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ON THE COMMUTATIVITY OF THE ALGEBRA OF INVARIANT DIFFERENTIAL OPERATORS ON CERTAIN NILPOTENT HOMOGENEOUS SPACES

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ABSTRACT. Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$. Let χ_β be the unitary character of H with differential $2\sqrt{-1}\pi\beta$ at the origin. Let $\tau \equiv Ind_H^G \chi_\beta$ be the unitary representation of G induced from the character χ_β of H. We consider the algebra $\mathcal{D}(G,H,\beta)$ of differential operators invariant under the action of G on the bundle with basis $H\backslash G$ associated to these data. We consider the question of the equivalence between the commutativity of $\mathcal{D}(G,H,\beta)$ and the finite multiplicities of τ. Corwin and Greenleaf proved that if τ is of finite multiplicities, this algebra is commutative. We show that the converse is true in many cases.

1. NOTATIONS AND FORMULATION OF THE QUESTION

Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$. For $l \in \mathfrak{g}^*$, we denote by $\mathfrak{g}(l)$ the Lie algebra of the stabilizer $G(l)$ of l under the co-adjoint action Ad^* of G on $\mathfrak{g}^*$. For $\beta \in \mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$, the homomorphism β induces a character χ_β of H with $2\sqrt{-1}\pi\beta$ as differential at the origin. We then form the unitary induced representation $\tau \equiv Ind_H^G \chi_\beta$ of G in $\mathcal{H}_\tau$ realized, in the usual way, as the completion of a vector subspace of $C^\infty(G,H,\beta)$, namely the vector space of the C^∞ complex functions f on G satisfying the following covariance relation:

$$f(hg) = \chi_\beta(h)f(g) \ \ \forall h \in H \ \forall g \in G. \tag{1.1}$$

The action of G is given by right translations:

$$\tau(g')(f)(g) = f(gg') \quad \forall(g;g') \in G \times G \quad \forall f \in C^\infty(G,H,\beta). \tag{1.2}$$

We denote by $\mathcal{K}(G,H,\tau)$ the subspace of $C^\infty(G,H,\beta)$ of elements with compact support modulo H. Then the norm $|| \ ||_\tau$ on $\mathcal{K}(G,H,\tau)$ is given by

$$|| \ f \ ||_\tau^2 = \int_{H\backslash G} | \ f(g) \ |^2 \ d\dot{g} \tag{1.3}$$

where $d\dot{g}$ denotes a right G-invariant measure on $H\backslash G$. The Hilbert space $\mathcal{H}_\tau$ is just the completion of $\mathcal{K}(G,H,\tau)$ relative to this norm. Moreover, the unitary

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representation of G in $\mathcal{H}_\tau$ decomposes in a continuous sum of unitary irreducible representations of G,

$$\tau \simeq \int_{\hat{G}}^{\oplus} m(\pi)\pi d\mu(\pi) \tag{1.4}$$

where $m(\pi)$ denotes the multiplicity of π and $d\mu$ a Plancherel measure on the unitary dual $\hat{G}$ of G. Then we know [3], [11] that for almost all π in $\hat{G}$, the multiplicities $m(\pi)$ appearing in (1.4) either are finite and admit a uniform bound, or are all infinite. In the first case, we say that τ is of finite multiplicities.

Finally, let $\mathcal{D}(G,H,\beta)$ be the algebra of linear differential operators leaving $C^\infty(G,H,\beta)$ invariant and commuting with τ, that is

$$D \in \mathcal{D}(G,H,\beta) \Leftrightarrow D(\tau(g)f) = \tau(g)(Df) \ \ \forall g \in G \ \ \forall f \in C^\infty(G,H,\beta). \tag{1.5}$$

Corwin and Greenleaf established in [5] the commutativity of $\mathcal{D}(G,H,\beta)$ when τ is of finite multiplicities and they asked the question:

($\star$) *Is τ of finite multiplicities if $\mathcal{D}(G,H,\beta)$ is commutative ?*

Before we turn to the study of this question, we first recall some facts about parametrization of unipotent actions on vector spaces [12].

2. Orbits of H in $\mathfrak{g}^*$

Suppose we are given a m-dimensional vector space V admitting a unipotent action of H. Let $\{Y_1;\cdots;Y_m\}$ be a basis of V such that the subspaces $V_j \equiv \bigoplus_{i=j+1}^m \mathbb{R}Y_i$ of V are H-stable for all $0 \le j \le m$ with $V_m = \{0\}$. We consider the multi-index $e(\psi)$ defined by $(e_0(\psi);\cdots;e_m(\psi))$ for all $\psi \in V$, where $e_j(\psi)$ is the dimension of the H-orbit of the projection of ψ on V/V_j. We then denote by Σ the set of all possible multi-indexes, that is

$$\Sigma = \{e \in \mathbb{N}^{m+1} \mid \exists \psi \in V, \ \ e = e(\psi)\}. \tag{2.1}$$

This defines a stratification of V in layers U_e of H-orbits, more precisely [4]:

$$V = \bigcup_{e\in\Sigma} U_e \ \text{ where for } e \in \Sigma \text{ and } U_e = \{\psi \in V \mid e(\psi) = e\}. \tag{2.2}$$

It happens that among these layers U_e, $e \in \Sigma$, there exists one, and only one, which is a non-empty Zariski open subset of V. We shall call it the generic layer (associated to the action of H on V), and we will denote it by V^{gene}. Note that V^{gene} is just the subset of V of elements for which the dimensions of H-orbits in V/V_j are maximal for $0 \le j \le m$. We shall say that a ψ in V^{gene} is generic in V, and the dimension of the orbit $H\cdot\psi$ will be called the generic dimension of H-orbits in V.

In the sequel, we will consider a particular V. More precisely, fix a sequence $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of subalgebras of $\mathfrak{g}$ satisfying the following conditions:

- the subalgebras $\mathfrak{g}_i$ are of dimension i; they are normalised by the action of H.
- for a certain index p, the subalgebra $\mathfrak{g}_p$ coincides with $\mathfrak{h}$.

We choose a weak Malcev basis $\{X_i,\ 1 \le i \le n\}$ of $\mathfrak{g}$ through $\mathfrak{h}$ associated to the sequence $(\mathfrak{g}_i,\ 0 \le i \le n)$ such that for all $i \in \{1,...,n\}$, the vectors $\{X_j,\ 1 \le j \le i\}$ form a basis of $\mathfrak{g}_i$. We say that $\{X_i,\ 1 \le i \le n\}$ (resp. $(\mathfrak{g}_i,\ 0 \le i \le n)$) is a weak Malcev basis (resp. sequence of subalgebras) of $\mathfrak{g}$ through $\mathfrak{h}$. As usual we denote by $\{X_i^*,\ 1 \le i \le n\}$ the dual basis of $\{X_i,\ 1 \le i \le n\}$. Then we put $V = \mathfrak{g}^*$ and

$V_j = \mathfrak{g}_j^\perp = \bigoplus_{k=j+1}^n \mathbb{R}X_k^*$ for $1 \leq j \leq n$. Here we consider the unipotent action of H on $V = \mathfrak{g}^*$ given by the restriction to H of the co-adjoint action of G on $\mathfrak{g}^*$. In particular, the layer V^{gene} defined before will be denoted by $\mathfrak{g}^{*,gene}$. This defines a stratification of $\mathfrak{g}^*$ in U_e-layers.

Next, denote by $\Omega_{G,H,\beta}$ the space of all continuations of β to V. There is in $\{U_e,\ e \in \Sigma\}$ a unique layer intersecting $\Omega_{G,H,\beta}$ in a non-empty Zariski open subset (Section 2 of [5]). We will call it the generic layer associated to the data G, H and β, and we will denote it by $\mathfrak{g}^{*,gene}_{G,H,\beta}$. We shall say that an element ψ in $\Omega_{G,H,\beta}$ contained in $\mathfrak{g}^{*,gene}_{G,H,\beta}$ is generic in $\Omega_{G,H,\beta}$.

Remark 1. It is important to note that the layer $\mathfrak{g}^{*,gene}$ does not necessarily intersect $\Omega_{G,H,\beta}$. Indeed, a family of simple examples where the condition $\mathfrak{g}^{*,gene} \cap \Omega_{G,H,\beta} \neq \emptyset$ is not satisfied is the Heisenberg group with $\mathfrak{h}$ containing the center and β vanishing on the center. More precisely, let $\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ with bracket relation $[X, Y] = Z$. If $\mathfrak{h} = \mathbb{R}X \oplus \mathbb{R}Z$ and $\beta = X^*$, then $\Omega_{G,H,\beta} = X^* + \mathbb{R}Y^*$ and $\mathfrak{g}^{*,gene} = \{xX^* + yY^* + zZ^* \in \mathfrak{g}^* \mid z \neq 0\}$, which shows that $\mathfrak{g}^{*,gene} \cap \Omega_{G,H,\beta} = \emptyset$.

Finally, for $l \in \mathfrak{g}^*$, we denote by B_l the antisymmetric bilinear form on $\mathfrak{g}$ given by $B_l(X, Y) = l([X, Y])$. It is well known [3], [11] that the two following conditions are equivalent:

$(C1)$ τ is of finite multiplicities.
$(C2)$ $\mathfrak{h} + \mathfrak{g}(l)$ is lagrangian in $\mathfrak{g}$ relative to B_l for all generic l in $\Omega_{G,H,\beta}$.

If the second conditon is satisfied, we say that $\mathfrak{h} + \mathfrak{g}(l)$ is generically lagrangian in $\mathfrak{g}$. It turns out that question 6 asked by Duflo in [7] is, in the case of a simply connected connected real nilpotent Lie group, exactly the same as the above question $(\star)$ of Corwin-Greenleaf.

More precisely, consider the following assertions:

(i) $\mathcal{D}(G, H, \beta)$ is a commutative algebra.
(ii) $\mathfrak{h} + \mathfrak{g}(l)$ is generically lagrangian in $\mathfrak{g}$.
(iii) $H \cdot l$ is generically a lagrangian submanifold of $G \cdot l$ relative to $B_l : (X, Y) \mapsto l([X, Y])$.

Note that $(ii) \Leftrightarrow (iii)$ is obvious, since if $\mathfrak{h}+\mathfrak{g}(l)$ is lagrangian in $\mathfrak{g}$, then $dim\ H \cdot l = \frac{1}{2}\ dim\ G \cdot l$. But $(ii) \Rightarrow (i)$ is a fundamental result proved by Corwin and Greenleaf in [5]. Baklouti and Ludwig have studied in [1] the implication $(i) \Rightarrow (ii)$ when $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. In the sequel, we shall study the implication $(i) \Rightarrow (ii)$ in more general cases.

3. A first result on the commutativity of $\mathcal{D}(G, H, \beta)$

The description of $\mathcal{D}(G, H, \beta)$ given in [6] in terms of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification of $\mathfrak{g}$ will be useful. Let $\mathfrak{a}_\beta$ be the vector subspace of $\mathcal{U}(\mathfrak{g})$ generated by the $X + 2\sqrt{-1}\pi\beta(X)$, $X \in \mathfrak{h}$, and let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ be the left sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\beta$. If $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ denotes the subalgebra of $\mathcal{U}(\mathfrak{g})$ defined by

$$\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta) = \{A \in \mathcal{U}(\mathfrak{g}) \mid \ \forall W \in \mathfrak{h},\ \ [A, W] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta\}, \tag{3.1}$$

then the left action L of $\mathcal{U}(\mathfrak{g})$, defined for Y in $\mathfrak{g}$ and f in $C^\infty(G)$ by

$$L(Y)(f)(g) = \frac{d}{dt} f(e^{-tY} g) \mid_{t=0}, \tag{3.2}$$

induces the algebra isomorphism L_β: $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta \simeq \mathcal{D}(G, H, \beta)$.

Next, in the usual way, we shall denote by $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$ and by $\sigma : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ the symmetrization map. We still denote by Ad (resp. ad) the natural continuation of the adjoint action of G (resp. $\mathfrak{g}$) in a G-action (resp. $\mathfrak{g}$-action) on $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$. Moreover, we shall identify an element of $S(\mathfrak{g})$ with a polynomial function on $\mathfrak{g}^*$.

Remark 2. F. P. Greenleaf has also obtained the following theorem with a different proof [10].

Theorem 1. *Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ such that $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$. We assume that the unitary representation $\tau = Ind_H^G \chi_\beta$ of G is of infinite multiplicities. Let G_0 be a connected subgroup of codimension one of G with Lie algebra $\mathfrak{g}_0$ containing $\mathfrak{h}$ and such that the unitary representation $\tau_0 = Ind_H^{G_0} \chi_\beta$ of G_0 is of finite multiplicities. If we suppose that there is an element W of $\mathcal{U}(\mathfrak{g},\mathfrak{h},\beta)$ such that $W \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$, then there exists an element T of $\mathcal{U}(\mathfrak{g}_0,\mathfrak{h},\beta)$ satisfying $[W,T] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$.*

In other words, if $\mathcal{D}(G_0,H,\beta)$ is properly imbeded in $\mathcal{D}(G,H,\beta)$, then $\mathcal{D}(G,H,\beta)$ is not commutative.

Proof. We shall frequently use in the sequel the following property: let $\mathfrak{k}$ be a subalgebra of codimension one of $\mathfrak{g}$, and l be in $\mathfrak{g}^*$, then the dimensions of $\mathfrak{g}(l)$ and of $\mathfrak{k}(l_{|\mathfrak{k}})$ differ by 1, and one of those subspaces is imbedded in the other, (see [2], Lemme 1.1.1, p. 49). In the case where $\mathfrak{g}(l) \subset \mathfrak{k}(l_{|\mathfrak{k}})$, one has that the dimension of $G.l$ is bigger by 2 than the dimension of $K.(l_{|\mathfrak{k}})$ where $K = exp(\mathfrak{k})$, and the dimensions of polarizations in $\mathfrak{g}$ and $\mathfrak{k}$ are the same. In the other case, the orbits $G.l$ and $K.(l_{|\mathfrak{k}})$ have the same dimension, whereas the dimension of polarizations in $\mathfrak{g}$ is bigger by 1 than the dimension of polarizations in $\mathfrak{k}$.

Let us recall that the finite multiplicities situation is characterised by the fact that generically on $\Omega_{G,H,\beta}$, the dimension of an H-orbit is half the dimension of a G-orbit [5] . Thus, under the assumptions of the theorem, we have $\mathfrak{g}(l) \subset \mathfrak{g}_0$.

Next, we proceed by induction on the dimension of G and the theorem is supposed to be true for all groups of dimension at most $n-1$, where n is the dimension of G. Let $\mathcal{Z}$ be the center of $\mathfrak{g}$. We shall consider two main cases depending on whether $\mathcal{Z}$ is included in $\mathfrak{h}$.

1) *Case: $\mathcal{Z} \subset \mathfrak{h}$ and $\mathcal{Z} \cap Ker(\beta) \neq \{0\}$.* In this case, we apply the induction hypothesis to the quotient group with Lie algebra $\mathfrak{g}/(\mathcal{Z} \cap Ker(\beta))$.

In the following cases 2), 3) and 4), we have $\mathcal{Z} \subset \mathfrak{h}$ and $\mathcal{Z} \cap Ker(\beta) = \{0\}$, so that the center $\mathcal{Z}$ of $\mathfrak{g}$ is necessarily one-dimensional. We put $\mathcal{Z} = \mathbb{R}Z$ with $\beta(Z) = 1$. Moreover, it is easy to check the existence of elements X of $\mathfrak{g}$ and Y of $\mathfrak{g}_0$ such that $[X,Y] = Z$. In the sequel, we shall denote by $\mathfrak{k}$ the centralizer of Y in $\mathfrak{g}$ and by K the connected subgroup of G with Lie algebra $\mathfrak{k}$.

2) *Case: $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap Ker(\beta) = \{0\}$, $\mathfrak{h} \subset \mathfrak{k}$ and $Y \in \mathfrak{h}$.* Let l be an element of $\mathfrak{g}^*$ satisfying $l(Z) \neq 0$, (since $\beta(Z) = 1$, this condition is satisfied by any element of $\Omega_{G,H,\beta}$). One has $\mathfrak{k}(l_{|\mathfrak{k}}) = \mathfrak{g}(l) \oplus \mathbb{R}Y$. And, as we noticed before, the dimension $[dim(\mathfrak{g}) + dim(\mathfrak{g}(l))]/2$ of a polarization of $\mathfrak{g}$ at a point $l \in \mathfrak{g}^*$ is the same as the dimension $[dim(\mathfrak{k}) + dim(\mathfrak{k}(l_{|\mathfrak{k}}))]/2$ of a polarization of $\mathfrak{k}$ at the point $l_{|\mathfrak{k}} \in \mathfrak{k}^*$. Moreover, we also have $\mathfrak{h} + \mathfrak{k}(l_{|\mathfrak{k}}) = \mathfrak{h} + \mathfrak{g}(l)$. Next, we choose a weak Malcev basis of

$\mathfrak{g}$ passing throught $\mathfrak{h}$ and $\mathfrak{k}$, and we consider a generic element l in $\Omega_{G,H,\beta}$. Then, the equivalent conditions $(C1)$ and $(C2)$ of Section 2 show that the multiplicities of the representations $\tau = Ind_H^G \chi_\beta$ of G and $\tau' = Ind_H^K \chi_\beta$ of K are of the same type, that is, both infinite. But $\tau_0 = Ind_H^{G_0} \chi_\beta$ is supposed to be of finite multiplicities, so one has $\mathfrak{g}_0 \neq \mathfrak{k}$.

On the other hand, for $l \in \Omega_{G,H,\beta}$, we have $(\mathfrak{g}_0 \cap \mathfrak{k})(l_{|_{\mathfrak{g}_0 \cap \mathfrak{k}}}) = \mathfrak{g}_0(l_{|_{\mathfrak{g}_0}}) \oplus \mathbb{R}Y$. Choosing a weak Malcev basis of $\mathfrak{g}$ passing through $\mathfrak{h}$, $\mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{g}_0$, we see that the unitary representation $\tau_0' = Ind_H^{G_0 \cap K} \chi_\beta$ of $G_0 \cap K$ is, as the representation $\tau_0 = Ind_H^{G_0} \chi_\beta$ of G_0, of finite multiplicities. Let us write the element W of the theorem in the form $W = \sum_{j=0}^{j=r} X^j U_j$ with U_j in $\mathcal{U}(\mathfrak{k})$. Since the element Y is in $\mathfrak{h}$, the elements $ad(-Y)^r(W) = r! Z^r U_r$ and U_r are in $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ for all $r \neq 0$. Thus we can suppose $W = U_0$ is in $\mathcal{U}(\mathfrak{k}, \mathfrak{h}, \beta)$. Finally, for $W \in \mathcal{U}(\mathfrak{k}, \mathfrak{h}, \beta)$, we apply the induction hypothesis to K and $G_0 \cap K$ with the representation $\tau' = Ind_H^K \chi_\beta$ and $\tau_0' = Ind_H^{G_0 \cap K} \chi_\beta$ respectively.

3) *Case:* $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap Ker(\beta) = \{0\}$, $\mathfrak{h} \subset \mathfrak{k}$ *and* $Y \notin \mathfrak{h}$. *i)* $\mathfrak{g}_0 = \mathfrak{k}$.

The element Y is in $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)$ and satisfy $[W, Y] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$. We take $T = Y$.

ii) $\mathfrak{g}_0 \neq \mathfrak{k}$ and $W \notin \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$.

We still can choose T to be the element Y.

iii) $\mathfrak{g}_0 \neq \mathfrak{k}$ and $W \in \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$.

We consider the inclusions: $(\mathfrak{g}_0 \cap \mathfrak{k}) \subset \mathfrak{g}_0 \subset \mathfrak{g}$ and $(\mathfrak{g}_0 \cap \mathfrak{k}) \subset \mathfrak{k} \subset \mathfrak{g}$. Let l be in $\mathfrak{g}^*$ with $l(Z) \neq 0$. We denote by d the dimension of $\mathfrak{g}(l)$. Under the assumption of the theorem, as we have already remarked at the beginning, we have $\mathfrak{g}(l) \subset \mathfrak{g}_0$, so that the dimension of $\mathfrak{g}_0(l_{|_{\mathfrak{g}_0}})$ is $d+1$. The dimension of $\mathfrak{k}(l_{|_\mathfrak{k}})$ is also $d+1$ because the element Y of $\mathfrak{k}(l_{|_\mathfrak{k}})$ does not belong to $\mathfrak{g}(l)$. Moreover, Y is in $(\mathfrak{g}_0 \cap \mathfrak{k})(l \mid_{(\mathfrak{g}_0 \cap \mathfrak{k})})$ but is not in $\mathfrak{g}_0(l_{|_{\mathfrak{g}_0}})$. In other words, the dimension of $(\mathfrak{g}_0 \cap \mathfrak{k})(l \mid_{(\mathfrak{g}_0 \cap \mathfrak{k})})$ is $d+2$. Thus, the representation $\tau' = Ind_H^K \chi_\beta$ of K is necessarily of infinite multiplicities. Indeed, the representation $\tau_0' = Ind_H^{K \cap G_0} \chi_\beta$ has finite multiplicities and, from the previous calculus of dimensions of stabilizers, we deduce that if the representation $\tau' = Ind_H^K \chi_\beta$ of K had finite multiplicities, the dimensions of $H-$orbits for generic $l \in \Omega_{G,H,\beta}$ would increase when passing from $K \cap G_0$ to K, which makes impossible the existence of an element $W \in \mathcal{U}(\mathfrak{k}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ that is not in $\mathcal{U}(\mathfrak{k} \cap \mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ as it is shown in [5]. Finally, we apply the induction hypothesis to K and $G_0 \cap K$ with the representation τ' and τ_0' respectively.

4) *Case:* $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap Ker(\beta) = \{0\}$ *and* $\mathfrak{h} \not\subset \mathfrak{k}$. Note that, since $\mathfrak{h} \subset \mathfrak{g}_0$, one has necessarily $\mathfrak{g}_0 \neq \mathfrak{k}$. Let us write $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathbb{R}X$, with $X \in \mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{R}X$, and denote by $\hat{\beta}$ the restriction of β to $\mathfrak{h} \cap \mathfrak{k}$. Let l be an element of $\Omega_{G,H,\beta}$. We have $l(Z) \neq 0$. Then $\mathfrak{k}(l_{|_\mathfrak{k}}) = \mathfrak{g}(l) \oplus \mathbb{R}Y$, so that $(\mathfrak{h} \cap \mathfrak{k}) \cap \mathfrak{k}(l_{|_\mathfrak{k}}) = \mathfrak{h} \cap \mathfrak{g}(l)$, since $\mathfrak{h} \not\subset \mathfrak{k}$. This forces the vector spaces $\mathfrak{h} + \mathfrak{g}(l)$ and $(\mathfrak{h} \cap \mathfrak{k}) + \mathfrak{k}(l_{|_\mathfrak{k}})$ to have the same dimension. It follows that if the first subspace is not lagrangian in $\mathfrak{g}$, the second is not lagrangian in $\mathfrak{k}$. Hence, choosing sequences of subalgebras and using equivalence of conditions $(C1)$ and $(C2)$, we obtain that the representation $\tau_1 = Ind_{H \cap K}^K \chi_{\hat{\beta}}$ of K is of infinite multiplicities. On the other hand, we have $(\mathfrak{g}_0 \cap \mathfrak{k})(l \mid_{\mathfrak{g}_0 \cap \mathfrak{k}}) = \mathfrak{g}_0(l_{|_{\mathfrak{g}_0}}) \oplus \mathbb{R}Y$ and $dim(\mathfrak{h} + \mathfrak{g}_0(l \mid_{\mathfrak{g}_0})) = dim((\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{g}_0 \cap \mathfrak{k})(l \mid_{\mathfrak{g}_0 \cap \mathfrak{k}}))$, which imply that the representation $\tau_2 = Ind_{H \cap K}^{G_0 \cap K} \chi_{\hat{\beta}}$ of $G_0 \cap K$ is of finite multiplicities.

Moreover, W can be supposed to belong to $\mathcal{U}(\mathfrak{k}, \mathfrak{h} \cap \mathfrak{k}, \hat{\beta})$ since $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{R}X$ with $X \in \mathfrak{h}$, and $X = -2\sqrt{-1}\pi\beta(X)$ modulo $\mathfrak{a}_\beta$. We apply the induction hypothesis to

K and $G_0 \cap K$ with the representation τ_1 and τ_2 respectively to obtain an element $\hat{T}$ in $\mathcal{U}(\mathfrak{g}_0 \cap \mathfrak{k}, \mathfrak{h} \cap \mathfrak{k}, \hat{\beta})$ such that $[W, \hat{T}] \notin \mathcal{U}(\mathfrak{k})\mathfrak{a}_{\hat{\beta}}$.

Next, let $X_1 = Y$ and $X_i, i \in \{2, \cdots, q\}$, be in $\mathfrak{g}_0 \cap \mathfrak{k}$, such that if we put $\tilde{\mathfrak{g}}_i = \mathfrak{h} \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \cdots \oplus \mathbb{R}X_i$ with $\mathfrak{g}_q = \mathfrak{g}_0$, then the sequence of subalgebras $(\tilde{\mathfrak{g}}_i)_{i=1,\cdots,q}$ of $\mathfrak{g}_0$ is Jordan-Hölder for the action of H on $\mathfrak{g}_0$. It is interesting to notice that the sequence $(\hat{\mathfrak{g}}_i)_{i=1,\cdots,q}$, with $\hat{\mathfrak{g}}_i = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_i$ is also Jordan-Hölder for the action of $H \cap K$ on $\mathfrak{g}_0 \cap \mathfrak{k}$. We put $\tilde{G}_i = exp(\tilde{\mathfrak{g}}_i)$ and $\hat{G}_i = exp(\hat{\mathfrak{g}}_i)$ for $i = 1, \cdots, q$. Then the dimension of generic H-orbits in $\Omega_{\tilde{G}_j,H,\beta}$ is bigger by one than that of generic $H \cap K$-orbits in $\Omega_{\hat{G}_j,H\cap K,\hat{\beta}}$.

On the other hand, since the representations $\tau_0 = Ind_H^{G_0}\chi_\beta$ of G_0 and $\tau_2 = Ind_{H\cap K}^{G_0\cap K}\chi_{\hat{\beta}}$ of $G_0 \cap K$ are of finite multiplicities, there are elements $\{\tilde{\gamma_1}, \cdots, \tilde{\gamma_r}\}$ of $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)$ and $\{\hat{\delta_0} = Y, \hat{\delta_1}, \cdots, \hat{\delta_r}\}$ of $\mathcal{U}(\mathfrak{g}_0 \cap \mathfrak{k}, \mathfrak{h} \cap \mathfrak{k}, \hat{\beta})$ given in [5], such that the families $\{\gamma_i = L(\tilde{\gamma_i}) \mid i = 1, \cdots, r\}$ and $\{\delta_i = L(\hat{\delta_i}) \mid i = 0, \cdots, r\}$ are rational generators of $\mathcal{D}(G_0, H, \beta)$ and $\mathcal{D}(G_0 \cap K, H \cap K, \hat{\beta})$ respectively. To simplify, we shall call the γ_i's and δ_i's, the Corwin-Greenleaf generators of $\mathcal{D}(G_0, H, \beta)$ and $\mathcal{D}(G_0 \cap K, H \cap K, \hat{\beta})$ respectively. Note that $\mathcal{D}(G_0, H, \beta)$ is contained in $\mathcal{D}(G_0 \cap K, H \cap K, \hat{\beta})$. Moreover, we may suppose that for the element $\hat{T}$ above, $L(\hat{T})$ is one of the Corwin-Greenleaf generators of $\mathcal{D}(G_0 \cap K, H \cap K, \hat{\beta})$. Thus, since $[W, Y] = 0$, we denote by δ_{j_0} the first element of $\{\hat{\delta_1}, \cdots, \hat{\delta_r}\}$ satisfying $[\hat{\delta}_{j_0}, W] \notin \mathcal{U}(\mathfrak{k})\mathfrak{a}_{\hat{\beta}}$. Then, from [5], one can find polynomials A, B and C of j_0 variables such that $A(\delta_0, \cdots, \delta_{j_0-1})\gamma_{j_0} = B(\delta_0, \cdots, \delta_{j_0-1})\delta_{j_0} + C(\delta_0, \cdots, \delta_{j_0-1})$, with $A(\delta_0, \cdots, \delta_{j_0-1})$ and $B(\delta_0, \cdots, \delta_{j_0-1})$ non-zero.

It turns out that $[\tilde{\gamma}_{j_0}, W] \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$. Indeed, because $\mathcal{D}(G, H, \beta)$ has no non-zero divisor of zero, we have $A(\delta_1, \cdots, \delta_{j_0-1})[\gamma_{j_0}, L(W)] = B(\delta_1, \cdots, \delta_{j_0-1})[\delta_{j_0}, L(W)] \neq 0$, so that $[\gamma_{j_0}, L(W)] \neq 0$. Hence, we can choose $T = \tilde{\gamma}_{j_0}$, that is $L(T)$ is the Corwin-Greenleaf generator γ_{j_0} of $\mathcal{D}(G_0, H, \beta)$.

5) *Case:* $\mathcal{Z} \not\subset \mathfrak{h}$. First, remember that an immediate consequence of the assumptions of the theorem is that $\mathcal{Z}$ is embeded in $\mathfrak{g}_0$. Next, let Z be in $\mathcal{Z}$ which does not belong to $\mathfrak{h}$. Denote by $\mathfrak{h}'$ the subalgebra $\mathfrak{h} \oplus \mathbb{R}Z$ of $\mathfrak{g}$ and by H' the connected subgroup of G with Lie algebra $\mathfrak{h}'$. Let ϕ be a generic element of $\Omega_{G,H,\beta}$ and put $\alpha = \phi(Z)$. Define, as usual, the character χ_ϕ of H' by $\chi_\phi(e^U) = e^{2\sqrt{-1}\pi\phi(U)}$ for all $U \in \mathfrak{h}'$, so that the unitary representation $\tau_0^\alpha = Ind_{H'}^{G_0}\chi_\phi$ of G_0 is of finite multiplicities. Let $\mathfrak{h} \subset \mathfrak{h} \oplus \mathbb{R}Z \subset \mathfrak{h} \oplus \mathbb{R}Z \oplus \mathbb{R}X_1 \subset \mathfrak{h} \oplus \mathbb{R}Z \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \cdots \subset \mathfrak{g}_0$ be a Jordan-Hölder sequence for the action of H on $\mathfrak{g}_0$, and consider the sequence $\mathfrak{h}' \subset \mathfrak{h}' \oplus \mathbb{R}X_1 \subset \mathfrak{h}' \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \cdots \subset \mathfrak{g}_0$ which is also a Jordan-Hölder sequence for the action of H' on $\mathfrak{g}_0$. Actually, since H and H' have the same orbits in $\mathfrak{g}_0$, if $\{\gamma_1 = L(Z), \gamma_2, \cdots, \gamma_q\}$ is a set of Corwin-Greenleaf generators of $D(G_0, H, \beta)$, then $\{\gamma_2, \cdots, \gamma_q\}$ is a set of Corwin-Greenleaf generators of $D(G_0, H', \phi)$. Any Corwin-Greenleaf generator of $D(G_0, H, \beta)$ can be represented in $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)/\mathcal{U}(\mathfrak{g}_0)\mathfrak{a}_\beta$ by an element $C = \sum_{\nu,\mu} a_{\nu,\mu} Z^\nu X_1^{\mu_1} \cdots X_p^{\mu_p}$ of $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)$. And observe that any element of $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)$ belongs to $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}', \phi)$. Actually, C acts on $C^\infty(G_0, H', \phi)$ as $C(\alpha) = \sum_{\nu,\mu} a_{\nu,\mu}(-2\sqrt{-1}\pi\alpha)^\nu X_1^{\mu_1} \cdots X_p^{\mu_p}$. On the other hand, the element W of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ acts on $C^\infty(G, H', \phi)$ as $W(\alpha)$, so that $[W(\alpha), C(\alpha)] = [W, C](\alpha)$ on $C^\infty(G, H', \phi)$. Moreover, one can choose α in such a way that $W = W(\alpha) + (W - W(\alpha)) = W(\alpha) + \tilde{W}[Z + 2\sqrt{-1}\pi\alpha]$, with $\tilde{W} \in \mathcal{U}(\mathfrak{g})$

and $W(\alpha) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\phi$. If $\mathcal{Z} \not\subset \mathfrak{h}^{'}$, taking an element in $\mathcal{Z}$ which does not belong to $\mathfrak{h}^{'}$, we apply the same procedure as above. After a finite number of steps, we get, instead of $\mathfrak{h}$, a subalgebra containing the center $\mathcal{Z}$ of $\mathfrak{g}$. In this case, we just apply the results of the previous cases and we choose for the element T one of the Corwin-Greenleaf generators. ☐

3.1. **The case where $(\mathfrak{g}; \mathfrak{h})$ is a reductive pair.** We say that $(\mathfrak{g}; \mathfrak{h})$ is a reductive pair, if there exists a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

Corollary 1. *Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ such that $\beta([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Suppose that $(\mathfrak{g}; \mathfrak{h})$ is a reductive pair. Then the assertions (i), (ii) and (iii) of Section 2 are equivalent.*

Proof. $(i) \Rightarrow (ii)$: Under the notation of Section 2, we take $V = \mathfrak{m}^*$, which could be identified to $\Omega_{G,H,\beta}$, so that we have the unipotent action Ad^* of H on V. Let $\mathfrak{g}_0$ be an ideal of codimension one of $\mathfrak{g}$ containing $\mathfrak{h}$ and $G_0 = \exp(\mathfrak{g}_0)$. One can suppose that $\tau_0 = Ind_H^{G_0} \chi_\beta$ is of finite multiplicities. Then, the H-orbits in V have generically the same dimension as the H-orbits in $(\mathfrak{m} \cap \mathfrak{g}_0)^*$ (that is the H-orbits in V are generically not saturated in the direction $(\mathfrak{m} \cap \mathfrak{g}_0)^{\perp}$). This implies the existence of an H-invariant homogeneous polynomial $\mathcal{P}$ on V which does not belong to $S(\mathfrak{m} \cap \mathfrak{g}_0)$ (Theorem of page 55 in [12]). On the other hand, since $(\mathfrak{g}; \mathfrak{h})$ is a reductive pair then it is easy to check that the symmetrization map σ is a vector space isomorphism between $S(\mathfrak{m})^H$ and $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$. In particular, $\sigma(\mathcal{P})$ is a non-zero element of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ satisfying $\sigma(\mathcal{P}) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$. So Theorem 1 above applies to $W \equiv \sigma(\mathcal{P})$.

For $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ see the end of Section 2. ☐

Remark 3. An interesting consequence of the Corollary 1 is that if H is one-dimensional then the assertions (i), (ii) and (iii) are equivalent. Indeed, if $\mathfrak{h}$ is a one-dimensional subalgebra of $\mathfrak{g}$, then it is easy to see that $(\mathfrak{g}; \mathfrak{h})$ is a reductive pair [5].

Remark 4. Another consequence of the Corollary 1 is the case where $Ker(\beta)$ is an ideal of $\mathfrak{g}$. In this case, we just apply the Remark 3 above to the one-dimensional quotient $\mathfrak{h}/Ker(\beta)$.

4. A second result on the commutativity of $\mathcal{D}(G, H, \beta)$

Here we explain, precisely, how to construct, in some cases, the element W of Theorem 1. We keep the previous notation. In particular, remember that $\sigma : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ denotes the symmetrization map, and L is the left action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G)$ defined by (3.2).

4.1. **Preliminary results.**

Lemma 1. *Let $\mathfrak{m}$ be an ideal of codimension one in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathbb{R}X$. If $P \in S(\mathfrak{m})$, then $\sigma(PX) = \sigma(P)X + Q$ where $Q \in \mathcal{U}(\mathfrak{m})$.*

Proof. Let $k \geq 1$. If $I_k = (i_1, \cdots, i_k) \in [1; m]^k$, we define $P_{I_k} = X_{i_1} \cdots X_{i_k}$. Let $P \in S(\mathfrak{m})$ be of degree d, such that $P = \sum_{k=1}^{d} \sum_{I_k \in [1;m]^k} a_{I_k} P_{I_k}$, with $a_{I_k} \in \mathbb{C}$.

Thus, we have

$$\sigma(PX) = \sum_{k=1}^{d} \sum_{I_k \in [1;m]^k} a_{I_k} \frac{1}{(k+1)!} \sum_{\mu \in \mathcal{S}_k} \sum_{j=0}^{k} T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) \tag{4.1}$$

where $\mathcal{S}_k$ denotes the symmetric group of k elements, and, for all $0 \leq j \leq k$, $T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) \equiv X_{\mu(i_1)} \cdots X_{\mu(i_j)} X X_{\mu(i_{j+1})} \cdots X_{\mu(i_k)}$. Remarking that

$$T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) = T_{j+1}(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) + q, q \in \mathcal{U}(\mathfrak{m}), \tag{4.2}$$

we get that

$$T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) = X_{\mu(i_1)} \cdots X_{\mu(i_k)} X + \tilde{q}, \quad \tilde{q} \in \mathcal{U}(\mathfrak{m}), \tag{4.3}$$

so

$$\begin{aligned} \sigma(PX) &= \sum_{k=1}^{d} \sum_{I_k \in [1;m]^k} a_{I_k} \frac{1}{k!} \sum_{\mu \in \mathcal{S}_k} X_{\mu(i_1)} \cdots X_{\mu(i_k)} X + Q \\ &= \sigma(P)X + Q, \quad Q \in \mathcal{U}(\mathfrak{m}). \end{aligned} \tag{4.4}$$

□

Now let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{g}$, such that $\{X_1, \cdots, X_p\}$ is a basis of $\mathfrak{h}$ and

$$[X_j, X_k] = \sum_{l=1}^{sup(j,k)-1} a_l(j,k) X_l \quad \text{ with } \ a_l(j,k) \in \mathbb{R}. \tag{4.5}$$

Moreover, for all f in $C^\infty(G, H, \beta)$, we define a function $f^\sharp$ on $\mathbb{R}^n$ by

$$f^\sharp(x_1, \cdots, x_n) = f(\exp(x_1 X_1) \cdots \exp(x_n X_n)). \tag{4.6}$$

If X_j is in $\mathfrak{g}$, we put

$$[L(X_j)]^\sharp(f^\sharp) = [L(X_j)(f)]^\sharp \quad \forall f \in C^\infty(G, H, \beta). \tag{4.7}$$

This definition extends naturally to $\mathcal{U}(\mathfrak{g})$.

Lemma 2. *We have*

$$[L(X_j)]^\sharp = \begin{cases} -2\sqrt{-1}\pi\beta(X_j)Id & \text{if } 1 \leq j \leq p, \\ -\frac{\partial}{\partial x_j} - \sum_{l=p+1}^{j-1} q_l \frac{\partial}{\partial x_l} - 2\sqrt{-1}\pi \sum_{l=1}^{p} q_l \beta(X_l) Id & \text{if } p+1 \leq j \leq n, \end{cases} \tag{4.8}$$

where the q_l are polynomials in variables $x_1, \cdots, x_{j-1}$ such that $q_l(0) = 0$.

Proof. Let $f \in C^\infty(G, H, \beta)$. By definition, we have

$$\begin{aligned} &[L(X_j)]f(\exp(x_1 X_1) \cdots \exp(x_n X_n)) \\ &\qquad = \frac{d}{dt} f(\exp(-tX_j)\exp(x_1 X_1) \cdots \exp(x_n X_n)) \mid_{t=0} . \end{aligned} \tag{4.9}$$

Then we have to consider the two cases $1 \leq j \leq p$ and $p+1 \leq j \leq n$.

Case: $1 \leq j \leq p$. Since f is H-covariant, it follows that

$$\begin{aligned} &[L(X_j)]f(\exp(x_1 X_1) \cdots \exp(x_n X_n)) \\ &\qquad = \frac{d}{dt} \exp(-2\sqrt{-1}\pi t \beta(X_j)) f(\exp(x_1 X_1) \cdots \exp(x_n X_n)) \mid_{t=0} \end{aligned} \tag{4.10}$$

which means that

$$[L(X_j)]^\sharp = -2\sqrt{-1}\pi\beta(X_j)Id \quad \forall 1 \leq j \leq p. \tag{4.11}$$

Case: $p+1 \leq j \leq n$. First note that

$$\begin{aligned}
&\exp(-tX_j)\exp(x_1X_1)\cdots\exp(x_nX_n)\\
&= [\prod_{k=1}^{j-1}\exp(Ad(\exp(-tX_j)(x_kX_k)))]\\
&\quad\times\exp((x_j-t)X_j)\exp(x_{j+1}X_{j+1})\cdots\exp(x_nX_n),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
Ad(\exp(-tX_j))(x_kX_k) &= [\exp(-tad(X_j))](x_kX_k)\\
&= x_kX_k - tx_k\sum_{l=1}^{j-1}P_l(k,t)X_l \quad \forall 1 \leq k \leq j-1,
\end{aligned} \tag{4.13}$$

where $P_l(k,t)$ is a polynomial in the variable t. Moreover, the Campbell-Hausdorff formula in [4] allows us to write that

$$\prod_{k=1}^{j-1}\exp(x_kX_k - tx_k\sum_{l=1}^{j-1}P_l(k,t)X_l) = \exp(\sum_{k=1}^{j-1}x_kX_k - t\sum_{l=1}^{j-1}q_l(j-1;t,x)X_l) \tag{4.14}$$

where $q_l(j-1;t,x)$ is a polynomial in the variables t and $x=(x_1,\cdots,x_{j-1})$ such that $q_l(j-1;t,0)=0$ for all $1 \leq l \leq j-1$. The idea is then to rewrite the right side of (4.14) as follows:

$$\begin{aligned}
&\exp(\sum_{k=1}^{j-1}x_kX_k - t\sum_{l=1}^{j-1}q_l(j-1;t,x)X_l)\\
&= \exp(\sum_{k=1}^{j-2}x_kX_k - t\sum_{l=1}^{j-2}q_l(j-1;t,x)X_l + (x_{j-1}-tq_{j-1}(j-1;t,x))X_{j-1})\\
&= \exp(\sum_{k=1}^{j-2}x_kX_k - t\sum_{l=1}^{j-2}q_l(j-1;t,x)X_l + (x_{j-1}-tq_{j-1}(j-1;t,x))X_{j-1})\\
&\times\exp(-(x_{j-1}-tq_{j-1}(j-1;t,x))X_{j-1})\exp((x_{j-1}-tq_{j-1}(j-1;t,x))X_{j-1}).
\end{aligned} \tag{4.15}$$

Again the Campbell-Hausdorff formula implies that

$$\begin{aligned}
&\exp(\sum_{k=1}^{j-1}x_kX_k - t\sum_{l=1}^{j-1}q_l(j-1;t,x)X_l)\\
&= \exp(\sum_{k=1}^{j-2}x_kX_k - t\sum_{l=1}^{j-2}q_l(j-2;t,x)X_l)\exp((x_{j-1}-tq_{j-1}(j-1;t,x))X_{j-1})
\end{aligned} \tag{4.16}$$

where $q_l(j-2;t,x)$ is a polynomial in the variables t and $x=(x_1,\cdots,x_{j-1})$, such that $q_l(j-2;t,0)=0$. We apply the same process to

$$\exp(\sum_{k=1}^{j-2} x_k X_k - t\sum_{j=1}^{j-2} q_l(j-2;t,x)X_l).$$

After $j-2$ steps, we obtain that

(4.17)

$$\exp(\sum_{k=1}^{j-1} x_k X_k - t\sum_{l=1}^{j-1} q_l(j-1;t,x)X_l) = \prod_{k=1}^{j-1} \exp((x_k - tq_k(k;t,x))X_k)$$

where, for all $1 \leq k \leq j-1$, $q_k(k;t,x)$ is a polynomial in the variables t and $x=(x_1,\cdots,x_{j-1})$ such that $q_k(k;t,0)=0$. Thus, we have

(4.18)

$$\begin{aligned}&[L(X_j)]^\sharp f^\sharp(x_1,\cdots,x_n)\\ &= \frac{d}{dt} f^\sharp(x_1 - tq_1(1;t,x),\cdots,x_{j-1}-tq_{j-1}(j-1;t,x),x_j - t,x_{j+1},\cdots,x_n)\mid_{t=0} .\end{aligned}$$

If we put $q_k(x)=q_k(k;0,x)$, $1\leq k\leq j-1$, we obtain the result using the H-covariance of f in $C^\infty(G,H,\beta)$. □

As we said before (Section 3), we view the symmetric algebra $S(\mathfrak{g})$ (resp. $S^m(\mathfrak{g})$) of $\mathfrak{g}$ as the algebra $\mathbb{C}[\mathfrak{g}^*]$ of polynomials (resp. polynomials of degree m) on $\mathfrak{g}^*$. Denote by $S(\mathfrak{g})^H$ (resp. $S^m(\mathfrak{g})^H$) its subalgebra of H-invariant polynomials on $\mathfrak{g}^*$ defined by

$$\mathbb{C}[\mathfrak{g}^*]^H = \{P\in\mathbb{C}[\mathfrak{g}^*] \mid Ad(h)(P)(l)=P(l)\ \forall h\in H\ \forall l\in\mathfrak{g}^*\}. \tag{4.19}$$

It is clear that any polynomial on $\mathfrak{g}^*$ can be written as a finite sum of homogeneous polynomials. Then we have

$$\forall m\in\mathbb{N}\ \ \forall Y\in H\ \ \forall P\in S^m(\mathfrak{g}),\quad ad(Y)(P)\in S^m(\mathfrak{g}) \tag{4.20}$$

so that

$$S(\mathfrak{g})^H = \bigoplus_{m\geq 0} S^m(\mathfrak{g})^H. \tag{4.21}$$

On the other hand, using the basis $\{X_1,\cdots,X_n\}$ defined by (4.5), we write for multi-indexes in $\mathbb{N}^n$, $(\nu,\alpha)=(\nu_1,\cdots,\nu_p,\alpha_{p+1},\cdots,\alpha_n)$. Then, following the Poincaré-Birkhoff-Witt Theorem, any element of $\mathcal{U}(\mathfrak{g})$ can be written as

$$\begin{aligned}&\sum_{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p} a_{\nu,\alpha} X_n^{\alpha_n}\cdots X_{p+1}^{\alpha_{p+1}} X_p^{\nu_p}\cdots X_1^{\nu_1}\\ &\equiv \sum_{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p} a_{\nu,\alpha} X^\alpha X^\nu \text{ with } a_{\nu,\alpha}\in\mathbb{C}.\end{aligned} \tag{4.22}$$

However, to avoid confusion between $\mathcal{U}(\mathfrak{g})$ and $S(\mathfrak{g})$, we shall use small letters for the basis of $\mathfrak{g}$ defined by (4.5) to write any polynomial on $\mathfrak{g}^*$ as

(4.23)

$$\sum_{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p} a_{\nu,\alpha} x_n^{\alpha_n}\cdots x_{p+1}^{\alpha_{p+1}} x_p^{\nu_p}\cdots x_1^{\nu_1} \equiv \sum_{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p} a_{\nu,\alpha} x^\alpha x^\nu \text{ with } a_{\nu,\alpha}\in\mathbb{C}.$$

As usual, if $\lambda \in \mathbb{N}^n$ is a multi-index, we shall denote its length as the number $|\lambda| = \sum_{k=1}^{n} \lambda_k$; so that the degree of the element $\sum_{(\nu,\alpha)\in\mathbb{N}^n} a_{\nu,\alpha} X^\alpha X^\nu$ of $\mathcal{U}(\mathfrak{g})$ is the number $|\alpha| + |\nu|$. In the sequel, we shall denote by $\mathcal{U}_m(\mathfrak{g})$ the vector subspace of $\mathcal{U}(\mathfrak{g})$ of the elements with degree at most m.

Lemma 3. *Let G be a simply connected connected nilpotent Lie group. Assume H is a commutative subgroup of G. Let $\mathcal{P}$ be an H-invariant homogeneous polynomial on $\mathfrak{g}^*$ such that $\mathcal{P}$ does not vanish identically on $\mathfrak{g}^*$. Then there exists a non-empty Zariski open subset $\mathcal{O}$ of $\mathfrak{h}^*$, such that for all β in $\mathcal{O}$, we have $(L_\beta \circ \sigma)(\mathcal{P}) \neq 0$ in $\mathcal{D}(G, H, \beta)$, where L_β is the isormorphism induced by* (3.2).

Proof. Suppose that $\mathcal{P}$ is a homogeneous polynomial of degree d which does not vanish identically on $\mathfrak{g}^*$. We can write as in (4.23):

$$\mathcal{P} = \sum_{\substack{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p \\ |\alpha|+|\nu|=d}} a_{\nu,\alpha} x^\alpha x^\nu \tag{4.24}$$

with $a_{\nu,\alpha}$ in $\mathbb{C}$. Then applying the symmetrization map to $\mathcal{P}$, we get

$$\sigma(\mathcal{P}) = \sum_{\substack{(\alpha,\nu)\in\mathbb{N}^{n-p}\times\mathbb{N}^p \\ |\alpha|+|\nu|=d}} (a_{\nu,\alpha} X^\alpha X^\nu + W_{\alpha,\nu}) \tag{4.25}$$

where

$$W_{\alpha,\nu} = \sum_{\substack{(\alpha',\nu')\in\mathbb{N}^{n-p}\times\mathbb{N}^p \\ |\alpha'|+|\nu'|<|\alpha|+|\nu|}} b_{\nu',\alpha'} X^{\alpha'} X^{\nu'}. \tag{4.26}$$

Actually, we can rewrite (4.25) as follows:

$$\sigma(\mathcal{P}) = \sum_{\alpha\in\mathbb{N}^{n-p}} X^\alpha \Big(\sum_{\substack{\nu\in\mathbb{N}^p \\ |\nu|=d-|\alpha|}} a_{\alpha,\nu} X^\nu + \sum_{\substack{\nu'\in\mathbb{N}^p \\ |\nu'|<d-|\alpha|}} b_{\alpha,\nu'} X^{\nu'} \Big). \tag{4.27}$$

Let us define the polynomial $\mathcal{P}_\alpha$ on $\mathfrak{h}^*$ as

$$\mathcal{P}_\alpha = \sum_{\substack{\nu\in\mathbb{N}^p \\ |\nu|=d-|\alpha|}} a_{\alpha,\nu} x^\nu + \sum_{\substack{\nu'\in\mathbb{N}^p \\ |\nu'|<d-|\alpha|}} b_{\alpha,\nu'} x^{\nu'}, \tag{4.28}$$

such that

$$(L_\beta \circ \sigma)(\mathcal{P}) = \sum_{\alpha\in\mathbb{N}^{n-p}} \mathcal{P}_\alpha(-2\sqrt{-1}\pi\beta) X^\alpha. \tag{4.29}$$

Define the subset $\mathcal{A}_\mathcal{P}$ of multi-indexes in $\mathbb{N}^{n-p}$ by

$$\mathcal{A}_\mathcal{P} = \{\alpha \in \mathbb{N}^{n-p} \mid \mathcal{P}_\alpha \not\equiv 0\}. \tag{4.30}$$

Since $\mathcal{P}$ does not vanish identically on $\mathfrak{g}^*$, there exists a multi-index α in N^{n-p} such that $\mathcal{P}_\alpha$ does not vanish identically on $\mathfrak{h}^*$, so that the subset $\mathcal{A}_\mathcal{P}$ is not empty. Next define the variety $\mathcal{M}_\mathcal{P}$ of $\mathfrak{h}^*$ by

$$\mathcal{M}_\mathcal{P} = \bigcap_{\alpha\in\mathcal{A}_\mathcal{P}} \{\mathcal{P}_\alpha = 0\}. \tag{4.31}$$

It is clear that $\mathcal{M}_\mathcal{P}$ is a non-empty Zariski closed subset of $\mathfrak{h}^*$ which differs from $\mathfrak{h}^*$. Then we define $\mathcal{O}_\mathcal{P}$ as the non-empty Zariski open subset of $\mathfrak{h}^*$:

$$\mathcal{O}_\mathcal{P} = \mathfrak{h}^* \setminus \mathcal{M}_\mathcal{P}. \tag{4.32}$$

On the other hand, for all linear forms l in $\mathfrak{h}^*$, define the subset $\mathcal{A}_{\mathcal{P},l}$ of $\mathcal{A}_{\mathcal{P}}$ by

$$\mathcal{A}_{\mathcal{P},l} = \{\alpha \in \mathcal{A}_{\mathcal{P}} \mid \mathcal{P}_\alpha(l) \neq 0\}. \tag{4.33}$$

Note that if l is in $\mathcal{O}_{\mathcal{P}}$, then $\mathcal{A}_{\mathcal{P},l}$ is not empty.

Finally, fix β in $\mathcal{O}_{\mathcal{P}}$. Let ξ be an element of maximal length in $\mathcal{A}_{\mathcal{P},\beta}$. We define a function ϕ_ξ in $C^\infty(G,H,\beta)$ as follows:

(4.34)
$$\phi_\xi(\exp(t_1X_1)\cdots\exp(t_nX_n)) = \chi_\beta(\exp(t_1X_1)\cdots\exp(t_pX_p))(-t_{p+1})^{\xi_{p+1}}\cdots(-t_n)^{\xi_n}.$$

ϕ_ξ is a homogeneous function of degree $\mid \xi \mid$ in the variables $t_{p+1},\cdots,t_n$. Using Lemma 2 together with (4.29), we obtain that

$$[(L_\beta \circ \sigma)(\mathcal{P})]^\sharp(\phi_\xi^\sharp)(0) = \mathcal{P}_\xi(-2\sqrt{-1}\pi\beta)t_{p+1}!\cdots t_n!. \tag{4.35}$$

We have $[(L_\beta \circ \sigma)(\mathcal{P})]^\sharp(\phi_\xi^\sharp) \neq 0$. Hence $[(L_\beta \circ \sigma)(\mathcal{P})](\phi_\xi) \neq 0$. This shows that $(L_\beta \circ \sigma)(\mathcal{P}) \neq 0$. □

4.2. **A second theorem.**

Theorem 2. *Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, H a connected closed commutative subgroup of G with Lie algebra $\mathfrak{h}$. Consider a weak Malcev basis passing through $\mathfrak{h}$. Then, if $\mathfrak{h}+\mathfrak{g}(l)$ is not lagrangian in $\mathfrak{g}$ for generic l in $\Omega_{G,H,\beta}$, the algebra $\mathcal{D}(G,H,\beta)$ is not commutative, for all β in a non-empty Zariski open subset of $\mathfrak{h}^*$.*

Proof. Let $\mathfrak{g}_0$ be an ideal of codimension one of $\mathfrak{g}$ containing $\mathfrak{h}$ and $G_0 = \exp(\mathfrak{g}_0)$. One can suppose that $\tau_0 = Ind_H^{G_0}\chi_\beta$ is of finite multiplicities. Under the notation of Section 2, we take $V = \mathfrak{g}^*$. So it is clear that $V^{gene} \cap \Omega_{G,H,\beta} \neq \emptyset$ for almost all β in $\mathfrak{h}^*$. Under the assumptions of the Theorem 2, the Pukanszky parametrization of the H-orbits in $\mathfrak{g}^*$, outlined in Section 2, gives a non-zero H-invariant polynomial $\mathcal{P}$ on $\mathfrak{g}^*$ such that $\mathcal{P} \notin S(\mathfrak{g}_0)$. Moreover, using $(4.20)-(4.21)$, one can suppose that $\mathcal{P}$ is homogeneous. Then, from Lemma 3, $\sigma(\mathcal{P}) \notin \mathcal{U}(\mathfrak{g}_0)+\mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ and $(L_\beta \circ \sigma)(\mathcal{P})$ is a non-zero element of $\mathcal{D}(G,H,\beta)$, for all β in $\mathcal{O}_{\mathcal{P}}$, as defined by (4.32). Thus, we apply Theorem 1 to get an element T of $\mathcal{U}(\mathfrak{g}_0,\mathfrak{h},\beta)$ such that $[(L_\beta \circ \sigma)(\mathcal{P}), L_\beta(T)] \neq 0$ in $\mathcal{D}(G,H,\beta)$. □

4.3. **The case where $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.**

Corollary 2. *Let G be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and H a connected closed normal subgroup of G with Lie algebra $\mathfrak{h}$. Then, for almost all β in $\mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$, the assertions (i), (ii) and (iii) of Section 2 are equivalent.*

Proof. $(i) \Rightarrow (ii)$: Under the notation of Section 2, we take $V = [\mathfrak{h},\mathfrak{h}]^\perp$ and we choose β in the fundamental layer of V to apply Theorem 2.

For $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ see the end of Section 2. □

5. Characterization of $\mathcal{D}(G,H,\beta)$ in terms of the algebra of $Ad^*(H)$-invariant rational functions on $\Omega_{G,H,\beta}$

We shall denote by π_l the representation associated to $l \in \Omega_{G,H,\beta}/H$ by the Kirillov map and by $d\tilde{\mu}$ the image on $\Omega_{G,H,\beta}/H$ of the Lebesgue measure on $\Omega_{G,H,\beta}$.

If $\phi = \int^{\oplus}_{\Omega_{G,H,\beta}/H} \phi_{\pi_l} d\tilde{\mu}(l)$, then

$$D\phi = \int^{\oplus}_{\Omega_{G,H,\beta}/H} \Theta^{\tau}(D,l)\phi_{\pi_l} d\tilde{\mu}(l) \;\; \forall D \in \mathcal{D}(G,H,\beta) \tag{5.1}$$

where $\Theta^{\tau}(D,.)$ belongs to $\mathbb{C}(\Omega_{G,H,\beta})^H$, the algebra of $Ad^*(H)$-invariant rational functions on $\Omega_{G,H,\beta}$. The application $\Theta^{\tau} : \mathcal{D}(G,H,\beta) \to \mathbb{C}(\Omega_{G,H,\beta})^H$ is an isomorphism between $\mathcal{D}(G,H,\beta)$ and a subalgebra of $\mathbb{C}(\Omega_{G,H,\beta})^H$. Actually Fujiwara proved that if there exists a common polarization for almost all linear forms on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is β or if $\mathfrak{h}$ is 1-dimensional, then Θ^{τ} is an isomorphism between $\mathcal{D}(G,H,\beta)$ and $\mathbb{C}[\Omega_{G,H,\beta}]^H$, the algebra of $Ad^*(H)$-invariant polynomials on $\Omega_{G,H,\beta}$ [8]. This gives a partial answer to a question of Corwin and Greenleaf [5], also asked by Duflo (Problème 3 of [7]) in a more general context.

In the particular cases studied above, we have

Corollary 3. *Let G be a connected simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$. The following assertions (a) and (b) are equivalent:*

- for all β in $\mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$ when $(\mathfrak{g},\mathfrak{h})$ is a reductive pair,

- for almost all β in $\mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$ when $\mathfrak{h}$ is commutative or $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

(a) $\mathcal{D}(G,H,\beta)$ is a commutative algebra.

(b) $\mathcal{D}(G,H,\beta)$ is isomorphic, via Θ^{τ}, to a subalgebra of $\mathbb{C}(\Omega_{G,H,\beta})^H$.

Proof. $(a) \Rightarrow (b)$: From Theorem 2 and Corollaries 1 and 2 if $\mathcal{D}(G,H,\beta)$ is commutative, then τ is of finite multiplicities, so that the results of [5] apply.

$(b) \Rightarrow (a)$ is obvious. □

Remark 5. Note that in the particular reductive case where H is one-dimensional (Remark 3) or if H is a normal subgroup of G (Corollary 2), then from [8], the image of $\mathcal{D}(G,H,\beta)$ under Θ^{τ} is, actually, the algebra $\mathbb{C}[\Omega_{G,H,\beta}]^H$ of $Ad^*(H)$-invariant polynomials on $\Omega_{G,H,\beta}$.

6. Examples

In the following examples $\mathfrak{g}$ will be the real nilpotent Lie algebra of dimension 7 generated by the vectors $\{X_i,\ 1 \le i \le 7\}$ with the following non-zero brackets:

$[X_1,X_3] = X_2$, $[X_1,X_4] = X_3$, $[X_1,X_5] = X_4$, $[X_1,X_7] = X_6$, $[X_4,X_5] = X_6$, $[X_5,X_6] = X_2$ and $[X_4,X_7] = -X_2$.

Moreover, in the following examples, $\tau = Ind_H^G \chi_\beta$ is of infinite multiplicities.

Example 1. Take $\mathfrak{h} = \mathbb{R}X_1$ and $\beta = \xi_1 X_1^*$. Put $l = \sum_{i=1}^{7} \xi_i X_i^*$ with $\xi_2 \neq 0$. Corollary 1 and Remark 3 apply in this situation. We take the Malcev basis ordered in the following way: X_1, X_2, X_3, X_4, X_6, X_5 and X_7. This defines a Jordan-Hölder sequence of subalgebras of $\mathfrak{g}$. It happens that

$$Ad^{\star}(\exp(-tX_1))(\sum_{i=1}^{7} \xi_i X_i^{\star}) = \sum_{i=1}^{7} \xi_i(t) X_i^{\star}$$

with

$\xi_1(t) = \xi_1, \quad \xi_2(t) = \xi_2, \quad \xi_3(t) = \xi_3 + t\xi_2,$

$\xi_4(t) = \xi_4 + t\xi_3 + \frac{1}{2}t^2\xi_2, \quad \xi_6(t) = \xi_6,$

$\xi_5(t) = \xi_5 + t\xi_4 + \frac{1}{2}t^2\xi_3 + \frac{1}{6}t^3\xi_2, \quad \xi_7(t) = \xi_7 + t\xi_6.$

We parametrize the H-orbits by $u = \xi_3 + t\xi_2$. The orbit of ℓ is of dimension 1 and is exactly the set $\{\ell(u) = \sum r_i(\ell, u)X_i^\star, u \in \mathbb{R}\}$ where

$r_1(\ell, u) = \xi_1, \quad r_2(\ell, u) = \xi_2,$

$r_3(\ell, u) = u$, at this step, dimension of orbits passes from 0 to 1

$r_4(\ell, u) = \frac{2\xi_2\xi_4 - \xi_3^2}{2\xi_2} + \frac{1}{2\xi_2}u^2, \quad r_6(\ell, u) = \xi_6,$

$r_5(\ell, u) = \frac{1}{6\xi_2^2}u^3 + \frac{2\xi_2\xi_4 - \xi_3^2}{2\xi_2^2}u + \frac{\xi_3^3 + 3\xi_2^2\xi_5 - 3\xi_2\xi_3\xi_4}{3\xi_2^2}, \quad r_7(\ell, u) = \frac{\xi_6}{\xi_2}u + \frac{\xi_2\xi_7 - \xi_3\xi_6}{\xi_2}.$

Thus, this gives us rational functions and then H-invariant polynomial functions that are written in terms of the variables ξ_i. The elements of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ obtained by symmetrization are: X_1, X_2, $2X_2X_4 - X_3^2$, X_6, $X_3^3 + 3X_2^2X_5 - 3X_2X_3X_4$ and $X_2X_7 - X_3X_6$.

We have $[X_3^3 + 3X_2^2X_5 - 3X_2X_3X_4, X_6] = 3X_2^3 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$ and $[X_2X_7 - X_3X_6, 2X_2X_4 - X_3^2] = 2X_2^3 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$. The left action of these elements on $C^\infty(G, H, \beta)$ gives elements of the algebra $\mathcal{D}(G, H, \beta)$, which is not commutative.

Example 2. Put $\mathfrak{h} = \mathbb{R}X_1 \oplus \mathbb{R}X_6$ with $\beta = \xi_1X_1^* + \xi_6X_6^*$ where $\xi_6 \neq 0$. Since $\mathfrak{h}$ is commutative, we apply Theorem 2. The condition $\xi_6 \neq 0$ ensures the coincidence of the fundamental and generic layers. Analogous calculations as those of Example 1 give the following elements of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ whose images under L belong to $\mathcal{D}(G, H, \beta)$: X_6, X_1, X_2, $X_2X_7 - X_3X_6$ and $2X_4X_6^2 - 2X_3X_6X_7 + X_2X_7^2$.

As in the previous example, the algebra $\mathcal{D}(G, H, \beta)$ is not commutative, since $[X_2X_7 - X_3X_6, 2X_4X_6^2 - 2X_3X_6X_7 + X_2X_7^2] = 2X_2^2X_6^2 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$.

Example 3. Take $\mathfrak{h} = \mathbb{R}X_1 \oplus \mathbb{R}X_6$ with $\beta = \xi_1X_1^* + \xi_6X_6^*$ where $\xi_6 = 0$. Analogous calculations as those of Example 1 give the following elements of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$: X_6, X_1, X_7, X_2 and $2X_2X_4 - X_3^2$.

Since $[2X_2X_4 - X_3^2, X_7] = -2X_2^2 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\beta$, the algebra $\mathcal{D}(G, H, \beta)$ is not commutative. Here it is interesting to note that our situation is degenerated. However, we observe that in this example the previous constructions give a non-commutative family of elements in $\mathcal{D}(G, H, \beta)$.

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