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CUNTZ-KRIEGER ALGEBRAS AND ENDOMORPHISMS OF FINITE DIRECT SUMS OF TYPE I_∞ FACTORS

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ABSTRACT. A correspondence between algebra endomorphisms of a finite sum of copies of the algebra of all bounded operators on a Hilbert space and representations of certain norm closed $*$-subalgebras of bounded operators generated by a finite collection of partial isometries is introduced. Basic properties of this correspondence are investigated after developing some operations on bipartite graphs that usefully describe aspects of this relationship.

INTRODUCTION

Consider an infinite dimensional Hilbert space $\mathcal{H}$ which decomposes into an orthogonal sum of two isomorphic subspaces $\mathcal{H}_1, \mathcal{H}_2$ so that there are unitary isomorphisms $S_i : \mathcal{H} \to \mathcal{H}_i$ of $\mathcal{H}$ $(i = 1, 2)$ with these subspaces. Viewed as self-maps of $\mathcal{H}$ the maps S_i are isometries of $\mathcal{H}$. The norm closed $*$-subalgebra of all bounded linear operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ generated by S_1 and S_2 is the Cuntz algebra $\mathcal{O}_2$. It is simple, and unique up to isomorphism of C^*-algebras ([7]). The particular isometries above may be viewed as defining a representation of $\mathcal{O}_2$ on $\mathcal{H}$. If A is a bounded linear operator on $\mathcal{H}$ the unitarily equivalent operators $S_iAS_i^*$ on $\mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, may also be viewed as operators on $\mathcal{H}$, where they may be added. This defines a self-map of the algebra $\mathcal{B}(\mathcal{H})$, namely $A \to \sum S_iAS_i^*$, which is a unital $*$-endomorphism. Conversely, by viewing any given unital $*$-endomorphism φ of $\mathcal{B}(\mathcal{H})$ as a representation of the algebra $\mathcal{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ and using standard techniques of unitary equivalence for representations, it follows that φ arises in this manner from a representation of some Cuntz algebra $\mathcal{O}_n$. By exploring the representation theory of Cuntz algebras one can analyse endomorphisms of $\mathcal{B}(\mathcal{H})$. Of particular interest are the shifts and ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$ — see [1], [16], [4], [12] and references therein for example.

Now consider isometries defined only on subspaces of $\mathcal{H}$, so that in essence part of $\mathcal{H}$ is sent to zero. Given a finite collection $\{S_1, \ldots, S_d\}$ of these so-called partial isometries, subject to the conditions that the range spaces are orthogonal, and their domain spaces decompose orthogonally in terms of the various range spaces, we can form the norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by them to obtain a Cuntz-Krieger algebra $\mathcal{O}_A$. Here A is a $\{0, 1\}$ valued $d \times d$ matrix describing how each initial, or domain subspace of a partial isometry decomposes into the various range spaces of the given partial isometries.

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In this note we extend the above correspondence to one between representations of $\mathcal{O}_A$ on a Hilbert space $\mathcal{H}$, and unital $*$-endomorphisms of a certain von Neumann algebra, namely the subalgebra of all operators which are reduced by the initial subspaces of the partial isometries. This subalgebra is isomorphic to a finite direct sum of finitely many copies of $\mathcal{B}(\mathcal{H})$. Since unital $*$-endomorphisms of algebras of this form can be described using matrices ([5]) the following question arises: how is the matrix for the endomorphism defined by a representation of $\mathcal{O}_A$ related to the matrix A?

Here a third correspondence, a geometrical one, is invoked to describe this relationship. The matrix A can be viewed as the vertex matrix for a certain bipartite graph, and it is by using certain operations on bipartite graphs that the relationship between these two matrices becomes clear. The graph viewpoint for endomorphisms was already part of the established context, going back to work of Bratteli ([3]) on homomorphisms between finite dimensional algebras; while a graphical viewpoint for Cuntz-Krieger algebras was more fully exploited only recently ([15]). These graph operations are also needed to describe how an arbitrary unital $*$-endomorphism of a finite sum of algebras isomorphic to $\mathcal{B}(\mathcal{H})$ arise from a representation of an appropriate Cuntz-Krieger algebra.

An interesting extension of these graph operations with an application to arbitrary Cuntz-Krieger algebras is found in [6].

The Cuntz and Cuntz-Krieger algebras have played and continue to play a significant role in several areas of investigation. Originally Cuntz and Krieger investigated these algebras from the perspective of providing dynamical invariants for the topological Markov chains, or subshifts of finite type, that are also associated with such matrices A. They showed that flow equivalent subshifts give rise to stably isomorphic algebras $\mathcal{O}_A$. In fact the K-groups of the algebras $\mathcal{O}_A$ are computed in [8], and using Frank's classification of topological Markov chains up to flow equivalence the algebra $\mathcal{O}_A$ can be shown to be classified by these groups. Details and references can be found in [21].

The argument in [9] involving flow equivalent subshifts rests on an alternate description of the algebras $\mathcal{O}_A$, namely that they arise stably as the crossed product by an automorphism of an AF subalgebra of $\mathcal{O}_A$. This subalgebra is itself constructed as a crossed product involving the algebra of continuous, complex valued functions vanishing at infinity on the unstable space of the subshift defined by A.

In [19] Putnam built on this dynamical aspect of Cuntz-Krieger algebras by describing and investigating several C^*-algebras associated with hyperbolic dynamical systems in general, in other words, with expansive homeomorphisms of compact metric spaces equipped with canonical coordinates. Outside of the usual crossed product algebra, these dynamical systems in addition give rise to groupoid C^*-algebras formed from stable and unstable equivalence relations, and to their associated Ruelle algebras, which are defined as crossed product algebras by an induced automorphism of the groupoid algebra. Those hyperbolic dynamical systems where the underlying space is zero dimensional give rise to topological Markov chains, and the Ruelle algebras in this case are the Cuntz-Krieger algebras up to stable isomorphism. Insights into these general dynamical algebras and also their underlying dynamical systems can be guided by knowledge of the Cuntz-Krieger algebras associated to topological Markov chains. An overview of recent progress and problems in this direction can be found in [14].

The possibilities of a groupoid approach to Cuntz-Krieger algebras were indicated in [8] but it first seems to have been exploited in [15]. Beginning with a locally finite directed graph one can associate with it a groupoid whose C^*-algebra is the Cuntz-Krieger algebra defined by the possibly infinite, locally finite edge matrix of the graph. This not only provided an approach to Cuntz-Krieger algebras for infinite, locally finite matrices but also introduced a context in which to understand the structure of the Morita equivalent C^*-algebras arising in Doplicher and Robert's duality theory for compact groups ([15]).

Other current lines of investigation are linked to Cuntz-Krieger algebras. In [18] Pimsner introduced a construction for C^*-algebras associated with Hilbert bimodules which simultaneously incorporated into a common context both the Cuntz-Krieger algebras, which arise from a bimodule over an abelian finite dimensional algebra, and crossed product algebras, which arise from a bimodule structure defined by an automorphism of an algebra.

The threads connecting the Cuntz-Krieger algebras with these approaches are also apparent in the continuous graph C^*-algebras of Deaconu ([10]), and more recently in the C^*-algebras associated with branched coverings ([11]). These C^*-algebras arise from groupoids described in [2], which reflect an underlying dynamics, and which restrict in the graph situation to the groupoids considered in [15]. These algebras are shown to be isomorphic to certain Cuntz-Pimsner algebras for appropriate Hilbert bimodules ([10], [11]). If the underlying dynamics come from an expansive homeomorphism, then these groupoid C^*-algebras are Morita equivalent to the Ruelle C^*-algebras.

It is hoped that the results described below will provide a useful tool to help explore the various connections mentioned above.

In summary, this paper introduces and investigates some aspects of a new correspondence between certain unital $*$-endomorphisms of finite direct sums of countably decomposable type I factors on the one hand and representations of finite dimensional Hilbert spaces of partial isometries on the other. This correspondence restricts to one between certain $*$-endomorphisms of finite direct sums of type I_∞ factors and representations of Cuntz-Krieger algebras. The well known and much investigated correspondence between unital $*$-endomorphisms of a type I_∞ factor with finite index n and representations of the Cuntz algebra $\mathcal{O}_n$ is included in this description. As one can imagine the step from O_n to O_A with A a square matrix of nonnegative integer entries introduces combinatorial complexities which either stand in the way of, or further enrich, development of a basic theory. It is hoped that some necessary tools for further pursuing questions concerning $*$-endomorphisms of the kind considered here are provided below.

The process of going from a representation of a Cuntz-Krieger algebra to a unital $*$-endomorphism of a certain von Neumann algebra directly mimics the procedure which takes representations of Cuntz algebras to endomorphisms of $\mathcal{B}(\mathcal{H})$. The concept of a Hilbert space parametrizing partial isometries generating a Cuntz algebra is useful in order to show that the endomorphism is independent of a particular basis of the parametrization space. In a similar fashion it is possible to use Hilbert spaces to parametrize some sets of partial isometries generating Cuntz-Krieger algebras. In section 1 we opt for a conventional approach which generalizes in a straightforward manner the original Roberts' Hilbert space of isometries for a Cuntz algebra. If A is a square n by n matrix of zeroes and ones, the parametrizing Hilbert space for $\mathcal{O}_A$ is of dimension n. Pimsner's bimodule approach to Cuntz-Krieger algebras is also conceivably available here, one that uses a Hilbert space of varying dimension, up to n^2, depending on the entries of A.

A quick overview of the paper follows. In section 1 we develop a Hilbert space of partial isometries. The basic object, called a coordinate system on a Hilbert space $\mathcal{E}$, is a finite collection $\boldsymbol{P}$ of commuting projections that cover $\mathcal{E}$. It is called orthogonal if any two of these projections are either equal or orthogonal. Certain linear maps $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ are called displays of a coordinate system. Basically a display fans out the projections in a given coordinate system to become orthogonal projections. One can show that a display is faithful, norm decreasing and maps unit vectors of $\mathcal{E}$ to certain linear combinations of partial isometries. To any coordinate system $\boldsymbol{P}$ along with an orthogonal system generating it one can associate a matrix with entries in $\{0, 1\}$. Using such a matrix A which is square we introduce the concept of a dual pair $\boldsymbol{P}$ and $\boldsymbol{Q}$ of coordinate systems along with the basic idea of a representation $\phi : (\boldsymbol{P}, A, \boldsymbol{Q}) \to \mathcal{B}(\mathcal{H})$ of a dual pair. Such a representation is a properly aligned display of the coordinate system $\boldsymbol{P}$. These representations are equivalent to representations of the Cuntz-Krieger algebra $\mathcal{O}_A$ if A satisfies condition I of Cuntz-Krieger. Using representations of a dual pair also allows a natural definition for the quasi-free automorphisms of a Cuntz-Krieger algebra $\mathcal{O}_A$.

In section 2 the process of assigning a unital $*$-endomorphism φ_ϕ of the commutant of a certain coordinate system to a given representation ϕ of a dual system is described. A natural line of investigation is to see how properties of the endomorphism are reflected in algebraic properties involving the representation. We show, among other things, that the fixed point algebra of the endomorphism φ_ϕ is the commutant of the $*$-algebra generated by the image of ϕ (Theorem 2.6). Thus the endomorphism is ergodic if and only if the representation is irreducible. A unital endomorphism φ is called a shift if the set of elements in common with all the ranges $\varphi^n, n \in \mathbb{N}$, is the set of scalars. We show (Theorem 2.7) that the endomorphism is a shift if the canonical AF subalgebra in the image of the representation is irreducible. We also show that the domain of φ_ϕ must be a sum of type I_∞ factors if the matrix A describing the dual system satisfies condition I of Cuntz-Krieger.

Square matrices with nonnegative integer entries are naturally associated with unital $*$-endomorphisms of finite direct sums of type I factors (cf. [5]). A natural line of investigation is to determine relationships between the matrices used in the description of the dual systems being represented and the matrices of the endomorphisms associated with the representation. Using bipartite graphs to picture the matrices is a convenient way to describe the relationships that arise.

Section 3 quickly describes the in-split and in-amalgamation graphs of a bipartite graph and introduces a partial order on finite bipartite graphs, or equivalently on square matrices with nonnegative integer entries. This partial order has minimal elements that determine intervals yielding a readily computable equivalence relation on such matrices. It is clear that equivalent matrices are also strong shift equivalent. The section closes with a small technical result showing that two equivalent matrices either both satisfy condition I of Cuntz-Krieger or neither does.

Section 4 includes results about the correspondence described in section 2. The two basic results are: that the endomorphism associated to a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ has the complete in-amalgamation of A as its matrix, and that any finitely embedded endomorphism φ of a finite direct sum of type I factors arises from a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ where A is the complete in-split of the matrix for φ.

Notation. Our Hilbert spaces have an inner product $\langle\ ,\ \rangle$ which is conjugate linear in the first variable and linear in the second variable. A basis of a Hilbert space $\mathcal{E}$ is understood to be an orthonormal basis and $\dim \mathcal{E}$ is the cardinality of such a basis. The identity map on $\mathcal{E}$ is $I_{\mathcal{E}}$, or just I if the context is clear. For a subset E of $\mathcal{E}$, $\operatorname{Sp} E$ or Span E denotes the linear subspace of $\mathcal{E}$ generated by E. If $\mathcal{S}$ is a collection of elements in $\mathcal{B}(\mathcal{H})$, the C^*-algebra of all bounded operators on a Hilbert space $\mathcal{H}$, then $\mathcal{S}'$ denotes the commutant of $\mathcal{S}$ and $C^*(\mathcal{S})$ is the C^*-algebra generated by $\mathcal{S}$.

The graphs we consider are bipartite graphs $\mathcal{G}(V,W)$ with finite initial state set V, finite final state set W and finite edge set $\mathcal{E}$. If e is an edge of $\mathcal{G}(V,W)$ then $i(e)$ denotes its initial vertex, an element of V, while $t(e)$ denotes the terminal vertex of e, an element of W. In order to skirt complications that are not central to the main aspects of this study we restrict attention to those bipartite graphs $\mathcal{G}(V,W)$ which have no stranded vertices; i.e., each element of V is $i(e)$ for some $e \in \mathcal{E}$ and each element of W is $t(e)$ for some $e \in \mathcal{E}$. This restriction is not necessary for our results on in-splits for example, but does play a role in in-amalgamations as described below (cf. [6]).

1. A spatial view of Cuntz-Krieger algebras

In this section a structure analogous to the Hilbert space underlying a Cuntz algebra is developed for Cuntz-Krieger algebras. This gives a new perspective on the structure of Cuntz-Krieger algebras that allows for a simplification of some aspects of these algebras and also further unifies them with the family of Cuntz algebras. In [18] Pimsner has earlier shown how both these families of algebras arise from a common context, namely from a Hilbert bimodule structure. This section develops a common spatial context that is different from that developed in [18], with what seems different applicabilities. However, it seems likely that by altering Pimsner's approach one could include this present framework. For the time being though, we adopt a direct generalization of the spatial view of Cuntz algebras.

In order to clarify the spatial description of Cuntz algebras it is helpful to keep the description of Cuntz-Krieger algebras as Hilbert spaces of isometries in mind. This structure was brought to use by Roberts in [20], and later usefully exploited in studying endomorphisms of $\mathcal{B}(\mathcal{H})$ by Arveson in [1]. Using the terminology of [1], a map $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a Hilbert space $\mathcal{E}$ of (finite) dimension m is a Cuntz system over $\mathcal{E}$ if

a) $\phi(v)^*\phi(w) = \langle v, w\rangle I$, $v, w \in \mathcal{E}$,
b) $\phi(\mathcal{E})\mathcal{H} = \mathcal{H}$.

These properties alone are enough to establish that ϕ is a linear isometry sending unit vectors to partial isometries with initial space $\mathcal{H}$ such that orthonormal vectors are sent to partial isometries with orthogonal ranges. It follows that $\sum \phi(e_j)\phi(e_j)^* = I$ for any basis $\{e_j \mid j = 1, \ldots, m\}$ of $\mathcal{E}$. Cuntz's uniqueness result shows that to any Cuntz system $(\phi, \mathcal{E})$ there is a representation π_ϕ of the Cuntz algebra $\mathcal{O}_m$ on $\mathcal{B}(\mathcal{H})$, where m is the dimension of the Hilbert space $\mathcal{E}$ and $C^*(\phi(\mathcal{E})) = \pi_\phi(\mathcal{O}_m)$.

The corresponding coordinate free approach for Cuntz-Krieger algebras $\mathcal{O}_A$ requires a more intricate framework. There is still a linear contraction $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$

from a Hilbert space $\mathcal{E}$ with $C^*(\phi(\mathcal{E})) = \pi_\phi(\mathcal{O}_A)$ under the usual condition I of Cuntz and Krieger on A, but ϕ will in general no longer send unit vectors to partial isometries. Instead unit vectors correspond to certain linear combinations of partial isometries. Also, a preferred dual set of subspaces of the Hilbert space $\mathcal{E}$ needs to be specified, and this structure must be reflected by the map ϕ. Note that if A is an $m \times m$ matrix of 0's and 1's, then the dimension of $\mathcal{E}$ is m.

We first introduce a key concept.

Definition 1.1. A *coordinate system* $\boldsymbol{P} = \{P_k \mid k \in \Sigma\}$ on a Hilbert space $\mathcal{E}$ is a finite collection of non-zero commuting projections P_k with $\bigvee P_k = I_{\mathcal{E}}$. Denote by $|\boldsymbol{P}|$ the cardinality of the index set Σ $(= \Sigma_{\boldsymbol{P}})$.

A coordinate system $\boldsymbol{P}$ on $\mathcal{B}(\mathcal{E})$ may be viewed as a linear map $\boldsymbol{P} : C(\Sigma_{\boldsymbol{P}}) \to \mathcal{B}(\mathcal{E})$ mapping the minimal projections δ_k $(k \in \Sigma_{\boldsymbol{P}})$, where $\delta_k(j) = \delta_{kj}$, $(j \in \Sigma_{\boldsymbol{P}})$, to non-zero commuting projections P_k in $\mathcal{B}(\mathcal{E})$. Thus the C^*-algebra generated by the image of $\boldsymbol{P}$, denoted $C^*(\boldsymbol{P})$, is abelian. Two coordinate systems $\boldsymbol{P}$, $\boldsymbol{Q}$ are said to commute if Image $\boldsymbol{Q}$ is contained in the commutant $\boldsymbol{P}$ $(C(\Sigma_{\boldsymbol{P}}))'$. In the case that $\dim \mathcal{E}$ is finite we say the coordinate system $\boldsymbol{P}$ is normalized if $|\Sigma_{\boldsymbol{P}}| = \dim \mathcal{E}$.

If $\boldsymbol{P}$ is a coordinate system define an equivalence relation on $\Sigma_{\boldsymbol{P}}$ by setting $k \sim l$ iff $P_k = P_l$. Setting $\psi(\delta_k) = \delta_{\pi(k)}$ for $k \in \Sigma_{\boldsymbol{P}}$ where $\pi_{\boldsymbol{P}} : \Sigma_{\boldsymbol{P}} \to X_{\boldsymbol{P}}$ is the quotient map onto the space $X_{\boldsymbol{P}}$ of equivalence classes defines a conditional expectation $\psi : C(\Sigma_{\boldsymbol{P}}) \to C(X_{\boldsymbol{P}})$. If the context is clear $\pi_{\boldsymbol{P}}$ is written π. We also write $[k]_{\boldsymbol{P}}$ for $\pi_{\boldsymbol{P}}(k)$. For $f = \sum f_k \delta_k$ we have $\psi(f) = \sum\limits_{j \in X} \left(\sum\limits_{k \in \pi^{-1}(j)} f_k \right) \delta_j$. Setting $\boldsymbol{P}_{red} \delta_{\pi(k)} = P_k$ and extending linearly yields a coordinate system $\boldsymbol{P}_{red} : C(X_{\boldsymbol{P}}) \to \mathcal{B}(\mathcal{E})$ with $\boldsymbol{P}_{red} \psi = \boldsymbol{P}$. We refer to $\boldsymbol{P}_{red}$ as the reduced coordinate system associated with $\boldsymbol{P}$. A coordinate system is called *reduced* if $\boldsymbol{P}_{red} = \boldsymbol{P}$, so in other words, if the projections of $\boldsymbol{P}$ are distinct. We also refer to the cardinality of $\pi^{-1}([k])$ as the multiplicity of the projection P_k. It is the number of times each projection P_k occurs in $\boldsymbol{P}$.

An important class of coordinate systems are the orthogonal systems. We say a coordinate system $\boldsymbol{P}$ is *orthogonal* if $P_i P_k = 0$ whenever $P_i \neq P_k$. An intrinsic description follows.

Proposition 1.2. *A coordinate system $\boldsymbol{P} : C(\Sigma_{\boldsymbol{P}}) \to \mathcal{B}(\mathcal{E})$ is orthogonal iff $\boldsymbol{P} = [\boldsymbol{P}] \circ \psi$ where $\psi : C(\Sigma_{\boldsymbol{P}}) \to C(X)$ is a conditional expectation onto a unital subalgebra $C(X)$ of $C(\Sigma_{\boldsymbol{P}})$ and $[\boldsymbol{P}]$ is an injective $*$-homomorphism of $C(X)$ to $\mathcal{B}(\mathcal{E})$.*

Proof. Suppose $\boldsymbol{P} = [\boldsymbol{P}]\psi$. Since $[\boldsymbol{P}]$ is a $*$-homomorphism we have $[\boldsymbol{P}](\psi(\delta_k)) = P_k = P_k^2 = [\boldsymbol{P}](\psi(\delta_k))^2$ and $[\boldsymbol{P}](\psi(\delta_k)) = P_k = P_k^* = [\boldsymbol{P}](\psi(\delta_k))^*$. Thus each $\psi(\delta_k)$ is a projection by the injectivity of $[\boldsymbol{P}]$. Denote by $\pi : \Sigma_{\boldsymbol{P}} \twoheadrightarrow X$ the quotient map dual to the inclusion $i : C(X) \to C(\Sigma_{\boldsymbol{P}})$. For $j \in X$ we have $i(\delta_j) = \delta_j \circ \pi = \sum\limits_{k \in \pi^{-1}(j)} \delta_k$, so $\psi(\delta_k)\delta_j = \psi(\delta_k i(\delta_j)) = \psi\left(\sum\limits_{l \in \pi^{-1}(j)} \delta_k \delta_l \right) = \psi(\delta_k)$ if $\pi(k) = j$ and zero otherwise. Since the only non-zero projection of $C(X)$ supported on j is δ_j we have $\psi(\delta_k) = \delta_{\pi(k)}$, $(k \in \Sigma_{\boldsymbol{P}})$ and so $P_k = [\boldsymbol{P}](\delta_{\pi(k)})$. It follows that $\boldsymbol{P}$ is orthogonal.

Conversely, let $\boldsymbol{P} = [\boldsymbol{P}]\psi$ where $[\boldsymbol{P}] = \boldsymbol{P}_{red}$ is the coordinate system described above and $\psi : C(\Sigma_{\boldsymbol{P}}) \to C(X_{\boldsymbol{P}})$ is the conditional expectation. To check that $[\boldsymbol{P}]$ is a homomorphism it is sufficient to show that $[\boldsymbol{P}](\delta_{\pi(k)}\delta_{\pi(l)}) = [\boldsymbol{P}](\delta_{\pi(k)})[\boldsymbol{P}](\delta_{\pi(l)})$.

However, since $\boldsymbol{P}$ is orthogonal the right side is $P_k P_l = P_k \delta_{\pi(k)\pi(l)}$, which is the left side. It follows that $[\boldsymbol{P}]$ is also injective. $\square$

It is clear that if $\boldsymbol{P}$ is orthogonal, then so is $\boldsymbol{P}_{red}$. It also follows that $\boldsymbol{P}$ is orthogonal if and only if $\boldsymbol{P}_{red}$ is a $*$-homomorphism.

By a *normalized orthogonal* system $\boldsymbol{P}$ we mean that $\boldsymbol{P}$ is not only normalized and orthogonal but in addition that the multiplicity of each element P_k of $\boldsymbol{P}$ is equal to its dimension. Thus rank $P_k = \left|\pi_{\boldsymbol{P}}^{-1}(k)\right|$. A normalized orthogonal system with multiplicity one for each element will also be referred to as a basis system.

Two systems $\boldsymbol{P}$ and $\boldsymbol{Q}$ are called *strongly equivalent* if there is a bijection ν between the index sets of $\Sigma_{\boldsymbol{P}}$ and $\Sigma_{\boldsymbol{Q}}$ with $P_k = Q_{\nu(k)}$, $k \in \Sigma_{\boldsymbol{P}}$, and *equivalent* if there is a unitary U in $\mathcal{B}(\mathcal{E})$ so that the coordinate system $U\boldsymbol{P}U^*$ is strongly equivalent to $\boldsymbol{Q}$.

We first investigate some elementary properties of coordinate systems.

The C^*-algebra $C^*(\boldsymbol{P})$ is finite dimensional, abelian, and contains the identity map on $\mathcal{E}$, $I_{\mathcal{E}}$. Since a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathcal{E})$ has dimension m, the algebra $C^*(\boldsymbol{P})$ has dimension $\leq m$. If $\boldsymbol{P}$ is a reduced orthogonal coordinate system, then the elements P_k of $\boldsymbol{P}$ are the minimal projections in $C^*(\boldsymbol{P})$. In general, if $\boldsymbol{E} = \{E_k \mid k \in \Sigma_E\}$ are the minimal projections in $C^*(\boldsymbol{P})$, then $\boldsymbol{E}$ is a reduced orthogonal coordinate system with $C^*(\boldsymbol{E}) = C^*(\boldsymbol{P})$. We call an orthogonal coordinate system $\boldsymbol{E}$ satisfying $C^*(\boldsymbol{E}) = C^*(\boldsymbol{P})$ an *orthogonal generator* for $\boldsymbol{P}$.

Define a partial order on coordinate systems by setting $\boldsymbol{P} \prec \boldsymbol{Q}$, read as $\boldsymbol{Q}$ is a refinement of $\boldsymbol{P}$, iff $\boldsymbol{P}(C(\Sigma_{\boldsymbol{P}})_+) \subseteq \boldsymbol{Q}(C(\Sigma_{\boldsymbol{Q}})_+)$. Note that if $\boldsymbol{P} \prec \boldsymbol{Q}$, then $\boldsymbol{P}(C(\Sigma_{\boldsymbol{P}})_+) \subseteq \boldsymbol{Q}(C(\Sigma_{\boldsymbol{Q}})) \subseteq \boldsymbol{Q}(C(\Sigma_{\boldsymbol{Q}}))'$, so that $\boldsymbol{P}$ and $\boldsymbol{Q}$ commute. We also have that if each projection P_k is a sum of projections Q_i, then $\boldsymbol{P} \prec \boldsymbol{Q}$. Although $\boldsymbol{P} \prec \boldsymbol{Q}$ implies that $C^*(\boldsymbol{P}) \subseteq C^*(\boldsymbol{Q})$ the converse is not true in general: for example if $\boldsymbol{P}$ consists of two orthogonal projections e and f and $\boldsymbol{Q} = \{e+f, f\}$, then $C^*(\boldsymbol{P}) = C^*(\boldsymbol{Q})$ but $\boldsymbol{Q}$ is not a refinement of $\boldsymbol{P}$. Since there is a conditional expectation ψ with $\boldsymbol{P} = \boldsymbol{P}_{red} \circ \psi$, it follows that $\boldsymbol{P} \prec \boldsymbol{Q}$ if and only if $\boldsymbol{P}_{red} \prec \boldsymbol{Q}_{red}$.

For commuting coordinate systems $\boldsymbol{P}$ and $\boldsymbol{Q}$ on $\mathcal{E}$ we may form the tensor product coordinate system $\boldsymbol{P} \cdot \boldsymbol{Q}$ on $\mathcal{E}$ where $\Sigma_{\boldsymbol{P}} \cdot \boldsymbol{Q} = \Sigma_{\boldsymbol{P}} \times \Sigma_{\boldsymbol{Q}}$ and $P \cdot Q_{(k,j)} = P_k \cdot Q_j$ for $(k,j) \in \Sigma_{\boldsymbol{P}} \times \Sigma_{\boldsymbol{Q}}$. We may also form the disjoint union $\boldsymbol{P} \cup \boldsymbol{Q}$, a coordinate system on $\mathcal{E}$ with index set the disjoint union $\Sigma_{\boldsymbol{P}} + \Sigma_{\boldsymbol{Q}}$.

Proposition 1.3. *If $\boldsymbol{P}$, $\boldsymbol{Q}$ are two coordinate systems on $\mathcal{E}$ that commute, then*

$$C^*(\boldsymbol{P}) \cdot C^*(\boldsymbol{Q}) = C^*(\boldsymbol{P} \cdot \boldsymbol{Q}) = C^*(\boldsymbol{P} \cup \boldsymbol{Q}).$$

Proof. Both $\boldsymbol{P}, \boldsymbol{Q} \prec \boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{P} \cdot \boldsymbol{Q}$, so $C^*(\boldsymbol{P}) \cdot C^*(\boldsymbol{Q}) \subseteq C^*(\boldsymbol{P} \cup \boldsymbol{Q}) \subseteq C^*(\boldsymbol{P} \cdot \boldsymbol{Q})$. Since $\boldsymbol{P} \cdot \boldsymbol{Q} \subseteq C^*(\boldsymbol{P}) \cdot C^*(\boldsymbol{Q})$ we also have $C^*(\boldsymbol{P} \cdot \boldsymbol{Q}) \subseteq C^*(\boldsymbol{P}) \cdot C^*(\boldsymbol{Q})$. $\square$

The partial order relation has stronger implications under orthogonality assumptions.

Proposition 1.4. *If $\boldsymbol{Q}$ is an orthogonal coordinate system, then $\boldsymbol{P} \prec \boldsymbol{Q}$ if and only if each projection of $\boldsymbol{P}$ is a sum of projections from $\boldsymbol{Q}$.*

Proof. It is enough to show this if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are both reduced. Since $\boldsymbol{P} \prec \boldsymbol{Q}$, each projection $P_l \in \boldsymbol{Q}(C(\Sigma)_+)$, so is of the form $\boldsymbol{Q}(\sum \alpha_i \delta_i)$ with $\alpha_i \in \mathbb{R}_+$. Since $\boldsymbol{Q}$ is reduced and orthogonal, it is a $*$-homomorphism and $\sum \alpha_i Q_i$ is a projection, where Q_i are distinct orthogonal projections. Thus α_i is either 0 or 1 for each i. The converse direction was already noted without the orthogonality hypothesis. $\square$

Proposition 1.5. *Let $\boldsymbol{P}$, $\boldsymbol{Q}$ be two commuting coordinate systems. If $\boldsymbol{Q}$ is orthogonal, then $\boldsymbol{P} \prec \boldsymbol{Q}$ if and only if $C^*(\boldsymbol{P}) \subseteq C^*(\boldsymbol{Q})$.*

Proof. It is enough to show that $C^*(\boldsymbol{P}) \subseteq C^*(\mathbf{Q})$ implies that $\boldsymbol{P} \prec \boldsymbol{Q}$. Since each projection P_k of $\boldsymbol{P}$ is a sum of minimal projections in $C^*(\boldsymbol{P})$, and each minimal projection of $C^*(\boldsymbol{P})$ is a projection in $C^*(\boldsymbol{Q})$, we see that P_k is a sum of projections in $C^*(\boldsymbol{Q})$. However $\boldsymbol{Q}_{red}$ is a set of orthogonal minimal projections in $C^*(\boldsymbol{Q})$ corresponding to the points of $X_{\boldsymbol{Q}}$. Thus any projection in $C^*(\boldsymbol{Q})$, and therefore in $\boldsymbol{P}$, is a sum of elements from $\boldsymbol{Q}$. □

Thus, if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are commuting systems that are both orthogonal, then $C^*(\boldsymbol{P}) = C^*(\boldsymbol{Q})$ if and only if $\boldsymbol{P} \prec \boldsymbol{Q}$ and $\boldsymbol{Q} \prec \boldsymbol{P}$.

Proposition 1.6. *If $\boldsymbol{P}$, $\boldsymbol{Q}$ are two commuting orthogonal and reduced coordinate systems, then $C^*(\boldsymbol{P}) = C^*(\boldsymbol{Q})$ if and only if $\boldsymbol{P}$ is strongly equivalent to $\boldsymbol{Q}$.*

Proof. It is enough to show that if $C^*(\boldsymbol{P}) = C^*(\boldsymbol{Q})$, then $\boldsymbol{P}$ is strongly equivalent to $\boldsymbol{Q}$. For each $k \in \Sigma_{\boldsymbol{P}}$, P_k is a minimal projection in $C^*(\boldsymbol{Q})$, since it is minimal in $C^*(\boldsymbol{P})$. Since $\boldsymbol{P}$, $\boldsymbol{Q}$ commute, and since $\boldsymbol{Q}$ is orthogonal and reduced, the projections P_kQ_j $(j \in \Sigma_{\boldsymbol{Q}})$ must all be zero, except for exactly one $j = \varphi(k)$ with $P_kQ_j = P_k$. Since Q_j is minimal in $C^*(\boldsymbol{Q})$, we have $Q_j = P_kQ_j = P_k$ for $j = \varphi(k)$. This defines a bijection φ giving the strong equivalence. □

If $\boldsymbol{E}$ and $\boldsymbol{F}$ are two reduced orthogonal systems generating the same coordinate system $\boldsymbol{P}$ on $\mathcal{E}$, then $C^*(\boldsymbol{E}) = C^*(\boldsymbol{P}) = C^*(\mathbf{F})$ so the previous proposition shows that $\boldsymbol{E}$ and $\boldsymbol{F}$ are strongly equivalent. If $\mathcal{E}$ is finite dimensional and if we normalize $\boldsymbol{E}$ by setting the multiplicity of each member to be its rank in $\mathcal{B}(\mathcal{E})$ we obtain a normalized orthogonal generating system for $\boldsymbol{P}$ which is also unique up to strong equivalence. Notice that the rank m_k of E_k is recoverable from $C^*(\boldsymbol{P})'E_k$, since this is isomorphic to a type I_{m_k} factor ([13]).

To each pair $(\boldsymbol{P}, \boldsymbol{B})$ of commuting coordinate systems with $\boldsymbol{B}$ orthogonal and $\boldsymbol{P} \prec \boldsymbol{B}$ assign a bipartite graph $\mathcal{G}(\Sigma_{\boldsymbol{P}}, \Sigma_{\boldsymbol{B}}) = G(\boldsymbol{P}, \boldsymbol{B})$ with edges e given by $i(e) = k \in \Sigma_{\boldsymbol{P}}$ and $t(e) = j$ in $\Sigma_{\boldsymbol{B}}$ if and only if $B_j \leq P_k$. We may also associate with such a pair $(\boldsymbol{P}, \boldsymbol{B})$ a matrix $A = [\boldsymbol{P}, \boldsymbol{B}]$ which is the transpose of the usual adjacency matrix of the graph $G(\boldsymbol{P}, \boldsymbol{B})$. Thus $[\boldsymbol{P}, \boldsymbol{B}]$ has entries in $\{0, 1\}$, has $\Sigma_{\boldsymbol{B}}$ rows and $\Sigma_{\boldsymbol{P}}$ columns, and $A(j, k) = 1$ if and only if $B_j \leq P_k$. We have $B_jP_k = A(j, k)B_j$ $(j \in \Sigma_{\boldsymbol{B}}, k \in \Sigma_{\boldsymbol{P}})$.

Notice that $P_j = \sum^{\perp} A(i, j)B_i$ and rank $(P_j) = \sum^{\perp} A(i, j)$ rank B_i, where $\sum^{\perp}$ is a sum over distinct orthogonal terms. If $\boldsymbol{B}$ is a normalized system, so that rank B_i is the multiplicity of B_i, then rank $(P_j) = \sum_{i \in \Sigma_{\boldsymbol{B}}} A(i, j)$. Since we assume that $P_k \neq 0$ for all k, each column of $[\boldsymbol{P}, \boldsymbol{B}]$ has at least one nonzero entry. Also since $\bigvee P_k = I$, each row of $[\boldsymbol{P}, \boldsymbol{B}]$ has at least one nonzero term. Conversely, given such a matrix A with entries in $\{0, 1\}$ there is a coordinate system $\boldsymbol{P}$ and an orthogonal system $\boldsymbol{E}$ with $\boldsymbol{P} \prec \boldsymbol{E}$ and $[\boldsymbol{P}, \boldsymbol{E}] = A$. For example let $\boldsymbol{E}$ be any basis system of $\mathcal{E}$, a Hilbert space of dimension equal to the number of rows of A, and set $P_k = \sum A(j, k)E_j$.

If $\boldsymbol{E}, \boldsymbol{B}$ are two commuting orthogonal systems with $\boldsymbol{E} \prec \boldsymbol{B}$, or equivalently with $C^*(\boldsymbol{E}) \subseteq C^*(\boldsymbol{B})$ then since any two elements of $\boldsymbol{E}$ are either equal or orthogonal, the matrix $D = [\boldsymbol{E}, \boldsymbol{B}]$ satisfies the following: if two columns of D share a 1 in the same row, the columns must be equal. Thus if $\boldsymbol{E}$ is a reduced orthogonal system there is exactly one 1 in each row.

For the next proposition notice that one may obtain a groupoid structure on bipartite graphs by composing. Thus $\mathcal{G}(V,W)\mathcal{G}(W,Y)$ is a bipartite graph $\mathcal{G}(V,Y)$ where the edges are paths ef, where e, f are edges in $\mathcal{G}(V,W)$, $\mathcal{G}(W,Y)$ respectively such that $t(e) = i(f)$. If A_{VW} denotes the matrix associated with $\mathcal{G}(V,W)$, then $A_{WY}A_{VW}$ is the matrix associated with $\mathcal{G}(V,W)\mathcal{G}(W,Y)$

Proposition 1.7. *Let $\boldsymbol{P}$ be a coordinate system with $\boldsymbol{E}$ a reduced orthogonal system such that $\boldsymbol{P} \prec \boldsymbol{E}$. If $\boldsymbol{B}$ is an orthogonal system with $\boldsymbol{E} \prec \boldsymbol{B}$, then $G(\boldsymbol{P},\boldsymbol{B}) = G(\boldsymbol{P},\boldsymbol{E})G(\boldsymbol{E},\boldsymbol{B})$ and $[\boldsymbol{P},\boldsymbol{B}] = [\boldsymbol{E},\boldsymbol{B}][\boldsymbol{P},\boldsymbol{E}]$.*

Proof. Since $\boldsymbol{E}$ is reduced, there is exactly one 1 in each row of $[\boldsymbol{E},\boldsymbol{B}]$. Thus the (i,j) entry of the product $[\boldsymbol{E},\boldsymbol{B}][\boldsymbol{P},\boldsymbol{E}] = \sum[\boldsymbol{E},\boldsymbol{B}]_{(i,k)}[\boldsymbol{P},\boldsymbol{E}]_{(k,j)} = 1$ iff there is a k such that $B_i \leq E_k$ and $E_k \leq P_j$, which is equivalent to $B_i \leq P_j$, i.e., $[\boldsymbol{P},\boldsymbol{B}]_{(i,j)} = 1$. $\square$

Example 1.8. If E is not reduced this can easily fail. Take $P_1 = I$, $P_2 = \mathrm{Sp}\{e_3\}$ on a three dimensional Hilbert space $\mathcal{E}$ with basis $\{e_1, e_2, e_3\}$, $E_1 = E_2 = \mathrm{Sp}\{e_1, e_2\}$, $E_3 = P_2$, and $B_i = \mathrm{Sp}\{e_i\}$. Then $[P,E] = [P,B] \neq [E,B][P,E]$.

Remark 1.9. If $\boldsymbol{E}$ is the reduced orthogonal system for $\boldsymbol{P}$, and $\boldsymbol{B}$ is an orthogonal system with $\boldsymbol{P} \prec \boldsymbol{B}$, then $C^*(\boldsymbol{E}) = C^*(\boldsymbol{P}) \subseteq C^*(\boldsymbol{B})$ and therefore by Proposition 1.5, $\boldsymbol{E} \prec \boldsymbol{B}$. Thus the hypotheses of Proposition 1.7 are fulfilled under these assumptions also.

For the following we assume $\mathcal{E}$ is finite dimensional.

Definition 1.10. For $\boldsymbol{E}$ and $\boldsymbol{B}$ normalized coordinate systems with $\boldsymbol{E} \prec \boldsymbol{B}$ we say $\boldsymbol{E}$ and $\boldsymbol{B}$ are *aligned* if $\boldsymbol{E} - \boldsymbol{B}$ is a positive map from $C(\Sigma)$ to $\mathcal{B}(\mathcal{E})$.

This just ensures that $B_k \leq E_k$ for all $k \in \Sigma$.

Lemma 1.11. *If $\boldsymbol{E}, \boldsymbol{B}$ are normalized orthogonal coordinate systems on $\mathcal{E}$ with $\boldsymbol{E} \prec \boldsymbol{B}$, then there is a system $\boldsymbol{B}'$ strongly equivalent to $\boldsymbol{B}$ so that $\boldsymbol{E}$ and $\boldsymbol{B}'$ are aligned. Equivalently, there is a system $\boldsymbol{E}'$ strongly equivalent to $\boldsymbol{E}$ with $\boldsymbol{E}'$ and $\boldsymbol{B}$ aligned.*

Proof. Let $\varphi : X_{\boldsymbol{B}} \to X_{\boldsymbol{E}}$ be the onto map defined by $\varphi([j]_B) = [i]_E$ if and only if $B_j \leq E_i$. Since each E_i is a sum of elements from $\boldsymbol{B}$, $\left|\pi_E^{-1}[i]\right| = \text{rank } E_i = \sum_{B_j \leq E_i}^{\perp} \text{rank } B_j = \sum\limits_{[j]\in\varphi^{-1}[i]} \left|\pi_B^{-1}[j]\right|$ and $\Sigma = \bigcup\limits_{[i]\in X_{\boldsymbol{E}}} \pi_E^{-1}[i] = \bigcup\limits_{[i]\in X_{\boldsymbol{E}}} \bigcup\limits_{[j]\in\varphi_{[i]}^{-1}} \pi_B^{-1}[j]$ where these are disjoint unions. Thus there is a permutation σ of Σ with $\sigma\left(\pi_E^{-1}[i]\right) = \bigcup\limits_{[j]\in\varphi^{-1}[i]} \pi_B^{-1}[j]$. Since $k \in \pi_E^{-1}[k]$ we have $\sigma(k) \in \pi_B^{-1}[j]$ for some $[j] \in \varphi^{-1}[k]$ and so $\pi_B(\sigma(k)) \in \varphi^{-1}([k]_E)$. Thus $\varphi[\sigma(k)]_B = [k]_E$, which is equivalent to $B_{\sigma(k)} \leq E_k$. Setting $B'_k = B_{\sigma(k)}$ yields a system $\boldsymbol{B}'$ aligned with $\boldsymbol{E}$, or letting $E'_k = E_{\sigma^{-1}(k)}$ yields a system $\boldsymbol{E}'$ aligned with $\boldsymbol{B}$. $\square$

Proposition 1.12. *Let $\boldsymbol{P}$ be a coordinate system, $\boldsymbol{E}$ and $\boldsymbol{B}$ normalized orthogonal systems with $\boldsymbol{P} \prec \boldsymbol{E} \prec \boldsymbol{B}$. If $\boldsymbol{E}$ and $\boldsymbol{B}$ are aligned then $[\boldsymbol{P},\boldsymbol{E}] = [\boldsymbol{P},\boldsymbol{B}]$ and $G(\boldsymbol{P},\boldsymbol{E}) = G(\boldsymbol{P},\boldsymbol{B})$.*

Proof. Since $[\boldsymbol{P},\boldsymbol{E}](k,j) = 1$ if and only if $E_k \leq P_j$, and $[\boldsymbol{P},\boldsymbol{B}](k,j) = 1$ if and only if $B_k \leq P_j$, it is enough to note that $E_k \leq P_j$ if and only if $B_k \leq P_j$ for the aligned pair $\boldsymbol{E}, \boldsymbol{B}$. $\square$

Lemma 1.13. *Let $\boldsymbol{P}$ be a coordinate system, $\boldsymbol{E}$ an orthogonal system with $\boldsymbol{P} \prec \boldsymbol{E}$ and $\boldsymbol{E}'$ the orthogonal system strongly equivalent to $\boldsymbol{E}$ defined by $E'_k = E_{\sigma(k)}$ for a permutation σ of $\Sigma_{\boldsymbol{E}} = \{1, \dots, |\boldsymbol{E}|\}$. Then $[\boldsymbol{P}, \boldsymbol{E}'](k,l) = [\boldsymbol{P}, \boldsymbol{E}](\sigma(k), l)$.*

Proof. It is enough to note that $E'_k \leq P_l$ if and only if $E_{\sigma(k)} \leq P_l$. □

We see now what the possible set of matrices $[\boldsymbol{P}, \boldsymbol{E}]$ is for a given fixed coordinate system $\boldsymbol{P}$ and $\boldsymbol{E}$ any normalized orthogonal system with $\boldsymbol{P} \prec \boldsymbol{E}$. The previous results show that, up to a permutation of the rows, there is only one possible matrix. For if $\boldsymbol{B}$ and $\boldsymbol{B}'$ are two normalized orthogonal systems with $\boldsymbol{P} \prec \boldsymbol{B}$ and $\boldsymbol{P} \prec \boldsymbol{B}'$, by Proposition 1.5 we have $\boldsymbol{F} \prec \boldsymbol{B}$ and $\boldsymbol{F} \prec \boldsymbol{B}'$ where $\boldsymbol{F}$ is any (normalized) orthogonal generator for $\boldsymbol{P}$. There are, by Lemma 1.11, two normalized orthogonal systems $\boldsymbol{E}$ and $\boldsymbol{E}'$ strongly equivalent to $\boldsymbol{F}$ such that $\boldsymbol{E}$ and $\boldsymbol{B}$ are aligned and $\boldsymbol{E}'$, $\boldsymbol{B}'$ are aligned. We have by Proposition 1.12 $[\boldsymbol{P}, \boldsymbol{B}] = [\boldsymbol{P}, \boldsymbol{E}]$ and $[\boldsymbol{P}, \boldsymbol{B}'] = [\boldsymbol{P}, \boldsymbol{E}']$. Now $\boldsymbol{E}$ and $\boldsymbol{E}'$ are strongly equivalent so by Lemma 1.13 the matrices $[\boldsymbol{P}, \boldsymbol{E}]$ and $[\boldsymbol{P}, \boldsymbol{E}']$ are equal up to a permutation of the rows.

Proposition 1.14. *Let $\boldsymbol{P}$ be a normalized coordinate system. Given any normalized orthogonal system $\boldsymbol{E}$ with $\boldsymbol{P} \prec \boldsymbol{E}$ there is a normalized coordinate system $\boldsymbol{Q}$ commuting with $\boldsymbol{P}$ such that*

1. $[\boldsymbol{P}, \boldsymbol{B}]^T = [\boldsymbol{Q}, \boldsymbol{B}]$,
2. $[\boldsymbol{P}, \boldsymbol{B}] = [\boldsymbol{P}, \boldsymbol{E}]$

for some orthogonal coordinate system $\boldsymbol{B}$ with $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}$.

Proof. Set $A = [\boldsymbol{P}, \boldsymbol{E}]$ and choose a basis $\{e_k\}$ of $\mathcal{E}$ with $e_k \in E_k$. If $\boldsymbol{B}_k = \mathrm{Sp}\{e_k\}$ then $(\boldsymbol{E}, \boldsymbol{B})$ is an aligned pair and $[\boldsymbol{P}, \boldsymbol{B}] = [\boldsymbol{P}, \boldsymbol{E}] = A$. Define $\boldsymbol{Q}$ by $Q_k = \sum A(k,i) B_i$, so $[\boldsymbol{Q}, \boldsymbol{B}] = A^T, \boldsymbol{Q}$ commutes with $\boldsymbol{P}$ and $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}$. □

Note that any orthogonal system $\boldsymbol{B}'$ with $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}'$ and $(\boldsymbol{B}', \boldsymbol{B})$ aligned will satisfy the above properties for $\boldsymbol{B}$ also.

Definition 1.15. Let $\boldsymbol{P}$ be a normalized coordinate system on $\mathcal{E}$ and $\boldsymbol{E}$ a given normalized orthogonal system with $\boldsymbol{P} \prec \boldsymbol{E}$. With $A = [\boldsymbol{P}, \boldsymbol{E}]$ and $\boldsymbol{Q}$ a normalized coordinate system on $\mathcal{E}$ commuting with $\boldsymbol{P}$ and satisfying $[\boldsymbol{P}, \boldsymbol{B}] = [\boldsymbol{Q}, \boldsymbol{B}]^T = A$ for some orthogonal system $\boldsymbol{B}$ with $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}$ we say $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a *dual system*, or that $\boldsymbol{P}, \boldsymbol{Q}$ are dual via A. Such a coordinate system $\boldsymbol{B}$ is said to implement the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$.

Note that $\boldsymbol{E}$ may be chosen to be the normalized orthogonal generator for $\boldsymbol{P}$ with $[\boldsymbol{P}, \boldsymbol{E}] = A$. Proposition 1.14 shows that given $\boldsymbol{P}$ and $[\boldsymbol{P}, \boldsymbol{E}] = A$ there is always a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. If $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system, then so is $(\boldsymbol{Q}, A^T, \boldsymbol{P})$.

The following proposition shows that all dual systems on $\mathcal{E}$ given by a matrix A are related by an automorphism of $\mathcal{B}(\mathcal{E})$.

Proposition 1.16. *Let $\boldsymbol{P}$ be a normalized coordinate system and $A = [\boldsymbol{P}, \boldsymbol{E}]$ for $\boldsymbol{E}$ a normalized orthogonal system with $\boldsymbol{P} \prec \boldsymbol{E}$. If $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system, then $(\boldsymbol{P}', A, \boldsymbol{Q}')$ is also a dual system if and only if there is a unitary U on $\mathcal{E}$ with $\boldsymbol{Q}' = U\boldsymbol{Q}U^*$ and $\boldsymbol{P}' = U\boldsymbol{P}U^*$.*

Proof. Since $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system there is a normalized orthogonal system $\boldsymbol{B}$ with $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}$ and $[\boldsymbol{P}, \boldsymbol{B}] = A = [\boldsymbol{Q}, \boldsymbol{B}]^T$.

Assume there is a unitary U on $\mathcal{E}$ with $\boldsymbol{Q}' = U\boldsymbol{Q}U^*$ and $\boldsymbol{P}' = U\boldsymbol{P}U^*$. Setting $\boldsymbol{B}' = U\boldsymbol{B}U^*$ we have $\boldsymbol{B}'$ is a normalized orthogonal system with $\boldsymbol{P}' \cup \boldsymbol{Q}' \prec \boldsymbol{B}'$.

Since $[\boldsymbol{P}', \boldsymbol{B}'](i,k) = 1$ iff $B'_i \leq P'_k$ which is equivalent to $B_i \leq P_k$, i.e., $A(i,k) = 1$, we have $[\boldsymbol{P}', \boldsymbol{B}'] = A$. Similarly $[\boldsymbol{Q}', \boldsymbol{B}'] = [\boldsymbol{Q}, \boldsymbol{B}]$, so $(\boldsymbol{P}', A, \boldsymbol{Q}')$ is a dual system.

Conversely suppose $(\boldsymbol{P}', A, \boldsymbol{Q}')$ is a dual system implemented by an orthogonal coordinate system $\boldsymbol{B}'$. Choose two orthogonal coordinate systems of rank one projections $\boldsymbol{E}, \boldsymbol{E}'$ aligned with $\boldsymbol{B}$ and $\boldsymbol{B}'$ respectively and define a unitary U on $\mathcal{E}$ with $UE_kU^* = E'_k$. Then $[\boldsymbol{P}, \boldsymbol{E}] = [\boldsymbol{P}, \boldsymbol{B}] = A = [\boldsymbol{P}', \boldsymbol{B}'] = [\boldsymbol{P}', \boldsymbol{E}']$ and $[\boldsymbol{Q}, \boldsymbol{E}] = [\boldsymbol{Q}, \boldsymbol{B}] = A^T = [\boldsymbol{Q}', \boldsymbol{B}'] = [\boldsymbol{Q}', \boldsymbol{E}']$. One can check that $P'_k = \sum A(i,k)E'_i = \sum A(i,k)UE_iU^* = U(\sum A(i,k)E_i)U^* = UP_kU^*$, and similarly $Q'_k = UQ_kU^*$. □

Corollary 1.17. *If $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system, then $(\boldsymbol{P}, A, \boldsymbol{Q}')$ is a dual system if and only if there is a unitary $U \in C^*(\boldsymbol{P})'$ with $\boldsymbol{Q}' = U\boldsymbol{Q}U^*$.*

Example 1.18. Let $\mathcal{E} = \mathbb{C}^2$ with basis $\{e_1, e_2\}$. Set $P_1 = I$, $P_2 = \mathbb{C}e_2$ and $E_i = \mathbb{C}e_i$. Then $[\boldsymbol{P}, \boldsymbol{E}] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $Q_1 = \mathbb{C}\,e_1$, $Q_2 = I$ defines the unique system $\boldsymbol{Q}$ with $[\boldsymbol{Q}, \boldsymbol{E}] = [\boldsymbol{P}, \boldsymbol{E}]^T$, since any unitary U commuting with the system $\boldsymbol{P}$, or equivalently with $\boldsymbol{E}$, must be diagonal, and so $\boldsymbol{Q} = U\boldsymbol{Q}U^*$. If we had chosen $\boldsymbol{E}'$ by $E'_1 = \mathbb{C}\,e_2$ and $E'_2 = \mathbb{C}\,e_1$ then $[\boldsymbol{P}, \boldsymbol{E}'] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{P}$ is self-dual, i.e., $\boldsymbol{P}$ is the unique system $\boldsymbol{P}$ with $[\boldsymbol{P}, \boldsymbol{E}'] = [\boldsymbol{P}, \boldsymbol{E}']^T$.

Definition 1.19. A *symmetry* of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a unitary in $C^*(\boldsymbol{P})' \cap C^*(\boldsymbol{Q})'$.

Proposition 1.20. *Let $(\boldsymbol{P}, A, \boldsymbol{Q})$ be a dual system of coordinates on $\mathcal{E}$ and $\boldsymbol{B}$ an orthogonal coordinate system implementing the dual system. Another orthogonal coordinate system $\boldsymbol{B}'$, implements $(\boldsymbol{P}, A, \boldsymbol{Q})$ if and only if $\boldsymbol{B}' = U\boldsymbol{B}U^*$ for a symmetry U of $(\boldsymbol{P}, A, \boldsymbol{Q})$.*

Proof. If U is a symmetry of $(\boldsymbol{P}, A, \boldsymbol{Q})$, then $\boldsymbol{B}' = U\boldsymbol{B}U^*$ is a normalized orthogonal system with $\boldsymbol{P} \cup \boldsymbol{Q} \prec \boldsymbol{B}'$. Arguing as in Proposition 1.16 we see that $[\boldsymbol{P}, \boldsymbol{B}'] = [\boldsymbol{P}, \boldsymbol{B}]$ and $[\boldsymbol{Q}, \boldsymbol{B}'] = [\boldsymbol{Q}, \boldsymbol{B}]$. In the converse direction apply Proposition 1.16 for the case $\boldsymbol{Q}' = \boldsymbol{Q}$, to obtain a symmetry U. □

Note that it is certainly possible for two coordinate systems $(\boldsymbol{P}, \boldsymbol{Q})$ to be dual via different matrices A and A'. By preceding remarks the matrices must however be equal up to a permutation of the rows. If $\boldsymbol{B}$ and $\boldsymbol{B}'$ are normalized orthogonal systems implementing $(\boldsymbol{P}, A, \boldsymbol{Q})$ and $(\boldsymbol{P}, A', \boldsymbol{Q})$ respectively, then by Proposition 1.20 this is equivalent to $\boldsymbol{B}'$ not being conjugate to $\boldsymbol{B}$ via a symmetry of $(\boldsymbol{P}, \boldsymbol{Q})$. In the following examples we let $\boldsymbol{B}$ be a basis of $\mathcal{E}$ and set $B'_k = B_{\sigma(k)}$ for σ a permutation of $\{1, \dots, m\}$.

Set A to be the matrix $[\boldsymbol{P}, \boldsymbol{B}]$ and $A' = [\boldsymbol{P}, \boldsymbol{B}']$. We have $Q_k = \sum[\boldsymbol{Q}, \boldsymbol{B}](i,k)B_i = \sum A(k,i)B_i$ and also

$$\begin{aligned} Q_k &= \sum[\boldsymbol{Q}, \boldsymbol{B}'](i,k)B'_i = \sum[\boldsymbol{P}, \boldsymbol{B}']^T(i,k)B_{\sigma(i)} = \sum[\boldsymbol{P}, \boldsymbol{B}'](k,i)B_{\sigma(i)} \\ &= \sum[\boldsymbol{P}, \boldsymbol{B}](\sigma(k), i)B_{\sigma(i)} = \sum A(\sigma(k), \sigma^{-1}(i))B_i. \end{aligned}$$

Thus $A(k,i) = A(\sigma(k), \sigma^{-1}(i))$ is needed in order to have $\boldsymbol{P}$ dual to $\boldsymbol{Q}$ via both $\boldsymbol{B}$ and $\boldsymbol{B}'$. For a trivial example, one can set $\boldsymbol{P} = \boldsymbol{B}$ so $A = Id$ and $\boldsymbol{Q} = \boldsymbol{P}$, i.e., $\boldsymbol{P}$ is self-dual. Then if $\sigma^2 = Id$, $A(k,i) = A(\sigma(k), \sigma^{-1}(i)) = Id$, and $\boldsymbol{P}$ is also self-dual via $\boldsymbol{B}'$. Note that any symmetry U must be a diagonal unitary, so $\boldsymbol{B}' \neq U\boldsymbol{B}U^*$ for any such U. For a second example, let $\mathcal{E} = \mathbb{C}^3$ with basis $\{e_1, e_2, e_3\}$ and set $B_i =$

$\mathbb{C}e_i$. Let $P_1 = \mathrm{Sp}\{e_1, e_3\}$, $P_2 = \mathrm{Sp}\{e_2, e_3\}$, $P_3 = \mathrm{Sp}\{e_1, e_2\}$, and $\sigma = (1\ 2\ 3)$. Again, any unitary commuting with $\boldsymbol{P}$, in particular any symmetry, must be a diagonal unitary. Then $[\boldsymbol{P}, \boldsymbol{B}] = A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ satisfies $A(k,i) = A(\sigma(k), \sigma^{-1}(i))$, $[\boldsymbol{P}, \boldsymbol{B}'] \neq A$ and $\boldsymbol{B}'$ is not conjugate to $\boldsymbol{B}$ via a symmetry. Here $\boldsymbol{P}$ is again self-dual.

To obtain a non-self-dual example in $\mathcal{E} = \mathbb{C}^3$, set $P_1 = \mathrm{Sp}\{e_1, e_3\}$, $P_2 = \mathrm{Sp}\{e_2, e_3\}$, $P_3 = \mathrm{Sp}\{e_3\}$ and let σ be the transposition $(1,2)$. The matrix $[\boldsymbol{P}, \boldsymbol{B}] = A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ satisfies $A(k,i) = A(\sigma(k), \sigma^{-1}(i))$ and $Q_1 = \mathrm{Sp}\{e_1\}$, $Q_2 = \mathrm{Sp}\{e_2\}$, $Q_3 = I$ defines the dual system $\boldsymbol{Q}$. The system $\boldsymbol{B}'$ is not conjugate to $\boldsymbol{B}$ via a symmetry, since a symmetry must again be a diagonal unitary in this example.

Definition 1.21. For $\boldsymbol{P}$ a coordinate system on $\mathcal{E}$, a map $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ is a *display* of $\boldsymbol{P}$ if there is a reduced orthogonal coordinate system $\rho : C(\Sigma_{\boldsymbol{P}}) \to \mathcal{B}(\mathcal{H})$ such that

1. $\phi(y)^*\phi(z)\rho(k) = \langle y, \boldsymbol{P}(k)z\rangle \rho(k)$, $k \in C(\Sigma_{\boldsymbol{P}})$; $y, z \in \mathcal{E}$,
2. $\overline{\phi(\mathcal{E})\mathcal{H}} = \mathcal{H}$.

Of course condition 1 may be restated in terms of the projections ρ_k and P_k as $\phi(y)^*\phi(z)\rho_k = \langle y, P_k z\rangle \rho_k$, $k \in \Sigma_{\boldsymbol{P}}$. We also say $(\phi, \rho, \mathcal{H})$ or just (ϕ, ρ) is a display of $\boldsymbol{P}$. We shall see that the usual facts true for Cuntz systems over $\mathcal{E}$ also hold in this setting.

Proposition 1.22. *If (ϕ, ρ) is a display of $\boldsymbol{P}$, then ϕ is a faithful norm decreasing linear map.*

Proof. To see that ϕ is linear, we compute for example that

$$[\phi(\lambda y + z)^* - \lambda\phi(y)^* - \phi(z)^*][\phi(\lambda y + z) - \lambda\phi(y) - \phi(z)]\rho_k = 0 \text{ for all } k.$$

Summing over k and using $\sum \rho_k = I_{\mathcal{H}}$ we have

$$(\phi(y+z) - \phi(y) - \phi(z))^*(\phi(y+z) - \phi(y) - \phi(z)) = 0.$$

Also $\phi(y)^*\phi(y)\rho_k = \langle y, P_k y\rangle \rho_k \leq \|y\|^2 \rho_k$. Summing over k we have $\phi(y)^*\phi(y) \leq \|y\|^2$ and thus $\|\phi\| \leq 1$.

To see that ϕ is faithful note that since $\bigvee P_k = I$, for any $y \in \mathcal{E}$ there is a k with $P_k y \neq 0$. Thus $\phi(y)^*\phi(y)\rho_k = \langle y, P_k y\rangle \rho_k = \|P_k y\|^2 \rho_k \neq 0$. □

Let $(\phi, \rho, \mathcal{H})$ be a display of a coordinate system $\boldsymbol{P}$ on $\mathcal{E}$. If $\boldsymbol{E}$ is an orthogonal system with $\boldsymbol{P} \prec \boldsymbol{E}$ and v a unit vector in E_s, then $\phi(v)^*\phi(v)\rho_k = \langle v, P_k v\rangle \rho_k = A(s,k)\|v\|^2 \rho_k = A(s,k)\rho_k$, where $A = [\boldsymbol{P}, \boldsymbol{E}]$. Thus $\phi(v)^*\phi(v) = \sum A(s,k)\rho_k$, a projection independent of the given unit vector v in E_s. Thus, if U is a unitary in $C^*(\boldsymbol{P})'$, then $A = [\boldsymbol{P}, \boldsymbol{E}']$ where $E'_s = UE_sU^*$ and $\phi(v)^*\phi(v) = \phi(w)^*\phi(w)$ for unit vectors $w \in E'_s$, $v \in E_s$.

Proposition 1.23. *Let $(\phi, \rho, \mathcal{H})$ be a display of a coordinate system $\boldsymbol{P}$ and $\boldsymbol{E}$ an orthogonal coordinate system with $\boldsymbol{P} \prec \boldsymbol{E}$. Then $\phi(v)$ is a partial isometry for any unit vector v contained in an element of $\boldsymbol{E}$. Its initial projection is independent of the particular v chosen in a fixed element of $\boldsymbol{E}$. If v, w are orthogonal unit vectors,* *each contained in some element of $\boldsymbol{E}$, then $\phi(v)$ and $\phi(w)$ have orthogonal final projections $\phi(v)\phi(v)^*$ and $\phi(w)\phi(w)^*$.*

Proof. For $v \in E_s$, $w \in E_r$ orthogonal unit vectors in $\mathcal{E}$, it remains to show that $\phi(v)^*\phi(w) = 0$. We have

$$\phi(v)^*\phi(w)\rho_k = \langle v, P_k w\rangle\rho_k = \begin{cases} \langle v, w\rangle\rho_k & \text{if } P_k w = w \\ 0 & \text{if } P_k w = 0 \end{cases} = 0$$

for any k, so summing over k yields the result. □

Remark 1.24. In particular if $\{v_i \mid i = 1, \ldots, m\}$ is a basis of $\mathcal{E}$ so that for each k, $v_k \in E_s$ for some s, then $\phi(v_k)\phi(v_k)^* = \mathrm{p}_k$ is an orthogonal family of projections in $B(\mathcal{H})$. We also have $\sum \mathrm{p}_k = I_{\mathcal{H}}$. Indeed,

$$\begin{aligned} \mathcal{H} &= \phi(\mathcal{E})\mathcal{H} = \{\sum \alpha_k\phi(v_k)\mathcal{H} \mid \alpha_k \in \mathbb{C}\} \\ &= \sum \phi(v_k)\mathcal{H} = \sum \phi(v_k)\phi(v_k)^*\phi(v_k)\mathcal{H} \subseteq \sum \phi(v_k)\phi(v_k)^*\mathcal{H} \\ &= \sum \mathrm{p}_k\mathcal{H}. \end{aligned}$$

Thus $\mathbf{p} = \{\mathrm{p}_k \mid k = 1, \ldots, m\}$ is a reduced orthogonal coordinate system on $\mathcal{H}$ associated to the display ϕ and such a basis of $\mathcal{E}$.

Remark 1.25. Let $(\boldsymbol{P}, A, \boldsymbol{Q})$ be a dual system and $(\phi, \rho, \mathcal{H})$ a display of $\boldsymbol{P}$. We associate with the dual system and the display ϕ a family of projections q_l, $l \in \{1, \ldots, m\}$ on $\mathcal{H}$, namely

$$\mathrm{q}_l = \sum A(l,k)\rho_k = \phi(v)^*\phi(v)$$

where $v \in B_l$ is any unit vector and $\boldsymbol{B} = \{B_l \mid l = 1, \ldots, m\}$ is any orthogonal system implementing $(\boldsymbol{P}, A, \boldsymbol{Q}$.) Since A is a matrix with a nonzero entry in each row and column, we see that $\bigvee \mathrm{q}_k = I_{\mathcal{H}}$, so $\mathbf{q} = \{\mathrm{q}_k \mid k = 1, \ldots, m\}$ is also a coordinate system on $\mathcal{H}$.

It is relatively straightforward to construct displays $(\phi, \rho, \mathcal{H})$ of a coordinate system $\boldsymbol{P}$. Choose any normalized orthogonal coordinate system $\boldsymbol{E}$ with $\boldsymbol{P} \prec \boldsymbol{E}$ and $\{e_i \mid i = 1, \ldots, m\}$ a basis of $\mathcal{E}$ with $e_i \in E_i$. Set $A = [\boldsymbol{P}, \boldsymbol{E}]$. Choose $\mathcal{H}$ a Hilbert space and a family ρ_k of orthogonal projections with sum $I_{\mathcal{H}}$ and partial isometries v_i with initial space $\mathrm{q}_k = \sum A(k,i)\rho_i$ and orthogonal range spaces $v_iv_i^*$ with $\sum v_iv_i^* = I_{\mathcal{H}}$. The form of the matrix A may of course force the rank of some of the ρ_i, and therefore also of $\mathcal{H}$, to be infinite.

Define $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ by $\phi(e_i) = v_i$ and extend linearly. Then $(\phi, \rho, \mathcal{H})$ is a display of $\boldsymbol{P}$. To check this it is enough to see that $\phi(e_j)^*\phi(e_l)\rho_k = \langle e_j, P_k e_l\rangle\rho_k$ for all j, k, l. However,the left side is $\delta_{jl}\mathrm{q}_l\rho_k = \delta_{jl}A(l,k)\rho_k$, while the right side is equal to $\langle e_j, A(l,k)e_l\rangle\rho_k = \delta_{jl}A(l,k)\rho_k$. Thus $(\phi, \rho, \mathcal{H})$ is a display of $\boldsymbol{P}$.

Definition 1.26. If $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system and $(\phi, \rho, \mathcal{H})$ is a display of $\boldsymbol{P}$, we say ϕ is a *representation of the dual system* $(\boldsymbol{P}, A, \boldsymbol{Q})$ if

$$\phi(Q_k\mathcal{E})\mathcal{H} = \mathrm{q}_k\mathcal{H} \qquad \forall k,$$

where $\mathrm{q}_k = \sum A(k,l)\rho_l$ is the family of projections on $\mathcal{H}$ associated with the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ described in Remark 1.25. The display ϕ of $\boldsymbol{P}$ is *Cuntz-Krieger realizable* if there is a system $\boldsymbol{Q}$ dual to $\boldsymbol{P}$ so that ϕ is a representation of the dual pair $(\boldsymbol{P}, \boldsymbol{Q})$, i.e., of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. We say ϕ is *infinite* if the projections

q_k are infinite in $\mathcal{B}(\mathcal{H})$ for all k. In this case the projections in any coordinate system $\mathbf{p}$ on $\mathcal{H}$ associated to ϕ as in Remark 1.24 are also all infinite.

Lemma 1.27. *Let $(\phi, \rho, \mathcal{H})$ be a display of a normalized coordinate system $\boldsymbol{P}$ on $\mathcal{E}$ that is also a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. Then $\phi(e_k)^*\phi(e_k) = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*$ for any basis $\{e_k \mid k = 1, \ldots, m\}$ of $\mathcal{E}$ with $e_k \in B_k$, where $\boldsymbol{B}$ is any orthogonal system implementing the duality $(\boldsymbol{P}, A, \boldsymbol{Q})$.*

Proof. Since ϕ is a representation of the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ we have that $\phi(Q_k\mathcal{E})\mathcal{H} = \mathrm{q}_k\mathcal{H}$ where $\mathrm{q}_k = \phi(e_k)^*\phi(e_k) = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\rho_i$. Thus

$$\begin{aligned}
\mathrm{q}_k\mathcal{H} = \phi(Q_k)\mathcal{H} &= \{\sum \alpha_i[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\mathcal{H} \mid \alpha_i \in \mathbb{C}\} \\
&= \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\mathcal{H} = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*\phi(e_i)\mathcal{H} \\
&\subseteq \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*\mathcal{H} \\
&\subseteq \sum[\boldsymbol{P}\boldsymbol{B}](k,i)\phi(e_i)\mathcal{H} = \phi(Q_k\mathcal{E})\mathcal{H} = \mathrm{q}_k\mathcal{H}.
\end{aligned}$$

By Remark 1.24 we know $\sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*$ is a projection, so it must then be the projection q_k. ☐

Theorem 1.28. *Let $(\phi, \rho, \mathcal{H})$ be a display of a normalized coordinate system $\boldsymbol{P}$ on $\mathcal{E}$. Let $(\boldsymbol{P}, A, \boldsymbol{Q})$ be a dual system, $\{e_k \mid k = 1, \ldots, m\}$ a basis of $\mathcal{E}$ with $e_k \in B_k$ where $\boldsymbol{B}$ implements $(\boldsymbol{P}, A, \boldsymbol{Q})$, and $\mathrm{p}_k = \phi(e_k)\phi(e_k)^*$ the orthogonal coordinate system $\mathbf{p}$ of $\mathcal{H}$ associated with ϕ and this basis, as in Remark 1.24. The following are equivalent:*

a) *ϕ is a representation of the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$.*
b) *$\sum A(k,i)\mathrm{p}_i = \sum A(k,i)\rho_i$ for each k.*
c) *$(\phi, \mathbf{p}, \mathcal{H})$ is a display of $\boldsymbol{P}$.*

Proof. a) $\Rightarrow$ b): By the comments after Proposition 1.22

$$\phi(e_k)^*\phi(e_k) = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\rho_i.$$

Lemma 1.27 finishes the claim.

b) $\Rightarrow$ c): Remark 1.24 shows $\sum \mathrm{p}_k = I_{\mathcal{H}}$, so it remains to show that $\phi(y)^*\phi(z)\mathrm{p}_k = \langle y, P_n z\rangle \mathrm{p}_k$ for $y, z \in \mathcal{E}$. Since ϕ is linear we may set $y = e_l$ and $z = e_i$. Then $\phi(y)^*\phi(z)\mathrm{p}_k = \delta_{l,i}\phi(e_l)^*\phi(e_i)\mathrm{p}_k$ by Proposition 1.23. This is equal to $\delta_{il}\phi(e_l)^*\phi(e_l)\mathrm{p}_k = \delta_{il}(\sum[\boldsymbol{P}, \boldsymbol{B}](l,j)\rho_j)\mathrm{p}_k$ by the comments after Proposition 1.22. By assumption, this equals $\delta_{il}(\sum[\boldsymbol{P}, \boldsymbol{B}](l,j)\mathrm{p}_j)\mathrm{p}_k$ which is $\delta_{il}[\boldsymbol{P}, \boldsymbol{B}](l,k)\mathrm{p}_k$. However $\langle y, P_k z\rangle \mathrm{p}_k = \langle e_l, P_k e_i\rangle \mathrm{p}_k = \langle e_l, [\boldsymbol{P}, \boldsymbol{B}](i,k)e_i\rangle \mathrm{p}_k = \delta_{il}[\boldsymbol{P}, \boldsymbol{B}](l,k)\mathrm{p}_k$ proving the claim.

c) $\Rightarrow$ a): To show $(\phi, \mathbf{p}, \mathcal{H})$ is a representation we need to establish that $\phi(Q_k\mathcal{E})\mathcal{H} = \phi(e_k)^*\phi(e_k)\mathcal{H}$. Since $(\phi, \mathbf{p}, \mathcal{H})$ is a display we see by the same arguments preceding Proposition 1.23 that

$$\phi(e_k)^*\phi(e_k) = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\mathrm{p}_i = \sum[\boldsymbol{P}, \boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*.$$

We have

$$\begin{aligned}\phi(Q_k\mathcal{E})\mathcal{H} &= \sum\{\alpha_i[\boldsymbol{P},\boldsymbol{B}](k,i)\phi(e_i)\mathcal{H} \mid \alpha_i \in \mathbb{C}\} \\ &= \sum[\boldsymbol{P},\boldsymbol{B}](k,i)\phi(e_i)\mathcal{H} \\ &= \sum[\boldsymbol{P},\boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*\phi(e_i)\mathcal{H} \\ &= \sum[\boldsymbol{P},\boldsymbol{B}](k,i)\phi(e_i)\phi(e_i)^*\mathcal{H} \\ &= \phi(e_k)^*\phi(e_k)\mathcal{H}.\end{aligned}$$

□

Notice that if A is the matrix with a 1 in each entry, $A(i,k) = 1$ for all i and k, then condition b) of the previous theorem is satisfied. In this case $\boldsymbol{P}$ must be the coordinate system $P_k = I_{\mathcal{E}}$ for all k, and so any display of $\boldsymbol{P}$ is a representation of the dual system $(\boldsymbol{P}, A, \boldsymbol{P})$.

Proposition 1.29. *Let $(\boldsymbol{P}, A, \boldsymbol{Q})$ be a dual system on $\mathcal{E}$, U a unitary in $\mathcal{B}(\mathcal{E})$ and $(\phi, \rho, \mathcal{H})$ a display of $\boldsymbol{P}$. If $\psi = \phi \circ U$, then ψ is a display of $U^*\ \boldsymbol{P}U$. If ϕ is also a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$, then ψ is a representation of $(U^*\boldsymbol{P}U, A, U^*\boldsymbol{Q}U)$.*

Proof. We have

$$\begin{aligned}\psi(y)^*\psi(z)\rho_k &= \phi(Uy)^*\phi(Uz)\rho_k = \langle Uy, P_kUz\rangle\rho_k \\ &= \langle Uy, UU^*P_kUz\rangle\rho_k = \langle y, U^*P_kUz\rangle\rho_k,\end{aligned}$$

which shows the first claim.

If $\boldsymbol{B}$ implements $(\boldsymbol{P}, A, \boldsymbol{Q})$ and $\{e_k,\ k = 1, \ldots, m\}$ is a basis of $\mathcal{E}$ with $e_k \in B_k$, then $f_n = U^*e_k$ is a basis of $\mathcal{E}$ with $f_k \in U^*B_kU$, where $U^*\boldsymbol{B}U$ implements $(U^*\boldsymbol{P}U, A, U^*\boldsymbol{Q}U)$. Since $\psi(f_k) = \phi(e_k)$ condition b) of Theorem 1.28 holds and so ψ is a representation of $(U^*\boldsymbol{P}U, A, U^*\boldsymbol{Q}U)$. □

The real case of interest here is if U is a symmetry of $(\boldsymbol{P}, A, \boldsymbol{Q})$. Then $\psi = \phi \circ U$ remains a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$ if ϕ is a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$. More importantly, the coordinate system $\boldsymbol{B}$ still implements the duality and we may choose the elements e_k for a basis of $\mathcal{E}$ in this case, not the f_k. Then as in the proof of Proposition 1.29, $\psi(y)^*\psi(z) = \phi(y)^*\phi(z)$ for $y, z \in \mathcal{E}$, so $\psi(e_k) = \phi(Ue_k)$ is a partial isometry with the same initial space as $\phi(e_k)$, namely q_k. We have $\psi(Q_k\mathcal{E})\mathcal{H} = \phi(UQ_k\mathcal{E})\mathcal{H} = \phi(Q_kU\mathcal{E})\mathcal{H} = \phi(Q_k\mathcal{E})\mathcal{H} = q_k\mathcal{H}$, so ψ is a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$.

Proposition 1.30. *Let $(\phi, \rho, \mathcal{H})$ be a display of a normalized coordinate system $\boldsymbol{P}$ on $\mathcal{E}$ and $U \in \mathcal{B}(\mathcal{H})$ a unitary. Set $\mathrm{r}_k = U\rho_kU^*$.*

a) *If $\psi(x) = U\phi(x)U^*$, then $(\psi, \mathrm{r}, \mathcal{H})$ is also a display of $\boldsymbol{P}$. If ϕ is a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$, then so is ψ.*
b) *If $\psi(x) = \phi(x)U^*$ then $(\psi, \mathrm{r}, \mathcal{H})$ is a display of $\boldsymbol{P}$.*

Proof. In both cases, to check that ψ is a display of $\boldsymbol{P}$ it is enough to check that $\psi(y)^*\psi(z)\mathrm{r}_k = \langle y, P_kz\rangle\mathrm{r}_k$. However, the left side is

$$U\phi(y)^*\phi(z)\rho_kU^* = U\langle y, P_kz\rangle\rho_kU^* = \langle y, P_kz\rangle\mathrm{r}_k.$$

The map ϕ is a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ iff $(\phi, \mathbf{p})$ is a display, where $\mathrm{p}_k = \phi(e_k)\phi(e_k)^*$ with $e_k \in B_k$ a basis of $\mathcal{E}$ and $\boldsymbol{B}$ implementing $(\boldsymbol{P}, A, \boldsymbol{Q})$. Thus

$(\psi, U\mathbf{p}U^*)$ is a display and since $\psi(e_k)\psi(e_k)^* = U\mathrm{p}_k U^*$, we have by c) of Theorem 1.28 that the display (ψ, r, H) of $\boldsymbol{P}$ must also be a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$. $\square$

Remark 1.31. With ϕ, U as above, $\boldsymbol{B}$ a coordinate system implementing a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ and $\psi(x) = \phi(x)U^*$ it is certainly possible for ψ to be a representation of the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ even if ϕ is not. For if $e_k \in B_k$ is a basis then $\mathrm{p}_k = \phi(e_k)\phi(e_k)^* = \psi(e_k)\psi(e_k)^*$. Thus if U is a unitary satisfying $\sum A(k,i)U\rho_i U^* = \sum A(k,i)\mathrm{p}_i$, then Theorem 1.28 c) implies that ψ is a representation of $(\boldsymbol{P}, A, \boldsymbol{Q})$, even though ϕ may not be.

We give a simple example of a display ϕ of a system $\boldsymbol{P}$ on $\mathcal{E}$ so that ψ, where $\psi(x) = \phi(x)U^*$ cannot be a representation of any dual pair $(\boldsymbol{P}, \boldsymbol{Q})$, for any unitary U.

Let $\mathcal{E}$ be a two dimensional Hilbert space with basis $\{e_1, e_2\}$. If $P_1 = I$, P_2 the projection onto Span $\{e_2\}$ then $[\boldsymbol{P}, \boldsymbol{E}] = A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ where E_i is the projection onto Span $\{e_i\}$. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and β_i, $i = 1, 2$, two orthogonal projections, each of infinite rank, with sum $I_{\mathcal{H}}$. Let ρ_i, $i = 1, 2$, be two orthogonal projections with sum $I_{\mathcal{H}}$ and rank ρ_2 finite. Choose v_i, $i = 1, 2$, two partial isometries on $\mathcal{H}$ with $v_i v_i^* = \beta_i$ and $v_1^* v_1 = \rho_1$, $v_2^* v_2 = I_{\mathcal{H}}$. Defining $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ by $\phi(e_i) = v_i$ and extending linearly we have that $(\phi, \rho, \mathcal{H})$ is a display of $\boldsymbol{P}$.

We claim that ψ cannot be a representation for any dual system $(\boldsymbol{P}, A', \boldsymbol{Q}')$. Any possible matrix A' for $\boldsymbol{P}$ is related to A by a permutation θ, namely $A'(k,l) = A(\theta(k), l)$ for all k, l, where $A' = [\boldsymbol{P}, \boldsymbol{E}']$ with E'_i the projection onto Span $\{e'_i\}$ and $e'_i = e_{\theta(i)}$. Then by Theorem 1.28b), ψ is a representation of $(\boldsymbol{P}, A', \boldsymbol{Q}')$ if and only if $\sum A'(k,i)U\rho_i U^* = \sum A'(k,i)\beta'_i$ where $\beta'_i = \psi(e'_i)\psi(e'_i)^* = \phi(e'_i)\phi(e'_i)^* = \beta_{\theta(i)}$. This amounts to requiring $\sum A(k,i)U\rho_i U^* = \sum A(k,i)\beta_{\theta(i)}$ for each k. For our A, this means that $U\rho_1 U^* = \beta_{\theta(1)}$ and $U(\rho_1 + \rho_2)U^* = \beta_{\theta(1)} + \beta_{\theta(2)}$. The last equality imposes no new condition since both sides are the identity operator on $\mathcal{H}$. However the first equality implies $U\rho_2 U^* = \beta_{\theta(2)}$, which contradicts the choice of rank for β_2 and ρ_2. For $(\phi, \rho, \mathcal{H})$ a display of a coordinate system $\boldsymbol{P}$ on $\mathcal{E}$ there are some very mild conditions on the orthogonal family of projections ρ_i that ensure that one can perturb ϕ to be a representation of some dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. The last example shows how the rank of the projections ρ_i may prevent this from being the case, but basically this is the main obstacle.

Proposition 1.32. *Let $(\phi, \rho, \mathcal{H})$ be a display of $\boldsymbol{P}$, and $(\boldsymbol{P}, A, \boldsymbol{Q})$ a dual system implemented by $\boldsymbol{B}$. Set $\mathrm{p}_k = \phi(e_k)\phi(e_k)^*$ where $\{e_k \mid k = 1, \dots, m\}$ is a basis of $\mathcal{E}$ with $e_k \in B_k$. There is a permutation θ and a unitary U such that*

$$U\left(\sum (A(k,i)\rho_i)U^*\right) = \sum A(k,i)\mathrm{p}_{\theta(i)}$$

iff ψ defined by $\psi(x) = \phi(x)U^*$ is a representation of the dual system $(\boldsymbol{P}, A', \boldsymbol{Q}')$ where $A'(k,i) = A(\theta(k), i)$.

Remark 1.33. In particular, if there is permutation θ so that rank ρ_i = rank $\mathrm{p}_{\theta(i)}$ for all i, then there is a unitary U with $U\rho_i U^* = \mathrm{p}_{\theta(i)}$ for all i, and the conditions of the proposition are fulfilled. Thus the display ϕ of $\boldsymbol{P}$ may be perturbed to yield a representation of a dual system involving $\boldsymbol{P}$.

Proof. Since $[\boldsymbol{P}, \boldsymbol{B}] = A$ we have $[\boldsymbol{P}, \boldsymbol{B}'] = A'$ where $B'_k = B_{\theta(k)}$. The projections $\psi(e'_i)\psi(e'_i)^* = \phi(e'_i)\phi(e'_i)^* = \mathrm{p}_{\theta(i)}$, where $e'_i = e_{\theta(i)}$. Since $(\phi, \rho, \mathcal{H})$ is a display of

$\boldsymbol{P}$, so is $(\psi, U\rho U^*, \mathcal{H})$ by Proposition 1.30. By Theorem 1.28 ψ is a representation of $(\boldsymbol{P}, A', \boldsymbol{Q}')$ iff $\sum A'(k,i)U\rho_i U^* = \sum A'(k,i)\mathrm{p}'_i$, which is equivalent to the stated condition. $\square$

Let A be a square $m \times m$ matrix with entries in $\{0,1\}$. If $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ is a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$, a Hilbert space of dimension m, then there is a basis e_k of $\mathcal{E}$ with $\phi(e_k)^*\phi(e_k) = \sum A(k,i)\phi(e_i)\phi(e_i)^*$ and the $\phi(e_k)$ are partial isometries. If A satisfies condition I of Cuntz-Krieger [9], then by their uniqueness result, the C^*-algebra generated by the image of the Hilbert space $\mathcal{E}$ under ϕ is $\mathcal{O}_A$. One may think of this as a particular representation $\pi_\phi : \mathcal{O}_A \to \mathcal{B}(\mathcal{H})$ of the Cuntz-Krieger algebra naturally associated with the dual system representation ϕ of $\mathcal{E}$ in $\mathcal{B}(\mathcal{H})$.

By the comments after Proposition 1.29, for A satisfying condition I of [9], there is therefore an automorphism α_U of $\mathcal{O}_A$ for every unitary U on the m-dimensional Hilbert space $\mathcal{E}$ that is a symmetry of $(\boldsymbol{P}, A, \boldsymbol{Q})$. The automorphism α_U is determined by $\alpha_U\phi(e_i) = \phi(Ue_i)$. One may view these automorphisms as the quasi-free automorphisms of $\mathcal{O}_A$, as they naturally extend the notion of such automorphisms for the Cuntz algebras.

Given a Cuntz-Krieger algebra $\mathcal{O}_A$ acting on a Hilbert space $\mathcal{H}$, with a specified set of generating partial isometries v_i, $i = 1, \dots, m$, satisfying the Cuntz-Krieger relations then this determines a representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on an m-dimensional Hilbert space $\mathcal{E}$ so that $\mathcal{O}_A = C^*(\phi(\mathcal{E}))$, the C^*-algebra generated by the subspace $\phi(\mathcal{E})$. The map ϕ is defined by setting $\phi(e_i) = v_i$ with $\{e_i \mid i = 1, \dots, m\}$ a basis for $\mathcal{E}$ and extending linearly. The coordinate systems $\boldsymbol{P}$ and $\boldsymbol{Q}$ are given by $P_k = \sum A(i,k)E_i$, $Q_k = \sum A(k,i)E_i$ with E_i the orthogonal rank one projections onto Span $\{e_i\}$. One can think of $\mathcal{E}$, or rather $\phi(\mathcal{E})$ as an underlying finite dimensional Hilbert space of partial isometries in the algebra $\mathcal{O}_A$, and one can refer to a generating Hilbert space $\mathcal{E}$ of partial isometries as meaning a representation $\phi : \mathcal{E} \to \mathcal{O}_A \subseteq \mathcal{B}(\mathcal{H})$ of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$ with $C^*(\phi(\mathcal{E})) = \mathcal{O}_A$.

2. Endomorphisms of finite direct sums of type I_∞ von Neumann algebras arising from representations of $\mathcal{O}_A$

There is a well known correspondence between $*$-representations π of the Cuntz algebra $\mathcal{O}_n$, $\pi : \mathcal{O}_n \to B(\mathcal{H})$, and unital $*$-endomorphisms α of $\mathcal{B}(\mathcal{H})$ [1]; namely $\alpha(a) = \sum^n \pi(v_i)a\pi(v_i)^*$, $(a \in \mathcal{B}(\mathcal{H}))$ where $\{v_i \mid i = 1, \dots, n\}$ are partial isometries in $\mathcal{O}_n$ with $v_k^*v_k = I = \sum v_l v_l^*$. In this section we explore an extension of this correspondence to one between representations of Cuntz-Krieger algebras and unital $*$-endomorphisms of finite direct sums of type I_∞ factors. In the following $(\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H}), \rho, \mathcal{H})$ is a display of a coordinate system $\boldsymbol{P}$ on an m-dimensional Hilbert space $\mathcal{E}$. If $\boldsymbol{E}$ is a normalized orthogonal coordinate system with $\boldsymbol{P} \prec \boldsymbol{E}$ and $[\boldsymbol{P}, \boldsymbol{E}] = A$, and $\{e_k \mid n = 1, \dots, m\}$ is a basis of $\mathcal{E}$ with $e_k \in E_k$, then by Remarks 1.24 and 1.25 $s_k = \phi(e_k)$ is a partial isometry with initial space $\mathrm{q}_k = \phi(e_k)^*\phi(e_k) = \sum A(k,i)\rho_i$; the final spaces $\phi(e_k)\phi(e_k)^* = \mathrm{p}_k$ are m orthogonal projections with sum $I_\mathcal{H}$. For each such display ϕ form the $*$-linear unital map $\varphi : x \to \sum s_k x s_k^*$ of $\mathcal{B}(\mathcal{H})$ to itself. Since $\bigvee \mathrm{q}_k = I_\mathcal{H}$ and the projections p_i are orthogonal it is clear that φ is injective. There is a maximal domain $\mathcal{R}$ of definition for φ so that φ becomes a $*$-homomorphism defined on $\mathcal{R}$. Let $\mathbf{q}$ denote the coordinate system

$\{q_k \mid k = 1, \ldots, n\}$ and $\mathbf{p}$ the orthogonal coordinate system $\{\mathrm{p}_k \mid k = 1, \ldots, m\}$ on $\mathcal{H}$.

Proposition 2.1. *The maximal domain $\mathcal{R}$ for φ so that φ is a $*$-homomorphism is the von Neumann algebra $\{\mathrm{q}_k \mid k = 1, \ldots, m\}'$.*

Proof. We first show $\{\mathrm{q}_k \mid k = 1, \ldots, m\}' \subseteq \mathcal{R}$, i.e., that $\varphi(a\ b) = \varphi(a)\varphi(b)$ for $a, b \in \{\mathrm{q}_k \mid k = 1, \ldots, m\}'$. The right side is $\sum s_k a s_k^* s_l b s_l^* = \sum s_k a \delta_{kl} \mathrm{q}_k b = \sum s_k a \mathrm{q}_k b s_k^* = \sum s_k \mathrm{q}_k a b s_k^* = \varphi(a\ b)$, since $s_k \mathrm{q}_k = s_k s_k^* s_k = s_k$.

Conversely suppose $\varphi(a\ b) = \varphi(a)\varphi(b)$. Then $\sum s_k a \mathrm{q}_k b s_k^* = \sum s_k a b s_k^*$, so multiplying by s_l^* on the left and s_l on the right we obtain the equation $\mathrm{q}_l a \mathrm{q}_l b \mathrm{q}_l = \mathrm{q}_l a b \mathrm{q}_l$ for each l, and for each $a, b \in \mathcal{R}$. If $a\mathrm{q}_l \neq \mathrm{q}_l a$ for some l and some $a \in \mathcal{R}$, then since $a\mathrm{q}_l = \mathrm{q}_l a \mathrm{q}_l + (1 - \mathrm{q}_l) a \mathrm{q}_l$ and $\mathrm{q}_l a = \mathrm{q}_l a \mathrm{q}_l + \mathrm{q}_l a (1 - \mathrm{q}_l)$ we must have $(1 - \mathrm{q}_l) a \mathrm{q}_l \neq \mathrm{q}_l a (1 - \mathrm{q}_l)$. Thus one of these, say $(1 - \mathrm{q}_l) a \mathrm{q}_l \neq 0$. Since $\{\mathrm{q}_k \mid k = 1, \ldots, m\}' \subseteq \mathcal{R}$ we also have that both $x = (1 - \mathrm{q}_l) a \mathrm{q}_l \in \mathcal{R}$ and $x^* \in \mathcal{R}$. Thus by setting $a = x^*$ and $b = x$ in the equation above, $\mathrm{q}_l x^* \mathrm{q}_l x \mathrm{q}_l = \mathrm{q}_l x^* x \mathrm{q}_l$. Since $\mathrm{q}_l x = 0$, this implies $0 = x^* x$. However $x \neq 0$ by assumption, so a must commute with each q_l. A similar contradiction follows if $\mathrm{q}_l a (1 - \mathrm{q}_l)$ is nonzero. □

Since

$$\mathrm{p}_k \varphi(a) = s_k s_k^* \sum s_l a s_l^* = \sum \delta_{lk} s_k s_k^* s_l a s_l^* = s_k a s_k^*$$

and

$$\varphi(a) \mathrm{p}_k = \sum s_l a s_l^* s_k s_k^* = \sum s_l a \delta_{lk} s_l^* s_k s_k^* = s_k a s_k^*,$$

we have that φ is a unital $*$-homomorphism from $\mathcal{R} = \{s_k^* s_k \mid k = 1, \ldots, m\}'$ to $\{s_k s_k^* \mid k = 1, \ldots, m\}'$. Note that φ is ultra weakly continuous. The domain algebra $\mathcal{R}$ has as its commutant the discrete abelian, and so type I algebra generated by the commuting projections q_k, $k = 1, \ldots, m$. It follows that $\mathcal{R}$ is unitarily equivalent to a finite direct sum of type I factors. Since the range of φ is contained in $\bigoplus \mathrm{p}_k (\mathcal{B}(\mathcal{H}) \mathrm{p}_k$, a finite direct sum of type I factors, φ is a unital $*$-homomorphism of a finite sum of type I factors on $\mathcal{H}$ to another finite direct sum of type I factors on $\mathcal{H}$.

If one now strengthens the assumption on the display ϕ and insist that $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ is a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ rather than just a display, we may then assume by Theorem 1.28 that $(\phi, \mathbf{p}, \mathcal{H})$ is a display. Thus $\mathrm{q}_k = \sum A(k, i) \mathrm{p}_i$ in this case and the algebra generated by the projections p_k therefore contains the algebra generated by the projections q_k, so $\mathcal{R} \supseteq \{\mathrm{p}_k \mid k = 1, \ldots, m\}'$. Thus the range of $\varphi \subseteq \mathcal{R}$ and φ becomes a unital $*$-endomorphism of $\mathcal{R} = \bigoplus \mathcal{R}_k$, a finite direct sum of type I von Neumann algebras $\mathcal{R}_k$. If ϕ is an infinite representation then $\mathcal{R}$ is a finite direct factor sum of type I_∞ factors.

Theorem 2.2. *Let $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ be a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ of coordinate systems on $\mathcal{E}$ and $s_k = \phi(e_k)$ where $\{e_k \mid k = 1, \ldots, m\}$ is a basis of $\mathcal{E}$ with $e_k \in B_k$ and $\boldsymbol{B}$ implementing the duality. The map*

$$\varphi : x \to \sum_{k=1}^{m} s_k x s_k^*, \qquad x \in \{s_k^* s_k \mid k = 1, \ldots, m\}' = \mathcal{R},$$

is a unital injective $*$-endomorphism of $\mathcal{R}$, a finite direct sum of type I *factors.*

Although a basis $\{e_k \mid k = 1, \ldots, m\}$ of $\mathcal{E}$ was chosen with $e_k \in B_k$ where $\boldsymbol{B}$ implements the duality in order to define the endomorphism φ we will show that φ does not depend on the particular choice of such a basis.

For φ a unital $*$-endomorphism of a von Neumann subalgebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$, let L_φ be the norm closed linear subspace defined by $\{T \in \mathcal{B}(\mathcal{H}) \mid \varphi(a)T = Ta, (a \in \mathcal{R})\}$. We will show that L_φ is a Hilbert $\mathcal{A}$-module where $\mathcal{A} = \mathcal{R}'$. First note that L_φ is a right $\mathcal{A}$-module, in fact a two-sided $\mathcal{A}$-module. For if $x \in \mathcal{A}$, $T \in L_\varphi$, then $\varphi(a)Tx = Tax = Txa$, $(a \in \mathcal{R})$ so $Tx \in L_\varphi$. Also, given, $T, S \in L_\varphi$ we have $T^*Sa = T^*\varphi(a)S = (\varphi(a)^*T)^*S = (\varphi(a^*)T)^*S = (Ta^*)^*S = aT^*S$, $(a \in \mathcal{R})$, so $T^*S \in \mathcal{A}$. Since $\|T\| = \|T^*T\|^{1/2}$ for $T \in \mathcal{B}(\mathcal{H})$ we see that L_φ is a Hilbert $\mathcal{A}$-module with the $\mathcal{A}$-valued inner product given by $\langle T, S\rangle = T^*S$ for $T, S \in L_\varphi$.

Proposition 2.3. *If φ is an endomorphism of $\mathcal{R} = C^*(\mathbf{q})'$ arising from a representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ then $L_\varphi = \operatorname{Span}_{\mathcal{A}} \phi(\mathcal{E})$, the right $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$ generated by $\phi(\mathcal{E})$.*

Proof. Let $\{e_k \mid k = 1, \ldots, m\}$ be a basis of $\mathcal{E}$ with $e_k \in B_k$, $\boldsymbol{B}$ a coordinate system implementing the dual system. Since $\varphi(a)\phi(e_k) = \varphi(a)\phi(e_k)\mathrm{q}_k = \sum \phi(e_j)a\phi(e_j)^*\phi(e_k)\mathrm{q}_k = \phi(e_k)a\mathrm{q}_k = \phi(e_k)\mathrm{q}_k a = \phi(e_k)a$ for $a \in \mathcal{R}$ we have that $\phi(E) \subseteq L_\varphi$. For $T \in L_\varphi$ we have $T = 1_{\mathcal{H}}T = \sum \phi(e_i)\phi(e_i)^*T = \sum \phi(e_i)\langle\phi(e_i), T\rangle$ which is in the right $\mathcal{A}$-module generated by $\phi(\mathcal{E})$. $\square$

Proposition 2.4. *Let $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ be a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$ and φ be the unital $*$-endomorphism of $\mathcal{R} = C^*(\mathbf{q})'$ described in Theorem 2.2. If $bT = Ta$ $(T \in \phi(\mathcal{E}))$ for some $a \in \mathcal{R}$, $b \in \mathcal{B}(\mathcal{H})$ then $b = \varphi(a)$*

Proof. Since $\phi(\mathcal{E}) \subset L_\varphi$ and $a \in \mathcal{R}$ we have $\varphi(a)T = Ta$, $(T \in \phi(\mathcal{E}))$. Thus $(b - \varphi(a))T = T(a - a) = 0$ for all $T \in \phi(\mathcal{E})$. Thus $(b - \varphi(a))(\phi(\mathcal{E})\mathcal{H}) = 0$ and since $\overline{\phi(\mathcal{E})\mathcal{H}} = \mathcal{H}$ we must have $b = \varphi(a)$. $\square$

To define the endomorphism φ given by a representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a given dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$ we first choose a basis $\{e_k \mid k = 1, \ldots, m\}$ of $\mathcal{E}$ implementing the duality $(\boldsymbol{P}, A, \boldsymbol{Q})$. We will show that φ is independent of such a chosen basis. If $\{e'_k \mid k = 1, \ldots, m\}$ is another basis of $\mathcal{E}$ implementing the duality then by Proposition 1.20 there is a symmetry U of $(\boldsymbol{P}, A, \boldsymbol{Q})$ with $e'_k = Ue_k$. If s'_k are the partial isometries $\phi(e'_k) = \phi(Ue_k)$, then the s'_k have the same initial projections q_k as the $s_k = \phi(e_k)$. Thus the domains of the unital $*$-endomorphisms φ' and φ coincide, where $\varphi' : x \to \sum s'_k x s'^*_k$. For $x \in \mathcal{R}$ we have $\varphi'(x)T = Tx$ for all $T \in \phi(U\mathcal{E}) = \phi(\mathcal{E})$, so by Proposition 2.4, $\varphi'(x) = \varphi(x)$. Thus the endomorphism φ depends only on the representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a given dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. We denote this endomorphism by φ_ϕ if the context is not clear.

Given two representations of the same dual system then their associated endomorphisms are basically the same.

Proposition 2.5. *Given two infinite representations $\phi, \psi : (\boldsymbol{P}, A, \boldsymbol{Q}) \to \mathcal{B}(\mathcal{H})$ of a dual system on $\mathcal{E}$ there is an inner automorphism β of $\mathcal{B}(\mathcal{H})$ and an inner automorphism α of range (φ_ϕ) so that $\varphi_\phi = \alpha \circ \beta \circ \varphi_\psi \circ \beta^{-1}$.*

Proof. Denote by $\mathrm{q}_i, \mathrm{p}_i$ the infinite initial and final projections respectively of the partial isometry $\phi(e_i)$, where $\{e_i \mid i = 1, \ldots, n\}$ is a basis of $\mathcal{E}$ implementing the duality, and $\mathrm{q}'_i, \mathrm{p}'_i$ the analogous projections for the partial isometry $\psi(e_i)$. Since the projections $\mathrm{p}_i, \mathrm{p}'_i$ are infinite with $\sum \mathrm{p}_i = \sum \mathrm{p}'_i = I_{\mathcal{H}}$, there is a unitary $U \in \mathcal{B}(\mathcal{H})$

with $\beta(\mathrm{p}'_i) = \mathrm{p}_i$, where $\beta = adU$. Thus $\beta(\psi(e_i))$ is a partial isometry with final projection p_i and initial projection $\beta(\mathrm{q}'_i) = \beta(\sum A(i,l)\mathrm{p}'_l) = \sum A(i,l)\beta(\mathrm{p}'_l) = \mathrm{q}_i$. Thus the domain of the endomorphism associated to the representation $\beta \circ \psi$ is that of the endomorphism φ_ϕ. It is clear that $\varphi_{\beta\circ\psi} = \beta \circ \varphi_\psi \circ \beta^{-1}$.

Since $\phi(e_i)$ and $\beta\psi(e_i)$ are partial isometries with the same initial and final spaces, and the final projections have sum $I_\mathcal{H}$, $\sum_{i=1}^n \phi(e_i)\beta\psi(e_i)^*$ defines a unitary V in $\mathcal{B}(\mathcal{H})$ with $V\beta(\psi(e_i))$ equal to the partial isometry $\phi(e_i)$. Since V commutes with the projections p_i for all i, V is in range (φ_ϕ). For $x \in \mathcal{M}$ we have $\sum V\beta(\psi(e_i))x\beta\psi(e_i)^*V^* = \sum \phi(e_i)x\phi(e_i)^* = \varphi_\phi(x)$, so $\varphi_\phi = \alpha \circ \beta \circ \varphi_\psi \circ \beta^{-1}$ with $\alpha = adV$. □

Notice that if ψ is an infinite representation of a dual system $(\boldsymbol{P}', A, \boldsymbol{Q}')$ on $\mathcal{E}$ defined by the same matrix A as a representation ϕ of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$, then Propositions 1.16 and 1.29 imply that there is a unitary W in $\mathcal{B}(\mathcal{E})$ so that $\psi \circ W$ is a representation of the dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$, and the hypothesis of the proposition above are satisfied.

The following theorem describes the fixed point algebra of the endomorphism φ_ϕ as the commutant of the $*$-algebra generated by the image of ϕ. If the matrix A satisfies condition I of Cuntz-Krieger [9], then of course this is the commutant of $\pi_\phi(\mathcal{O}_A)$, where π_ϕ is the representation of $\mathcal{O}_A$ on $\mathcal{H}$ defined by ϕ. This again extends the known situation for the Cuntz algebras $\mathcal{O}_n$ ([1], [16]).

Theorem 2.6. *Let $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ be a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$. If φ is the unital $*$-endomorphism of the von Neumann algebra $\mathcal{R} = C^*(\mathbf{q})'$ defined by ϕ, then*

$$\{a \in \mathcal{R} \mid \varphi(a) = a\} = (\phi(\mathcal{E}) \cup \phi(\mathcal{E})^*)' \cap \mathcal{R}.$$

Proof. If $a \in \phi(\mathcal{E})'$ then a commutes with $\phi(\mathcal{E})$, so $\varphi(a) = \sum \phi(e_i)a\phi(e_i)^* = a\sum \phi(e_i)\phi(e_i)^* = a \cdot I = a$. Here $\{e_k \mid k \in \Sigma\}$ is a basis of $\mathcal{E}$ with $e_k \in B_k$ where $\boldsymbol{B}$ implements $(\boldsymbol{P}, A, \boldsymbol{Q})$.

Conversely, if $\varphi(a) = a$, then $\varphi(a^*) = \varphi(a)^* = a^*$. Now $\{T \in \mathcal{B}(\mathcal{H}) \mid \varphi(a)T = Ta \ (a \in \mathcal{R})\} = \mathrm{Span}_\mathcal{A}\phi(\mathcal{E})$ by Proposition 2.3, so both a and a^* commute with $\mathrm{Span}_\mathcal{A}\phi(\mathcal{E})$. Thus a commutes with both $\phi(\mathcal{E})$ and $\phi(\mathcal{E})^*$. □

For the next result we make use of some notation in [9]. Let $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ be a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$. Choose a basis $\{e_k \mid k \in \Sigma\}$ of $\mathcal{E}$ with $e_k \in B_k$ where $\boldsymbol{B}$ implements $(\boldsymbol{P}, A, \boldsymbol{Q})$. For $\mu = (i_1, \dots, i_k)$ a multi-index with $i, j \in \Sigma$, set $|\mu| = k$, the length of μ. Write $s_\mu = \phi(e_{i_1})\phi(e_{i_2})\dots\phi(e_{i_k})$ and p_μ, q_μ the range and support projections of s_μ, respectively. Let $\mathcal{M}_A$ denote the set of multi-indices μ with $s_\mu \neq 0$. Set $\mathcal{F}_k$ to be the finite dimensional C^*-algebra generated by $\{s_\mu \mathrm{p}_i s_\nu^* \mid i \in \Sigma, |\mu| = |\nu| = k\}$, an increasing sequence of algebras, and $\mathcal{F}_A$ to be the norm closure of $\bigcup_{k=0}^{\infty} \mathcal{F}_k$ ([9]).

Theorem 2.7. *Let $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ be a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ on $\mathcal{E}$. If φ is the unital $*$-endomorphism of $\mathcal{R} = C^*(\mathrm{q})'$ defined by ϕ, then*

$$\bigcap_{k\geq 0} \varphi^k(\mathcal{R}) = {\mathcal{F}'}_A.$$

Proof. Since ${\mathcal{F}'}_A = \bigcap_{k\geq 0} {\mathcal{F}'}_k$ it is enough to establish that $\varphi^{k+1}(\mathcal{R}) \subseteq {\mathcal{F}'}_k \subseteq \varphi^k(\mathcal{R})$ for all $k \geq 0$. Recall from [9] that $E^i_{\mu\nu} = s_\mu \mathrm{p}_i s_\nu^*$ $(i \in \Sigma, \mu, \nu \in \mathcal{M}_A$ such that

$|\mu| = |\nu| = k)$ are a system of matrix units for $\mathcal{F}_k$. By Proposition 2.3 $\phi(\mathcal{E}) \subseteq \{T \in \mathcal{B}(\mathcal{H}) \mid \varphi(a)T = Ta,\ (a \in \mathcal{R})\}$, so for μ with $|\mu| = k$ it follows that $\varphi^{k+1}(a)s_\mu = s_\mu\varphi(a)$, $(a \in \mathcal{R})$. Thus $\varphi^{k+1}(a)E^i_{\mu\nu} = s_\mu\varphi(a)\mathrm{p}_i s^*_\nu = s_\mu \mathrm{p}_i\varphi(a)s^*_\nu = E^i_{\mu\nu}\varphi^{k+1}(a)$, the second equality following from the fact that range $\varphi \subseteq \{\mathrm{p}_i \mid i \in \Sigma\}'$. Thus $\varphi^{k+1}(R) \subseteq \mathcal{F}'_k$.

Since $\mathcal{F}_k$ is the direct sum of matrix algebras $\bigoplus_{i\in\Sigma} C^*\{E^i_{\mu\nu} \mid |\mu| = |\nu| = k\}$, the commutant $\mathcal{F}'_k$ is the direct sum $\bigoplus_{i\in\Sigma} C^*\{E^i_{\mu\nu} \mid |\mu| = |\nu| = k\}'$. Since the commutant of a matrix algebra consists of diagonal matrices we see that any $y \in \mathcal{F}'_k$ must have the form $\sum_{i\in\Sigma}\{\sum_{|\nu|=k} E^i_{\nu\mu}(E^i_{\mu\mu}yE^i_{\mu\mu})E^i_{\mu\nu}$ where μ is chosen so that $E^i_{\mu\mu} \neq 0$. Set $x = \sum_{i\in\Sigma}(s_\mu \mathrm{p}_i)^*E^i_{\mu\mu}yE^i_{\mu\mu}(s_\mu\mathrm{p}_i)$ and note that $x \in C^*(\{\mathrm{p}_i \mid i \in \Sigma\})' \subseteq C^*(\{\mathrm{q}_i \mid i \in \Sigma\})'$ $= \mathcal{R}$. We compute $\varphi^k(x) = \sum_{|\nu|=k} s_\nu x s^*_\nu = \sum_{|\nu|=k} s_\nu \left(\sum_{i\in\Sigma} \mathrm{p}_i s^*_\mu E^i_{\mu\mu}yE^i_{\mu\mu}s_\mu\mathrm{p}_i\right) s^*_\nu =$ $\sum_{|\nu|=k,i\in\Sigma} s_\nu\mathrm{p}_i(s_\mu\mathrm{p}_i)^*E^i_{\mu\mu}yE^i_{\mu\mu}(s_\mu\mathrm{p}_i)(s_\nu\mathrm{p}_i)^* = \sum_{|\nu|=k,i\in\Sigma} E^i_{\nu\mu}E^i_{\mu\mu}yE^i_{\mu\mu}E^i_{\mu\nu} = y$. Thus $\mathcal{F}'_k \subseteq \varphi^k(\mathcal{R})$. □

It is clear that φ restricts to an automorphism of $\mathcal{F}'_A$.

We recall ([9]) that a square 0-1 matrix A satisfies condition I of Cuntz-Krieger if the compact space $X_A = \{(x_k)_{k\in\mathbb{N}} \in \Sigma^{\mathbb{N}} \mid A(x_k, x_{k+1}) = 1, (k \in \mathbb{N})\}$ has no isolated points. This space may be viewed as a one sided vertex shift space. We will return to this space later in section 3.

Theorem 2.8. *With the assumptions of Theorem 2.2, if the matrix A satisfies condition* I *of Cuntz-Krieger then the domain $\mathcal{R}$ of the $*$-endomorphism φ is a finite direct sum of* I_∞ *factors. In particular, the representation ϕ must be infinite.*

Proof. Suppose that $\mathcal{R} = \bigoplus \mathrm{r}_k\mathcal{R}\mathrm{r}_k$ has a finite type I factor summand $\mathcal{R}_k = \mathrm{r}_k\mathcal{R}\mathrm{r}_k$ where $\mathrm{r} = \{\mathrm{r}_l \mid l \in \Sigma\}$ is a reduced orthogonal generator for $\mathbf{q} = \{\mathrm{q}_k \mid k = 1,\dots,m\}$. Since $\mathrm{p} = \{\mathrm{p}_l \mid l = 1,\dots,m\}$ is an orthogonal system with $\mathbf{q} \prec \mathbf{p}$ we have by Proposition 1.5 that $\mathbf{r} \prec \mathbf{p}$. Thus by Proposition 1.6, $\mathrm{p}_j\mathcal{R}\mathrm{p}_j$ is a type I factor for each j, and so there are projections p_j with $\mathrm{p}_j\mathcal{R}\mathrm{p}_j$ a finite type I factor. Choose one, say p_{j_0} with $\mathrm{p}_{j_0}\mathcal{R}\mathrm{p}_{j_0}$ a type I_n factor with n finite and minimal.

Since $s_{j_0} = \phi(e_{j_0})$ is a partial isometry mapping q_{j_0} to p_{j_0}, the endomorphism φ maps the algebra $\mathrm{q}_{j_0}\mathcal{R}\mathrm{q}_{j_0}$ into $\mathrm{p}_{j_0}\mathcal{R}\mathrm{p}_{j_0}$. Since $\mathrm{q}_{j_0} = \sum A(j_0,k)\mathrm{p}_k$, each algebra $\mathrm{p}_k\mathcal{R}\mathrm{p}_k$ with $A(j_0,k) \neq 0$ is mapped injectively into $\mathrm{p}_{j_0}\mathcal{R}\mathrm{p}_{j_0}$. Since n is minimal, there is a unique j_1 with $A(j_0,j_1) = 1$ and $A(j_0,k) = 0$ for $k \neq j_1$, and $\mathrm{p}_{j_1}\mathcal{R}\mathrm{p}_{j_1}$ a type I_n factor.

If $j_0 = j_1$ then A cannot satisfy condition I of Cuntz-Krieger so $j_0 \neq j_1$ and $\mathrm{p}_{j_1}\mathcal{R}\mathrm{p}_{j_1}$ is another type I_n factor. Using the above argument again, there is a unique $j_2, j_1 \neq j_2$ with $A(j_1,k) = 0$ for $k \neq j_2$ and $A(j_1,j_2) = 1$. The point j_2 can also not be j_0, since otherwise A will not satisfy condition I of Cuntz-Krieger. Thus j_0, j_1, j_2 are distinct. Continuing in this manner forces A to be a permutation matrix, which again contradicts the assumption on A. Thus R cannot have finite type I factor summands. □

We note some known facts concerning normal unital $*$-homomorphisms $\varphi : \mathcal{M} \to \mathcal{N}$ where both $\mathcal{M} = \bigoplus_{k=1}^{m} \mathcal{M}_k$ and $\mathcal{N} = \bigoplus_{k=1}^{n} \mathcal{N}_k$ are finite direct sums of type I factors

[5]. With such a map φ associate a matrix φ_*, with (i,j) entry equal to $\varphi_*(i,j)$, the multiplicity of $\varphi(\mathcal{M}_j)$ in $\mathcal{N}_i$. Although in [5] this is defined for $\mathcal{M}$ and $\mathcal{N}$ both finite direct sums of type I_∞ factors, it is possible to modify the presentation slightly to accommodate finite direct sums of any type I factors. With this convention one has that $\varphi_*\psi_* = (\varphi\circ\psi)_*$ for $\varphi : \mathcal{M} \to \mathcal{N}$ and $\psi : \mathcal{L} \to \mathcal{M}$ two such $*$-homomorphisms. Other results of [5] will still hold, so for example, if $\varphi_{1*} = \varphi_{2*}$ with $\varphi_i : \mathcal{M} \to \mathcal{N}$, $(i = 1, 2)$, then there is an inner automorphism of $\mathcal{N}$ with $\alpha\varphi_1 = \varphi_2$. Also, given an $n \times m$ matrix D of nonnegative integers with at least one nonzero entry in each row and column, there is an injective unital $*$-homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ with $\varphi_*(i,j) = D_{ij}$, as long as we assume in addition that $n_i = \sum D_{ij} m_j$, where $\mathcal{M}_k$ is a type I_{m_k} factor and $\mathcal{N}_k$ is a type I_{n_k} factor. This additional caveat is unnecessary if all the factors involved are type I_∞. With $\varphi : \mathcal{R} \to \mathcal{R}$ the unital injective $*$-endomorphism arising from our context, namely from a representation of a dual system with A a square 0-1 valued matrix, we will later see that φ_* is a square matrix with values in $\mathbb{N}$. Also since φ is unital and injective, φ_* has a non-zero entry in each row and column. In the situation of a single type I_∞ factor with a unital $*$-endomorphism φ the matrix φ_* is of course just a single number referred to as the index of the endomorphism. In a similar vein we may refer to the matrix φ_* as the index of the endomorphism φ.

The question naturally arises about a relationship between the matrix A and the matrix φ_*. There are also questions about the nature of the correspondence between representations of the Cuntz-Krieger algebras and unital $*$-endomorphisms of finite direct sums of I_∞ factors. In order to answer these questions we first describe some concepts from a graph theoretical perspective.

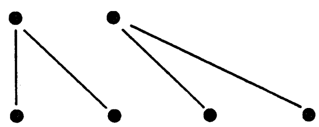
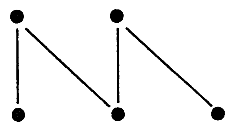
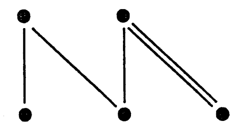
3. Operations on bipartite graphs

For bipartite graphs we describe various in-split and in-amalgamation procedures and determine their elementary properties. These are slight variations of the usual concepts for graphs ([17]). This allows one to define a partial order on (finite) bipartite graphs which one can translate to a partial order on square matrices with nonnegative integer entries. This context also allows one to understand the connection between endomorphisms of finite sums of type I factors and their graphs.

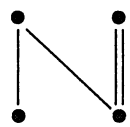
Recall that V, W are respectively the initial and final states of a bipartite graph $\mathcal{G}(V,W)$. A bipartite graph $\mathcal{G}(V,\tilde{W})$ is a *final state in-split* of $\mathcal{G}(V,W)$ if it can be formed by the following procedure from $\mathcal{G}(V,W)$: for $J \in W$, partition $\mathcal{E}_J$, the set of edges with terminal state J, into disjoint sets $\mathcal{E}_J^1, \dots, \mathcal{E}_J^{m(J)}$. Replace the vertex $J \in W$ with $m(J)$ vertices $J_1, \dots, J_{m(J)}$, and the edges of $\mathcal{G}(V,\tilde{W})$ with terminal point J_k are the edges in $\mathcal{E}_J^k$. If this is done for all $J \in W$ and if the partition of $\mathcal{E}_J$ in each case is the maximal one consisting of one element sets then the resulting graph $\mathcal{G}(V,\tilde{W})$ is referred to as the complete final state in-split of $\mathcal{G}(V,W)$.

We also deal with graphs $\mathcal{G}(V,V)$. In this case, we can form what is known as an *in-split* $\mathcal{G}(\tilde{V},\tilde{V})$ of $G(V,V)$. For $J \in V$ partition $\mathcal{E}_J$, the set of edges with terminal state J into disjoint sets as before, and form the final state in-split $\mathcal{G}(V,\tilde{V})$ as before. Replace the point $J \in V$ with $m(J)$ points J_k, $k = 1, \dots, m(J)$. Then for each edge e in $\mathcal{G}(V,\tilde{V})$ with initial vertex J and final vertex I, there is for each $k = 1, \dots, m(J)$, an edge (e,k) in $\mathcal{G}(\tilde{V},\tilde{V})$ with initial vertex J_k, $k = 1, \dots, m(J)$, and final vertex I. The graph $\mathcal{G}(V,\tilde{V})$ is referred to as the corresponding intermediate final state split for the in-split $\mathcal{G}(\tilde{V},\tilde{V})$.

Example 3.1. The graphs

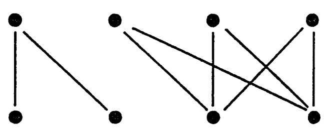
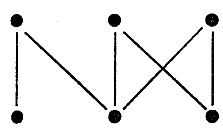
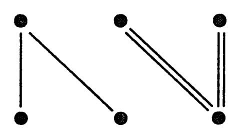
 ,  and 

are all final state in-splits of the graph

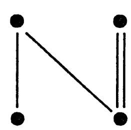


The first graph is the complete final state in-split.

The graphs

 ,  and 

are all in-splits of the graph



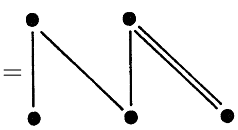
The first one is the complete in-split.

Given a graph $\mathcal{G}(V,W)$ we say that a vertex I in V is connected to J in W with multiplicity m if there are exactly m edges with initial vertex I and final vertex J. If the associated matrix for a graph $\mathcal{G}$ has entries consisting of 0's and 1's only, we say $\mathcal{G}$ is a multiplicity free graph. If $\mathcal{G}(V,W)$ is a graph it is clear that there is always a final state in-split $\mathcal{G}(V,\widetilde{W})$ which is multiplicity free. If $\mathcal{G}=\mathcal{G}(V,V)$, then it follows that there is always an in-split of $\mathcal{G}$ which is multiplicity free.

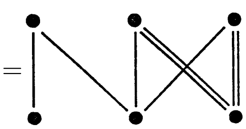
We describe other operations on a graph $\mathcal{G}(V,W)$, namely an *initial state amalgamation*, and if $V=W$ an *in-amalgamation* of $\mathcal{G}(V,V)$. For each $m\in\mathbb{N}$, $J\in W$ let $S_J^m=\{I\in V \mid I \text{ is connected to } J \text{ with multiplicity } m\}$ and let $\{T_1,\dots,T_k\}=\mathcal{T}$ be the partition of V defined by the finite number of nonempty subsets S_J^m, $(m\in\mathbb{N}, J\in W)$. We refer to this partition as the 'base' partition of V. Note that if a vertex I in T_j is connected to some vertex J in W with multiplicity m, then every point in T_j is connected to J with multiplicity m. For each set T_k consider a partition of T_k into $m(k)$ sets, $T_k^1,\dots,T_k^{m(k)}$. Form a new graph $\mathcal{G}(\underset{\sim}{V},W)$ where each set of initial states T_k^j, $j=1,\dots,m(k)$, $k=1,\dots,l$, is collapsed to a single vertex I_k^j in V and the edges of multiplicity m with initial vertex in T_k^j and final vertex J in W are replaced with one edge of multiplicity m with initial vertex I_k^j and final vertex J. The graph $\mathcal{G}(\underset{\sim}{V},W)$ is called an initial state amalgamation of $\mathcal{G}(V,W)$. If $V=W$ then by further collapsing the vertices in W in the same way that V was formed and leaving the edges in $\mathcal{G}(\underset{\sim}{V}, W)$ otherwise intact, one obtains the partial in-amalgamation $\mathcal{G}(\underset{\sim}{V}, \underset{\sim}{V})$ of $\mathcal{G}(V, V)$. The amalgamation is complete if the entire set T_j is collapsed to a vertex for each $j = 1, \ldots, k$. The graph $\mathcal{G}(\underset{\sim}{V}, V)$ is referred to as the corresponding intermediate initial state amalgamation of the in-amalgamation $\mathcal{G}(\underset{\sim}{V}, \underset{\sim}{V})$.

Note that, except for the trivial in-amalgamation of a graph $\mathcal{G}(V, V)$, the vertex set $\underset{\sim}{V}$ of an in-amalgamated graph has cardinality strictly less than the cardinality of V, assuming that V is a finite set.

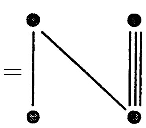
Example 3.2. The graph

$\mathcal{G}(\underset{\sim}{V}, V) =$ 

is an initial state amalgamation of

$\mathcal{G}(V, V) =$  .

It is complete. Also

$\mathcal{G}_c =$ 

is a complete in-amalgamation of $\mathcal{G}(V, V)$.

The following observations are straightforward. If $\mathcal{G}(V, V)$ has an in-split graph $\mathcal{G}(\tilde{V}, \tilde{V})$ with $\mathcal{G}(V, \tilde{V})$ the corresponding intermediate final state split then $\mathcal{G}(V, V)$ is an in-amalgamation of $\mathcal{G}(\tilde{V}, \tilde{V})$ with corresponding intermediate initial state amalgamation $\mathcal{G}(V, \tilde{V})$. Conversely, if $\mathcal{G}(\underset{\sim}{V}, \underset{\sim}{V})$ is an in-amalgamation of $\mathcal{G}(V, V)$ with corresponding intermediate initial state amalgamation $\mathcal{G}(\underset{\sim}{V}, V)$, then $\mathcal{G}(V, V)$ is an in-split of $\mathcal{G}(\underset{\sim}{V}, \underset{\sim}{V})$ with corresponding final state split $\mathcal{G}(\underset{\sim}{V}, V)$. Thus $\mathcal{G}(\tilde{V}, \tilde{V})$ is an in-split of $\mathcal{G}(V, V)$ if and only if $\mathcal{G}(V, V)$ is an in-amalgamation of $\mathcal{G}(\tilde{V}, \tilde{V})$

Also, if $\mathcal{G}_1 = \mathcal{G}(V_1, V_1)$ is the complete in-split of $\mathcal{G} = \mathcal{G}(V, V)$, then $\mathcal{G}$ is the complete in-amalgamation of $\mathcal{G}_1$. To see this, note that the intermediate final state split $\mathcal{G}(V, V_1)$ has exactly one edge e_J with terminal state $t(e_J) = J$ for each vertex J of V_1. Thus the complete initial state amalgamation of $\mathcal{G}_1 = \mathcal{G}(V_1, V_1)$ is $\mathcal{G}(V, V_1)$ and the complete in-amalgamation of $\mathcal{G}_1$ is $\mathcal{G}$. The converse of this is false; namely if $\mathcal{G}_c$ is the complete in-amalgamation of $\mathcal{G}$, then the complete in-split of $\mathcal{G}_c$ is not in general $\mathcal{G}$. For example, the complete in-amalgamation $\mathcal{G}_c$ of the graph $\mathcal{G}$ given in Example 3.2 has complete in-split different from $\mathcal{G}$. We shall see however that the complete in-split of $\mathcal{G}_c$ is always an in-split of $\mathcal{G}$.

We fix some notation regarding partitions $\mathcal{T}$ of an arbitrary set V. We say $\mathcal{T}_1 \prec \mathcal{T}_2$ if and only if each element of $\mathcal{T}_2$ is contained in some element of $\mathcal{T}_1$.

The collection $\mathcal{P}$ of partitions of V not only forms a partially ordered set but a complete lattice. To see this note that any collection $\mathcal{S}$ of subsets covering V determines a partition, namely that given by the equivalence relation $x \sim y$ if and only if $x \in \mathrm{S}$ for some $\mathrm{S} \in \mathcal{S}$ implies $y \in \mathrm{S}$. For $\mathcal{T}_\alpha \in \mathcal{P}$, $\alpha \in A$ define $\bigvee \mathcal{T}_\alpha$ to be the partition determined by the collection of subsets $\mathcal{S} = \bigcup_{\alpha \in A} \{\mathrm{T} \mid \mathrm{T} \in \mathcal{T}_\alpha\}$ and set $\bigwedge \mathcal{T}_\alpha = \bigvee \{\mathcal{T} \mid \mathcal{T} \prec \mathcal{T}_\alpha (\alpha \in A)\}$.

For $\mathcal{G} = \mathcal{G}(V,V)$ let $\mathcal{T}_\mathcal{G}$ be the 'base' partition $\{T_k \mid k = 1, \dots, l\}$ of V defined by the nonempty subsets S_J^m, $(m \in \mathbb{N},\ J \in V)$ described above for forming in-amalgamations, and $\mathcal{T}_w$ the partition of V consisting of the singleton subsets of V. Then $\mathcal{T}_\mathcal{G} \prec \mathcal{T}_w$, and any in-amalgamation $\mathcal{G}_1 = \mathcal{G}(V_1, V_1)$ of $\mathcal{G}$ is defined using a partition $\mathcal{T}_1$ of V with $\mathcal{T}_\mathcal{G} \prec \mathcal{T}_1 \prec \mathcal{T}_w$ and $\mathrm{V}_1 = V/\mathcal{T}_1$. The trivial in-amalgamation $\mathcal{G}$ itself corresponds to the partition $\mathcal{T}_w$ while the complete in-amalgamation of $\mathcal{G}$ corresponds to the partition $\mathcal{T}_\mathcal{G}$.

Lemma 3.3. *If the graphs $\mathcal{G}_k = \mathcal{G}(V_k, V_k)$ are in-amalgamations of $\mathcal{G}$ corresponding to two partitions $\mathcal{T}_k$ of V, $(k = 1, 2)$, then $\mathcal{T}_1 \prec \mathcal{T}_2$ implies that $\mathcal{G}_1$ is an in-amalgamation of $\mathcal{G}_2$.*

Proof. If $\mathcal{T}_\mathcal{G} = \{T_k \mid k = 1, \dots, l\}$ as above then recall that if a vertex I in T_k is connected to a vertex J with multiplicity m, then every vertex in T_k is connected to J with multiplicity m. Since $\mathcal{T}_\mathcal{G} \prec \mathcal{T}_1 = \{T_k^1 \mid k = 1, \dots, l_1\}$ the same is true for the vertices in T_k^1. To form an in-amalgamation of $\mathcal{G}_2(V_2, V_2)$ where $V_2 = V/\mathcal{T}_2$ one needs the 'base' partition $\mathcal{T}_{\mathcal{G}_2}$ of V_2. When viewed as a partition of V we have that $\mathcal{T}_{\mathcal{G}_2} \prec \mathcal{T}_1$. Thus $\mathcal{T}_1$ also defines a partition of V_2 with $\mathcal{T}_{\mathcal{G}_2} \prec \mathcal{T}_1$ and $\mathcal{G}_1$ is an in-amalgamation of $\mathcal{G}_2$. □

Corollary 3.4. *The complete amalgamation $\mathcal{G}_c$ of $\mathcal{G}$ is itself an in-amalgamation of any other in-amalgamation $\mathcal{G}_1$ of $\mathcal{G}$.*

Corollary 3.5. *If $\mathcal{G}_1$, $\mathcal{G}_2$ are two in-amalgamations of $\mathcal{G} = \mathcal{G}(V,V)$, then there is a graph $\mathcal{G}_1 \wedge \mathcal{G}_2$ which is an in-amalgamation of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}$, and any in-amalgamation of $\mathcal{G}$, which is also an in-amalgamation of $\mathcal{G}_1$ and $\mathcal{G}_2$ is an in-amalgamation of $\mathcal{G}_1 \wedge \mathcal{G}_2$.*

Proof. Let $\mathcal{T}_\mathcal{G}$ be the 'base' partition of V used to form in-amalgamations of $\mathcal{G}$. Let $\mathcal{T}_k$ be the partitions of V corresponding to the in-amalgamations $\mathcal{G}_k$, $k = 1, 2$, so that $\mathcal{T} \prec \mathcal{T}_k \prec \mathcal{T}_w$, $k = 1, 2$. Then $\mathcal{T} \prec \mathcal{T}_1 \wedge \mathcal{T}_2 \prec \mathcal{T}_k \prec \mathcal{T}_w$, $k = 1, 2$, so the in-amalgamation corresponding to $\mathcal{T}_1 \wedge \mathcal{T}_2$, named $\mathcal{G}_1 \wedge \mathcal{G}_2$, is an in-amalgamation of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}$ with the required properties. □

Definition 3.6. Define a partial order $\prec$ on the set of all finite graphs $\mathcal{G}(V,V)$ (with V finite) by setting $\mathcal{G}_0 \prec \mathcal{G}$ if there is a finite collection of graphs $\mathcal{G}_k$, $k = 0, \dots, l$, with $\mathcal{G}_k$ an in-amalgamation of $\mathcal{G}_{k+1}$ $(k = 0, 1, \dots, l-1)$ and $\mathcal{G}_l = \mathcal{G}$.

Theorem 3.7. *For a given graph $\mathcal{G} = \mathcal{G}(V,V)$ there is a unique graph $\underset{\sim}{\mathcal{G}}$ with*

1. $\underset{\sim}{\mathcal{G}} \prec \mathcal{G}$,
2. *$\underset{\sim}{\mathcal{G}}$ is a minimal element.*

Proof. To show existence start with $\mathcal{G}_0 = \mathcal{G}$ and inductively form a sequence $\mathcal{G}_k$, with $\mathcal{G}_{k+1}$ the complete in-amalgamation of $\mathcal{G}_k$. Since the set V is finite there is a

k_0 with $\mathcal{G}_k = \mathcal{G}_{k_0}$ for $k \geq k_0$. Set $\underset{\sim}{\mathcal{G}} = \mathcal{G}_{k_0}$. It is clear that $\underset{\sim}{\mathcal{G}}$ is minimal by Corollary 3.4 and the note preceding Example 3.2.

It remains to show uniqueness. Suppose $\mathcal{J}$ is a graph with $\mathcal{J} \prec \mathcal{G}$ and $\mathcal{J}$ minimal. Since $\mathcal{J} \prec \mathcal{G}$ there is a finite sequence $\mathcal{J}_0, \dots, \mathcal{J}_p$ with $\mathcal{J}_0 = \mathcal{G}$, $\mathcal{J}_p = \mathcal{J}$ and $\mathcal{J}_{k+1}$ an in-amalgamation of $\mathcal{J}_k$, $k = 0, \dots, p-1$. We claim that $\mathcal{G}_k \prec \mathcal{J}_k$ for all k. For $k = 1$, this is clear by Corollary 3.4 since $\mathcal{J}_1$ is an in-amalgamation of $\mathcal{G}$ while $\mathcal{G}_1$ is a complete in-amalgamation of $\mathcal{G}$. Now proceed by induction and show that $\mathcal{G}_{k+1} \prec \mathcal{J}_{k+1}$ if $\mathcal{G}_k \prec \mathcal{J}_k$. Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be graphs with $\mathcal{H}_1 = \mathcal{J}_k$, and $\mathcal{H}_r = \mathcal{G}_k$ and $\mathcal{H}_{j+1}$ an in-amalgamation of $\mathcal{H}_j$ for all j. Set $\mathcal{H}'_1 = \mathcal{J}_{k+1}$. Then by Corollary 3.5 $\mathcal{H}'_2 = \mathcal{H}_2 \wedge \mathcal{H}'_1$ is defined, since $\mathcal{H}_2$ and $\mathcal{J}_{k+1}$ are both in-amalgamations of $\mathcal{J}_k$. Again, since $\mathcal{H}_3$ and $\mathcal{H}'_2$ are both in-amalgamations of $\mathcal{H}_2$, set $\mathcal{H}'_3 = \mathcal{H}_3 \wedge \mathcal{H}'_2$, and recursively, $\mathcal{H}'_j = \mathcal{H}_j \wedge \mathcal{H}'_{j-1}$ for $j = 1, \dots, r$. Thus $\mathcal{H}'_j$ is an in-amalgamation of $\mathcal{H}'_{j-1}$ and $\mathcal{H}'_r = \mathcal{G}_k \wedge \mathcal{H}'{}_{r-1}$. By Corollary 3.4 it follows that $\mathcal{G}_{k+1}$ is an in-amalgamation of $\mathcal{H}'_r$. If we set $\mathcal{H}'_{r+1} = \mathcal{G}_{k+1}$ we obtain a chain $\mathcal{H}'_j$ of length $r+1$ of in-amalgamations, showing that $\mathcal{G}_{k+1} \prec \mathcal{J}_{k+1}$.

Choosing k large enough we obtain $\underset{\sim}{\mathcal{G}} \prec \mathcal{J}$. Since $\mathcal{J}$ is minimal, we conclude that $\underset{\sim}{\mathcal{G}} = \mathcal{J}$. □

This last result allows one to use the partial order on the set of all finite bipartite graphs $\mathcal{G}(V,V)$ to partition this set and thus obtain an equivalence relation on finite bipartite graphs $\mathcal{G}(V,V)$. Indeed if $\mathcal{J}_1$ and $\mathcal{J}_2$ are minimal elements, then the intervals of the partial order, $I_{\mathcal{J}_k} = \{\mathcal{G} = \mathcal{G}(V,V) \mid \mathcal{J}_k \prec \mathcal{G}(V,V)\}$ for $k = 1, 2$ are either disjoint or equal. The intervals $I_{\mathcal{J}}$ with $\mathcal{J}$ a minimal element are also the directed subsets of the set of all finite bipartite graphs with respect to this partial order.

A similar approach is also available for studying in-splits of graphs. For $\mathcal{G} = \mathcal{G}(V,V)$, let $S_{\mathcal{G}}$ be the partition of the edge set $\mathcal{E}$ given by $\{\mathcal{E}_J \mid J \in V\}$ and S_w the partition of $\mathcal{E}$ given by the singleton subsets of $\mathcal{E}$. Then $S_{\mathcal{G}} \prec S_w$, and any in-split $\mathcal{G}_1$ of $\mathcal{G}$ is defined by using a partition S_1 of $\mathcal{E}$ with $S_G \prec S_1 \prec S_w$. The trivial in-split $\mathcal{G}$ itself corresponds to the partition $S_{\mathcal{G}}$, while the complete in-split of $\mathcal{G}$, denoted $\mathcal{G}_w$, corresponds to the partition S_w.

Lemma 3.8. *If the graphs $\mathcal{G}_k = \mathcal{G}(V_k, V_k)$ are in-splits of $\mathcal{G} = \mathcal{G}(V,V)$ corresponding to the partitions S_k of the edge set $\mathcal{E}$ of $\mathcal{G}$ $(k = 1, 2)$, then $S_1 \prec S_2$ implies that $\mathcal{G}_2$ is an in-split of $\mathcal{G}_1$.*

Proof. Recall that for e an edge of a graph $\mathcal{G}(V,W)$, $i(e)$ denotes the initial vertex of e and $t(e)$ denotes the terminal vertex of e. Also, for $J \in V$, $\mathcal{E}_J = \{e \in \mathcal{E} \mid t(e) = J\}$. Since $S_1 \prec S_2$ we have that S_1 is the partition $\mathcal{E}^k_J$, $1 \leq k \leq m(J)$, $J \in V$ of $\mathcal{E}$, with $\mathcal{E}_J = \dot{\bigcup}_{k=1}^{m(J)} \mathcal{E}^k_J$, while S_2 is the partition of $\mathcal{E}$ defined by $\mathcal{E}^{k,l}_J$, $J \in V$, $1 \leq k \leq m(J)$ and $1 \leq l \leq m_k(J)$, with $\mathcal{E}^k_J = \dot{\bigcup}_{l=1}^{m_k(J)} \mathcal{E}^{k,l}_J$. For $\mathcal{G}$, the vertex set $V_1 = \dot{\bigcup}_{J \in V} \{(J,k) \mid 1 \leq k \leq m(J)\}$ and for $\mathcal{G}_2$ the vertex set $V_2 = \dot{\bigcup}_{(J,k) \in V_1} \{(J,k,l) \mid 1 \leq l \leq m_k(J)\}$. For the edge set $\mathcal{E}^1$ of $\mathcal{G}_1$ we have $\mathcal{E}^1 = \bigcup_{(J,k) \in V_1} \mathcal{E}^1_{(J,k)}$ with $\mathcal{E}^1_{(J,k)} = \bigcup_{e \in \mathcal{E}^k_J} \{(e,j) \mid 1 \leq j \leq m(i(e))\}$, where (e,j) has initial vertex $(i(e), j)$ in V_1 and terminal vertex

(J,k). For $\mathcal{G}_2$ the edge set $\mathcal{E}^2 = \bigcup_{(J,k,l)\in V_2} \mathcal{E}^2_{(J,k,l)}$ with $\mathcal{E}^2_{(J,k,l)} = \bigcup_{e\in\mathcal{E}^{(k,l)}_J} \{(e,j,h) \mid 1 \le j \le m(i(e)), 1 \le h \le m_j(i(e))\}$ where (e,j,h) has initial vertex $(i(e),j,h)$ and terminal vertex (J,k,l).

To form an in-split of $\mathcal{G}_1$ use the partition $\mathcal{E}^{1l}_{(J,k)}$, $1 \le l \le m_k(J)$ of $\mathcal{E}^1_{(J,k)}$, where $\mathcal{E}^{1l}_{(J,k)} = \{(e,j) \mid 1 \le j \le m(i(e)), e \in \mathcal{E}^{kl}_J\}$. This is the partition of $\mathcal{E}^1_{(J,k)}$ determined by the partition of $\mathcal{E}^k_J$ by the sets $\mathcal{E}^{kl}_J$, $1 \le l \le m_K(J)$. It is clear that the vertex set formed by this in-split of $\mathcal{G}_1$ is the same as V_2 with $m(J,k) = m_k(J)$.

The edges $\tilde{\mathcal{E}}$ for this in-split of $\mathcal{G}_1$ are $\tilde{\mathcal{E}} = \bigcup_{(J,k,l)\in V_2} \tilde{\mathcal{E}}_{(J,k,l)}$ with $\tilde{\mathcal{E}}_{(J,k,l)} = \bigcup_{\tilde{e}\in\mathcal{E}^{1l}_{(J,k)}} \{(\tilde{e},h) \mid 1 \le h \le m(i(\tilde{e}))\} = \bigcup_{(e,j)\in\mathcal{E}^{1l}_{(J,k)}} \{(e,j,h) \mid 1 \le j \le m(i(e)), 1 \le h \le m_j(i(e))\}$ which is clearly $\mathcal{E}^2_{(J,k,l)}$. Thus $\mathcal{G}_2$ is an in-split of $\mathcal{G}_1$. □

Corollary 3.9. *The complete in-split $\mathcal{G}_w$ of $\mathcal{G} = \mathcal{G}(V,V)$ is itself an in-split of any other in-split $\mathcal{G}_1$ of $\mathcal{G}$.*

Corollary 3.10. *If $\mathcal{G}_1, \mathcal{G}_2$ are two in-splits of $\mathcal{G} = \mathcal{G}(V,V)$, then there is a graph $\mathcal{G}_1 \vee \mathcal{G}_2$ which is an in-split of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}$ such that any in-split of $\mathcal{G}$ which is also an in-split of $\mathcal{G}_1$ and $\mathcal{G}_2$ is then an in-split of $\mathcal{G}_1 \vee \mathcal{G}_2$.*

Proof. Let $S_{\mathcal{G}}$ be the partition $\mathcal{E}_J$, $J \in V$, of the edge set $\mathcal{E}$ of $\mathcal{G} = \mathcal{G}(V,V)$ and let S_k be the partitions of $\mathcal{E}$ corresponding to the graphs $\mathcal{G}_k$, $k = 1,2$, so that $S_{\mathcal{G}} \prec S_k \prec S_w$, $k = 1,2$. Define $\mathcal{G}_1 \vee \mathcal{G}_2$ to be the in-split of $\mathcal{G}$ given by the partition $S_1 \vee S_2$ of $\mathcal{E}$. □

Corollary 3.11. *Let $\mathcal{G}$ be a graph with an in-amalgamation $\mathcal{G}_1$. There is a multiplicity free graph $\mathcal{H}$ so that both $\mathcal{G}$ and $\mathcal{G}_1$ are in-amalgamations of $\mathcal{H}$.*

Proof. We will show something slightly stronger. Let $\mathcal{H}_1$ be any multiplicity free in-split of $\mathcal{G}_1$. These certainly exist since the complete in-split of $\mathcal{G}_1$ is multiplicity free. Then one can form $\mathcal{H} = \mathcal{G} \vee \mathcal{H}_1$, since $\mathcal{G}$ is also an in-split of $\mathcal{G}_1$. Note that if $\mathcal{H}_1$ is the complete in-split of $\mathcal{G}_1$, then $\mathcal{H} = \mathcal{H}_1$. Now apply Corollary 3.10. □

Before the relationship between representations of $\mathcal{O}_A$ and the associated endomorphisms is investigated further we translate our partial order on graphs to one on nonnegative integer valued square matrices, and show how the compact space X_A associated to a matrix A behaves under this partial order.

Recall that with a graph $\mathcal{G}(V,W)$ we associate a $|W| \times |V|$ matrix $A_{\mathcal{G}}$ with nonnegative integer entries: $A_{\mathcal{G}}(J,I) =$ number of edges from $I \in V$ to $J \in W$. This is, as mentioned in section 1, the transpose of the usual adjacency matrix of the graph. Conversely, if A is an $m \times n$ matrix with nonnegative integer entries then $\mathcal{G}_A$ is the graph $\mathcal{G}(V,W)$ with $V = \{1,\dots,n\}$, $W = \{1,\dots,m\}$ with $A(j,i)$ edges from i in V to j in W.

Definition 3.12. We say a matrix $A_1 = A_{\mathcal{G}_1}$ associated with a graph $\mathcal{G}_1$ is an in-split (or in-amalgamation) of $A = A_{\mathcal{G}}$ iff $\mathcal{G}_1$ is an in-split (or in-amalgamation) of $\mathcal{G}$. We also translate the partial order $\prec$ defined previously on graphs $\mathcal{G}(V,V)$ to a partial order on square matrices with nonnegative integer entries.

The equivalence relation on the set of finite bipartite graphs defined by the minimal elements of the partial order then gives rise to an equivalence relation on the set of square nonnegative integer valued matrices with non-zero rows and columns. Two such matrices are equivalent if they both determine the same minimal element. It follows that two such matrices A and B are equivalent if there is a finite chain of matrices $A_0, \dots , A_n$ so that A_i is either an in-split or an in-amalgamation of A_{i+1} for each $i \in \{0, \dots , n-1\}$. It is clear that this is a stronger equivalence relation than the strong shift equivalence relation on square matrices, so that A is equivalent to B implies that A is strong shift equivalent to B. Computationally it is a straightforward task to calculate the minimal element of our partial order determined by any given matrix, so one can quickly and easily determine whether two given matrices are equivalent or not, in contrast to the strong shift equivalence relation.

Recall ([9]) that for an $n \times n$ matrix A with nonnegative integer entries and satisfying the analogue of condition I of [9], the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined to be $\mathcal{O}_{A_w}$, where A_w is the square matrix with entries in 0-1 with the larger index set $\{(i, k, j) \mid i, j \in \{1, \dots , n\}, 1 \le k \le A(i, j)\}$;

$$A_w(i, k, j)(i', k', j') = \begin{cases} 1 & \text{if } j = i', \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that this matrix A_w defined in [9] is actually the matrix of the complete in-split $\mathcal{G}_w$ of the graph $\mathcal{G} = \mathcal{G}_A$. For example, the vertex i in $\mathcal{G}_A$ is split into $\sum_j A(i, j)$ vertices, corresponding to the new vertices $\{(i, k, j) \mid j \in \{1, \dots , n\}, 1 \le k \le A(i, j)\}$. The vertices $\{(i, k, j) \mid 1 \le k \le A(i, j)\}$ correspond to the $A(i, j)$ edges from j to i in $\mathcal{G}$. By a generating Hilbert space of partial isometries for $\mathcal{O}_A$ we will then mean a representation $\phi : \mathcal{F} \to \mathcal{O}_{A_w}$ where A_w is the complete in-split of A.

For A a square matrix with nonnegative integer entries we may define A to have condition I if the complete in-split matrix A_w, a 0-1 matrix, satisfies condition I. To be consistent it should be checked that if A is a 0-1 square matrix and B an in-split of A, then A satisfies condition I if and only if B does. Restated, we must have that the one-sided vertex shift space X_A has isolated points if and only if X_B has isolated points. It is known that the two-sided edge shift spaces for matrices A and B, one matrix the in-split of the other, must be topologically conjugate ([17]). We show this also to be the case for the spaces X_A.

Note that since A is a 0-1 matrix then the one-sided vertex shift space $X_A = \{(x_k)_{k\in\mathbb{N}} \mid A(x_k, x_{k+1}) = 1\}$ may be identified with the one-sided edge shift space $\{(e_k)_{k\in\mathbb{N}} \in \prod_{\mathbb{N}} \mathcal{E}_A \mid t(e_{k+1}) = i(e_k), k \ge 0\}$. Note that the paths are in this direction because of our convention that $e \in \mathcal{E}_A$ if and only if $A(t(e), i(e)) = 1$.

Proposition 3.13. *If A, B are square 0-1 matrices with B an in-split of A, then X_A is homeomorphic with X_B.*

Proof. We provide a sketch of the proof only. An edge in $\mathcal{G}_B$ is the form (e, j) for some j, $1 \le j \le m(i(e))$, and for e an edge in $\mathcal{G}_A$. This yields a continuous surjection of the edge set $\mathcal{E}_B$ of $\mathcal{G}_B$ to $\mathcal{E}_A$ which defines a continuous map $\psi : X_B \to X_A$, namely the product of this surjection over $\mathbb{N}$, restricted to X_B. A map $\phi : X_A \to X_B$ is defined by taking the product over $\mathbb{N}$ of a map that takes pairs of edges ef with $e, f \in \mathcal{E}_A$ and $t(f) = i(e)$ to the edge (e, k) in $\mathcal{E}_B$, where f is in the k-th partition $\mathcal{E}^k_{t(f)} \subseteq \mathcal{E}_B$. A straightforward check shows $\phi \circ \psi = Id_{X_B}$ and $\psi \circ \phi = Id_{X_A}$. $\square$

Corollary 3.14. *If A is a square 0-1 matrix and if B is an in-split of A, then A satisfies condition I of Cuntz-Krieger if and only if B does.*

Proof. Proposition 3.13 implies that X_A has isolated points if and only if X_B does. $\square$

There are other results concerning this partial order. For example if A, B are matrices with A_c, B_c denoting their respective complete in-amalgamations, then $A \prec B$ implies that $A_c \prec B_c$. To see this it is enough to show that if A is an in-amalgamation of B then A_c is an in-amalgamation of B_c, which follows from two applications of Corollary 3.4. The first shows that B_c must be an in-amalgamation of A while the second concludes that A_c must be an in-amalgamation of B_c. Similarly, by using Corollary 3.9, if A_w and B_w denote the complete in-splits of A and B respectively, then $A \prec B$ implies $A_w \prec B_w$. It follows that if A and B are two equivalent square matrices with nonnegative integer entries, then A satisfies condition I if and only if B does.

4. Endomorphisms and their extensions

In this section we investigate relationships between endomorphisms induced by representations of dual systems $(\boldsymbol{P}, A, \boldsymbol{Q})$ in terms of the partial order on square matrices A. For example, if $A \prec B$ then roughly speaking the endomorphism associated with a representation of a dual system involving B is the restriction of the endomorphism coming from a representation of a dual system involving A. A more restricted version of the converse, which may however be applied inductively, also holds and is proved at the end of the section.

Proposition 4.1. *Let A, B be two square* 0-1 *matrices with $A \prec B$ or $B \prec A$. If $\phi : \mathcal{K} \to \mathcal{B}(\mathcal{H})$ is a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$, then there is a representation $\psi : \tilde{\mathcal{K}} \to \mathcal{B}(\mathcal{H})$ of a dual system $(\tilde{\boldsymbol{P}}, B, \tilde{\boldsymbol{Q}})$ so that the $*$-algebras generated by $\phi(\mathcal{K})$ and $\psi(\tilde{\mathcal{K}})$ are equal and the AF C^*-algebras $\mathcal{F}_A$ and $\mathcal{F}_B$ are equal. The associated unital $*$-endomorphism φ_ψ is the restriction of φ_ϕ if $A \prec B$ and vice versa if $B \prec A$.*

Proof. We assume $A \prec B$ as the proof for $B \prec A$ is similar. In this case there is a finite sequence of in-splits from A to B, so we may assume that B is itself an in-split of A.

Let $\mathcal{G}_A = \mathcal{G}(V, V)$ be the graph associated with A where $V = \{1, \dots, n\}$, $n = \dim \mathcal{K}$. Also, as usual, denote by $\mathcal{E}_J$ the elements of the edge set $\mathcal{E}$ of $\mathcal{G}_A$ with terminal vertex J. Choose a basis $\{e_K \mid K \in V\}$ of $\mathcal{K}$ so that $S_K = \phi(e_K)$ is a partial isometry with initial projection $q_K = \sum A(K, I)p_I$ and final projection p_K as in the comments preceding Proposition 2.1 and Theorem 2.2. Let $\{\mathcal{E}_J^k \mid J \in V, 1 \le k \le m(J)\}$ be the partition of $\mathcal{E}$ defining the in-split graph $\mathcal{G}_B(W, W)$ of $\mathcal{G}_A$, where $\mathcal{G}_B$ has associated matrix B and edge set $\tilde{\mathcal{E}}$. Write $W = \{J_k \mid J \in V, 1 \le k \le m(J)\}$ and set $\tilde{\mathcal{K}}$ to be a Hilbert space of dimension $\sum_{J \in V} m(J) = |W|$. It will be enough to show that there are partial isometries T_I, $I \in W$, in the $*$-algebra generated by the S_K, $K \in V$, that also generate this $*$-algebra and satisfy $T_K^* T_K = \sum B(K, I) T_I T_I^*$.

Since A is a 0-1 matrix, the graph $\mathcal{G}_A$ is multiplicity free and there is a bijection between the edges e in $\mathcal{E}_J$ and the vertices I in V with I connected to J, i.e., with $I = i(e)$ and $J = t(e)$ for some $e \in \mathcal{E}$. Since B is an in-split of A, B is also a matrix with entries in $\{0, 1\}$. Using this bijection we then write that a vertex $I \in \mathcal{E}_J$

even though, strictly speaking, $\mathcal{E}_J$ consists of edges. Thus $S_J^*S_J = \sum_{I\in\mathcal{E}_J} S_IS_I^*$ since $A(J,I)=1$ iff $I\in\mathcal{E}_J$.

For $J_k \in W$ define $T_{J_k} = S_J\left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)$. Since T_{J_k} is of the form vp with v a partial isometry and p a projection with $p\le v^*v$, we have that T_{J_k} is a partial isometry. It is clear that the $*$-algebra generated by the partial isometries T_{J_k}, $(J_k \in W)$, is contained in the $*$-algebra generated by the S_I, $I\in V$. However $S_I = S_IS_I^*S_I = S_I\sum_{K\in\mathcal{E}_I} S_KS_K^* = S_I\sum_{j=1}^{m(I)}\sum_{K\in\mathcal{E}_I^j} S_KS_K^* = \sum_{j=1}^{m(I)} T_{I_j}$, so $\phi(\mathcal{K})\subseteq\psi(\tilde{\mathcal{K}})$ and the two $*$-algebras are equal.

We next need to show that $T_{J_k}^*T_{J_k} = \sum_{P_l\in\tilde{\mathcal{E}}_{J_k}} T_{P_l}T_{P_l}^*$. The left side

$$\begin{aligned}
&= \left(\sum_{J\in\mathcal{E}_J^k} S_IS_I^*\right) S_J^*S_J\left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)\\
&= \left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)\left(\sum_{I\in\mathcal{E}_J} S_IS_I^*\right)\left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)\\
&= \sum_{I\in\mathcal{E}_J^k} S_IS_I^*.
\end{aligned}$$

Moreover the right side $= \sum_{P_l\in\tilde{\mathcal{E}}_{J_k}} S_P\left(\sum_{I\in\mathcal{E}_P^l} S_IS_I^*\right)S_P^*$. Now if $P_l\in\tilde{\mathcal{E}}_{J_k}$ for some l then it is true for all l with $1\le l\le m(P)$. Since $\mathcal{E}_P = \bigcup_{l=1}^{m(P)}\mathcal{E}_P^l$ we have that this last expression $= \sum_{P\in\mathcal{E}_J^k} S_P\left(\sum_{I\in\mathcal{E}_P} S_IS_I^*\right)S_P^* = \sum_{P\in\mathcal{E}_J^k} S_P(S_P^*S_P)S_P^* = \sum_{P\in\mathcal{E}_J^k} S_PS_P^*$, which is equal to the left side. Now note that $S_I^*S_I = \sum_{k,j=1}^{m(I)} T_{I_k}^*T_{I_j} = \sum_{k=1}^{m(I)} T_{I_k}^*T_{I_k}$ so it follows that $\mathrm{dom}(\varphi_\psi)\subseteq\mathrm{dom}(\varphi_\phi)$. For $a\in\mathrm{dom}(\varphi_\psi)$, Proposition 2.4 shows $\varphi_\psi(a)T = Ta$ for all $T\in\psi(\tilde{\mathcal{K}})$. Since $\phi(\mathcal{K})\subseteq\psi(\tilde{\mathcal{K}})$ this also holds for all $T\in\phi(\mathcal{K})$, so Proposition 2.4 implies that $\varphi_\psi(a)$ must be $\varphi_\phi(a)$.

It remains to show that $\mathcal{F}_A = \mathcal{F}_B$ where $\mathcal{F}_A = \bigcup_{k=1}^{\infty}\mathcal{F}_{kA}$ and $\mathcal{F}_{kA}$ is the finite dimensional C^*-algebra generated by $\{S_\mu P_I S_\nu \mid I\in V,\ |\nu| = |\mu| = k\}$ described before Theorem 2.6, and similarly for $\mathcal{F}_B$. Since $S_I = \sum_{j=1}^{m(I)} T_{I_j}$ it is clear that $\mathcal{F}_{kA}\subseteq\mathcal{F}_{kB}$ and so $\mathcal{F}_A\subseteq\mathcal{F}_B$. To see the other inclusion we first notice that $\mathcal{F}_{0B}\subseteq\mathcal{F}_{1A}$ since the final projection $T_kT_{J_k}^* = S_J\left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)S_J^*\in\mathcal{F}_{1A}$. We then have that $T_{J_k}\left(T_{I_l}T_{I_l}^*\right)T_{L_r}^* \subseteq S_J\left(\sum_{I\in\mathcal{E}_J^k} S_IS_I^*\right)\mathcal{F}_{1A}\left(\sum_{I\in\mathcal{E}_L^r} S_IS_I^*\right)S_L^* \subseteq S_J\mathcal{F}_{1A}S_L^*\subseteq\mathcal{F}_{2A}$, so $\mathcal{F}_{1B}\subseteq\mathcal{F}_{2A}$. In general $\mathcal{F}_{kB}\subseteq\mathcal{F}_{(k+1)A}$, and so $\mathcal{F}_A = \mathcal{F}_B$. $\square$

Theorem 4.2. *Let A be a square matrix with nonnegative integer entries satisfying condition* I *of Cuntz and Krieger. Let $\phi:\mathcal{E}\to\pi(\mathcal{O}_A)\subseteq\mathcal{B}(\mathcal{H})$ be an underlying*

Hilbert space of partial isometries for a representation π of $\mathcal{O}_A$ on a Hilbert space $\mathcal{H}$. If A is equivalent to B there is a representation $\tilde{\pi} : \mathcal{O}_B \to \mathcal{B}(\mathcal{H})$ with $\psi : (\mathcal{K}) \to \tilde{\pi}(\mathcal{O}_B)$ an underlying Hilbert space of partial isometries so that the $*$-algebras generated by $\phi(\mathcal{E})$ and $\psi(\mathcal{K})$ are equal. In particular $\pi(\mathcal{O}_A) = \tilde{\pi}(\mathcal{O}_B)$.

Proof. Since there is a finite sequence of in-amalgamations and in-splits from B to A it is enough to show this if A is an in-amalgamation of B. Let A_w and B_w be the complete in-splits of A and B respectively. By definition $\mathcal{O}_A = \mathcal{O}_{A_w}$ and $\mathcal{O}_B = \mathcal{O}_{B_w}$. By Corollary 3.9, A_w is an in-split of B , since B is an in-split of A. Applying Corollary 3.9 once again, we see that B_w is an in-split of A_w. The result follows from Proposition 4.1. ☐

Using a notion of when a finite direct sum of type I factors is included in another such sum we define a fairly weak notion of when an endomorphism ψ *extends* an endomorphism φ. Note that this is not the same as requiring φ to be a restriction of ψ.

Definition 4.3. Given $\mathcal{M} = \bigoplus_{i \in V} \mathcal{M}_i$, $\mathcal{N} = \bigoplus_{i \in W} \mathcal{N}_i$ finite direct sums of countably decomposable type I factors, we say a unital $*$-homomorphism $d : \mathcal{M} \to \mathcal{N}$ is a unital inclusion if the matrix d_* is a 0-1 matrix such that

a) each row has at least one nonzero entry,
b) each column has exactly one nonzero entry.

Such a 0-1 matrix is known as a division matrix [17]. Recall that $d_*(i, j)$ is by definition the multiplicity of $d(\mathcal{M}_j)$ in $\mathcal{N}_i$. Thus, if $\mathcal{E}$ is the edge set of the graph $\mathcal{G}(V, W)$ of the matrix d_*, then $|\mathcal{E}| = |V|$ by b) and $|W| \leq |\mathcal{E}|$ by a). Thus $|W| \leq |V|$. We remark that if $d : \mathcal{M} \to \mathcal{N}$ is a unital inclusion, then $d(\mathcal{M})' \cap \mathcal{N}$ is abelian. If $\mathcal{N}$ is represented in $\mathcal{B}(\mathcal{H})$ via a unital inclusion, then $d(\mathcal{M})'$ in $\mathcal{B}(\mathcal{H})$ is also abelian.

Let φ, ψ be unital $*$-endomorphisms of $\mathcal{M}$, $\mathcal{N}$ respectively with $|W| \leq |V|$ and D a division matrix with $|W|$ rows and $|V|$ columns. If the entries of D satisfy a dimension condition, $n_i = \sum D_{ij} m_j$, $(i \in W)$, and if $\psi_* D = D\varphi_*$, then by Propositions 1.1 and 1.2 of [5] there is a unital inclusion $d : \mathcal{M} \to \mathcal{N}$ with $\alpha \circ d \circ \varphi = \psi \circ d$. Here $\mathcal{M}_i$ is a type I_{m_i} factor and $\mathcal{N}_i$ is a type I_{n_i} factor. If $\mathcal{M}, \mathcal{N}$ are both direct sums of I_∞ factors only, the dimension condition on the entries D_{ij} of D is superfluous.

Definition 4.4. With $\mathcal{M}, \mathcal{N}$ as above and φ, ψ unital $*$-endomorphisms of $\mathcal{M}$ and $\mathcal{N}$ respectively, we say ψ extends φ if there is a unital inclusion $d : \mathcal{M} \to \mathcal{N}$ and α an inner $*$-automorphism of $\mathcal{N}$ such that $\psi \circ d = \alpha \circ d \circ \varphi$.

It is straightforward to show that the product of two division matrices, when defined, is again a division matrix. This can be used to show that if $\varphi : \mathcal{M} \to \mathcal{M}$, $\psi : \mathcal{N} \to \mathcal{N}$ and $\rho : \mathcal{R} \to \mathcal{R}$ are unital $*$-endomorphisms of $\mathcal{M}, \mathcal{N}, \mathcal{R}$ respectively, each a finite direct sum of type I factors, and if ψ extends φ and ρ extends ψ, then ρ also extends φ. To see this let $d_1 : \mathcal{M} \to \mathcal{N}$ and $d_2 : \mathcal{N} \to \mathcal{R}$ be unital inclusions with $\psi \circ d_1 = \alpha \circ d_1 \circ \varphi$ and $\rho \circ d_2 = \beta \circ d_2 \circ \psi$, where α, β are inner automorphisms of $\mathcal{N}$ and $\mathcal{R}$ respectively. Then $\rho \circ d_2 \circ d_1 = \beta \circ d_2 \circ \psi \circ d_1 = \beta \circ d_2 \circ \alpha \circ d_1 \circ \varphi$. However, it is evident that there is an inner automorphism $\tilde{\alpha}$ of $\mathcal{R}$ with $\tilde{\alpha} \circ d_2 = d_2 \circ \alpha$, yielding $\rho \circ d_2 \circ d_1 = \beta \circ \tilde{\alpha} \circ d_2 \circ d_1 \circ \psi$. Since $(d_2 \circ d_1)_* = d_{2*} \circ d_{1*}$ is again a division matrix, $d_2 \circ d_1$ is a unital inclusion and ρ extends φ.

The following fact ([17]) is used in the next theorem. For A, B square matrices with nonnegative integer entries, B is an in-split of A, or A is an in-amalgamation of B, if and only if there is a division matrix D and a matrix E with nonnegative integer entries so that $A = DE$ and $B = ED$.

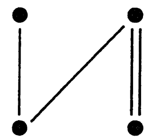
Theorem 4.5. *Let $\varphi : \mathcal{M} \to \mathcal{M}$, $\psi : \mathcal{N} \to \mathcal{N}$ be unital $*$-endomorphisms, where $\mathcal{M}$, $\mathcal{N}$ are finite direct sums of countably decomposable type* I_∞ *factors. If $\psi_* \prec \varphi_*$ then ψ extends φ.*

Proof. Since there are a finite number of in-amalgamation procedures required to arrive at ψ_* from φ_* the comments following Definition 4.4 show that it is enough to show this if the matrix ψ_* is an in-amalgamation of the matrix φ_*. By the above-mentioned fact there is a division matrix D and a nonnegative integer valued matrix E so that $\psi_* = DE$ and $\varphi_* = ED$. By Proposition 1.2 of [5] there is a unital $*$-homomorphism $d : M \to N$ with $d_* = D$. Thus d is a unital inclusion. The matrix $(\psi \circ d)_* = \psi_* D = DED = D\varphi_* = (d \circ \varphi)_*$, so Proposition 1.1 of [5] implies that there is an inner automorphism α of $\mathcal{N}$ with $\alpha \circ d \circ \varphi = \psi \circ d$. □

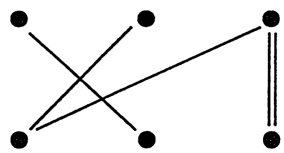
One may extend the result of Theorem 4.5 to the more general context of finite direct sums of type I factors by carefully keeping track of the dimensions of the finite type I factors that may occur. If φ is a unital $*$-endomorphism of $\mathcal{M} = \bigoplus_{i=1}^{m} \mathcal{M}_i$ with M_i a type I_{m_i} factor, then $m_i = \sum \varphi_*(i,j) m_j$. Denoting the finite part of $\mathcal{M}$, namely the sum of the finite type I factors occurring in $\mathcal{M}$, by $\mathcal{M}_f$ it is clear that $\varphi^{-1}(\mathcal{M}_f) \subseteq \mathcal{M}_f$. In fact if $m_i < \infty$, then $\varphi^{-1}(\mathcal{M}_i) \subseteq \bigoplus_{m_j < m_i} \mathcal{M}_j$ or $\varphi^{-1}(\mathcal{M}_i) = \mathcal{M}_j$ with $m_j = m_i$. If $\mathcal{M}_f$ is actually invariant under φ, so that $\varphi(\mathcal{M}_f) \subseteq \mathcal{M}_f$, and if φ is injective, as is the case for endomorphisms arising from representations of Cuntz-Krieger algebras, or also more generally of dual systems, then $\varphi|_{\mathcal{M}_f}$ is an isomorphism of $\mathcal{M}_f$. In this situation the associated matrix $(\varphi|_{\mathcal{M}_f})_*$ is a permutation matrix, or rather the identity matrix modified with some possible permutation matrix subblocks. Thus the graph of $\varphi|_{\mathcal{M}_f}$ is a minimal element in our partial order; it has no further possible in-amalgamations so remains fixed in any in-amalgamation process applied to φ. In general however, $\mathcal{M}_f$ may not be invariant under φ as finite type I summands of $\mathcal{M}_f$ may be mapped to type I_∞ factor summands of $\mathcal{M}$. If $\mathcal{T} = \{T_k \mid k = 1, \ldots, l\}$ is the base partition of $\{1, \ldots, m\}$ used for forming any in-amalgamation of the graph of φ_*, then a set T_k must consist solely of finite vertices, namely vertices j with $m_j < \infty$, if there is a finite vertex i and a $j \in T_k$ with $\varphi_*(i,j) \neq 0$. By keeping track of the sums of the m_j that occur in the subsets of the partition of T_k used in forming a particular in-amalgamation of the graph of φ_* it is possible to extend Theorem 4.5 to this more general context. One needs to further require that each n_i, where $\mathcal{N} = \bigoplus_{i=1}^{n} \mathcal{N}_i$, $\mathcal{N}_i$ a type I_{n_i} factor, is the sum of the m_k associated with n_i in the course of the finite number of in-amalgamation steps needed.

The converse of Theorem 4.5 is false, namely there are unital $*$-endomorphisms φ and ψ of finite direct sums of type I_∞ factors so that ψ extends φ in our weak sense yet $\psi_* \not\prec \varphi_*$. In fact ψ_* and φ_* can determine two different minimal elements of the partial order, so they may not even be equivalent under the relation determined by in-amalgamations and in-splits.

For example, define ψ to be the unital $*$-endomorphism of $\mathcal{N}$, a sum of two type I_∞ factors, with $\psi_* = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The graph of ψ_* is



Let φ be the unital $*$-endomorphism of $\mathcal{M}$, a sum of three type I_∞ factors with $\varphi_* = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, so the graph of φ_* is



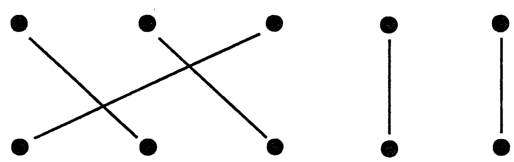
Both of these are minimal elements in our partial order, so ψ_* cannot be an in-amalgamation of φ_*. However if $d : \mathcal{M} \to \mathcal{N}$ is the unital inclusion with $d_* = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\psi_* d_* = d_* \varphi_* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and so ψ extends φ via d. It should be noted that φ is not the restriction of ψ, i.e., that $\psi \circ d$ is not $d \circ \varphi$, so that the inner automorphism in our definition of extension is a weakening of the concept of restriction.

In fact the notion of extension used here is quite weak. We see below that one way this is reflected is in the isomorphism classes of the direct limit C^*-algebras $\varinjlim(\mathcal{M}, \varphi_n)$ where $\varphi_n = \varphi$, $n \in \mathbb{N}$, for φ a unital injective $*$-endomorphism of a finite direct sum of type I_∞ factors, with φ_* a matrix with nonnegative integer entries. We first quickly show that the isomorphism class of these C^*-algebras is unaffected when φ is replaced by an equivalent endomorphism ψ.

Proposition 4.6. *Let $\varphi : \mathcal{M} \to \mathcal{M}$ and $\psi : \mathcal{N} \to \mathcal{N}$ be injective unital $*$-endomorphisms of finite direct sums of type I_∞ factors where φ_*, ψ_* are matrices with nonnegative integer entries. If $\psi_* \prec \varphi_*$ then $\varinjlim(\mathcal{M},\varphi) \simeq \varinjlim(\mathcal{N}, \psi)$.*

Proof. It is enough to show this if ψ_* is an in-amalgamation of φ_*. There is a division matrix D and a matrix E with nonnegative integer entries with $\varphi_* = ED$ and $\psi_* = DE$. Since the matrices φ_*, ψ_*, and D all have a nonzero element in each row and column, the same is true for E. By Proposition 1.2 of [5] there is a unital inclusion $d : \mathcal{M} \to \mathcal{N}$ and an injective unital $*$-homomorphism $e : \mathcal{N} \to \mathcal{M}$ with $d_* = D$ and $e_* = E$. Since $\varphi_* = e_* \circ d_*$ there is an inner automorphism α_1 of $\mathcal{M}$ with $\varphi = \alpha_1 \circ e \circ d$. Continuing in this manner one arrives at two sequences of compatible injective unital $*$-homomorphisms defining injective unital $*$-homomorphisms $\varepsilon : \varinjlim(\mathcal{N}, \psi) \to \varinjlim(\mathcal{M}, \varphi)$ and $\delta : \varinjlim(\mathcal{M}, \varphi) \to \varinjlim(\mathcal{N}, \psi)$ which are inverse to each other. $\square$

If ψ were an extension of φ, so $d_* \varphi_* = \psi_* d_*$ with $d : \mathcal{M} \to \mathcal{N}$ a unital inclusion, then there is always a unital C^*-homomorphism $\delta : \varinjlim(\mathcal{M}, \varphi) \to \varinjlim(\mathcal{N}, \psi)$. Since $d_* : \mathbb{Z}^m \to \mathbb{Z}^n$ is surjective, in fact $d_* : \mathbb{N}^m \to \mathbb{N}^n$ is surjective, where m, n are the number of type I_∞ factor summands of $\mathcal{M}$ and $\mathcal{N}$ respectively, we obtain a surjective monoid homomorphism from $\varinjlim(\mathbb{N}^m, \varphi_*)$ to $\lim(\mathbb{N}^n, \psi_*)$. In some cases δ is an isomorphism, even if ψ_* is not an in-amalgamation of φ_*. For example with $\psi_* = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $\varphi_* = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ as above, these limit algebras are isomorphic since $\Gamma A = B$ where A is the matrix for the complete in-split of φ_*, B is the matrix for the complete in-split of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, itself an in-split of ψ_*, and Γ is the permutation matrix for the graph

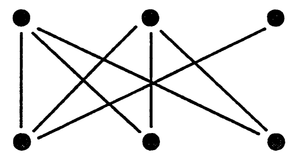


For an example where ψ extends φ but the limit algebras are non-isomorphic consider the matrix $\psi_* = [2]$ and $\varphi_* = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. The division matrix $D = [1, 1]$ satisfies $D\varphi_* = \psi_* D$, however $\varinjlim(\mathbb{Z}^2, \varphi_*) \not\simeq \varinjlim(\mathbb{Z}, \psi_*)$, which by [5] implies that the limit C^*-algebras are non-isomorphic. Here $\varinjlim(\mathbb{Z}, \psi_*) = \mathbb{Z}[1/2]$, while $\varinjlim(\mathbb{Z}^2, \varphi_*) = \{(s, s+t) \mid s \in \mathbb{Z}, t \in \mathbb{Z}[1/2]\} \simeq \mathbb{Z} \oplus \mathbb{Z}[1/2]$.

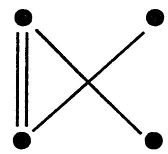
The next results show that the relationship between a representation ϕ of a dual system described by a square 0-1 matrix A and the corresponding unital $*$-endomorphism $\varphi_\phi = \varphi$ can be understood geometrically in terms of the bipartite graphs associated with the matrix A and the matrix φ_*. We first illustrate this relationship by considering a simple, though explicit, example.

Let $\mathcal{E}$ be a three dimensional Hilbert space and $\boldsymbol{P} = \{P_1, P_2, P_3\}$ a normalized coordinate system such that the matrix $A = [\boldsymbol{P}, \boldsymbol{E}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ for a normalized orthogonal system $\boldsymbol{E}, \boldsymbol{P} \prec \boldsymbol{E}$. Thus $P_1 = P_2 = I_{\mathcal{E}}$ and $P_3 = E_1$. If $\boldsymbol{Q}$ is the normalized coordinate system with $Q_1 = I_{\mathcal{E}}$, $Q_2 = Q_3 = E_1 + E_2$, then $(\boldsymbol{P}, A, \boldsymbol{Q})$ is a dual system. If $(\phi, \rho, \mathcal{H})$ is a representation of this dual system and e_k is unit vector in E_k then the partial isometries $s_k = \phi(e_k)$ have orthogonal final spaces p_k with sum $I_{\mathcal{H}}$, and initial spaces q_k with $\mathrm{q}_1 = I_{\mathcal{H}}$, $\mathrm{q}_2 = \mathrm{q}_3 = \mathrm{p}_1 + \mathrm{p}_2$. Since the matrix A satisfies condition I of Cuntz and Krieger these three partial isometries determine a representation of the Cuntz-Krieger algebra $\mathcal{O}_A$ on $\mathcal{H}$, and the von Neumann algebra $\mathcal{R} = \{\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3\}'$ is $\mathcal{R}_1 \oplus \mathcal{R}_2$, a sum of two type I_∞ factors, namely $\mathcal{R}_1 = \mathrm{q}_2 \mathcal{B}(\mathcal{H}) \mathrm{q}_2$ and $\mathcal{R}_2 = \mathrm{p}_3 \mathcal{B}(\mathcal{H}) \mathrm{p}_3$.

The unital $*$-endomorphism φ of $\mathcal{R}$ maps an element $a \oplus b$ of $\mathcal{R}$ to the sum $\sum s_k (a \oplus b) s_k^* = (s_1(a+b)s_1^* + s_2 a s_2^*) \oplus s_3 a s_3^*$. Since the (i,j) entry of the matrix φ_* is the multiplicity of $\varphi(\mathcal{R}_j)$ in $\mathcal{R}_i$ it is evident that $\varphi_* = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Note that the graph associated with A, namely



has as its complete in-amalgamation the graph associated with φ_*, namely



This relationship between a representation of a dual system described by the matrix A and the matrix of the induced $*$-endomorphism holds in general.

Theorem 4.7. *Let φ be the unital injective $*$-endomorphism of $\mathcal{R} = \{\mathrm{q}_k \mid k = 1, \dots, m\}'$ associated with $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$, a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ as in Theorem 2.2 where A is a* 0-1 *matrix. Then φ_* is the complete in-amalgamation of A.*

Proof. Let $\mathcal{G} = \mathcal{G}(V, V)$ be the graph, multiplicity free, associated with the 0-1 matrix A. If $\mathcal{T}_{\mathcal{G}} = \{T_1, \dots, T_l\}$ is the partition of V needed to form the complete in-amalgamation of $\mathcal{G}$ then $r_k = \sum\limits_{j \in T_k} s_j s_j^*$, $1 \le k \le l$, are orthogonal projections generating the same $*$-algebra as the not necessarily orthogonal projections $\mathrm{q}_k = s_k^* s_k = \sum\limits_{j \in \mathcal{E}_k} s_j s_j^*$, $1 \le k \le m$. Here, as in the proof of Proposition 4.1, $\mathcal{E}_k$ the edges of $\mathcal{G}$ with terminal vertex k, are identified with the vertices connected to k, since A is a 0-1 matrix. Setting $\mathcal{R}_k$ to be the type I factor $r_k(\mathcal{B}(\mathcal{H}))r_k$, we have $\mathcal{R} = \bigoplus \mathcal{R}_k$. Now $\varphi(x) = \sum s_j x s_j^*$ for $x \in \mathcal{R}$, and since $\varphi_*(k, l)$ is the multiplicity of $\varphi(\mathcal{R}_l)$ in $\mathcal{R}_k$ and the image of φ is actually contained in the $*$-subalgebra $\{s_k s_k^* \mid k = 1, \dots, m\}'$ of $\mathcal{R}$, it follows that φ_* is the complete in-amalgamation of A. □

Note that the endomorphism φ arising from a representation of a dual system $(\boldsymbol{P}, A, \boldsymbol{Q})$ is thus a finite embedding, namely φ_* is a matrix with entries in $\mathbb{N}$. Also, the number of type I factors in the domain of φ is n, where φ_* is an $n \times n$ matrix.

The following result is the analogue for Cuntz-Krieger algebras of the result which states that unital $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ arise from representations of the appropriate Cuntz algebra.

Theorem 4.8. *Let $\mathcal{R}$ be a finite direct sum of countably decomposable type* I *factors and $\varphi : \mathcal{R} \to \mathcal{R}$ an injective unital $*$-endomorphism of $\mathcal{R}$ with $\varphi_* = A$ a matrix with values in $\mathbb{N}$. For every unital inclusion $\Gamma : \mathcal{R} \to \mathcal{B}(\mathcal{H})$ there is a representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a dual system $(\boldsymbol{P}, A_w, \boldsymbol{Q})$ where A_w is the complete in-split of A, so that the unital $*$-endomorphism φ_ϕ associated with ϕ is φ.*

Proof. Let $\mathcal{R} = \bigoplus\limits_{l=1}^{m} \mathcal{R}_l$ with $\mathcal{R}_l$, $l = 1, \dots, m$, the type I factors. Since Γ is a unital inclusion the map $\Gamma_l : \mathcal{R}_l \to \mathcal{B}(\mathcal{H}_l)$ defined by $\Gamma_l(x) = \Gamma(e_l)\Gamma(x)\Gamma(e_l)$ is an irreducible representation of $\mathcal{R}_l$ on $\Gamma(e_l)\mathcal{H} = \mathcal{H}_l$, where e_l is the unit of $\mathcal{R}_l$.

The map of $\mathcal{R}_k$ to $\Gamma_l(\mathcal{R}_l)$ defined by $x \to \Gamma_l(e_l\varphi(x))$ is a representation of the I factor $\mathcal{R}_k$ on $\mathcal{B}(\mathcal{H}_l)$, so is unitarily equivalent to $\pi_{lk} \oplus 0$, where π_{lk} is a uniform multiplicity $A(l,k)$ representation of $\mathcal{R}_k$ on a subspace $\mathcal{H}_{lk}$ of $\mathcal{H}_l$ and 0 is the zero representation of $\mathcal{R}_k$ on $\mathcal{H}_{lk}^{\perp}$ with $H_l = H_{lk} \oplus H_{lk}^{\perp}$. Now π_{lk} is unitarily equivalent to the direct sum of $A(l,k)$ irreducible representations ρ_{lik}, $i = 1, \dots, A(l,k)$ of $\mathcal{R}_k$ on the mutually orthogonal subspaces $\mathcal{H}_{lk}^i$ of $\mathcal{H}_{lk}$, with $\mathcal{H}_{lk} = \bigoplus_{i=1}^{A(l,k)} \mathcal{H}_{lk}^i$. However the representations ρ_{lik}, $i = 1, \dots, A(l,k)$, are all unitarily equivalent, so there are unitary maps V_{lik} from $\mathcal{H}_{lk}^1$ to $\mathcal{H}_{lk}^i$ with

$$V_{lik}\rho_{l1k}(x)V_{lik}^* = \rho_{lik}(x) \quad \text{for} \quad x \in \mathcal{R}_k,$$

and $\pi_{lk}(x) = \sum_{i=1}^{A(l,k)} V_{lik}\rho_{l1k}(x)V_{lik}^*$, $x \in \mathcal{R}_k$.

Since Γ_k is an irreducible representation of $\mathcal{R}_k$ on $\mathcal{H}_k$ it is unitarily equivalent to ρ_{l1k}, and there is a unitary map U_{lk} from $\mathcal{H}_k$ to $\mathcal{H}_{lk}^1$ with $\rho_{l1k}(x) = U_{lk}\Gamma_k(x)U_{lk}^*$ for $x \in \mathcal{R}_k$. The unitary $V_{l1k}U_{lk}$ maps $\mathcal{H}_k$ to $\mathcal{H}_{lk}^i$ and may be viewed as a partial isometry s_{lik} in $\mathcal{B}(\mathcal{H})$ with unital space $\mathcal{H}_k$ and final space $\mathcal{H}_{lk}^i$.

We have

$$\begin{aligned} s_{pql}^* s_{pql} &= \mathcal{H}_l = \bigoplus_k \{\mathcal{H}_{lk} \mid A(l,k) \neq 0\} \\ &= \sum_{i,k} A(l,k) s_{lik} s_{lik}^* = \sum_{i,k} A_w((p,q,l),(l,i,k)) s_{lik} s_{lik}^* \\ &= \sum_{(h,i,k) \in V_w} A_w((p,q,l),(h,i,k)) s_{hik} s_{hik}^* \end{aligned}$$

where $\mathcal{G}(V_w, V_w)$ is the graph for the complete in-split A_w of A. This yields partial isometries s_{lik}, $(l,i,k) \in V_w$ in $\mathcal{H}$ satisfying the Cuntz-Krieger relations for O_{A_w}, so gives a representation $\phi : \mathcal{E} \to \mathcal{B}(\mathcal{H})$ of a dual system $(\boldsymbol{P}, A_w, \boldsymbol{Q})$. The domain of the endomorphism defined by ϕ is $\bigoplus_{k=1}^{m} \mathcal{B}(\mathcal{H}_k)$, which is isomorphic to $\mathcal{R}$ via the map Γ. For $x \in \mathcal{R}$, we compute that

$$\begin{aligned} \Gamma\varphi(x) &= \Gamma\left(\sum_k \varphi(e_k x)\right) = \Gamma\left(\sum_{l,k} e_l \varphi(e_k x)\right) \\ &= \sum_{l,k} \Gamma_l(e_l\varphi(e_n x)) = \sum_{l,k} \sum_{i=1}^{A(l,k)} \rho_{lik}(e_k x) \\ &= \sum_{(l,i,k) \in V_w} V_{lik}\rho_{l1k}(e_k x) V_{lik}^* \end{aligned}$$

$$= \sum_{(l,i,k)\in V w} V_{lik} U_{lk} \Gamma_k(e_k x) U_{lk}^* V_{lik}^*$$

$$= \sum_{(l,i,k)\in V_w} s_{lik} \left(\sum_j \Gamma_j(e_j x) \right) s_{lik}^*$$

$$= \sum_{(l,i,k)\in V_w} s_{lik} \left(\Gamma(x)) s_{lik}^* \right),$$

since the initial space of s_{lik} is $\mathcal{H}_k$. Thus the endomorphism defined by ϕ is φ. $\square$

It is worth noting that by using Theorem 4.8 a few more facts can be gleaned from the argument of Proposition 4.1. If A is a 0-1 matrix and B is an in-split of A, then using the notation of the proof of Proposition 4.1, the domain $\mathcal{N}$ of φ_ψ is $\{T_{J_k}^* T_{J_k} \mid J_k \in W\}'$, and $\mathcal{N}'$ is abelian. We also saw that the projection $T_{J_k}^* T_{J_k}$ was $\sum_{I\in\mathcal{E}_J^k} S_I S_I^*$, so $\operatorname{dom}(\varphi_\psi)$ contains $\{S_I S_I^* \mid I \in V\}'$, which in turn contains the range of the endomorphism φ_ϕ. The following proposition is a converse to this.

Proposition 4.9. *Let $\mathcal{M}, \mathcal{N}$ be finite direct sums of type* I *factors with $d : \mathcal{N} \to \mathcal{M}$ a unital inclusion and $\psi : \mathcal{M} \to \mathcal{M}$ an injective unital $*$-endomorphism of $\mathcal{M}$ with* $\operatorname{ran}(\psi) \subseteq d(\mathcal{N})$. *Then $\psi|_{d(\mathcal{N})} = \varphi$ is an injective unital $*$-endomorphism of $\mathcal{N}$ and φ_* is an in-split of ψ_*.*

Proof. View $\mathcal{N} \subseteq \mathcal{M}$. Since range of ψ is contained in $\mathcal{N}$, it is clear that $\varphi(\mathcal{N}) \subseteq \mathcal{N}$. Represent $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$ via a unital inclusion and use Theorem 4.8 to write ψ as φ_ϕ, the $*$-endomorphism associated to a representation $\phi : (\boldsymbol{P}, A, \boldsymbol{Q}) \to \mathcal{B}(\mathcal{H})$ of a dual system with A a 0-1 matrix, namely the complete in-split of the matrix ψ_*. Thus ψ_* is the complete in-amalgamation of A. We will show that φ_* is an in-amalgamation of A, which by Corollary 3.4 is enough to finish the proof. Note that $\mathcal{N}'$ in $\mathcal{B}(\mathcal{H})$ is abelian, since as noted earlier a composition of unital inclusions is a unital inclusion. With the notation of Theorem 2.2, so that $\mathbf{p}$, $\mathbf{q}$ are coordinate systems on $\mathcal{H}$ with ϕ defining the partial isometries from q_i to p_i we have $\mathcal{M} = C^*(\mathbf{q})'$, and $\operatorname{ran}\psi \subseteq C^*(\mathbf{p})'$. Thus $C^*(\mathbf{p})' \subseteq \mathcal{N}$, since A is the complete in-split of ψ_* and so $C^*(\mathbf{p})'$ is the smallest von Neumann algebra containing $\operatorname{ran}\psi$ with abelian commutant. If w_i is the unit of $\mathcal{N}_i$, where $\mathcal{N} = \bigoplus \mathcal{N}_i$, let $\mathbf{w}$ be the reduced orthogonal coordinate system defined by the projections w_i. We have $C^*(\mathbf{q})' \supseteq C^*(\mathbf{w})' \supseteq C^*(\mathbf{p})'$, so since the C^*-algebras are finite dimensional we have $C^*(\mathbf{q}) \subseteq C^*(\mathbf{w}) \subseteq C^*(\mathbf{p})$ and by Proposition 1.5, $\mathbf{q} \prec \mathbf{w} \prec \mathbf{p}$. If $\mathcal{T} = \{T_1, \dots, T_l\}$ is the partition needed to form the complete in-amalgamation of A, then $\mathrm{r}_k = \sum_{j\in T_k} \mathrm{p}_j$ are orthogonal projections defining a reduced orthogonal generator $\mathbf{r}$ for $\mathbf{q}$ with $\mathcal{M} = \bigoplus \mathrm{r}_k \mathcal{M} \mathrm{r}_k$, as in Theorem 4.7. Thus $\mathbf{r} \prec \mathbf{w} \prec \mathbf{p}$, and it follows that there is a partition of T_k into $m(k)$ sets, $T_k^1, \dots, T_k^{m(k)}$ so that each projection w_j is a sum of projections $\{\mathrm{p}_j \mid j \in T_k^i\}$ for some k and i, $1 \le i \le m(k)$. Relabel these projections w_k^i, so $\mathrm{w}_k^i = \sum_{j\in T_k^i} \mathrm{p}_j$, and $\mathrm{r}_k = \sum_{i=1}^{m(k)} \mathrm{w}_k^i$. Since $\varphi = \psi|_{\mathcal{N}}$ and $\mathcal{N} = \bigoplus \mathcal{N}_{k,i}$ with $\mathcal{N}_{k,i} = \mathrm{w}_k^i \mathcal{N} \mathrm{w}_k^i = \mathrm{w}_k^i \mathcal{M} \mathrm{w}_k^i$, it follows that φ_* is the in-amalgamation of A corresponding to the partition $\{T_k^i \mid k = 1, \dots, l,\ 1 \le i \le m(k)\}$ of the vertices of the graph of A. $\square$

Theorem 4.8 completes a picture of the correspondence $\phi \to \varphi_\phi$ between representations $\phi : (\boldsymbol{P}, A, \boldsymbol{Q}) \to \mathcal{B}(\mathcal{H})$ of a dual system, with A a 0-1 matrix, and injective unital $*$-endomorphisms φ_ϕ of a finite sum of type I factors $\mathcal{M}, \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. Given φ an injective unital $*$-endomorphism of $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ with $\varphi_* = B$, there is a representation ϕ_0 of a dual system $(\boldsymbol{P}, B_w, \boldsymbol{Q})$ with $\varphi_{\phi_0} = \varphi$, where B_w is the complete in-split of B. If the endomorphism φ arose from a representation $\phi : (\boldsymbol{P}, A, \boldsymbol{Q}) \to \mathcal{B}(\mathcal{H})$, so that $\varphi = \varphi_\phi$, then by Theorem 4.7 B is also the complete in-amalgamation of A. By Corollary 3.9, B_w must then be an in-split of A, and so, by Proposition 4.1 there is a representation ψ of a dual system with matrix B_w so that its associated endomorphism φ_ψ is the restriction of φ_ϕ. Now $\varphi_\phi = \varphi = \varphi_{\phi_0}$, so φ_ψ is the restriction to $\operatorname{dom} \varphi_\psi$ of φ_{ϕ_0}. In particular, $\operatorname{dom} \varphi_\psi \subseteq \operatorname{dom} \varphi_{\phi_0}$. However both ψ and ϕ_0 are representations of dual systems involving the matrix B_w, so $(\varphi_\psi)_* = (\varphi_{\phi_0})_* = B$ and so the number of type I factor summands in $\operatorname{dom} \varphi_\psi$ and $\operatorname{dom} \varphi_{\phi_0}$ must be the same. Thus $\operatorname{dom} \varphi_\psi = \operatorname{dom} \varphi_{\phi_0}$ and $\varphi = \varphi_\psi$ also.

We remark that this correspondence associates the infinite representations ϕ with unital $*$-endomorphisms of finite direct sum of type I_∞ factors. It also restricts to a correspondence between representations of Cuntz-Krieger algebras $\mathcal{O}_A$, A satisfying condition I of Cuntz-Krieger, and injective unital $*$-endomorphisms φ of finite direct sums of type I_∞ factors with φ_* a matrix satisfying condition I of Cuntz-Krieger.

This correspondence, and our knowledge of the partial order can be used to work with $*$-endomorphisms in a fairly straightforward manner. For example let $\varphi : \mathcal{M} \to \mathcal{M}$ with $\varphi_* = B$ be an injective unital $*$-endomorphism of $\mathcal{M}$, a finite direct sum of type I factors, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. If $A \prec B$, then by the comments at the end of section 3, $A_w \prec B_w$ where A_w, B_w are the complete in-splits of A and B respectively. In fact $A_w \prec B_w$ via a finite sequence of 0-1 matrices. Theorem 4.8 shows that φ is the $*$-endomorphism φ_ϕ associated with a representation ϕ of a dual system $(\boldsymbol{P}, B_w, \boldsymbol{Q})$. By applying Proposition 4.1 a finite member of times, there is a representation ψ of a dual system $(\boldsymbol{P}', A_w, \boldsymbol{Q}')$ with the image of ψ containing that of ϕ, the $*$-algebras generated by ϕ and ψ are equal, and with φ_ϕ the restriction of the $*$-endomorphism φ_ψ associated to ψ. Thus by Proposition 4.6, $\varinjlim(\mathcal{M}, \varphi) \simeq \varinjlim(\mathcal{N}, \varphi_\psi)$, where $\mathcal{N}$ is the domain of φ_ψ.

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