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BEREZIN TRANSFORM ON REAL BOUNDED SYMMETRIC DOMAINS

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Abstract. Let $\mathbb{D}$ be a bounded symmetric domain in a complex vector space $V_{\mathbb{C}}$ with a real form V and $D = \mathbb{D} \cap V = G/K$ be the real bounded symmetric domain in the real vector space V. We construct the Berezin kernel and consider the Berezin transform on the L^2-space on D. The corresponding representation of G is then unitarily equivalent to the restriction to G of a scalar holomorphic discrete series of holomorphic functions on $\mathbb{D}$ and is also called the canonical representation. We find the spectral symbol of the Berezin transform under the irreducible decomposition of the L^2-space.

Introduction

The main purpose of the present paper is to calculate the spectral symbol of the Berezin transform on real bounded symmetric domains. To explain our results and motivations we let $\mathbb{D}$ be the unit disk in the complex plane with the Lebesgue measure $dm(z)$. We consider the weighted Bergman space H^ν $(\nu > 1)$ of holomorphic functions on $\mathbb{D}$ square integrable with respect to the measure $(1 - |z|^2)^{\nu-2} dm(z)$. It has up to some positive constant the reproducing kernel $K_w(z) = K(z, w) = (1 - z\bar{w})^{-\nu}$. Moreover the group $G_c = SU(1, 1)$ of fractional transformations of $\mathbb{D}$ acts on the space H^ν via $f(z) \mapsto f(gz)g'(z)^{\frac{\nu}{2}}$ and it forms an irreducible unitary (projective) representation. Consider the subgroup $SO(1, 1)$ consisting of transformations of the form $z \mapsto \frac{az+b}{bz+a}$ with $a, b \in \mathbb{R}$ and $a^2 - b^2 = 1$. Thus it is of interest to study the irreducible decomposition of the weighted Bergman space under the subgroup $G = SO(1, 1)$. For that purpose we consider the unit interval $D = \mathbb{D} \cap \mathbb{R} = (-1, 1)$ as a trivial symmetric space $G/K = SO(1, 1)/\{\pm 1\}$ and the restriction of holomorphic functions in H^ν to the interval D. More precisely, consider $R : H^\nu \to C^\infty(D)$,

$$Rf(x) = f(x)(1 - x^2)^{\frac{\nu}{2}}, \quad x \in D.$$

Let $L^2(D, d\mu_0)$ be the L^2 space on D with the $SO(1, 1)$-invariant measure $d\mu_0(x) = \frac{dx}{(1-x^2)}$, whose decomposition under $SO(1, 1)$ can be done via the Mellin transform (see below). The restriction R is a bounded operator from H^ν into the space $L^2(D, d\mu_0)$ with dense image, and intertwines the respective actions of $SO(1, 1)$,

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see Section 1. To get a unitary intertwining operator we consider the polar decomposition of R, $R=|R|U$. Thus $U=|R|^{-1}R$ is a unitary intertwining operator from H^ν onto $L^2(D, d\mu_0)$, with $|R|^2=RR^*$ given by

$$B_\nu f(x)=\frac{\Gamma(\nu)}{2^{\nu-1}\Gamma(\frac{\nu}{2})^2}\int_D f(y)\frac{(1-x^2)^{\frac{\nu}{2}}(1-y^2)^{\frac{\nu}{2}}}{(1-xy)^\nu}d\mu_0(y)$$

which we call the Berezin transform on D.

Now the $L^2(D, d\mu_0)$, after performing the Cayley transform $x\mapsto\frac{1-x}{1+x}$ mapping D to the half line $\mathbb{R}^+=(0,\infty)$, can be decomposed under $SO(1,1)$ via the Mellin transform,

$$f(x)\mapsto\hat{f}(\lambda)=\int_0^\infty f(x)e_\lambda(x)\frac{dx}{x},$$

with $e_\lambda(x)=x^\lambda$ and $\lambda\in i\mathbb{R}$. On the half real line it becomes

$$B_\nu f(x)=\frac{\Gamma(\nu)}{\Gamma(\frac{\nu}{2})^2}\int_0^\infty f(y)\frac{x^{\frac{\nu}{2}}y^{\frac{\nu}{2}}}{(x+y)^\nu}\frac{dy}{y}$$

on the space $L^2(\mathbb{R}^+,\frac{dx}{x})$. B_ν is then a function of the self-adjoint operator $L=ix\frac{d}{dx}$. In terms of the eigenfunction $e_\lambda(x)$, $B_\nu e_\lambda=b_\nu(\lambda)e_\lambda$,

$$b_\nu(\lambda)=\frac{\Gamma(\frac{\nu}{2}+\lambda)\Gamma(\frac{\nu}{2}-\lambda)}{\Gamma(\frac{\nu}{2})^2};$$

$b_\nu(\lambda)$ is obtained by calculating the integral $B_\nu e_\lambda(x)$ at $x=1$. Thus $B_\nu=b_\nu(iL)$.

One can formulate the above problem for any real bounded symmetric domain, where the operator L is replaced by a system of generators. The intertwining operator R for a general bounded symmetric domain was introduced earlier in [12]. The main purpose of the present paper is to find the symbol b_ν. Moreover, we prove that the Berezin transform defines a bounded operator on the L^p-space $(1\le p\le\infty)$ on D with the invariant measure when ν is in a certain interval, and thus justifies the operator theoretic meaning of the symbol. In particular, this proves the boundedness of the intertwining operator R from the analytic continuation of the holomorphic discrete series to the L^2-space on D. The exact interval of ν depends on different root systems. For type BC and C the interval of ν is such that H^ν is a discrete series of $\mathbb{D}$; however, for other types of domains the range of ν is larger than the holomorphic discrete series, but still above the reducible points.

Before explaining our methods of calculation we make some general remarks and put our results into perspective. The problem of finding the irreducible decomposition of a unitary representation (π, H) of a Lie group H under a subgroup $G\subset H$ has been studied for a long time, both in mathematical physics and in representation theory, and is called the "Branching rule". We can also put the Berezin transform on the unit disk $\mathbb{D}$, originally introduced by Berezin [2] for any complex Kähler manifold, in this context. Consider the tensor product $H^\nu\otimes\overline{H^\nu}$ of the weighted Bergman space on $\mathbb{D}$ with its conjugate realized as space of functions $f(z,w)$ holomorphic in z and anti-holomorphic in w. It forms a unitary representation of the group $G_c\times G_c$. Then G_c can be considered as the subgroup $\{(g,g); g\in G_c\}$ of $G_c\times G_c$. To find the irreducible decomposition of the tensor product under G_c we consider also the restriction map of the function$f(z,w)$ to its diagonal $f(z,z)$;

more precisely, $Rf(z) = f(z,z)(1-|z|^2)^\nu$. In this way we also obtain the Berezin transform on $\mathbb{D}$, $B_\nu = RR^*$,

$$B_\nu f(z) = \int_{\mathbb{D}} f(w) \frac{(1-|z|^2)^\nu (1-|w|^2)^\nu}{|(1-z\bar{w})^\nu|^2} \frac{dm(w)}{(1-|w|^2)^2}. \tag{0.1}$$

See [13]. The symbol of the Berezin transform for a general bounded symmetric domain was found by Unterberger and Upmeier [17]. The spectral symbol of the Berezin transform then gives the spectral decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ of G_c for large ν. (See also [21] where the tensor product $H^\nu \otimes \overline{H^\kappa}$ for different ν and κ is considered and the corresponding Berezin symbol is calculated.) The exact formula plays a decisive role in decomposing the tensor product for small ν. Interestingly there appears discrete parts in the decomposition; cf [14] for the case of the unit disk $\mathbb{D} = SU(1,1)/U(1)$ and [6] for the matrix ball $SU(p,q)/S(U(p) \times U(q))$.

We proceed to explain our methods of calculation. We will use the idea of Unterberger and Upmeier [17], applying the theory of Jordan triples and Gindikin Gamma function. Let $V = V_2 \oplus V_1$ be the Peirce decomposition of V with respect to a maximal tripotent e. The Jordan algebra V_2 has further a Cartan decomposition $V_2 = A \oplus B$. The bounded domain $D = G/K$ can be realized as a real Siegel domain in V. The Harish-Chandra $e_{\underline{\lambda}}$-function has a rather explicit form, in terms of the determinant function on V_2; see (2.6). The symbol $b_\nu(\underline{\lambda})$ is then an integral on the Siegel domain. The integral reduces further to one integration on V_1, one on B and one on the symmetric cone Ω in A. In the case of the complex Siegel domains, $B = iA$ and the integration on B can be obtained directly by the Laplace transform on Ω. However, for real domains B is not very much related to A, and we calculate the integral by using a method of Shimura [16], writing the integral as one on the cone Ω in A; see Proposition 3.1.

The symbols of the Berezin transform have also been calculated previously by Neretin [10] for classical domains, and by van Dijk and Hille [18] for classical rank one domains. As we will see in this paper, the integrals involved are generalization of the classical Beta-integral. For certain special cases they were also studied earlier by Hua [7].

The Berezin transform formally defines a positive operator, namely $(B_\nu f, f) \geq 0$ for all $f \in C_0^\infty(D)$. Thus the problem of finding the irreducible decomposition of H^ν under G can also be formulated as finding the decomposition defined by the positive definite kernel $\frac{(1-|z|^2)^\nu(1-|w|^2)^\nu}{|(1-z\bar{w})^\nu|^2}$. In this context it is also called canonical representation, introduced and named by Vershik, Gel'fand and Graev [20].

Finally, we remark that even though our main goal is the symbol of the Berezin transform, some results in this paper are of independent interests. In particular, our Propositions 3.1 and 4.1 generalize the formula (2.13) in [16]. The exact value of those integrals is critical in studying the analytic continuation of Bessel functions and Eisenstein series; see loc. cit. We hope to study applications of our results to Bessel functions in future publications.

The paper is organized as follows. In Section 1 we introduce real bounded symmetric domains and the Berezin transform. In Section 2 we consider the Harish-Chandra e-function defined abstractly in terms of the Iwasawa decomposition $G = NAK$. We give explicit formulas for positive root vectors, and thus give a formula of the e-function in terms of the conical functions. The symbol of the Berezin transform is calculated in Sections 3, 4 and 5 for the different root systems. In

Section 6 we prove the L^p-bounded properties of the Berezin transform, therefore justifying the operator theoretic meaning of the symbol b_ν of the Berezin transform.

After this paper was finished the author received a preprint of [19] by van Dijk and M. Pevzner, where they also found the symbol for real tube domains using slightly different methods.

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For the reader's convenience we list the main symbols used in the paper:

1. $\mathbb{D} = G_c/K_c$, an irreducible bounded symmetric domain in a complex vector space $V_{\mathbb{C}} = \mathbb{C}^n$, V a real form of $V_{\mathbb{C}}$ and $D = \mathbb{D} \cap V$ a real bounded symmetric domain in V;
2. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace of $\mathfrak{p}$;
3. $D(z, \bar{w})v = \{z\bar{w}v\}$, the Jordan triple product;
4. $B(z, w) = I - D(z, \bar{w}) + Q(z)Q(\bar{w})$, the Bergman operator;
5. (z, w), normalized scalar product on V so that a minimal tripotent has norm 1;
6. p the genus of domain $\mathbb{D}$ and p_0 the genus of D (see Definition 2.2);
7. $h(z, w)$ an irreducible polynomial on $V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}}$, and $\det B(z, w) = h(z, w)^p$;
8. $V = V_2 \oplus V_1$ Peirce decomposition with respect to a maximal tripotent e, $V_2 = A \oplus B$ the Cartan decomposition of V_2 with respect to the involution $z \mapsto Q(e)\bar{z}$;
9. Δ the determinant function of V_2 and δ the determinant function of A;
10. $\mathcal{S}$ the Siegel domain realization of D.

1. Real bounded symmetric domains and restriction of holomorphic functions

We recall first some preliminary results on real bounded symmetric domains and fix notation. Our presentation is mainly based on Loos [9].

Let $\mathbb{D} = G_c/K_c$ be an irreducible bounded symmetric domain in a complex vector space $V_{\mathbb{C}} = \mathbb{C}^n$. The space V has then a Jordan triple structure. We denote $\{x\bar{y}z\}$ the Jordan triple product. Let V be a real form of $V_{\mathbb{C}}$ and τ be the conjugation with respect to V. Suppose $\tau(\mathbb{D}) = \mathbb{D}$, namely, τ fixes the bounded symmetric domain. Then the real form $D = \mathbb{D} \cap V$ will be called a *real bounded symmetric domain*. In this case the triple product $D(x, \bar{y})z = \{x\bar{y}z\}$ restricted on V defines also a triple product on V. A complete list of real bounded symmetric domains D is given in [9]. As a Riemannian symmetric space, $D = G/K$, where G is the subgroup of G_c consisting biholomorphic transformations of $\mathbb{D}$ which keep D invariant. The coset space G_c/G is called a causal symmetric space, a complete list of the pairs (G_c, G) can be found in e.g. [11], [5] and [6].

We describe briefly some algebraic and geometric structures of the domain D.

Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The Lie algebra of G_c will be realized as completely integrable holomorphic vector fields on $\mathbb{D}$. The elements in $\mathfrak{g}$ can then be written as $\mathfrak{g} = \{\xi_v(z) = v - Q(z)\bar{v}; v \in V\}$ where $Q(z)\bar{v} = \frac{1}{2}\{z\bar{v}z\}$ is the quadratic operator.

We fix a K-invariant inner product $(\cdot, \cdot)$ on V so that a minimal tripotent has norm 1, and let $\langle \cdot, \cdot \rangle$ be the corresponding Hermitian inner product on $V_{\mathbb{C}}$. (A minimal tripotent of $V_{\mathbb{C}}$ has norm $\frac{\sqrt{2}}{2}$ if the root system is of Type BC or C, otherwise it is of norm 1; see below.) We let dz be the corresponding Lebesgue measure.

Denote further $B(x, \bar{y})$ the Bergman operator

$$B(x, y) = 1 - D(x, \bar{y}) + Q(x)Q(\bar{y}).$$

The Bergman kernel of $\mathbb{D}$ is up to a constant $h(z, w)^{-p}$ where $h(z, w)$ is an irreducible polynomial and p is the genus of $\mathbb{D}$. For simplicity we write $h(z) = h(z, z)$. The Bergman metric of $\mathbb{D}$ is

$$\langle B(z, z)^{-1}u, v\rangle.$$

The domain D is a totally real and totally geodesic submanifold of $\mathbb{D}$ and its Riemannian metric is the above Bergman metric; with this metric it is also a Riemannian symmetric space, $D = G/K$. The G-invariant measure on D is

$$h(z)^{-\frac{p}{2}}dm(z),$$

and G acts unitarily on $L^2(D, h^{-\frac{p}{2}}dm)$ via change of variables.

Consider the weighted Bergman space $H^\nu(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ so that

$$\int_{\mathbb{D}} |f(z)|^2 h(z)^\nu \frac{dm(z)}{h(z)^p} < \infty.$$

The group G_c acts unitarily on $H^\nu(\mathbb{D})$ via

$$\pi_\nu f(z) = J_{g^{-1}}(z)^{\frac{\nu}{p}} f(g^{-1}z),$$

and it forms a projective representation of G_c. Let R be the restriction map $R : H^\nu \to C^\infty(D)$ by

$$Rf(x) = f(x)h(x)^{\frac{\nu}{2}}. \tag{1.1}$$

Consider the restriction of the group action π_ν of G_c to its subgroup G. Then R is an G-intertwining map, as one can easily check from the transformation properties of $h(x, x)$. Consider its formal conjugate operator R^* from $L^2(D, d\mu_0)$ to H^ν and form the operator R^*R on $L^2(D, \omega)$. It is of the form, up to a constant,

$$B_\nu f(z) = c_\nu \int_D f(w) \frac{h(z)^{\frac{\nu}{2}} h(w)^{\frac{\nu}{2}}}{h(z, w)^\nu} \frac{dm(w)}{h(w)^{\frac{p}{2}}},$$

where the constant c_ν is a constant normalized so that $B_\nu 1 = 1$.

We note that the kernel $h(z, w)$ actually is positive, so there is no ambiguity about its power $h(z, w)^\nu$. Indeed by the transformation rule of $h(z, w)$ on $\mathbb{D} \times \mathbb{D}$

$$h(gz, gw) = J_g(z)^{\frac{1}{p}} h(z, w) \overline{J_g(z)^{\frac{1}{p}}}$$

for $g \in G_c$ and $(z, w) \in \mathbb{D} \times \mathbb{D}$, and that $h(0, w) = 1$ we know that $h(z, w)$ is nonvanishing on $\mathbb{D} \times \mathbb{D}$. Moreover, $h(z, w)$ on $D \times D$ is a real-valued function. Thus being a continuous nonvanishing real-valued function on $D \times D$, it must be positive. Consequently, we can write B_ν as

$$B_\nu f(z) = c_\nu \int_D \frac{h(z, z)^{\frac{\nu}{2}} h(w, w)^{\frac{\nu}{2}}}{|h(z, w)^{\frac{\nu}{2}}|^2} f(w) \frac{dm(w)}{h(w)^{\frac{p}{2}}}.$$

2. Siegel domain realization of real bounded symmetric domains and the Harish-Chandra $e_{\underline{\lambda}}$-functions

The complex bounded symmetric domain $\mathbb{D}$ can be also realized as a Siegel domain in $V_{\mathbb{C}}$ via Cayley transform. The Cayley transform at the same time maps D onto an unbounded domain in V, which will be its Siegel domain realization. The advantage with the unbounded realization is that the Harish-Chandra functions $e_{\underline{\lambda}}$ have explicit form.

Let $\{e_1, \cdots, e_r\}$ be a frame of V and $e = e_1 + \cdots + e_r$; e is then a maximal tripotent of V and $V_{\mathbb{C}}$. Let

$$V = V_2 \oplus V_1 \tag{2.1}$$

be the Peirce decomposition of V with respect to e, where

$$V_j = \{z \in V; D(e,e)z = 2z\}, \quad j = 1, 2.$$

The space V_2 is a Jordan algebra with the product $z \circ w = \frac{1}{2}\{z\bar{e}w\}$ and with unit e. Denote x^{-1} the inverse of $x \in V_2$ with respect to this product. The Cayley transform is defined by

$$\gamma_e(x) = (e + x_2) \circ (e - x_2)^{-1} \oplus \sqrt{2}\{(e - x_2)^{-1}\bar{e}x_1\}.$$

To describe the image of γ_e we consider the involution $Q(e) : x \mapsto x^* = Q(e)\bar{x}$. Let A and B be the eigenspaces of the involution with eigenvalues 1 and -1, respectively. The subspace A is an Euclidean Jordan algebra. We will write $\Re(x)$ as the A-part of an element $x \in V_2$. Let Ω be the cone of positive elements in A. Denote $F(x_1, x_1) = D(e, \bar{e})x_1$. It maps V_1 into A. The image of D under γ_e is the unbounded domain

$$\mathcal{S} = \{x = x_2 \oplus x_1 \in V = V_2 \oplus V_1; x_2 + x_2^* - F(x_1, x_1) \in \Omega\}.$$

We observe that the differential of γ_e at $z = 0$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

under the decomposition of $V = V_2 \oplus V_1$.

We will denote the conjugation of the group G also by G. Thus $\mathcal{S} = G/K$ with $e \in \mathcal{S}$ being the base point. We will hereafter fix this realization of G/K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The elements $D(e_j, e_j)$, $j = 1, \ldots, r$, span then a maximal subspace $\mathfrak{a}$ of $\mathfrak{p}$ of dimension r. Let $\gamma_j \in \mathfrak{a}^*$ be the linear functional on $\mathfrak{a}$ defined by

$$\gamma_j(D(e_k, e_k)) = 2\delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker symbol. Then the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of the following type:

(A_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(B_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\frac{\gamma_j}{2}, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(C_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(BC_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \pm\frac{\gamma_j}{2}, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(D_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\frac{\gamma_j \pm \gamma_k}{2}\}$.

Let us arrange an ordering of the roots so that

$$\gamma_1 < \gamma_2 < \cdots < \gamma_r.$$

For type C and BC we further let $\gamma_1 > 0$. The root space decomposition can be explicitly done in terms of the Peirce decomposition of V. We present below formulas for the positive root spaces, which will be sufficient for our purposes. For the complex bounded symmetric domain $\mathbb{D}$ and its Siegel domain realization, the corresponding root spaces are calculated in [13] and [17].

Let

$$V = \sum_{1\le j\le k} V_{jk} \oplus \sum_{j=1}^{r} V_{0j}$$

be the joint Peirce decomposition with respect to $e_1, \cdots, e_j$, and

$$V_{jk} = A_{jk} \oplus B_{jk}$$

the Cartan decomposition of V_{jk}, $1 \le j \le k$, under the involution $z \mapsto Q(e)\bar{z}$. With the notation in (2.1) we have

$$V_2 = \sum_{1\le j\le k} V_{jk}, \quad V_1 = \sum_{j=1}^{r} V_{0j}$$

and $V_2 = A \oplus B$,

$$A = \sum_{1\le j\le k} A_{jk}, \quad B = \sum_{1\le j\le k} B_{jk}.$$

We give the explicit formulas for the positive root spaces, in the bounded realization, it is essentially in [9], Propositions 11.18 and 9.19.

Proposition 2.1. *The positive root spaces are given by*

$$\mathfrak{g}_{\gamma_j} = B_{j,j},$$

$$\mathfrak{g}_{\frac{\gamma_j}{2}} = \{v + D(e, \bar{v}); v \in V_{0,j}\},$$

$$\mathfrak{g}_{\frac{\gamma_j - \gamma_k}{2}} = \{D(v, e_k); v \in A_{kj}\}, \quad j > k,$$

$$\mathfrak{g}_{\frac{\gamma_j + \gamma_k}{2}} = B_{kj}, \quad j > k.$$

The type D_2 is somewhat special and will be treated separately in Section 5. For all other types we define the invariants

$$\iota = \dim V_{jj},\ a = \dim A_{jk},\ b = \dim V_{j0}, \tag{2.2}$$

where $0 < j < k$. They are independent of j, k and the choice of the frame. Observe that $\dim A_{jj} = 1$ and $\dim B_{jj} = \iota - 1$; $\dim B_{jk} = \dim A_{jk} = a$ for type BC, C and D_r ($r \ge 3$), and $\dim B_{jk} = 0$ for type A. Also $\iota = 1$ for type B_r and D_r ($r \ge 3$). The dimensions of A, B, V_2 can then be calculated in terms of those invariants

$$n_A = \dim A = r + \frac{a}{2} r(r-1), \quad n_B = \dim B = r(\iota - 1) + \frac{a}{2} r(r-1), \tag{2.3}$$

and $n_2 = \dim V_2 = n_A + n_B$. Recall that the genus p of the complex bounded symmetric domain D is defined

$$p = p(\mathbb{D}) = \frac{1}{r(\mathbb{D})}(2n_2 + n_1)$$

where $r(\mathbb{D})$ is the rank of $\mathbb{D}$.

Definition 2.2. The *genus* p_0 of D is defined by

$$p_0 = p(D) = \frac{1}{2r}(2n_2 + n_1).$$

Note that the genus $p_0 = p(D) = \frac{1}{2}p(\mathbb{D})$ if it is of type A, B or D, and $p(D) = p(\mathbb{D})$ if it is of type BC or C. (The rank of the complex domain $\mathbb{D}$ is $2r$ if D is of type BC or C.)

For $\underline{\lambda} \in (\mathfrak{a}^*)^{\mathbb{C}}$ we write

$$\underline{\lambda} = \sum_{j=1}^{r} \lambda_j \gamma_j$$

and we will identify $\underline{\lambda}$ with its coordinates $(\lambda_1, \dots, \lambda_r)$.

We let $\underline{\rho} = \sum_{j=1}^{r} \rho_j \gamma_j$ be the half sum of positive roots. Then

$$\rho_j = \frac{1}{2}((\iota - 1 + a(j-1) + \frac{b}{2}), \quad 1 \le j \le r, \tag{2.4}$$

for type B, BC, C, D_r $(r \ge 3)$, and

$$\rho_j = \frac{a}{4}(2j - (r+1)) \tag{2.5}$$

for type A.

Let $G = NAK$ be the corresponding Iwasawa decomposition of G. The Harish-Chandra $e_{\underline{\lambda}}$ on $\mathcal{S} = G/K$ is defined by

$$e_{\underline{\lambda}}(z) = e^{(\underline{\lambda}+\rho)A(g)}$$

where $g \in G$ is such that $z = gK$ and $A(g)$ is the $\mathfrak{a}$-part of $g \in G$ in its Iwasawa decomposition: $g = n\exp(A(g))k$.

We will need an explicit formula for the function $e_{\underline{\lambda}}$. For that purpose we introduce the conical functions. Let Ω be the symmetric cone of positive elements in A. Then the Harish-Chandra e-function on Ω can be expressed in terms of the conical functions on A. So let $\delta(x)$ be the determinant function of the Jordan algebra A. Put $e^{(j)} = e_1 + \cdots + e_j$ and consider the Peirce decomposition of A with respect to $e^{(j)}$, $A = A_2(e^{(j)}) \oplus A_1(e^{(j)}) \oplus A_0(e^{(j)})$. Let δ_j be the determinant function for the Jordan algebra $A_2(e^{(j)})$ with identity $e^{(j)}$ and extend it to a polynomial function of A via the projection from A onto $A_2(e^{(j)})$. For $\underline{\alpha} = (\alpha_1, \cdots, \alpha_r) \in \mathbb{C}^r$, we denote

$$\delta_{\underline{\alpha}}(x) = \delta_1(x)^{\alpha_1 - \alpha_2} \cdots \delta_{r-1}^{\alpha_{r-1}-\alpha_r}(x)\delta_r(x)^{\alpha_r}, \quad x \in \Omega.$$

The $\delta_{\underline{\alpha}}$ is the Harish-Chandra e-function on Ω. To find the e-function on $\mathcal{S}$ we let Δ be the determinant function of Jordan algebra V_2. On $\Omega \subset A \subset V_2$, $\Delta = \delta$ and has degree r if $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type A_r, B_r or D_r; $\Delta = \delta^2$ and is of degree $2r$ for type BC_r and C_r; see e.g. [8]. We define similarly the functions $\Delta_{\underline{\alpha}}$ on V_2.

Lemma 2.3. *The Harish-Chandra $e_{\underline{\lambda}}$-function on $\mathcal{S}$ is given by*

$$\begin{aligned} e_{\underline{\lambda}}(z) &= \Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1)) \\ &= \Delta_{\underline{\alpha}}(\frac{1}{2}(z_2 + z_2^*) - \frac{1}{2}F(z_1, z_1)), \quad z = z_2 \oplus z_1 \in \mathcal{S}, \end{aligned} \tag{2.6}$$

with

$$\underline{\alpha} = \underline{\lambda} + \underline{\rho} \tag{2.7}$$

if $\Sigma(\mathfrak{g}, \mathfrak{a})$ *is of type A, B, or D, and*

$$\underline{\alpha} = \frac{1}{2}(\underline{\lambda} + \underline{\rho}) \tag{2.8}$$

for type BC or C.

Proof. Indeed the function $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ and $e_{\underline{\lambda}}$ coincides on $A \cdot e = \exp(\mathfrak{a}) \cdot e$. To prove that they are the same we only need to know that $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ is invariant under N. By Proposition 3.1 it is clear that the function is invariant under the subgroups $\exp(\mathfrak{g}_{\gamma_j})$ and $\exp(\mathfrak{g}_{\frac{\gamma_j+\gamma_k}{2}})$, since $\Re(z_2) - \frac{1}{2}F(z_1, z_1)$ does not depend on the B-part of z_2. We consider now $\mathfrak{g}_{\frac{\gamma_j-\gamma_k}{2}}$, $j > k$. In view of Proposition 2.1, we see that the elements in the space, when considered as elements acting on $V = V_2 \oplus V_1 = A \oplus B \oplus V_1$, annihilate V_1 and keep the subspace A invariant. When restricted to A, the elements in $\mathfrak{g}_{\frac{\gamma_j-\gamma_k}{2}}$ are in the Lie algebra of the corresponding Iwasawa N-group of the symmetric cone Ω and thus keep the conical function $\Delta_\alpha(x)$ invariant (see [3], Chapter XI. Section 2). Consequently, $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ is invariant under $\exp(\mathfrak{g}_{\frac{\gamma_j-\gamma_k}{2}})$. Finally, consider an element $v + D(e, \bar{v}) \in \mathfrak{g}_{\frac{\gamma_j}{2}}$, $v \in V_{0j}$. It follows from [9], Lemma 10.7 (and its proof) that

$$\exp(v + D(e, \bar{v})) = \exp(v + \frac{1}{2}F(v, v)) \exp(D(e, \bar{v}))$$

where the element $\exp(v + \frac{1}{2}F(v, v))$ acts on vectors in V via translation, and that

$$\exp(D(e, \bar{v}))(z_2 \oplus z_1) = (z_2 + F(z_1, v)) \oplus z_1.$$

Thus

$$\exp(v + D(e, \bar{v}))(z_2 \oplus z_1) = (z_2 + F(z_1, v) + \frac{1}{2}F(v, v)) \oplus (z_1 + v).$$

From this formula we see that the action $\exp(v + D(e, \bar{v}))$ on $(z_2 \oplus z_1)$ keeps $\Re(z_2) - \frac{1}{2}F(z_1, z_1)$, and consequently $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$, invariant. This completes the proof. □

The Berezin transform on $\mathcal{S}$ is then of the form

$$B_\nu f(z) = C_\nu \int_{\mathcal{S}} f(w) \frac{\Delta(\Re z_2 - \frac{1}{2}F(z_1, z_1))^{\frac{\nu}{2}} \Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{\nu}{2}}}{(\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1)))^\nu} d\mu_0(w)$$

where

$$d\mu_0(w) = \frac{dm(w)}{\Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{p}{2}}},$$

is the G-invariant measure on $\mathcal{S}$. Here C_ν is a constant normalized so that $B_\nu 1 = 1$, its exact value will be given in the following sections.

As will be proved in Section 7, the operator B_ν is a positive bounded operator on $L^2(\mathcal{S}, d\mu_0)$ when ν is in a certain interval. Then the Harish-Chandra e-function $e_{\underline{\lambda}}$ is an eigenfunction of B_ν with eigenvalue $b_\nu(\underline{\lambda})$, in the sense of spectral calculus. Namely,

$$B_\nu e_{\underline{\lambda}}(z) = b_\nu(\underline{\lambda}) e_{\underline{\lambda}}(z). \tag{2.9}$$

Taking $z = e$ as the base point we get formally $b_\nu(\underline{\lambda}) = B_\nu e_{\underline{\lambda}}(e)$. Denote

$$I(\nu, \underline{\alpha}) = \int_{\mathcal{S}} e_{\underline{\lambda}}(w) \frac{\Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{\nu}{2}}}{(\Delta(\frac{1}{2}(e + w_2^*)))^\nu} d\mu_0(w). \tag{2.10}$$

Here $\underline{\alpha}$ and $\underline{\lambda}$ are related as in (2.7) and (2.8). Then $b_\nu(\underline{\lambda}) = C_\nu I(\nu, \underline{\alpha})$. So our main goal is the calculation of the integral $I(\nu, \underline{\alpha})$. Note that the normalization constant $C_\nu = I(\nu, 0)^{-1}$, and will be found at the same time.

We will prove that the integral is convergent for $\underline{\lambda}$ in a certain domain in $(\mathfrak{a}^*)^{\mathbb{C}}$. However, the function $e_{\underline{\lambda}}$ is N-invariant and transforms under A by the character $e^{\underline{\lambda}}$ and NA acts transitively on $\mathcal{S}$. Thus, if the integral (2.10) is absolutely convergent, that is, if the integral defining the Berezin transform $B_\nu e_{\underline{\lambda}}(z)$ at $z = e$ is absolutely convergent, then by a simple change of variable argument we know that the integral $B_\nu e_{\underline{\lambda}}(z)$ is absolutely convergent for all $z \in \mathcal{S}$; see [1].

We recall finally the Gindikin Gamma function on a symmetric cone; see e.g. [3], Chapter VII. Consider the Laplace transform of the function $\delta_{\underline{\alpha}}(x)\delta(x)^{-\frac{n_A}{r}}$,

$$\int_\Omega e^{-(u,x)} \delta_{\underline{\alpha}}(x) \delta(x)^{-\frac{n_A}{r}} dx,$$

for $u \in \Omega$. The integral is convergent if and only if $\Re(\alpha_j) > (j-1)a/2$ for $j = 1, 2, \cdots, r$. Moreover,

$$\int_\Omega e^{-(u,x)} \delta_{\underline{\alpha}}(u) \delta(x)^{-\frac{n_A}{r}} dx = \Gamma_\Omega(\underline{\alpha}) \delta^*_{\underline{\alpha}^*}(u), \tag{2.11}$$

where

$$\Gamma_\Omega(\underline{\alpha}) = (2\pi)^{(n_A - r)/2} \prod_{j=1}^r \Gamma(\alpha_j - (j-1)a/2),$$

and $\underline{\alpha}^* = (-\alpha_r, \dots, -\alpha_1)$ and $\delta^*_{\underline{\beta}}(x)$ for $\underline{\beta} = (\beta_1, \dots, \beta_r)$ is the conical function defined via Peirce decomposition with respect to $e_r, e_r + e_{r-1}, \dots, e_r + e_{r-1} + \cdots + e_1$.

3. Calculations of the symbol: Type BC and C

The type C root systems can be viewed as special cases of the type BC with the root multiplicity of $\frac{\gamma_j}{2}$ being 0 (and the other types can be viewed as degenerate cases). So we consider the type BC. Notice that the genus $p_0 = \iota + a(r-1) + \frac{b}{2}$.

The determinant function Δ of V_2 when restricted to the Jordan subalgebra A is the square of the determinant function of that of A. To distinguish them we write as before $\delta(z)$ the determinant function of A. Thus

$$\Delta(x) = \delta(x)^2, \quad x \in A.$$

We present first some integral formulas.

Proposition 3.1. *Suppose $\nu > \iota - 2 + \frac{1}{2}(1 + a(r-1))$. The following integral formula*

$$I_B(\nu) = \int_B \frac{d\eta}{\Delta(e - \eta)^\nu} = 4^{n_A - r(\nu - (\iota - 2))} \sqrt{\pi}^{r(\iota-2) + 2n_A} \frac{\Gamma_\Omega(2\nu - 2(\iota - 2) - \frac{n_A}{r})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu - \frac{\iota - 2}{2})}$$

holds and the integral is absolutely convergent.

Notice that

$$\begin{aligned}\Delta(e-\eta)^2 &= \Delta(e-\eta)\Delta(e+\eta) = \Delta(e-\eta^2)\\ &= \Delta(e+\eta\circ\eta^*) = \delta(e+\eta\circ\eta^*)^2.\end{aligned} \tag{3.1}$$

The integral $I_B(\nu)$ is now

$$I_B(\nu) = \int_B \frac{d\eta}{\delta(e+\eta\circ\eta^*)^\nu}. \tag{3.2}$$

To prove the integral formula we need the following result; see [3], Chapter VII, Ex. 5.

Lemma 3.2. *Suppose $\beta > \frac{1}{2}(1+a(r-1))$. The integral formula*

$$I_A(\beta) = \int_A \frac{dt}{|\delta(e+it)|^{2\beta}} = 4^{n_A - r\beta}\pi^{n_A}\frac{\Gamma_\Omega(2\beta-\frac{n_A}{r})}{\Gamma_\Omega(\beta)^2}$$

holds and the integral is absolutely convergent.

The integrals $I_A(\beta)$ and $I_B(\nu)$ above can also be written as an integration on the symmetric cone Ω in A; by comparing the two formulas we will then be able to calculate $I_B(\nu)$.

Indeed, we write $|\delta(e+it)|^2 = \delta(e+it)\overline{\delta(e+it)} = \delta(e+it)\delta(e-it) = \delta(e+t^2)$, and use the Gindikin Gamma function formula (2.11)

$$\delta(e+t^2)^{-\beta} = \frac{1}{\Gamma_\Omega(\beta)}\int_\Omega e^{-(e+t^2,u)}\delta(u)^{\beta-\frac{n_A}{r}}\,du;$$

$$\begin{aligned}I_A(\beta) &= \int_A \frac{dt}{|\delta(e+it)|^{2\beta}} = \frac{1}{\Gamma_\Omega(\beta)}\int_A\int_\Omega e^{-(e+t^2,u)}\delta(u)^{\beta-\frac{n_A}{r}}\,dudt\\ &= \frac{1}{\Gamma_\Omega(\beta)}\int_\Omega\int_A e^{-(e,u)}e^{-(t^2,u)}\delta(u)^{\beta-\frac{n_A}{r}}\,dtdu.\end{aligned} \tag{3.3}$$

Namely,

$$I_A(\beta) = \frac{1}{\Gamma_\Omega(\beta)}\int_\Omega G_A(u)e^{-(e,u)}\delta(u)^{\beta-\frac{n_A}{r}}\,dtdu, \tag{3.4}$$

with

$$G_A(u) = \int_A e^{-(t^2,u)}dt, \quad u\in\Omega,$$

being a Gaussian type integral. Let $u = \sum_{j=1}^r u_jc_j$ be the Peirce decomposition of u with c_j being minimal orthogonal tripotents, and $u_j > 0$ since $u\in\Omega$. Let $A = \sum A_{jk}$ be the joint Peirce decomposition and write $t = \sum t_{jk}$. Then we find that, by the Peirce rule (see [9], Theorem 3.15, Corollary 3.16)

$$(t^2,u) = \sum_{j<k}\frac{u_j+u_k}{2}(t_{jk},t_{jk}) + \sum_j u_j(t_{jj},t_{jj}).$$

Thus

$$\begin{aligned}G_A(u) &= \prod_{j=1}^r\int_{A_{jj}} e^{-u_j(t,t)}\prod_{j<k}\int_{A_{jk}} e^{-\frac{1}{2}(u_j+u_k)(t,t)}dt\\ &= \sqrt{\pi}^{\,r+\frac{a}{2}r(r-1)}(\prod_{j=1}^r u_j)^{-1}(\prod_{j<k}\frac{u_j+u_k}{2})^{-a}.\end{aligned} \tag{3.5}$$

Next we prove Proposition 3.1.

Proof. Using (3.2) and performing the same calculation as above (with A replaced by B) we get

$$I_B(\nu) = \frac{1}{\Gamma_\Omega(\nu)} \int_\Omega G_B(u) e^{-(e,u)} \delta(u)^{\nu - \frac{n_A}{r}} dt du, \tag{3.6}$$

with

$$G_B(u) = \int_B e^{(t^2,u)} dt = \int_B e^{-(t \circ t^*, u)} dt.$$

Consider as above the Peirce decomposition of u and B. We can then calculate $G_B(u)$ and obtain

$$G_B(u) = \sqrt{\pi}^{r(\iota-1)+\frac{a}{2}r(r-1)} (\prod_{j=1}^r u_j)^{-(\iota-1)} (\prod_{j<k} \frac{u_j+u_k}{2})^{-a}. \tag{3.7}$$

Therefore, comparing $G_B(u)$ with $G_A(u)$ given in (3.5) it follows that

$$G_B(u) = \sqrt{\pi}^{r(\iota-2)} (\prod_{j=1}^r u_j)^{-(\iota-2)} G_A(u) = \sqrt{\pi}^{r(\iota-2)} \delta(u)^{-(\iota-2)} G_A(u).$$

Substituting this into (3.6) we find

$$I_B(\nu) = \sqrt{\pi}^{r(\iota-2)} \frac{1}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(e,u)} \delta(u)^{\nu-\frac{n_A}{r}} \delta(u)^{-(\iota-2)} G_A(u) du.$$

Comparing this with (3.4) we find that the integral $I_B(\nu)$ is of the same type as $I_A(\beta)$ with $\beta = \nu - (\iota - 2)$; more precisely,

$$I_B(\nu) = \sqrt{\pi}^{r(\iota-2)} \frac{\Gamma_\Omega(\nu - (\iota-2))}{\Gamma_\Omega(\nu)} I_A(\nu - (\iota - 2)).$$

which again by Lemma 3.2, is

$$I_B(\nu) = 4^{n_A - r(\nu-(\iota-2))} \sqrt{\pi}^{r(\iota-2)+2n_A} \frac{\Gamma_\Omega(2\nu - 2(\iota-2) - \frac{n_A}{r})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu - \frac{\iota-2}{2})}.$$

□

We can now calculate the integral $I(\nu, \underline{\alpha})$.

Proposition 3.3. *Suppose $\nu > p_0 - 1 = p - 1$. The integral $I(\nu, \underline{\alpha})$ is absolutely convergent if*

$$-\frac{1}{2}((\nu - p_0 + 1) + \frac{a}{2}(r-j)) < \alpha_j < \frac{1}{2}(\nu - \frac{a}{2}(r-j)), \quad 1 \le j \le r,$$

and its value is given by

$$I(\nu, \underline{\alpha}) = 2^{2r\nu} \sqrt{2\pi}^{rb} I_B(\nu) \frac{\Gamma_\Omega(\nu - p_0 + \frac{n_A}{r} + 2\underline{\alpha}) \Gamma_\Omega(\nu + p_0 + 1 + \frac{n_2}{r} - \frac{b}{2} + 2\underline{\alpha}^*)}{\Gamma_\Omega(2\nu - \frac{n_B}{r})}.$$

Proof. We write $w = w_2 + w_1$ and $w_2 = x + y$ according to the Peirce decomposition of $V = V_2 \oplus V_1$ and the Cartan decomposition $V_2 = A \oplus B$. We perform the change of variables $\xi = x - \frac{1}{2}F(w_1, w_1)$. Thus $\mathcal{S}$ is parametrized by the product

$\Omega\times B\times V_1$. In terms of the new coordinates, $w_2=x+y=\xi+\frac{1}{2}F(w_1,w_1)+y$ and $w_2^*=\xi+\frac{1}{2}F(w_1,w_1)-y$. The integral can be rewritten as

$$2^{2r\nu}\int_{\Omega\times V_1\times B}\Delta_{\underline{\alpha}}(\xi)\frac{\Delta(\xi)^{\frac{\nu}{2}-\frac{p}{2}}}{\Delta(e+\xi+\frac{1}{2}F(w_1,w_1)-y)^{\nu}}dy\,dw_1\,d\xi.$$

(Here the constant $2^{2r\nu}$ appears because Δ is of degree $2r$, and $\Delta(\frac{e+w_2^*}{2})=2^{-2r}\Delta(e+w_2^*)$.) We consider first the integral with respect to y. Denote temporarily $u=e+\xi+\frac{1}{2}F(w_1,w_1)\in\Omega$ and note that

$$\begin{aligned}&\Delta(e+\xi+\frac{1}{2}F(w_1,w_1)-y)\\ &=\Delta(u-y)=\Delta(P(u^{\frac{1}{2}})e-P(u^{\frac{1}{2}})(P(u^{-\frac{1}{2}})(y))\\ &=\Delta(u)\Delta(e-P(u^{-\frac{1}{2}})(y))=\delta(u)^2\Delta(e-P(u^{-\frac{1}{2}})(y)).\end{aligned}\tag{3.8}$$

Change variables $\eta=P(u^{-\frac{1}{2}})(y)$. Then $dm(\eta)=\delta(u)^{-\frac{n_2-n_A}{r}}dy$ since $\dim B=n_2-n_A$. The integration with respect to y becomes,

$$\delta(u)^{\frac{n_2-n_A}{r}-2\nu}\int_B\Delta(e-\eta)^{-\nu}d\eta=\delta(u)^{\frac{n_2-n_A}{r}-2\nu}I_B(\nu)$$

and $I_B(\nu)$ is given by Proposition 3.1.

We consider now the integral with respect to w_1, which appears in u. We have

$$\delta(u)^{\frac{n_2-n_A}{r}-2\nu}=\frac{1}{\Gamma(2\nu-\frac{n_2-n_A}{r})}\int_\Omega e^{-(u,\zeta)}\delta(\zeta)^{2\nu-\frac{n_2}{r}}d\zeta$$

and $(u,\zeta)=(e+\xi,\zeta)+\frac{1}{2}(F(w_1,w_1),\zeta)$. Thus

$$\begin{aligned}&\int_{V_1}\delta(u)^{\frac{n_2-n_A}{r}-2\nu}dw_1\\ &=\frac{1}{\Gamma(2\nu-\frac{n_2-n_A}{r})}\int_{V_1}\int_\Omega e^{-(e+\xi,\zeta)}e^{-\frac{1}{2}(F(w_1,w_1),\zeta)}\delta(\zeta)^{2\nu-\frac{n_2}{r}}d\zeta dw_1\\ &=\frac{1}{\Gamma(2\nu-\frac{n_2-n_A}{r})}\int_\Omega e^{-(e+\xi,\zeta)}\delta(\zeta)^{2\nu-\frac{n_2-n_A}{r}}\left(\int_{V_1}e^{-\frac{1}{2}(F(w_1,w_1),\zeta)}dw_1\right)d\zeta,\end{aligned}\tag{3.9}$$

and the change of the order of integration will be justified by our method calculation. Writing

$$\begin{aligned}(F(w_1,w_1),\zeta)&=(F(w_1,w_1),P(\zeta^{\frac{1}{2}})e)=(P(\zeta^{\frac{1}{2}})F(w_1,w_1),e)\\ &=(F(\zeta^{\frac{1}{2}})\circ w_1,\zeta^{\frac{1}{2}}\circ w_1,e)\end{aligned}$$

by [9], Proposition 10.11. Here $\zeta^{\frac{1}{2}}\circ w_1=\frac{1}{2}D(\zeta^{\frac{1}{2}},e)w_1$ is the Jordan product on V. Performing a change of variables $v=\zeta^{\frac{1}{2}}\circ w_1$, we have $dv=\delta(\zeta)^{\frac{b}{2}}dw_1$ and

$$\int_{V_1}e^{-\frac{1}{2}(F(w_1,w_1),\zeta)}dw_1=\delta(\zeta)^{-\frac{b}{2}}\int_{V_1}e^{-\frac{1}{2}(F(v,v),e)}dv;$$

the last integral is of Gaussian type and can be easily evaluated, observing

$$\frac{1}{2}(F(v,v),e)=\frac{1}{2}(D(e,v)v,e)=\frac{1}{2}(v,D(v,e)e)=\frac{1}{2}(v,D(e,e)v)=\frac{1}{2}(v,v),$$

so that

$$\int_{V_1}e^{-\frac{1}{2}(F(v,v),e)}dv=\sqrt{2\pi}^{rb}.$$

Continuing the formula (3.9), we find it is, disregarding the Gamma factor,

$$\sqrt{2\pi}^{rb}\int_{\Omega} e^{-(e+\xi,\zeta)}\delta(\zeta)^{2\nu-\frac{n_2}{r}-\frac{b}{2}}d\zeta,$$

which is integrable if

$$2\nu-\frac{n_2}{r}-\frac{b}{2}>\frac{a}{2}(r-1),$$

namely, if $\nu>\frac{1}{2}(\frac{n_2}{r}+\frac{b}{2}+\frac{a}{2}(r-1))$ which is satisfied by our assumption of ν. This consequently justifies the change of order of integration in (3.9).

Our integral in question is

$$2^{r\nu}\sqrt{2\pi}^{rb}\frac{1}{\Gamma(2\nu-\frac{n_2-n_A}{r})}I_B(\nu)$$
$$\times\int_{\Omega}\int_{\Omega} e^{-(e,\zeta)}e^{-(\xi,\zeta)}\Delta_{\underline{\alpha}}(\xi)\Delta(\xi)^{\frac{\nu}{2}-\frac{p}{2}}\delta(\zeta)^{2\nu-\frac{n_2-n_A}{r}-\frac{b}{2}}d\xi d\zeta.$$

We calculate the integral with respect to ξ. Recall the formula (2.11) and that $\Delta(\xi)=\delta(\xi)^2$,

$$\int_{\Omega} e^{-(\xi,\zeta)}\Delta_{\underline{\alpha}}(\xi)\Delta(\xi)^{\frac{\nu}{2}-\frac{p}{2}}d\xi=\Gamma_\Omega(2\underline{\alpha}+\nu-p+\frac{n_A}{r})\delta^*_{2\underline{\alpha}^*-\nu+p-\frac{n_A}{r}}(\zeta),$$

which is absolutely convergent if

$$2\underline{\alpha}_j+\nu-p+\frac{n_A}{r}>\frac{a}{2}(j-1),\quad j=1,\dots,r. \tag{3.10}$$

The remaining integral, disregarding the constant, is then

$$\int_{\Omega} e^{-(e,\zeta)}\delta^*_{2\underline{\alpha}^*-\nu+p-\frac{n_A}{r}}(\zeta)\delta(\zeta)^{2\nu-\frac{n_2}{r}-\frac{b}{2}}d\zeta=\Gamma_\Omega(2\underline{\alpha}^*+\nu+p-\frac{n_2}{r}-\frac{b}{2}).$$

But since $p=\frac{n_2}{r}+\frac{b}{2}$, the above is $\Gamma_\Omega(2\underline{\alpha}^*+\nu)$. Moreover, the above integral is absolutely convergent if

$$2(\underline{\alpha}^*)_j-\nu<\frac{a}{2}(j-1),\quad j=1,\dots,r. \tag{3.11}$$

The inequalities (3.10) and (3.11) combined give our condition $\underline{\alpha}$. This finishes the proof. □

4. Calculations of the symbol: Types B_r $(r\geq 1)$ and D_r $(r\geq 3)$

In this section we calculate the integral $I(\nu,\underline{\alpha})$ for type B_r $(r\geq 1)$ and D_r $(r\geq 3)$. The type D_r $(r\geq 3)$ can be viewed as a degenerate case of type B_r with the root multiplicity of $\frac{\gamma_j}{2}$ being 0. The method of calculation is basically the same as in the previous section. So we will only indicate the necessary changes needed in the calculations.

Proposition 3.1 in this case takes the following form; we remark that, as we did for the function $h(z,w)$ in Section 2, the function $\Delta(e-\eta)$ is positive on B.

Proposition 4.1. *Suppose $\nu>a(r-1)-1$. The integral*

$$I_B(\nu)=\int_B\frac{d\eta}{\Delta(e-\eta)^\nu}$$

is absolutely convergent and its values are given by

$$I_B(\nu)=\sqrt{\pi}^{n_2}4^{n_A-r(\frac{\nu}{2}+\frac{1}{2})}\frac{\Gamma_\Omega(\nu+1-\frac{n_A}{r})}{\Gamma_\Omega(\frac{\nu}{2})\Gamma_\Omega(\frac{\nu}{2}+\frac{1}{2})}.$$

The value of the integral $I(\nu, \underline{\alpha})$ is given in the following; we note that in comparison with Types BC and C, the convergence region of ν improves to $\nu > p-2$. (The discrete series are for $\nu > p-1$.)

Proposition 4.2. *Suppose $\nu > 2p_0 - 2 = p-2$, that is $\frac{\nu}{2} > p_0 - 1 = \frac{p}{2} - 1$. The integral $I(\nu, \underline{\alpha})$ is absolutely convergent if*

$$-\frac{1}{2}((\nu - 2p_0 + 2) + a(r-j)) < \alpha_j < \frac{1}{2}(\nu - a(r-j)), \quad 1 \le j \le r.$$

and its values are given by

$$I_{\mathcal{S}}(\nu, \underline{\alpha}) = \sqrt{2\pi}^{rb} I_B(\nu) \frac{\Gamma_\Omega(\underline{\alpha} + \frac{\nu}{2} - \frac{p}{2} + \frac{n_A}{r})\Gamma_\Omega(\underline{\alpha}^* + \frac{\nu}{2})}{\Gamma_\Omega(\nu - \frac{n_B}{r})}.$$

Example 4.3. We consider the simple case of the unit disk D in $\mathbb{R}^2$, considered as a real bounded symmetric domain in the unit ball $\mathbb{D}$ of $\mathbb{C}^2$. It is of type B_1 with roots $\pm\frac{\gamma}{2}$. The ball $\mathbb{D}$ is a realization of the Hermitian symmetric space $SU(2,1)/S(U(2) \times U(1))$, and the real disk of $SO(2,1)/SO(2)$. The genus p of $\mathbb{D}$ is now 3 and the weighted Bergman space H^ν considered in Section 1 now has reproducing kernel $(1 - \langle z, w\rangle)^{-\nu}$ with $\nu > p - 1 = 2$. The Berezin transform on D is now

$$B_\nu f(y) = \frac{\nu - 1}{2\pi} \int_D f(x) \frac{(1 - \|x\|^2)^{\frac{\nu}{2}}(1 - \|y\|^2)^{\frac{\nu}{2}}}{(1 - (x, y))^\nu} \frac{dx}{(1 - \|x\|^2)^{\frac{3}{2}}}, \tag{4.1}$$

where $x = (x_1, x_2)$ and $(x, y) = x_1 y_1 + x_2 y_2$. However, the unit disk $D = SO(2,1)/SO(2)$ in $\mathbb{R}^2$ itself has a Hermitian structure defined by the Lie algebra of $SO(2)$, and in this realization it *is not* the standard Hermitian structure on $\mathbb{R}^2$ defined by $(u, v) \mapsto (-v, u)$ and $\frac{du^2 + dv^2}{(1-u^2-v^2)^2}$. However, those two structures *are equivalent*; denoting the unit disk in $\mathbb{C} = \mathbb{R}^2$ with the standard Hermitian structure by $D_{\mathbb{C}}$, a biholomorphic mapping from $D_{\mathbb{C}}$ onto $D = SO(2,1)/SO(2)$ is given by the Hua transformation

$$z = u + iv \mapsto \frac{2}{1 + |z|^2}(u, v).$$

See [4], Chapter X, Exercise D2. The corresponding Berezin transform on $D_{\mathbb{C}}$, obtained from (4.1) by conjugating the transformation, is then

$$B_\nu f(w) = \frac{2(\nu - 1)}{\pi} \int_{D_{\mathbb{C}}} f(z) \frac{(1 - |z|^2)^\nu (1 - |w|^2)^\nu}{|(1 - z\bar{w})^\nu|^2} (1 + |\frac{z - w}{1 - z\bar{w}}|^2)^{-\nu} \frac{dm(z)}{(1 - |z|^2)^2}.$$

The symbol of B_ν in this case is, with $\lambda\gamma$ identified with λ

$$b_\nu(\lambda) = \frac{\Gamma(\frac{\nu}{2} - \frac{1}{4} + \lambda)\Gamma(\frac{\nu}{2} - \frac{1}{4} - \lambda)}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu}{2} - \frac{1}{2})}.$$

It is interesting to see that the Berezin transform so obtained is quite close to the classical Berezin transform (0.1), whose symbol, after normalization, is

$$\frac{\Gamma(\nu - \frac{1}{2} + \frac{\lambda}{2})\Gamma(\nu - \frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\nu)\Gamma(\nu - 1)};$$

see [17] and [15].

5. Calculations of the symbol: Type A and D_2

The integral $I(\nu,\underline{\alpha})$ for Type A is very simple. The real Siegel domain $\mathcal{S}=\Omega$, and our integral is

$$2^{r\nu}\int_{\mathcal{S}}\Delta_\alpha(w)\frac{\Delta(w)^{\frac{\nu}{2}}}{\Delta(e+w)^\nu}\frac{dw}{\Delta(w)^{\frac{p}{2}}}$$

which is a Beta-type integral. The result is as follows.

Proposition 5.1. *Suppose $\nu>p-2=2p_0-2$. The integral $I(\nu,\underline{\alpha})$ is absolutely convergent if*

$$\frac{1}{2}(\nu-a(j-1))<\alpha_j<\frac{1}{2}(\nu-a(r-j)),\quad 1\le j\le r,$$

and its value is given by

$$2^{r\nu}\frac{\Gamma_\Omega(\frac{\nu}{2}+\underline{\alpha})\Gamma_\Omega(\frac{\nu}{2}+\underline{\alpha}^*)}{\Gamma_\Omega(\nu)}.$$

We now consider type D_2, the Jordan triple is denoted by $IV_n^{\mathbb{R},m}$ $(2\le m\le[\frac{n}{2}])$ in [9]. So we introduce the necessary notation. Let $V=\mathbb{R}^n$, and consider the quadratic operator $Q(x)$ on V defined by

$$Q(x)y=q(x,y)x-q(x)y$$

where $q(x)=x_1^2+\cdots+x_m^2-x_{m+1}^2-\cdots-x_n^2$ and $q(x,y)$ is its polarization $q(x,y)=q(x+y)-q(x)-q(y)$. The involution $\bar{x}$ of x is $\bar{x}=(x_1,\cdots,x_m,-x_{m+1},\cdots,-x_n)$. Let $e_1=\frac{1}{2}(1,0,\cdots,0,1)$ and $e_2=\frac{1}{2}(1,0,\cdots,0,\dots,-1)$ and $e=e_1\oplus e_2$. Then e_1 and e_2 form a frame of tripotents and e is a maximal tripotent. The joint Peirce decomposition of V is $V=V_{11}\oplus V_{22}\oplus V_{12}$ with $V_{jj}=\mathbb{R}e_j$, $j=1,2$. The Cartan involution $x\mapsto Q(e)\bar{x}$ gives a decomposition of V, $V=A\oplus B$ with $A=V_{11}\oplus V_{22}\oplus A_{12}$,

$$A_{12}=\{x\in V; x_n=x_1=x_2=\cdots=x_m=0\}$$

and

$$B=B_{12}=\{x\in V; x_1=x_{m+1}=\cdots=x_n=0\},$$

$\dim A_{12}=n-m-1$, $\dim B_{12}=m-1$. The determinant function $\Delta(x)=x_1^2+\cdots+x_n^2$. In terms of the notation in Section 2, the positive roots are $\frac{\gamma_2+\gamma_1}{2}$, $\frac{\gamma_2-\gamma_1}{2}$ with root multiplicities $m-1$ and $n-m-1$ respectively, and

$$\underline{\rho}=\frac{1}{4}((2m-n)\gamma_1+(n-2)\gamma_2).$$

The symmetric cone in A consists of all elements $x=(x_1,0,\dots,0,x_{m+1},\dots,x_n)$ so that $q(x)=x_1^2-x_{m+1}^2-\cdots-x_n^2>0$ and $x_1>0$, namely it is the forward light cone.

Observe that with our normalization of the Jordan triple system, we have $\|x\|^2=2(x_1^2+\cdots+x_n^2)$. Correspondingly $dx=\sqrt{2}^n dx_1\dots dx_n$. Also the genus $p=n$. The integral $I(\nu,\underline{\alpha})$ is, after the change of variables as in Section 3,

$$I(\nu,\underline{\lambda})=\int_\Omega\Delta_\alpha(\xi)\Delta(\xi)^{\frac{\nu}{2}-\frac{n}{2}}\left(\int_B\frac{1}{\Delta(e+\xi-y)^\nu}dy\right)d\xi.$$

The integration on B can be evaluated directly. Observe that $\Delta(x-y)=\Delta(x)+\Delta(y)$ if $x+y\in A\oplus B$. Performing a change of variable $\eta=\Delta(e+\xi)^{-\frac{1}{2}}y$ we have

$$\begin{aligned}\int_B \frac{1}{\Delta(e+\xi-y)^{\nu}}dy &= \Delta(e+\xi)^{-\nu+\frac{m-1}{2}}\int_B \frac{1}{\Delta(e-\eta)^{\nu}}d\eta\\ &= \sqrt{2}^{m-1}\Delta(e+\xi)^{-\nu+\frac{m-1}{2}}\int_{\mathbb{R}^{m-1}}\frac{1}{(1+\eta_2^2+\cdots+\eta_m^2)^{\nu}}d\eta_2\dots d\eta_m\\ &= \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\Delta(e+\xi)^{-\nu+\frac{m-1}{2}}.\end{aligned}$$

Eventually we find that

(5.1)

$$\begin{aligned}I(\nu,\underline{\lambda}) &= \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\int_\Omega \Delta_\alpha(\xi)\Delta(\xi)^{\frac{\nu}{2}-\frac{n}{2}}\Delta(e+\xi)^{-\nu+\frac{m-1}{2}}d\xi\\ &= \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\frac{\Gamma_\Omega(\alpha+\frac{\nu}{2}-\frac{n}{2}+\frac{n-m+1}{2})\Gamma_\Omega(\alpha+\frac{\nu}{2}+\frac{n}{2}-\frac{n-m+1}{2}}{\Gamma_\Omega(\nu-\frac{m-1}{2})}\end{aligned}$$

where the Gamma function Γ_Ω in this case is

$$\Gamma_\Omega(\beta)=\sqrt{2\pi}^{n-m-1}\prod_{j=1}^{2}\Gamma(\beta_j-\frac{n-m-1}{2}(j-1)).$$

Proposition 5.2. *Suppose* $\nu>p-2=2p_0-2$,

$$-\frac{1}{2}(\nu-m+1)<\alpha_1<\frac{1}{2}(\nu-n+2m+2)$$

and

$$-\frac{1}{2}(\nu-n)<\alpha_2<\frac{1}{2}(\nu+m-1).$$

The integral $I(\nu,\underline{\alpha})$ *is absolutely convergent and its value is given by*

$$I(\nu,\underline{\lambda})=\sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\frac{\Gamma_\Omega(\alpha+\frac{\nu}{2}-\frac{m-1}{2})\Gamma_\Omega(\underline{\alpha}^*+\frac{\nu}{2}+\frac{m-1}{2})}{\Gamma_\Omega(\nu-\frac{m-1}{2})}.$$

6. The L^p-bounded properties of the Berezin transform

We let $C_\nu=I(\nu,0)^{-1}$, which is the normalization constant. Denote

$$b_\nu(\underline{\lambda})=C_\nu I(\nu,\underline{\alpha}). \tag{6.1}$$

Recall the definition of the genus p_0 of D. We see that, after simplifying,

$$b_\nu(\underline{\lambda})=\frac{\prod_{j=1}^r\Gamma(\nu-\frac{p_0-1}{2}+\lambda_j)\Gamma(\nu-\frac{p_0-1}{2}-\lambda_j)}{\prod_{j=1}^r\Gamma(\nu-\frac{p_0-1}{2}+\rho_j)\Gamma(\nu-\frac{p_0-1}{2}-\rho_j)}$$

if it is of type BC or C,

$$b_\nu(\underline{\lambda})=\frac{\prod_{j=1}^r\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}+\lambda_j)\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}-\lambda_j)}{\prod_{j=1}^r\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}+\rho_j)\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}-\rho_j)}$$

for other types except D_2. So that $b_\nu(\underline{\lambda})$ has a rather general form in terms of p_0. For type D_2 it is

$$b_\nu(\underline{\lambda})=\frac{\prod_{j=1}^2\Gamma(\frac{\nu}{2}-\frac{n-2}{4}+\lambda_j)\Gamma(\frac{\nu}{2}-\frac{n-2m}{4}-\lambda_j)}{\prod_{j=1}^2\Gamma(\frac{\nu}{2}-\frac{n-2}{4}+\rho_j)\Gamma(\frac{\nu}{2}-\frac{n-2m}{4}-\rho_j)}.$$

Proposition 6.1. *Suppose $\nu > p-1 = 2p_0 - 1$ for type BC or C and $\frac{\nu}{2} > p_0 - 1$ for other types. The Berezin transform B_ν defines a bounded positive operator on $L^p(D, d\mu_0)$ for all $1 \le p \le \infty$.*

We recall that the functions $e_{\underline{\lambda}}$ transform under a character of NA. Observe that by a change of variables in the formula $B_\nu e_{\underline{\lambda}}(z) = b_\nu(\underline{\lambda}) e_{\underline{\lambda}}(z)$ for $z = e$ one obtains

$$B_\nu e_{\underline{\lambda}}(z) = b_\nu(\underline{\lambda}) e_{\underline{\lambda}}(z)$$

for all $z \in \mathcal{S}$; see [1].

In particular, taking $\underline{\lambda} + \underline{\rho} = 0$ we get

$$C_\nu \int_{\mathcal{S}} \frac{\Delta(\Re z)^{\frac{\nu}{2}} \Delta(\Re w)^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu} d\mu_0(w) = 1. \tag{6.2}$$

Proof. We use interpolation. For $p = 1$ we have, in view of (6.2),

$$\begin{aligned} \|B_\nu f\|_{L^1} &\le C_\nu \int_{\mathcal{S}} \int_{\mathcal{S}} |f(w)| \frac{\Delta(\Re z)^{\frac{\nu}{2}} \Delta(\Re w)^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu} d\mu_0(w) d\mu_0(z) \\ &= \int_{\mathcal{S}} |f(w)| \left(C_\nu \int_{\mathcal{S}} \frac{\Delta(\Re z)^{\frac{\nu}{2}} \Delta(\Re w)^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu} d\mu_0(z) \right) d\mu_0(w) \qquad (6.3) \\ &= \int_{\mathcal{S}} |f(w)| d\mu_0(w) = \|f\|_{L^1}, \end{aligned}$$

that is, B_ν is L^1-bounded. However, B_ν is a formally self-adjoint operator, thus it is L^∞ bounded. By interpolation we see that it is bounded on all L^p, for $1 \le p \le \infty$. □

We summarize our main results in the following

Theorem 6.2. *Suppose $\nu > p - 1 = 2p_0 - 1$ for type BC or C and $\frac{\nu}{2} > p_0 - 1$ for other types. The Berezin transform B_ν is a positive bounded operator on $L^2(D, d\mu_0)$ and its spectral symbol is given by $b_\nu(\underline{\lambda})$ under the decomposition of L^2-space into irreducible representations of G.*

Remark 6.3. Recall the definition (1.1) of the restriction operator R. The operator $B_\nu = RR^*$ is a bounded operator, which in turn implies that R is bounded when ν satisfies the condition above.

There are various implications of our results. We plan to pursue them in the future.

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