TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 9, Pages 3833–3834
S 0002-9947(01)02841-0
Article electronically published on April 23, 2001

# CORRECTION TO "RELATIVE COMPLETIONS OF LINEAR GROUPS OVER $\mathbb{Z}[t]$ AND $\mathbb{Z}[t, t^{-1}]$"

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The main theorem of the paper (Theorem 4.2) can be proved only for $n \geq 4$. We do not know if the result is true for $n = 3$. Moreover, the proof depends upon Corollary 3.2, which is false. The author apologizes for the mistake.

The gap in the argument may be filled as follows. Recall that we are considering the split extension

$$1 \longrightarrow K(R) \longrightarrow SL_n(R) \xrightarrow{\text{mod } \mathfrak{m}_R} SL_n(\mathbb{Z}) \longrightarrow 1,$$

for $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$, $\mathfrak{m}_R = (t), (t-1)$, respectively. The group $K(R)$ consists of matrices congruent to the identity modulo $\mathfrak{m}_R$; it has a filtration $K^\bullet(R)$ defined by

$$K^r(R) = \{X \in K(R) : X \equiv I \mod \mathfrak{m}_R^r\}.$$

The graded quotients all satisfy

$$K^r(R)/K^{r+1}(R) \cong \mathfrak{sl}_n(\mathbb{Z}).$$

Consider the extension

$$1 \longrightarrow \mathcal{U} \longrightarrow SL_n(\mathbb{Q}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Q}) \longrightarrow 1.$$

The group $\mathcal{U}$ has a filtration $\mathcal{U}^\bullet$ by powers of $T$ and we have injections $SL_n(R) \to SL_n(\mathbb{Q}[[T]])$ given by sending $t \mapsto T$ and $t \mapsto 1 + T$, respectively. Theorem 3.6 asserts that the map $K(R) \to \mathcal{U}$ is the Malcev completion. The proof given in the paper is incorrect.

*Proof of Theorem 3.6.* We have $\mathcal{U} = \varprojlim \mathcal{U}/\mathcal{U}^r$ and $\mathcal{U}/\mathcal{U}^r$ is the Malcev completion of $K(R)/K^r(R)$. The proof given in the paper asserts that $K^r(R) = \Gamma^r K(R)$ (where $\Gamma^\bullet$ denotes the lower central series), whence the result. This, in turn, relies on Corollary 3.2 which asserts in this case that

$$\varprojlim K(R)/\Gamma^r K(R) \xrightarrow{\cong} \varprojlim K(R)/K^r(R).$$

This is false. The above map is surjective, but its kernel, $\varprojlim K^r/\Gamma^r$, is nontrivial.

To prove the theorem, we need only show that $K^r/\Gamma^r$ is torsion. Assuming this, the composition $K/\Gamma^r \longrightarrow K/K^r \longrightarrow \mathcal{U}/\mathcal{U}^r$ is the Malcev completion of $K(R)$.

To see that $K^r/\Gamma^r$ is torsion we use some computations of $K$-groups of truncated polynomial rings [3]. Recall that an *elementary transvection* is a matrix of the form $I + e_{ij}(x)$, $i \neq j$, where $e_{ij}(x)$ has $x$ in the $i, j$ position and zeroes elsewhere. If $A$ is a ring and $J$ an ideal of $A$, set $GL_n(A, J) = \{X \in GL_n(A) : X \equiv I \mod J\}$ and set $SL_n(A, J) = SL_n(A) \cap GL_n(A, J)$. Let $E_n(A, J)$ denote the normal

Received by the editors February 1, 2001.
1991 *Mathematics Subject Classification.* Primary 55P60, 20G35, 20H05; Secondary 20G10, 20F14.

subgroup of $GL_n(A)$ generated by the transvections $I + e_{ij}(x)$ for $x \in J$. Note that $E_n(A, J) \subseteq GL_n(A, J)$. We now recall the following result (see e.g., [4]).

**Theorem 1.** *Let $n \geq 3$ and let $H$ be a normal subgroup of $GL_n(A)$. Then there is a unique ideal $J$ with $E_n(A, J) \subseteq H \subseteq GL_n(A, J)$.* □

The ideal $J$ may be constructed explicitly:

$$J = \{x \in A : I + e_{12}(x) \in H\}.$$

Now consider the groups $\Gamma^r K(R)$, $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$. We have

$$\Gamma^r K(R) \subseteq K^r(R) = SL_n(R, \mathfrak{m}_R^r)$$

We claim that $E_n(R, \mathfrak{m}_R^r) \subseteq \Gamma^r K(R)$. To see this note that $I + e_{12}(t^r) \in \Gamma^r K(\mathbb{Z}[t])$ and $I + e_{12}((t-1)^r) \in \Gamma^r K(\mathbb{Z}[t, t^{-1}])$ (here we use $n \geq 3$) so that $\mathfrak{m}_R^r \subseteq J$. But since any $I + e_{12}(x) \in \Gamma^r$ satisfies $t^r|x$ (resp. $(t-1)^r|x$), we have $x \in \mathfrak{m}_R^r$; that is, $J = \mathfrak{m}_R^r$.

We now need the following result.

**Theorem 2** ([1])**.** *If $A$ is a Noetherian ring, then the canonical map*

$$SL_n(A, J)/E_n(A, J) \longrightarrow SK_1(A, J)$$

*is surjective for $n \geq \dim(A) + 1$ and bijective for $n \geq \dim(A) + 2$.* □

Note that the rings $\mathbb{Z}[t]$ and $\mathbb{Z}[t, t^{-1}]$ are two-dimensional. Thus for $n \geq 4$ we have

$$SL_n(R, \mathfrak{m}_R^r)/E_n(R, \mathfrak{m}_R^r) \xrightarrow{\cong} SK_1(R, \mathfrak{m}_R^r).$$

Now let $n \geq 4$. By Theorems 1 and 2 it suffices to show that the group $SK_1(R, \mathfrak{m}_R^r)$ is torsion. There is an exact sequence of $K$-groups

$$K_2(R) \longrightarrow K_2(R/\mathfrak{m}_R^r) \longrightarrow SK_1(R, \mathfrak{m}_R^r) \longrightarrow SK_1(R).$$

We have $R/\mathfrak{m}_R^r \cong \mathbb{Z}[s]/s^r$. By [3], the group $K_2(R/\mathfrak{m}_R^r)$ is torsion, and since $SK_1(R) = 0$, we are done. This completes the proof of Theorem 3.6. □

Corollary 3.10 must now be modified as follows.

**Corollary 3.** *If $n \geq 4$, then $H_1(K(R), \mathbb{Q}) \cong \mathfrak{sl}_n(\mathbb{Q})$.*

*Proof.* We have an exact sequence

$$0 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K(R), \mathbb{Z}) \longrightarrow \mathfrak{sl}_n(\mathbb{Z}) \longrightarrow 0.$$

□

The cohomology calculations in the final section of [2] are correct, but apply only for $n \geq 4$.

## References

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