can identify $\mathfrak{s} = M_n \oplus M_n^* = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Here, $\mathfrak{s}_- = M_n^*$ denote the dual of $\mathfrak{s}_+ = M_n$ via the trace form (or Killing form), and $\mathfrak{s}_+$ is identified with the upper right M_n in (2.1). The complexification $K_{\mathbb{C}} = GL_n \times GL_n$ of K acts on $\mathfrak{s}$ by the restriction of the adjoint action, and the above notation is also compatible with the action of $K_{\mathbb{C}}$, i.e., $\mathfrak{s}_\pm$ are both stable under $K_{\mathbb{C}}$, and $\mathfrak{s}_- = M_n^*$ is the contragredient representation of $\mathfrak{s}_+ = M_n$.

Similarly, we choose a maximal compact subgroup $K' = U(p) \times U(q) \subset G'$, and a complexified Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$. We identify $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_- = M_{p,q} \oplus M_{q,p}$, where $M_{p,q} = M_{p,q}(\mathbb{C})$ denotes the space of all $p \times q$ matrices, and we make the identification $M_{p,q}^* = M_{q,p}$ similarly.

We define two *moment maps* φ and ψ as follows. Put $W = M_{p+q,2n}$ and take

$$X = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \in W \quad (x, z \in M_{p,n}; y, w \in M_{q,n}). \tag{2.2}$$

Then they are defined as

$$\begin{aligned} \varphi \quad &: W \to \mathfrak{s}, \quad \varphi(X) = ({}^t x z, {}^t({}^t y w)) = (a, b) \in M_n \oplus M_n, \\ \psi \quad &: W \to \mathfrak{s}', \quad \psi(X) = (x\, {}^t y, {}^t(z\, {}^t w)) = (c, d) \in M_{p,q} \oplus M_{q,p}. \end{aligned}$$

We will define an action of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ on W in such a way that it makes φ and ψ $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant maps. Note that $K_{\mathbb{C}}$-action on $\mathfrak{s}$ is given by the adjoint action, while $K'_{\mathbb{C}}$-action on $\mathfrak{s}$ is trivial. Similar remarks are applicable to the action of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ on $\mathfrak{s}'$. Explicitly, the action of $K_{\mathbb{C}} \times {K_{\mathbb{C}}}'$ on W is given as follows: for $X \in W$ as in (2.2),

$$((k_1, k_2), (k'_1, k'_2)) \cdot \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} k'_1 x\, {}^t k_1 & ({}^t k'_1)^{-1} z k_2^{-1} \\ ({}^t k'_2)^{-1} y k_1^{-1} & k'_2 w\, {}^t k_2 \end{pmatrix},$$

where

$$\begin{aligned} &(k_1, k_2) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) = K_{\mathbb{C}}, \qquad \text{and} \\ &(k'_1, k'_2) \in GL_p(\mathbb{C}) \times GL_q(\mathbb{C}) = {K_{\mathbb{C}}}'. \end{aligned}$$

Let φ^* and ψ^* be the induced algebra homomorphisms of regular function rings. Thus for example, we have the algebra homomorphism $\varphi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]$, and in terms of matrix entry coordinates, it is given by

$$\varphi^*(a_{ij})(X) = ({}^t x z)_{ij} = \sum_{k=1}^{p} x_{ki} z_{kj}, \;\; \varphi^*(b_{ij})(X) = {}^t({}^t y w)_{ij} = \sum_{l=1}^{q} w_{lj} y_{li},$$

where $a_{ij} \in \mathbb{C}[\mathfrak{s}_+]$ is the linear functional on $\mathfrak{s}_+$ taking $a = (a_{ij})_{n\times n} \in \mathfrak{s}_+$ to the (i,j)-th entry, and similarly for $b_{ij} \in \mathbb{C}[\mathfrak{s}_-]$. Classical invariant theory [24] then tells us that

$$\operatorname{Image} \varphi^* = \mathbb{C}[W]^{K'_{\mathbb{C}}} \quad \text{and} \quad \operatorname{Image} \psi^* = \mathbb{C}[W]^{K_{\mathbb{C}}}.$$

This means that both φ and ψ are geometric quotient maps from W onto their images.

Let $W = W_+ \oplus W_-$ be a decomposition of W, where

$$W_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in M_{p+q,n} \right\} \quad \text{and} \quad W_- = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in M_{p+q,n} \right\}$$

in the notation of (2.2). Let

$$\psi_+ : W_+ = M_{p+q,n} \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x\, {}^t y \in M_{p,q} = {\mathfrak{s}'}_+$$

be the restriction of ψ to the "holomorphic" half of W. We put

$$\mathfrak{N}_{p,q} = \psi_+^{-1}(0) = \psi^{-1}(0) \cap W_+ \subset W_+$$

and call it the *null cone*. Note that $K'_{\mathbb{C}} = GL_p \times GL_q$ and GL_n, which is the first component of $K_{\mathbb{C}} = GL_n \times GL_n$, act simultaneously on W_+.

We quote the following result of Kostant.

Theorem 2.1 ([8]). *Assume that $p+q \le n$. Then the null cone $\mathfrak{N}_{p,q}$ is irreducible, and its defining ideal $\mathbf{I}(\mathfrak{N}_{p,q})$ is generated by GL_n-invariant polynomials of W_+ of positive degree. Moreover, the regular function ring of $\mathfrak{N}_{p,q}$ is naturally isomorphic to $\mathcal{H}_{p,q}$, the space of GL_n-harmonic polynomials of W_+. We have*

$$\mathbb{C}[W_+] \simeq \mathbb{C}[\mathfrak{N}_{p,q}] \otimes \mathbb{C}[W_+]^{GL_n} \simeq \mathcal{H}_{p,q} \otimes \mathbb{C}[{\mathfrak{s}'}_+]$$

as $GL_n \times K'_{\mathbb{C}}$-modules.

The above theorem tells us that the action of GL_n on W_+ is completely determined by its action on the null cone $\mathfrak{N}_{p,q}$. Thus we are interested in the $GL_n \times K'_{\mathbb{C}}$-module structure of $\mathbb{C}[\mathfrak{N}_{p,q}] \simeq \mathcal{H}_{p,q}$. This is described below by the well-known result of Kashiwara and Vergne [7]. See also [4].

We recall some notations from the Introduction. To make it more transparent, we shall now denote the irreducible finite dimensional representation of GL_n with highest weight $\lambda \in \Lambda_n^+$ by $\tau_\lambda^{(n)}$.

Theorem 2.2 ([7], [4])**.** *Assume that $p+q \leq n$. As a $GL_n \times K'_{\mathbb{C}} = GL_n \times (GL_p \times GL_q)$-module, we have*

$$\mathbb{C}[\mathfrak{N}_{p,q}] \simeq \mathcal{H}_{p,q} \simeq \sum_{\lambda \in \mathcal{P}_p, \mu \in \mathcal{P}_q}^{\oplus} \tau_{\lambda \odot \mu}^{(n)\,*} \boxtimes (\tau_\lambda^{(p)*} \boxtimes \tau_\mu^{(q)}).$$

Remark 2.3. The above isomorphism is a graded isomorphism if we assign the grading on the homogeneous component $\tau_{\lambda \odot \mu}^{(n)\,*} \boxtimes (\tau_\lambda^{(p)*} \boxtimes \tau_\mu^{(q)})$ by $|\lambda| + |\mu|$, where $|\lambda| = \sum_i \lambda_i$ (resp., $|\mu| = \sum_j \mu_j$) is the size of λ (resp., μ). Similar remarks apply to isomorphisms in Theorem 3.6 and Theorem 3.9.

3. Theta lift of orbits

For a subset S in $\mathfrak{g}$, we denote by $\mathcal{N}(S)$ the subset of nilpotent elements in S. It is known that $K_{\mathbb{C}}$ acts on $\mathcal{N}(\mathfrak{s})$ with finitely many orbits, and that the $K_{\mathbb{C}}$-orbits in $\mathcal{N}(\mathfrak{s})$ correspond bijectively to G-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ (Kostant-Sekiguchi correspondence), where $\mathfrak{g}_{\mathbb{R}}$ denotes the Lie algebra of G (over $\mathbb{R}$). It is easy to see that the moment maps preserve nilpotent elements. Namely we have

$$\varphi(\psi^{-1}(\mathcal{N}(\mathfrak{s}'))) \subset \mathcal{N}(\mathfrak{s}) \quad \text{and} \quad \psi(\varphi^{-1}(\mathcal{N}(\mathfrak{s}))) \subset \mathcal{N}(\mathfrak{s}').$$

The following result is key to the notion of the theta lift of nilpotent orbits. It may be known to the experts. For lack of suitable references we provide a sketched proof which is based on simple and explicit calculations. Recently, we have learned more general and sophisticated proof from Takuya Ohta (unpublished).

Proposition 3.1. *Assume that $p+q \leq n$. Take a $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathcal{N}(\mathfrak{s}')$. Then $\Xi = \psi^{-1}(\overline{\mathcal{O}'})$ is the closure of a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit in W. As a consequence, the variety Ξ is irreducible, hence $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O} \in \mathcal{N}(\mathfrak{s})$.*

Proof. Recall that associated to a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathfrak{s}'$, there is a signed Young diagram Y of signature (p, q), i.e., a Young diagram of size $p+q$ which has p boxes of plus sign, and q boxes of minus sign. We put the numbers $\{1, \ldots, p\}$ in the plus-boxes, and $\{p+1, \ldots, p+q\}$ in the minus-boxes of Y. We denote the number in the (i, j)-th box of Y by $\lambda_{i,j}$. For an arbitrary basis

$$\{v_1, \ldots, v_p\} \subset V^+ = \mathbb{C}^p, \quad \text{and} \quad \{v_{p+1}, \ldots, v_{p+q}\} \subset V^- = \mathbb{C}^q,$$

we have a representative T in $\mathcal{O}'$ given by

$$T v_{\lambda_{i,j}} = v_{\lambda_{i,j+1}}.$$

Here $v_{\lambda_{i,j+1}}$ is understood as zero if there is no $(i, j+1)$-th box in Y.

Consider an element X in the fiber $\psi^{-1}(T)$ of $T \in \mathcal{O}'$, and express it as in (2.2):

$$X = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \in W \quad (x, z \in M_{p,n}; y, w \in M_{q,n}).$$

Then we have

$$T = \psi(X) = \begin{pmatrix} 0 & x\,{}^t y \\ w\,{}^t z & 0 \end{pmatrix} \overset{\text{put}}{=} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}.$$

It is convenient to consider the following (commutative) quiver diagram:

$$\begin{tikzcd}
& U^+ = \mathbb{C}^n \arrow[dl, "x"'] & \\
V^+ = \mathbb{C}^p \arrow[rr, shift right, "d"'] \arrow[dr, "{}^t z"'] & & \mathbb{C}^q = V^- \arrow[ll, shift right, "c"'] \arrow[ul, "{}^t y"'] \\
& \mathbb{C}^n = U^- \arrow[ur, "w"'] &
\end{tikzcd}$$

We first consider the full rank case, namely $\operatorname{rank} x = \operatorname{rank} {}^t z = p$ and $\operatorname{rank} {}^t y = \operatorname{rank} w = q$. Note that $x : \mathbb{C}^n \to \mathbb{C}^p$ is surjective, while ${}^t y : \mathbb{C}^q \to \mathbb{C}^n$ is injective by the rank condition.

We choose a basis $\{u_1, \dots, u_n\}$ of $U^+ = \mathbb{C}^n$ as follows. For $1 \le k \le q$, put $u_k = {}^t y(v_{p+k})$. Now for each v_i $(1 \le i \le p)$ which is not in $\operatorname{Im} c$, choose u_k such that $x(u_k) = v_i$. Note that there are $(p - \operatorname{rank} c)$ of such v_i's. After adding these u_k's to the already chosen $\{u_1, \dots, u_q\}$, we get $\{u_1, \dots, u_{p+q-\operatorname{rank} c}\}$. These u_k's are seen to be linearly independent. Finally, we choose $\{u_{p+q-\operatorname{rank} c+1}, \dots, u_n\}$ from $\ker x$ in such a way that $\{u_1, \dots, u_n\}$ forms a basis of $U^+ = \mathbb{C}^n$ (which is possible).

With respect to this basis of U^+ and bases of V^+, V^- fixed at the begining, we can (after some reordering) express x and y as

$$y = \begin{pmatrix} 1_q & 0_{q,n-q} \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 0_{p,q-\operatorname{rank} c} & 1_p & 0_{p,n+\operatorname{rank} c-(p+q)} \end{pmatrix},$$

where $0_{s,t}$ denotes the zero matrix of size $s \times t$. A similar argument may be applied to z and w. This implies that $\psi^{-1}(T)$ has an open dense $K_{\mathbb{C}}$-orbit. The $K'_{\mathbb{C}}$-translation of it will then give an open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathbb{O}$ in $\psi^{-1}(\overline{\mathcal{O}'})$ (one just needs to use the same argument as above for the boundary orbits in $\overline{\mathcal{O}'} \setminus \mathcal{O}'$). Therefore we conclude that $\Xi = \psi^{-1}(\overline{\mathcal{O}'}) = \overline{\mathbb{O}}$, and Ξ is irreducible.

Since φ is an affine geometric quotient map, it is closed. Because $\varphi(\Xi)$ is irreducible, closed and $K_{\mathbb{C}}$-stable, it is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}$. □

Remark 3.2. The above proof shows that, if the nilpotent elements in $\mathcal{O}'$ are k-step nilpotent, then those in $\mathcal{O}$ are $(k+1)$-step nilpotent.

Definition 3.3. We call the $K_{\mathbb{C}}$-orbit $\mathcal{O}$ which is open dense in $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ the *theta lift* of $\mathcal{O}'$.

Proposition 3.4. *Let $\mathcal{O}$ be the theta lift of $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathcal{N}(\mathfrak{s}')$. Then the closure $\overline{\mathcal{O}}$ is a geometric quotient of $\Xi = \psi^{-1}(\overline{\mathcal{O}'})$ by $K'_{\mathbb{C}}$, i.e., $\overline{\mathcal{O}} \simeq \Xi /\!/ K'_{\mathbb{C}}$. In particular, we have $\mathbb{C}[\overline{\mathcal{O}}] \simeq \mathbb{C}[\Xi]^{K'_{\mathbb{C}}}$.*

Proof. Since $\varphi : W \to \varphi(W)$ is a geometric quotient map, and Ξ is a $K'_{\mathbb{C}}$-stable closed subvariety of W, the proposition follows from the general arguments on geometric quotients. See [14] and [15] for details. $\square$

3.1. **Theta lift of the trivial orbit.** First, we consider the simplest case, where the orbit $\mathcal{O}'$ is trivial. We write $\Xi = \Xi^{\mathbf{1}}_{p,q} = \psi^{-1}(\{0\})$. Then we clearly have

$$\Xi = \Xi^{\mathbf{1}}_{p,q} = \mathfrak{N}_{p,q} \times \mathfrak{N}_{q,p} \subset W_+ \times W_-,$$

where the null cone $\mathfrak{N}_{q,p}$ is defined similarly as $\mathfrak{N}_{p,q}$. Since $\mathfrak{N}_{p,q} \simeq \mathfrak{N}_{q,p}$ as varieties, and the action of $GL_n \times K'_{\mathbb{C}}$ on $\mathfrak{N}_{q,p}$ is dual to that on $\mathfrak{N}_{p,q}$, we sometimes denote $\mathfrak{N}_{q,p}$ by $\mathfrak{N}^*_{p,q}$. In particular, we have $\mathbb{C}[\mathfrak{N}_{q,p}] \simeq \mathbb{C}[\mathfrak{N}_{p,q}]^*$ as a representation of $GL_n \times K'_{\mathbb{C}}$.

We denote the theta lift of the trivial orbit $\mathcal{O}' = \{0\}$ by $\mathcal{O}^{\mathbf{1}}_{p,q}$. The closure $\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}$ is the geometric quotient of $\mathfrak{N}_{p,q} \times \mathfrak{N}_{q,p}$ by the action of $K'_{\mathbb{C}}$. It is a two-step nilpotent orbit with Jordan normal form represented by the partition $2^{p+q} \cdot 1^{2n-2(p+q)}$.

We include the following simple proposition for its intrinsic interest (see [14] and [15]).

Proposition 3.5. *Let $\mathcal{O}^{\mathbf{1}}_{p,q}$ $(p+q \le n)$ be the theta lift of the trivial $K'_{\mathbb{C}}$-orbit in $\mathcal{N}(\mathfrak{s}')$. Then*
(1) *every two-step nilpotent $K_{\mathbb{C}}$-orbit in $\mathcal{N}(\mathfrak{s})$ for $G = U(n,n)$ is of the form $\mathcal{O}^{\mathbf{1}}_{p,q}$ for some $p+q \le n$. Two orbits $\mathcal{O}^{\mathbf{1}}_{p_1,q_1}$ and $\mathcal{O}^{\mathbf{1}}_{p_2,q_2}$ generate the same $G_{\mathbb{C}}$-orbits if and only if $p_1+q_1 = p_2+q_2$.*
(2) *The dimension and the closure of the orbit $\mathcal{O}^{1}_{p,q}$ are given by*

$$\dim \mathcal{O}^{\mathbf{1}}_{p,q} = (p+q)(2n-(p+q)), \quad \overline{\mathcal{O}^{\mathbf{1}}_{p,q}} = \coprod_{r \le p, s \le q} \mathcal{O}^{\mathbf{1}}_{r,s}.$$

(3) *The variety $\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}$ is normal. If $p+q<n$, then we have $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}] = \mathbb{C}[\mathcal{O}^{\mathbf{1}}_{p,q}]$.*

The regular function ring $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ has a beautiful $K_{\mathbb{C}}$ structure, which is described by the following theorem. It can be thought of as a generalization of the well-known decomposition of $\mathbb{C}[M_{n,n}]$ as a $GL_n \times GL_n$-module. We shall use this to determine the associated cycle of the theta lift of a unitary character of G' to G.

For $\alpha \in \mathcal{P}_p$ and $\beta \in \mathcal{P}_q$, we put $\alpha \odot_n \beta = (\alpha, 0, \ldots, 0, \beta^*) \in \Lambda^+_n$. Denote

$$\Lambda^+_n(p,q) = \{\alpha \odot_n \beta \mid \alpha \in \mathcal{P}_p, \beta \in \mathcal{P}_q\}. \tag{3.1}$$

Theorem 3.6. *Assume that $p+q \le n$, and let $\mathcal{O}^{\mathbf{1}}_{p,q} \subset \mathcal{N}(\mathfrak{s})$ be the theta lift of the trivial nilpotent $K'_{\mathbb{C}}$-orbit in $\mathcal{N}(\mathfrak{s}')$. Then we have*

$$\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}] \simeq \sum_{\lambda \in \Lambda^+_n(p,q)}^{\oplus} {\tau_\lambda}^* \boxtimes \tau_\lambda,$$

as a $K_{\mathbb{C}} = GL_n \times GL_n$-module.

Proof. The proof is similar to that of Theorem 3.9 below. See the remark after the proof of Theorem 3.9. $\square$

Remark 3.7. If $p+q=n$, one can show that $\mathbb{C}[\mathcal{O}^{\mathbf{1}}_{p,q}] \simeq \Sigma^{\otimes}_{\lambda \in \Lambda^+_n} {\tau_\lambda}^* \boxtimes \tau_\lambda$. Therefore, $\mathbb{C}[\mathcal{O}^{\mathbf{1}}_{p,q}]$ is strictly larger than $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ for $p+q=n$.

3.2. **Theta lift of the dense holomorphic orbit.** Note that $\mathfrak{s}'_- \subset \mathcal{N}(\mathfrak{s}')$. Since $\mathfrak{s}'_-$ is irreducible and $K'_{\mathbb{C}}$-stable, it has the dense open $K'_{\mathbb{C}}$-orbit $\mathcal{O}'_{\mathrm{hol}}$, which consists of the matrices in $M_{q,p} = \mathfrak{s}'_-$ of the maximal possible rank $\min(p, q)$.

Let $\mathcal{O}^{\mathrm{hol}}_{p,q} \subset \mathcal{N}(\mathfrak{s})$ be the theta lift of $\mathcal{O}'_{\mathrm{hol}}$. Since $\mathfrak{s}'_- = \overline{\mathcal{O}'_{\mathrm{hol}}}$, we have $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}} = \varphi(\psi^{-1}(\mathfrak{s}'_-))$ by definition. The elements in $\mathcal{O}^{\mathrm{hol}}_{p,q}$ are three-step nilpotent, and their Jordan normal forms are represented by the partition $3^p \cdot 2^{q-p} \cdot 1^{2n-p-2q}$ for $q \geq p$.

We refer the following proposition again to [14] and [15].

Proposition 3.8. *Let $\mathcal{O}^{\mathrm{hol}}_{p,q}$ be the theta lift of the open dense $K'_{\mathbb{C}}$-orbit in $\mathfrak{s}'_-$. Then*

(1) *the closure $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}$ $(p + q \leq n)$ is a normal variety. The dimension of the orbit is* $\dim \mathcal{O}^{\mathrm{hol}}_{p,q} = (p + q)(2n - (p + q)) + pq$.

(2) $\mathcal{O}^{\mathrm{hol}}_{p_1,q_1}$ *and* $\mathcal{O}^{\mathrm{hol}}_{p_2,q_2}$ *generate the same complex $G_{\mathbb{C}}$-orbit in $\mathcal{N}(\mathfrak{g})$ if and only if* $(p_2, q_2) = (p_1, q_1)$ *or* (q_1, p_1).

Theorem 3.9. *As a $K_{\mathbb{C}} = GL_n \times GL_n$-module, we have*

$$\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}] \simeq \sum^{\oplus}_{\substack{\alpha,\gamma\in\mathcal{P}_p\\ \beta,\delta\in\mathcal{P}_q}} c^{\,\gamma\odot\delta}_{\alpha,\,\beta^*}\ (\tau_{\alpha\odot\beta})^* \boxtimes \tau_{\gamma\odot\delta},$$

where $c^{\,\lambda}_{\mu,\,\nu}$ denotes the Littlewood-Richardson coefficient defined in (1.4).

Proof. We put $\Xi_{p,q} = \psi^{-1}(\mathfrak{s}'_-) \subset W$. Then, clearly we have $\Xi_{p,q} = \mathfrak{N}_{p,q} \times W_-$. Since $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}$ is the geometric quotient of $\Xi_{p,q}$ by $K'_{\mathbb{C}}$, we see

$$\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}] \simeq \mathbb{C}[\Xi_{p,q}]^{K'_{\mathbb{C}}} \simeq \Big(\mathbb{C}[\mathfrak{N}_{p,q}] \otimes \mathbb{C}[W_-]\Big)^{K'_{\mathbb{C}}}.$$

Since $W_- = M^*_{p,n} \oplus M_{q,n}$, we get (as a $GL_n \times K'_{\mathbb{C}}$-module)

$$\begin{aligned}
\mathbb{C}[W_-] &\simeq \mathbb{C}[M^*_{p,n}] \otimes \mathbb{C}[M_{q,n}] \\
&\simeq \sum^{\oplus}_{\lambda\in\mathcal{P}_p,\mu\in\mathcal{P}_q} (\tau^{(p)}_\lambda \boxtimes \tau^{(n)}_\lambda) \otimes (\tau^{(q)*}_\mu \boxtimes \tau^{(n)*}_\mu) \\
&\simeq \sum^{\oplus}_{\lambda\in\mathcal{P}_p,\mu\in\mathcal{P}_q} (\tau^{(p)}_\lambda \boxtimes \tau^{(q)*}_\mu) \boxtimes \sum^{\oplus}_{\nu\in\Lambda^+_n} c^{\,\nu}_{\lambda,\,\mu^*}\ \tau^{(n)}_\nu \\
&\simeq \sum^{\oplus}_{\lambda,\mu,\nu} c^{\,\nu}_{\lambda,\,\mu^*}\ \tau^{(n)}_\nu \boxtimes (\tau^{(p)}_\lambda \boxtimes \tau^{(q)*}_\mu).
\end{aligned}$$

In the last summation, note that it is sufficient to consider $\nu = \gamma\odot\delta$ ($\gamma \in \mathcal{P}_p$, $\delta \in \mathcal{P}_q$), because otherwise the Littlewood-Richardson coefficient $c^{\,\nu}_{\lambda,\,\mu^*}$ vanishes. The module structure of $\mathbb{C}[\mathfrak{N}_{p,q}]$ is given in Theorem 2.2, namely,

$$\mathbb{C}[\mathfrak{N}_{p,q}] \simeq \sum^{\oplus}_{\alpha\in\mathcal{P}_p,\beta\in\mathcal{P}_q} (\tau^{(n)}_{\alpha\odot\beta})^* \boxtimes (\tau^{(p)*}_\alpha \boxtimes \tau^{(q)}_\beta).$$

Taking $K'_{\mathbb{C}} = GL_p \times GL_q$-invariants in the tensor product of these two spaces, we obtain

$$\mathbb{C}[\Xi_{p,q}]^{K'_{\mathbb{C}}} \simeq \sum^{\oplus}_{\alpha,\beta,\nu} c^{\,\nu}_{\alpha,\,\beta^*}\ (\tau^{(n)}_{\alpha\odot\beta})^* \boxtimes \tau^{(n)}_\nu,$$

where $\nu = \gamma\odot\delta$ is as mentioned above. □

Remark 3.10. We comment on the proof of Theorem 3.6. Recall that $\overline{\mathcal{O}^{\mathbf{1}}_{p,q}} = \varphi(\Xi^{\mathbf{1}}_{p,q})$, where $\Xi^{\mathbf{1}}_{p,q} = \mathfrak{N}_{p,q} \times \mathfrak{N}_{q,p}$. Thus we get

$$\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}] = \mathbb{C}[\Xi^{\mathbf{1}}_{p,q}]^{K'_{\mathbb{C}}} \simeq (\mathbb{C}[\mathfrak{N}_{p,q}] \otimes \mathbb{C}[\mathfrak{N}_{p,q}]^*)^{K'_{\mathbb{C}}}.$$

The rest of the proof remains the same as above.

4. Theta lifting associated to the dual pair $(U(n,n), U(p,q))$

4.1. **Howe's maximal quotient.** Let $(G, G') \subseteq \mathbb{G} = Sp(2N, \mathbb{R})$ be a reductive dual pair, and let Ω be a fixed oscillator representation of $\widetilde{\mathbb{G}}$, the metaplectic cover of $\mathbb{G}$. Often when no confusion should arise, we shall not distinguish Ω with its Harish-Chandra module.

Denote by $\mathrm{Irr}(\mathfrak{g}', \widetilde{K'})$ the infinitesimal equivalent classes of irreducible admissible $(\mathfrak{g}', \widetilde{K'})$-modules, and $R(\mathfrak{g}', \widetilde{K'}; \Omega)$ the subset of those in $\mathrm{Irr}(\mathfrak{g}', \widetilde{K'})$ which can be realized as quotients by $(\mathfrak{g}', \widetilde{K'})$-invariant subspaces of Ω. According to [5], for each $\pi' \in R(\mathfrak{g}', \widetilde{K'}; \Omega)$ there exists a quasi-simple admissible $(\mathfrak{g}, \widetilde{K})$-module $\Omega(\pi')$ of finite length satisfying

$$\Omega/I(\pi') \simeq \Omega(\pi') \otimes \pi',$$

where

$$I(\pi') = \bigcap_{\phi \in \mathrm{Hom}'} \mathrm{Ker}(\phi), \qquad \mathrm{Hom}' = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi').$$

Furthermore, $\Omega(\pi')$ has a unique irreducible quotient, denoted by $\theta(\pi')$. $\Omega(\pi')$ is called *Howe's maximal quotient* of π', and $\theta(\pi')$ the *theta lift* of π'.

Note that any $\phi \in \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')$ factors through a map $\widetilde{\phi} : \Omega/I(\pi') \mapsto \pi'$ and will therefore define an element in the algebraic dual of $\Omega(\pi')$. This association is $(\mathfrak{g}, \widetilde{K})$-equivariant, and so we have (by taking $\widetilde{K}$-finite vectors)

Lemma 4.1. *We have the isomorphism*

$$\Omega(\pi')^* \simeq \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}\text{-finite}},$$

where $\Omega(\pi')^*$ is the dual in the category of Harish-Chandra modules.

We specialize to the case $(G, G') = (U(n,n), U(p,q)) \subseteq Sp(4n(p+q), \mathbb{R})$. We shall need to use the following

Proposition 4.2. *Let $(G, G') = (U(n,n), U(p,q))$. Suppose π' is the $(\mathfrak{g}', \widetilde{K'})$-module of*

(1) *a unitary character and $p + q \leq n$; or*

(2) *a discrete series representation of $U(p,q)$ and $p + q \leq 2n$,*

then the maximal quotient $\Omega(\pi')$ is irreducible. Hence we have $\theta(\pi') = \Omega(\pi')$.

Proof. (1) is a special case of Proposition 2.1 of [25]. (2) follows from (the proof of) Proposition 2.4 of [13], where it is shown that $\theta(\pi') \otimes \pi'$ occurs as an irreducible summand of $\Omega|_{\widetilde{G} \cdot \widetilde{G'}}$. ◻

Recall that associated to the dual pair $(U(n), U(p,q))$, the covering map $U(n)^\sim \to U(n)$ splits if and only if $p+q$ is even. When $p+q$ is odd, $U(n)^\sim$ can be identified with the half determinant cover, namely

$$U(n)^\sim = \{(u, c) \in U(n) \times \mathbb{C}^\times | c^2 = \det(u)\}.$$

Let $\det^{\frac{1}{2}}$ be the character of $U(n)^\sim$ defined by $U(n)^\sim \ni (u,c) \mapsto c$. Similar notation applies to the characters of $U(p,q)^\sim$.

Let

$$\Lambda_n^+(p,q) = \{\alpha \odot_n \beta \in \Lambda_n^+ | \alpha \in \mathcal{P}_k, \beta \in \mathcal{P}_l, k \le p, l \le q, k+l \le n\}.$$

Note that this definition coincides with our previous notation of $\Lambda_n^+(p,q)$ for $p+q \le n$ (see (3.1)).

Recall also the decomposition of an oscillator representation for a pair of compact type [7, 5]. For the pair $(U(n), U(p,q)) \subseteq Sp(2(p+q)n, \mathbb{R})$, denote ω the associated oscillator representation. Then as a $U(n)^\sim \times U(p,q)^\sim$-module, we have

$$\omega \simeq \sum_{\lambda \in \Lambda_n^+(p,q)}^{\oplus} (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes L(\tau_\lambda^{(n)}), \tag{4.1}$$

where $\chi_{p,q} = \det^{\frac{p-q}{2}}$. The irreducible representation $L(\tau_\lambda^{(n)})$ is a unitary highest weight module of $U(p,q)^\sim$ with minimal $\widetilde{K'}$-type $(\tau_\alpha^{(p)} \otimes \det^{\frac{n}{2}}) \boxtimes (\tau_\beta^{(q)} \otimes \det^{\frac{n}{2}})^*$.

Proposition 4.3. *Let $\pi' \in R(\mathfrak{g}', \widetilde{K'}; \Omega)$. Then*

$$\begin{aligned}\Omega(\pi')^*\big|_{\widetilde{K}} \simeq & \sum_{\lambda,\mu \in \Lambda_n^+(p,q)}^{\oplus} \dim \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^*, \pi') \\ & \times \big((\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})^*\big),\end{aligned} \tag{4.2}$$

or equivalently

$$\begin{aligned}\Omega(\pi')\big|_{\widetilde{K}} \simeq & \sum_{\lambda,\mu \in \Lambda_n^+(p,q)}^{\oplus} \dim \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^*, \pi') \\ & \times \big((\tau_\lambda^{(n)} \otimes \chi_{p,q})^* \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})\big).\end{aligned}$$

Proof. We consider the see-saw pair ([9, 3]):

$$\begin{array}{ccc} G = U(n,n) & & U(p,q) \times U(p,q) = L' \\ \cup & \times & \cup \quad \text{diagonal} \\ K = U(n) \times U(n) & & U(p,q) = G' \end{array}$$

By the functoriality of the oscillator representation, we have

$$\Omega \simeq \omega \otimes \omega^*$$

as $\widetilde{K} \times \widetilde{L'}$-modules. Here the first factor $U(n)$ of K acts on the first factor of $\omega \otimes \omega^*$ via ω, while the second factor $U(n)$ of K acts on the second factor of $\omega \otimes \omega^*$ via the dual of ω.

Thus as $\widetilde{K} \times (\mathfrak{g}', \widetilde{K'})$-modules, we have

$$\begin{aligned}\Omega \simeq \omega \otimes \omega^* \simeq & \Big(\sum_{\lambda \in \Lambda_n^+(p,q)}^{\oplus} (\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes L(\tau_\lambda^{(n)}) \Big) \\ & \otimes \Big(\sum_{\mu \in \Lambda_n^+(p,q)}^{\oplus} (\tau_\mu^{(n)} \otimes \chi_{p,q})^* \boxtimes L(\tau_\mu^{(n)})^* \Big) \\ \simeq & \sum_{\lambda,\mu \in \Lambda_n^+(p,q)}^{\oplus} \big((\tau_\lambda^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})^*\big) \boxtimes \big(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^*\big).\end{aligned}$$

In view of the isomorphism $\Omega(\pi')^* \simeq \mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(\Omega,\pi')_{\widetilde{K}\text{-finite}}$, the assertion follows. □

4.2. **Explicit $\widetilde{K}$-type formulas.** We first discuss some generalities on $(\mathfrak{g},K)$-modules, where $\mathfrak{g}$ is the complexification of the Lie algebra of a semisimple Lie group G and K is a maximal compact subgroup of G.

Let H be a $(\mathfrak{g},K)$-module which is K-admissible, i.e., $\dim \mathrm{Hom}_K(H,\tau) < \infty$ for any $\tau \in \mathrm{Irr}(K)$. We also assume that H is locally K-finite, which means that $H = H_K$ where H_K denotes the space of K-finite vectors in H. Then, $H^* = \mathrm{Hom}_{\mathbb{C}}(H,\mathbb{C})_K$ is also K-admissible and we have a canonical isomorphism $H \simeq (H^*)^*$. Note that H^* does *not* denote the algebraic dual of H. A straightforward argument gives

Lemma 4.4. *Let H_1 be a $(\mathfrak{g},K)$-module which is locally K-finite. Then, for any K-admissible $(\mathfrak{g},K)$-module H_2 which is locally K-finite, we have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_1\otimes H_2^*,\mathbf{1}) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(H_1,H_2).$$

By applying Lemma 4.4 twice, we obtain

Corollary 4.5. *Let H_1 be a $(\mathfrak{g},K)$-module which is locally K-finite. Assume that H_2 and π are K-admissible $(\mathfrak{g},K)$-modules which are locally K-finite, and $H_2^*\otimes\pi^* \simeq (H_2\otimes\pi)^*$. Then we have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_1\otimes H_2^*,\pi) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(H_1,H_2\otimes\pi).$$

We note that the hypothesis of the above corollary is satisfied if π is a finite dimensional $(\mathfrak{g},K)$-module, or if both H_2 and π are unitary highest weight modules.

Combining Proposition 4.2, Proposition 4.3 and Lemma 4.4, we conclude the following result first obtained in [10].

Theorem 4.6. *We have*

$$\Omega(\det{}^k)\big|_{\widetilde{K}} \simeq \sum_{\lambda\in\Lambda_n^+(p,q)}^{\oplus} \big(\tau_\lambda\otimes\chi_{p,q}\big)^*\boxtimes\big(\tau_\lambda\otimes\chi_{p,q}\big), \qquad k=0,$$

and for $k\neq 0$, $p+q\leq n$,

$$\Omega(\det{}^k)\big|_{\widetilde{K}} \simeq \begin{cases} \displaystyle\sum_{\lambda\in\Lambda_n^+(p,q)}^{\oplus} \big(\tau_{(\lambda+k\mathbb{I}_p\odot k\mathbb{I}_q)}\otimes\chi_{p,q}\big)^*\boxtimes\big(\tau_\lambda\otimes\chi_{p,q}\big), & k>0,\\ \displaystyle\sum_{\lambda\in\Lambda_n^+(p,q)}^{\oplus} \big(\tau_\lambda\otimes\chi_{p,q}\big)^*\boxtimes\big(\tau_{(\lambda+|k|\mathbb{I}_p\odot|k|\mathbb{I}_q)}\otimes\chi_{p,q}\big), & k<0.\end{cases}$$

In particular, this gives the $\widetilde{K}$-type decomposition of the theta lift $\theta(\det^k)$, for $p+q\leq n$.

Remark 4.7. For $k\neq 0$, we have $\Omega(\det^k)=0$ if $p+q>n$.

For $\eta\in\Lambda_{2n}^+$ and $\mu,\nu\in\Lambda_n^+$, define branching coefficients $b_{\mu,\nu}^\eta$ by

$$\tau_\eta^{(2n)}\big|_{GL_n\times GL_n} \simeq \sum_{\mu,\nu\in\Lambda_n^+}^{\oplus} b_{\mu,\nu}^\eta\,\tau_\mu^{(n)}\boxtimes\tau_\nu^{(n)}. \tag{4.3}$$

The following proposition is a special case of Howe's reciprocity theorem [3]. We give an argument for the sake of completeness. Similar arguments will be omitted later.

Proposition 4.8. *For μ, $\nu \in \Lambda_n^+(p,q)$, we have*

$$L(\tau_\mu^{(n)}) \otimes L(\tau_\nu^{(n)}) \simeq \sum_{\eta \in \Lambda_{2n}^+(p,q)}{}^{\oplus} \; b_{\mu,\nu}^{\eta} L(\tau_\eta^{(2n)}).$$

Proof. For the moment, we let $(G, G') = (U(2n), U(p,q)) \subseteq Sp(4n(p+q), \mathbb{R})$, and let Φ be an associated oscillator representation. We have the see-saw pair

$$\begin{array}{ccc} G = U(2n) & & U(p,q) \times U(p,q) = L' \\ \cup & \times & \cup \;\; \text{diagonal} \\ K = U(n) \times U(n) & & U(p,q) = G' \end{array}$$

Functoriality of the oscillator representation implies that

$$\Phi \simeq \omega \otimes \omega$$

as $\widetilde{K} \times \widetilde{L}'$-modules, where as before ω is an oscillator representation associated to the dual pair

$$(U(n), U(p,q)) \subseteq Sp(2(p+q)n, \mathbb{R}).$$

Thus we have

$$\begin{aligned} \Phi &\simeq \Big(\sum_{\mu \in \Lambda_n^+(p,q)}{}^{\oplus} \; (\tau_\mu^{(n)} \otimes \chi_{p,q}) \boxtimes L(\tau_\mu^{(n)}) \Big) \otimes \Big(\sum_{\nu \in \Lambda_n^+(p,q)}{}^{\oplus} \; (\tau_\nu^{(n)} \otimes \chi_{p,q}) \boxtimes L(\tau_\nu^{(n)}) \Big) \\ &\simeq \sum_{\mu,\nu}{}^{\oplus} \big((\tau_\mu^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\nu^{(n)} \otimes \chi_{p,q}) \big) \boxtimes \big(L(\tau_\mu^{(n)}) \otimes L(\tau_\nu^{(n)}) \big). \end{aligned} \tag{4.4}$$

On the other hand, we have (as $\widetilde{G} \times \widetilde{G'}$-modules)

$$\Phi \simeq \sum_{\eta \in \Lambda_{2n}^+(p,q)}{}^{\oplus} \; (\tau_\eta^{(2n)} \otimes \chi_{p,q}) \boxtimes L(\tau_\eta^{(2n)}).$$

From the definition of the branching coefficient (4.3), we have

$$\Phi \simeq \sum_{\mu,\nu}{}^{\oplus} \big((\tau_\mu^{(n)} \otimes \chi_{p,q}) \boxtimes (\tau_\nu^{(n)} \otimes \chi_{p,q}) \big) \boxtimes \Big(\sum_{\eta \in \Lambda_{2n}^+(p,q)}{}^{\oplus} \; b_{\mu,\nu}^{\eta} L(\tau_\eta^{(2n)}) \Big). \tag{4.5}$$

Comparing (4.4) and (4.5), we get the desired formula. □

From now on, we assume that the pair (G, G') is in the stable range with G' the small member, namely $p + q \leq n$. Then $L(\tau_\lambda^{(n)})$ is a holomorphic discrete series for each $\lambda \in \Lambda_n^+(p,q)$.

Let χ_0 be the following character of $U(p,q)^\sim$:

$$\chi_0 = \begin{cases} 1, & n \text{ even}, \\ \det^{-\frac{1}{2}}, & n \text{ odd}, \end{cases}$$

and for $\lambda \in \Lambda_n^+(p,q)$, let

$$\widetilde{L}(\tau_\lambda^{(n)}) = \chi_0 \otimes L(\tau_\lambda^{(n)}). \tag{4.6}$$

Thus $\widetilde{L}(\tau_\lambda^{(n)})$ defines a true representation of $U(p,q)$.

Recall also that associated to the dual pair $(U(n,n), U(p,q))$, the covering $U(p,q)^\sim \to U(p,q)$ splits. Note that since we are assuming that the pair is in the stable range, any unitary representation of $U(p,q)$ is in the domain of theta correspondence. See [12].

For $\lambda = \alpha \odot_n \beta \in \Lambda_n^+(p,q)$, denote

$$\widetilde{\lambda} = \alpha \odot_{2n} \beta \in \Lambda_{2n}^+(p,q) \tag{4.7}$$

by inserting n extra zeroes. We note that each $\eta \in \Lambda_{2n}^+(p,q)$ is of the form $\widetilde{\lambda}$ for some $\lambda \in \Lambda_n^+(p,q)$.

Theorem 4.9. *Consider the dual pair in the stable range*

$$(G,G') = (U(n,n), U(p,q)) \qquad (p+q \leq n).$$

Let $\widetilde{L}(\tau_\nu^{(n)})$ be a holomorphic discrete series of $U(p,q)$, where $\nu \in \Lambda_n^+(p,q)$. Then its maximal quotient $\Omega(\widetilde{L}(\tau_\nu^{(n)}))$ is irreducible and gives the theta-lift $\theta(\widetilde{L}(\tau_\nu^{(n)})) \in$ $\mathrm{Irr}(U(n,n)^\sim)$. We have $\widetilde{K}$-type decompositions

$$\theta(\widetilde{L}(\tau_\nu^{(n)}))\big|_{\widetilde{K}} \simeq \sum_{\lambda,\mu\in\Lambda_n^+(p,q)}^{\oplus} b_{\mu,\nu}^{\widetilde{\lambda}} (\tau^{(n)}_{(\lambda+\frac{n}{2}\mathbb{I}_p\odot_n\frac{n}{2}\mathbb{I}_q)} \otimes \chi_{p,q})^* \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})$$

for n even, and

$$\theta(\widetilde{L}(\tau_\nu^{(n)}))\big|_{\widetilde{K}} \simeq \sum_{\lambda,\mu\in\Lambda_n^+(p,q)}^{\oplus} b_{\mu,\nu}^{\widetilde{\lambda}} (\tau^{(n)}_{(\lambda+\frac{n-1}{2}\mathbb{I}_p\odot_n\frac{n+1}{2}\mathbb{I}_q)} \otimes \chi_{p,q})^* \boxtimes (\tau_\mu^{(n)} \otimes \chi_{p,q})$$

for n odd. Here the branching coefficient $b_{\mu,\nu}^{\eta}$ is defined in (4.3).

Proof. If n is even, then by Corollary 4.5 and Proposition 4.8, we have

$$\begin{aligned}&\mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^*, L(\tau_\nu^{(n)}))\\ &\qquad \simeq \sum_{\eta\in\Lambda_{2n}^+(p,q)}^{\oplus} b_{\mu,\nu}^{\eta} \mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\tau_\lambda^{(n)}), L(\tau_\eta^{(2n)})).\end{aligned}$$

If $\eta = \widetilde{\xi}$ for $\xi \in \Lambda_n^+(p,q)$, then we have

$$L(\tau_{\widetilde{\xi}}^{(2n)}) \simeq L(\tau^{(n)}_{(\xi+\frac{n}{2}\mathbb{I}_p\odot_n\frac{n}{2}\mathbb{I}_q)}),$$

by comparing the minimal $\widetilde{K'}$-types. Note that when n is odd, we have

$$L(\tau_{\widetilde{\xi}}^{(2n)}) \simeq \det{}^{1/2} \otimes L(\tau^{(n)}_{(\xi+\frac{n-1}{2}\mathbb{I}_p\odot_n\frac{n+1}{2}\mathbb{I}_q)}).$$

Thus

$$\dim \mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\tau_\lambda^{(n)}) \otimes L(\tau_\mu^{(n)})^*, L(\tau_\nu^{(n)})) = b_{\mu,\nu}^{\widetilde{\xi}},$$

where $\lambda = \xi + \frac{n}{2}\mathbb{I}_p \odot_n \frac{n}{2}\mathbb{I}_q$ and $\xi \in \Lambda_n^+(p,q)$. In view of Proposition 4.2 and Proposition 4.3, the desired result follows. The case of odd n is similar. $\square$

4.3. **Theta lifting and associated cycles.** For the moment, let G be a non-compact semi-simple Lie group of Hermitian type and K a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a complexified Cartan decomposition and further let $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_-$ be the $\mathrm{Ad}\, K$-stable decomposition of $\mathfrak{s}$. An irreducible unitary representation π of G is said to be holomorphic if there are non-zero K-finite vectors v in the space of π such that $\pi(\mathfrak{s}_-)(v) = 0$. Then the space of such vectors v is irreducible under K. This is the (unique) minimal K-type of π, and it determines the representation π completely. We denote an irreducible holomorphic unitary representation with the minimal K-type $\sigma \in \mathrm{Irr}(K)$ by $\pi(\sigma)$.

We give a result on tensor product of a holomorphic unitary representation and a holomorphic discrete series representation with a scalar minimal K-type.

Proposition 4.10. *Let π be a holomorphic unitary representation of G, and $\pi(\chi)$ be the holomorphic discrete series representation of G with the one-dimensional minimal K-type χ. Suppose that π has the following K-type decomposition:*

$$\pi\big|_K \simeq \sum_{\tau \in \mathrm{Irr}(K)} m(\tau)\tau,$$

where $m(\tau)$ is the multiplicity of τ. Then

$$\pi \otimes \pi(\chi) \simeq \sum_{\tau \in \mathrm{Irr}(K)} m(\tau)\pi(\tau \otimes \chi). \tag{4.8}$$

Proof. Since $\pi(\chi)$ is a holomorphic discrete series with the scalar minimal K-type χ, see that for any K-type τ of π, $\pi(\tau \otimes \chi)$ is also a holomorphic discrete series representation. Thus we have

$$\pi(\chi)\big|_K \simeq \chi \otimes S[\mathfrak{s}_+],$$

and

$$\pi(\tau \otimes \chi)\big|_{K'} \simeq (\tau \otimes \chi) \otimes S[\mathfrak{s}_+],$$

where $S[\mathfrak{s}_+]$ is the symmetric algebra over $\mathfrak{s}_+$.

Therefore the left and right-hand sides of (4.8) are isomorphic as K-modules.

On the other hand, it is clear that $\pi \otimes \pi(\chi)$ is the direct sum of irreducible holomorphic unitary representations (cf. Proposition 4.8 for the case which concerns us), and each K-type occurs with finite multiplicity. General theory for holomorphic representations tells us that their G-module decompositions (in the Grothendieck group) are determined by the weight space decompositions with respect to the compact Cartan subgroup $T \subseteq K$. We thus conclude that the isomorphism of the left and right-hand sides of (4.8) as K-modules in fact induce an isomorphism as G-modules. This proves the proposition. ☐

We are now back to the dual pair

$$(G, G') = (U(n,n), U(p,q)) \ (p + q \leq n)$$

in the stable range. For $m, l \in \mathbb{Z}^+$, denote

$$\nu(m,l) = m\mathbb{I}_p \odot_n l\mathbb{I}_q \in \Lambda_n^+(p,q).$$

Then $\widetilde{L}(\tau^{(n)}_{\nu(m,l)})$ is a holomorphic discrete series of $U(p,q)$ with the scalar minimal K'-type

$$\chi(m,l) = \begin{cases} (\det^{\frac{n}{2}} \boxtimes \det^{-\frac{n}{2}}) \otimes (\det^{m} \boxtimes \det^{-l}), & n \text{ even}, \\ (\det^{\frac{n-1}{2}} \boxtimes \det^{-\frac{n+1}{2}}) \otimes (\det^{m} \boxtimes \det^{-l}), & n \text{ odd}. \end{cases}$$

Proposition 4.11. *For $\mu \in \Lambda^+_n(p,q)$, we have*

$$L(\tau^{(n)}_{\mu}) \otimes L(\tau^{(n)}_{\nu(m,l)}) \simeq \sum^{\oplus}_{\alpha\in\mathcal{P}_p,\beta\in\mathcal{P}_q} c^{\mu}_{\alpha,\beta^*} L(\tau^{(2n)}_{(\alpha+m\mathbb{I}_p)\odot_{2n}(\beta+l\mathbb{I}_q)}).$$

Proof. We consider the see-saw pair

$$\begin{array}{ccc} U(n) & & U(p)\times U(q) = K' \\ \cap & \times & \cap \\ U(n)\times U(n) & & U(p,q) = G' \end{array}$$

Howe's reciprocity theorem [3] implies that

$$L(\tau^{(n)}_{\mu})\big|_{K'} \simeq (\det^{-\frac{n}{2}} \boxtimes \det^{\frac{n}{2}}) \otimes \Big(\sum^{\oplus}_{\alpha\in\mathcal{P}_p,\beta\in\mathcal{P}_q} c^{\mu}_{\alpha,\beta^*} \tau^{(p)}_{\alpha} \boxtimes (\tau^{(q)}_{\beta})^* \Big).$$

Proposition 4.10 then implies the result. □

By Proposition 4.8 and Proposition 4.11, we have the following

Corollary 4.12. *For $\mu \in \Lambda^+_n(p,q)$, and $\alpha \in \mathcal{P}_p, \beta \in \mathcal{P}_q$, we have*

$$c^{\mu}_{\alpha,\beta^*} = b^{(\alpha+m\mathbb{I}_p)\odot_{2n}(\beta+l\mathbb{I}_q)}_{\mu,\nu(m,l)}, \qquad m,l \in \mathbb{Z}^+.$$

Theorem 4.9 and Corollary 4.12 now imply the $\widetilde{K}$-type formula of $\theta(\pi'_{\mathrm{hol}})$ in Theorem 1.2 of the Introduction.

We recall the notion of associated variety and associated cycle for a Harish-Chandra module V, which are denoted by $\mathcal{AV}(V)$ and $\mathcal{AC}(V)$ (see [23], and also [16]). We need the following lemma.

Lemma 4.13. *The associated varieties of the theta lifts $\theta(\det^k)$ and $\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))$ are given as*

$$\mathcal{AV}(\theta(\det^k)) = \overline{\mathcal{O}^{\mathbf{1}}_{p,q}}, \qquad \text{and} \qquad \mathcal{AV}(\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))) = \overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}.$$

Proof. The assertion for $\det^k$ is proved by Przebinda ([18, Corollary 7.10], cf. [19, Theorem 1.4]) in a more general form. In general, the containment of the complex associated variety of the primitive ideals is proved also by Przebinda ([17, Theorem 7.1]). However, since we cannot find a convenient statement which assures the second assertion of the lemma, we will give an outline of the proof.

Recall that we are identifying dual spaces $\mathfrak{s}^*, \mathfrak{s}'^*$ with $\mathfrak{s}, \mathfrak{s}'$ respectively by an invariant bilinear form. Then, the elements in $\mathfrak{s}'_-$ annihilate $\overline{\mathcal{O}'_{\mathrm{hol}}} = \mathfrak{s}'_-$, and they generate its annihilator ideal. By the definition of theta lifting of nilpotent orbits, we see that $f(\cdot,\cdot) \in \mathbb{C}[\mathfrak{s}]$ belongs to the annihilator ideal of $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}$ if and only if $f({}^t xz, {}^t wy) = 0$ for $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \in \psi^*(\mathfrak{s}'_-)$ (see (2.2) for notations). On the other hand, it is clear that the maximal quotient $\Omega(\pi'_{\mathrm{hol}}) \otimes \pi'_{\mathrm{hol}}$ is annihilated by

$\psi^*(\mathfrak{s}'_-)$ after taking suitable degree-gradation. This means that the associated variety of $\theta(\pi'_{\mathrm{hol}})$ should be contained in the closure of the theta lift $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}$. However, Przebinda's result ([18, Theorem 7.9(a)]) says that the dimensions of both varieties are the same, and we can therefore conclude the equality. Note that the condition $(*)$ in [18, Theorem 7.9] is satisfied in our case. □

Remark 4.14. The associated variety of the theta lift of any holomorphic discrete series of $U(p,q)$ is the same, namely $\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}$. The same remark is valid for the theta lift of any irreducible finite dimensional unitary representation of $U(p,q)$. Of course such an irreducible finite dimensional unitary representation is a unitary character of $U(p,q)$ unless $pq = 0$.

The main result of this section is the statement on associated cycles of $\theta(\det^k)$ and $\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))$ in Theorem 1.1 and Theorem 1.2 of the Introduction.

Theorem 4.15. *Consider the following dual pair in the stable range*

$$(G, G') = (U(n,n), U(p,q)) \qquad (p+q \leq n).$$

Then for $k \in \mathbb{Z}$, $m, l \in \mathbb{Z}^+$, we have

$$\mathcal{AC}\,(\theta(\det^k)) = [\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}], \quad \textit{and} \quad \mathcal{AC}\,(\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))) = [\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}].$$

Proof. We compare the $\widetilde{K}$-module structure of $\theta(\det^k)$ (resp., $\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)})))$ with the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ (resp., $\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}]$).

For $\det^k$, the $\widetilde{K}$-module structure of $\theta(\det^k)$ coincides with the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}]$ up to the obvious determinant shift, and the shift in the parameter $\lambda = \alpha \odot_n \beta \to (\alpha + k\mathbb{I}_p) \odot_n (\beta + k\mathbb{I}_q)$. See Theorem 4.6 and Theorem 3.6.

For $\widetilde{L}(\tau^{(n)}_{\nu(m,l)})$, since $c^{\mu}_{\alpha,\beta^*} = b^{(\alpha+m\mathbb{I}_p)\odot_{2n}(\beta+l\mathbb{I}_q)}_{\mu,\nu(m,l)}$ for $\mu \in \Lambda^+_n(p,q)$, and $\alpha \in \mathcal{P}_p, \beta \in \mathcal{P}_q$, we see that the $\widetilde{K}$-module structure of $\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))$ coincides with the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}]$ up to the obvious determinant shift and the shift in the parameter $\alpha \odot_n \beta \to (\alpha + a(m)\mathbb{I}_p) \odot_n (\beta + b(l)\mathbb{I}_q)$. Here $a(m)$ and $b(l)$ are given in (1.5). See Theorem 4.9 and Theorem 3.9.

Thus in either case, the two Hilbert polynomials (associated to $K_{\mathbb{C}}$-stable filtrations) have the same degrees and the same leading terms. In particular we have

$$\mathrm{Deg}\,(\theta(\det^k)) = \deg(\overline{\mathcal{O}^{\mathbf{1}}_{p,q}}), \quad \text{and} \quad \mathrm{Deg}\,(\theta(\widetilde{L}(\tau^{(n)}_{\nu(m,l)}))) = \deg(\overline{\mathcal{O}^{\mathrm{hol}}_{p,q}}).$$

Our assertion follows from the equality of these degrees. See [16, Th. 1.4]. □

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