INVARIANT DISTRIBUTIONS SUPPORTED ON
THE NILPOTENT CONE OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \) and \( \mathcal{D}(g) \) be the algebra of differential operators with polynomial coefficients on \( g \). If \( g_0 \) is a real form of \( g \), we give the decomposition of the semisimple \( \mathcal{D}(g)^G \)-module of invariant distributions on \( g_0 \) supported on the nilpotent cone.

0. Introduction

Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \). Choose a Cartan subalgebra \( \mathfrak{h} \) of \( g \) and let \( W \) be the associated Weyl group. Denote by \( W(h) \) the set of isomorphism classes of irreducible \( W \)-modules and by \( H(h) \) the graded vector space of \( W \)-harmonic polynomials on \( h \). For \( \chi \in W(h) \), set

\[
b(\chi) = \inf \{ j \in \mathbb{N} : [H^j(h) : \chi] \neq 0 \}
\]

and choose a \( W \)-submodule \( V_\chi \subset H^{b(\chi)}(h^*) \) in the class of \( \chi \). Denote by \( d(\chi) \) the dimension of \( V_\chi \).

Let \( S(g^*) \) be the algebra of polynomial functions on \( g \) and \( \mathcal{D}(g) \) be the algebra of differential operators on \( g \), with coefficients in \( S(g^*) \). The group \( G \) acts on \( g \), via the adjoint action, and hence has an induced action on \( S(g^*) \), \( S(g) \) and \( \mathcal{D}(g) \). Denote the differential of this action by \( \tau : g \to \mathcal{D}(g) \). Let \( S_+(g)^G \) and \( S_+(g^*)^G \) be the set of invariant elements without constant term. Recall that \( N(g) \), the nilpotent cone of \( g \), is the variety of zeroes of the ideal \( S_+(g^*)^G S(g^*) \).

Let \( g_0 \) be a real form of \( g \) with adjoint group \( G_0 \subset G \). Denote by \( \text{Db}(g_0) \) the \( \mathcal{D}(g) \)-module of distributions on \( g_0 \). Then, the subspace of invariant distributions \( \text{Db}(g_0)^{Go} \) is a \( \mathcal{D}(g)^G \)-module, containing the submodule of invariant distributions supported on the nilpotent cone

\[
\text{Db}(g_0)^{Go}_{nil} = \{ \Theta \in \text{Db}(g_0)^{Go} : \text{Supp} \Theta \subset N(g_0) \}
\]

where \( N(g_0) = N(g) \cap g_0 \) is the nilpotent cone of \( g_0 \). The structure of \( \text{Db}(g_0)^{Go}_{nil} \) as a vector space is well understood, see, for example, [1][5]. Let \( \{ \mathfrak{h}_1, \ldots, \mathfrak{h}_r \} \) be the conjugacy classes of Cartan subalgebras of \( g_0 \). For each \( j \), let \( \varepsilon_{1,j} : W(h_j) \to \{ \pm 1 \} \) be taken

\[\]
Theorem A. The $D(\mathfrak{g})^G$-module $Db(\mathfrak{g}_0)_{nil}^{G_0}$ decomposes as

$$Db(\mathfrak{g}_0)_{nil}^{G_0} \cong \bigoplus_{\chi \in W^c} m_{\chi} M_{\chi}^G$$

where $m_{\chi} = \sum_{j=1}^{r} \dim V_{\chi}^{\epsilon_{1,j}}$.

This theorem is proved by combining the isomorphism $(\ast)$ and the properties, established in 18 11 12 13, of the Harish-Chandra homomorphism

$$\delta : D(\mathfrak{g})^G \longrightarrow D(\mathfrak{h})^W.$$  

In the particular case where $\mathfrak{g}_0$ is a complex Lie algebra $\mathfrak{g}_1$ (viewed as a real Lie algebra), Theorem A was proved by N. Wallach 18. In this case, $\mathfrak{g} \simeq \mathfrak{g}_1 \times \mathfrak{g}_1$, $W \simeq W_1 \times W_1$ where $W_1$ is the Weyl group of $\mathfrak{g}_1$. Then, each $M_{\chi}$ occurring in the decomposition of $Db(\mathfrak{g}_0)_{nil}^{G_0}$ is of the form $M_{\phi} \boxtimes M_{\phi}$ with $\chi = \phi \boxtimes \phi$, $\phi \in W_1^c$, and one has $m_{\chi} = 1$. Hence $Db(\mathfrak{g}_0)_{nil}^{G_0} \cong \bigoplus_{\phi \in W_1^c} M_{\phi}^{G_1} \boxtimes M_{\phi}^{G_1}$ as a $D(\mathfrak{g})^G$-module.

The next corollary is an easy consequence of Theorem A.

Corollary B. Let $\chi \in W^c$, then, $M_{\chi} \cong D(\mathfrak{g}).\Theta$ for some $\Theta \in Db(\mathfrak{g}_0)$ if, and only if, $V_{\chi}^{\epsilon_{1,j}} \neq 0$ for some $j \in \{1, \ldots, r\}$.

In Remark 3.7, we apply this result to give examples of modules $M_{\chi}$ which cannot be generated by a distribution on any real form of $\mathfrak{g}$.

1. Preliminary results

We retain the notation of the introduction. Denote by $\Delta$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and fix a system $\Delta^+$ of positive roots. Set $n = \dim \mathfrak{g}$, $\ell = \dim \mathfrak{h}$ and $\nu = \# \Delta^+$, hence $n = 2\nu + \ell$. Let $\pi$ be the product of positive roots and recall that $x \in \mathfrak{g}$ is called generic if $\pi(x) \neq 0$. If $a \subseteq \mathfrak{g}$, we denote by $a'$ the set of generic elements in $a$.

For $q \in S(\mathfrak{g})$, let $\partial(q) \in D(\mathfrak{g})$ be the corresponding differential operator with constant coefficients. Let $\{e_i\}_{1 \leq i \leq n}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form $\kappa$ such that $\{e_i\}_{1 \leq i \leq \ell}$ is a basis of $\mathfrak{h}$. Denote by $x_i \in S(\mathfrak{g}^*)$,
1 \leq i \leq n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\mathfrak{g}$ and $\mathfrak{h}$ by $E_\mathfrak{g} = \sum_{i=1}^n x_i \partial_i$ and $E_\mathfrak{h} = \sum_{i=1}^\ell x_i \partial_i$.

We now give some notation and results from \[11\] \[12\] \[13\] \[18\]. Recall first that the algebra homomorphism, defined by Harish-Chandra,

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$$

extends the Chevalley isomorphisms $\mathcal{S}(\mathfrak{g})^G \cong \mathcal{S}(\mathfrak{h})^W$ and $\mathcal{S}(\mathfrak{g}^*)^G \cong \mathcal{S}(\mathfrak{h}^*)^W$. The map $\delta$ is surjective and its kernel is $\mathcal{I} = (\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$. This enables one to identify, through $\delta$, modules over $A(\mathfrak{g}) := \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$ with $\mathcal{D}(\mathfrak{h})^W$-modules.

**Lemma 1.1.** Let $D \in \mathcal{D}(\mathfrak{g})^G$. Then $D = P + Q$ with $P \in \mathcal{C}(\mathcal{S}(\mathfrak{g})^G, \mathcal{S}(\mathfrak{g}^*)^G)$ and $Q \in \mathcal{I}$.

**Proof.** By \[11\], we know that $\mathcal{D}(\mathfrak{h})^W = \mathcal{C}(\mathcal{S}(\mathfrak{h})^W, \mathcal{S}(\mathfrak{h}^*)^W)$. The lemma is therefore a consequence of the properties of $\delta$ previously recalled. \[\square\]

Recall that the $(\mathcal{D}(\mathfrak{h})^W, W)$-module $\mathcal{S}(\mathfrak{h}^*)$ decomposes as

$$\mathcal{S}(\mathfrak{h}^*) \cong \bigoplus_{\chi \in W^*} V_\chi \otimes_\mathbb{C} V_\chi$$

where $V_\chi = \text{Hom}_W(V_\chi, \mathcal{S}(\mathfrak{h}^*))$ is a simple $\mathcal{D}(\mathfrak{h})^W$-module. Let $\{v_1^\chi, \ldots, v_{d(\chi)}^\chi\}$ be a basis of $V_\chi$, then $V_\chi \cong \mathcal{D}(\mathfrak{h})^W. v_j^\chi$ for all $j$ and \[11\] implies that

$$\mathcal{S}(\mathfrak{h}^*) = \bigoplus_{\chi \in W^*} \bigoplus_{j=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W. v_j^\chi.$$

Now, set $N = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} \mathcal{S}(\mathfrak{h}^*)$ and $N_\chi = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} V_\chi$. We have

$$N = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})\mathcal{S}(\mathfrak{g}^*)^G$$

and, using \[11\],

$$N = \bigoplus_{\chi \in W^*} N_\chi \otimes_\mathbb{C} V_\chi.$$

Then each $N_\chi$ is a simple (holonomic) $\mathcal{D}(\mathfrak{g})$-module \[13\] and, therefore, $N$ is a semisimple $\mathcal{D}(\mathfrak{g})$-module (see also \[9\]). Let $\mathcal{C}(N)$ be the full subcategory of finitely generated $\mathcal{D}(\mathfrak{g})$-modules of the form $\bigoplus_{\chi \in W^*} m_\chi N_\chi$, $m_\chi \in \mathbb{N}$. From \[13\] we know that the category $\mathcal{C}(N)$ is equivalent to the category $W$-mod (of finite dimensional $W$-modules) via the functor

$$\text{Sol} : \mathcal{C}(N) \longrightarrow W\text{-mod}, \quad \text{Sol}(N) = \text{Hom}_{\mathcal{D}(\mathfrak{h})^W}(N^G, \mathcal{S}(\mathfrak{h}^*))$$

where $W$ acts on $\text{Sol}(N)$ through its natural action on $\mathcal{S}(\mathfrak{h}^*)$.

The Killing form $\kappa$ induces a $G$-isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and an algebra automorphism $\kappa$ of $\mathcal{D}(\mathfrak{g})$, defined by $\kappa(\partial(v)) = \kappa(v)$, $\kappa(\kappa(v)) = -\partial(v)$, for all $v \in \mathfrak{g}$. Hence, in coordinates, $\kappa(\partial_j) = x_j$, $\kappa(x_j) = -\partial_j$. Set $i = \sqrt{-1} \in \mathbb{C}$ and denote by $i$ the automorphism of $\mathcal{D}(\mathfrak{g})$ given by $i(\partial_j) = -i\partial_j$, $i(x_j) = ix_j$. Define then the “Fourier transformation” $F_\mathfrak{g} \in \text{Aut} \mathcal{D}(\mathfrak{g})$ by $F_\mathfrak{g} = i \circ \kappa = \kappa \circ i^{-1}$; thus $F_\mathfrak{g}(x_j) = i\partial_j$, $F_\mathfrak{g}(\partial_j) = ix_j$. One easily checks that $\kappa(\tau(x)) = F_\mathfrak{g}(\tau(x)) = \tau(x)$. 

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for all \( x \in \mathfrak{g} \); moreover, \( \kappa \) and \( F_\mathfrak{g} \) are \( G \)-equivariant. Similarly, since \( \kappa \) is non-degenerate and \( W \)-invariant on \( \mathfrak{h} \), one can define \( W \)-equivariant automorphisms \( \kappa \) and \( F_\mathfrak{h} = 1 \circ \kappa \) in \( \text{Aut} \ D(\mathfrak{h}) \).

**Lemma 1.2.** One has \( \delta \circ F_\mathfrak{g} = F_\mathfrak{h} \circ \delta \).

**Proof.** A direct computation shows that \( \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) \) when \( P \) belongs to \( S(\mathfrak{g})^G \) or \( S(\mathfrak{g}^*)^G \). Since \( \delta \) is a homomorphism, it follows that \( \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) \) for all \( P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G) \). Now, let \( D \in D(\mathfrak{g})^G \) and write \( D = P + Q \) as in Lemma 1.1. Then, since \( F_\mathfrak{g}(I) = I \), we have \( \delta(F_\mathfrak{g}(D)) = \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) = F_\mathfrak{h}(\delta(D)) \).

Recall that \( \mathcal{H}(\mathfrak{h}^*) \) is the vector space of \( W \)-harmonic polynomials on \( \mathfrak{h} \). Hence

\[
\mathcal{H}(\mathfrak{h}^*) = \{ f \in S(\mathfrak{h}^*) : \partial(q).f = 0 \text{ for all } q \in S_+(\mathfrak{h})^W \}
\]

and, as a \( W \)-module, \( \mathcal{H}(\mathfrak{h}^*) \) identifies with the regular representation of \( W \). The vector space \( \mathcal{H}(\mathfrak{h}^*) \) is a graded subspace of \( S(\mathfrak{h}^*) \) and we set \( \mathcal{H}^j(\mathfrak{h}^*) = S^j(\mathfrak{h}^*) \cap \mathcal{H}(\mathfrak{h}^*) \), \( 0 \leq j \leq \nu \). Define the harmonic elements of \( S(\mathfrak{h}) \) by \( \mathcal{H}(\mathfrak{h}) = F_\mathfrak{h}(\mathcal{H}(\mathfrak{h}^*)) = \bigoplus_{j=0}^\nu \mathcal{H}^j(\mathfrak{h}) \). (We could as well have set \( \mathcal{H}(\mathfrak{h}) = \kappa(\mathcal{H}(\mathfrak{h}^*)) \), since \( \mathcal{H}^j(\mathfrak{h}^*) \) is stable under \( \kappa \).)

Since \( V_\chi \subset \mathcal{H}^k(\chi)(\mathfrak{h}^*) \), we have \( (E_\chi - b(\chi)).v^j_\chi = 0 \). For all \( d \in L := \text{ann}_{D(\mathfrak{h})^W}(v^j_\chi) \), we have \( [E_\chi - b(\chi), d] = [E_\chi, d] \in L \). It follows that \( L = \bigoplus_{k \in \mathbb{Z}} L \cap D^k(\mathfrak{h})^W \), where \( D^k(\mathfrak{h}) = \{ d \in D(\mathfrak{h}) : [E_\chi, d] = kd \} \). Equivalently, \( L \) is stable under the \( \mathbb{C}^* \)-action on \( D(\mathfrak{h}) \) given by \( f \mapsto \lambda f, \partial(v) \mapsto \lambda^{-1} \partial(v) \), \( f \in \mathfrak{h}^*, v \in \mathfrak{h} \). In particular, we see that \( F_\mathfrak{h}(L) = \kappa(L) \).

Let \( R \) be a ring and \( \alpha \in \text{Aut}(R) \). If \( M \) is an \( R \)-module, we define the \( R \)-module \( M^\alpha \) to be the abelian group \( M \) with action of \( a \in R \) on \( x \in M \) given by \( ax = \alpha(a)x \). This applies to the modules \( N_\chi \) and the automorphism \( \alpha = F_\mathfrak{g}^{-1} \). Define

\[
M = N_{\mathfrak{h}^*}^{F_\mathfrak{g}}, \quad M_\chi = N_{\chi}^{F_\mathfrak{g}}.
\]

Thus, from (1.2) and (1.3), we obtain

\[
M = D(\mathfrak{g})/\langle D(\mathfrak{g})\tau(\mathfrak{g}) + D(\mathfrak{g})S_+(\mathfrak{g}^*)^G \rangle \cong \bigoplus_{\chi \in W^+} M_{\chi} \otimes_C V_\chi.
\]

**Remark.** In (1.3) one defines \( M_\chi \) to be \( \mathfrak{N}_{\chi}^{\mathfrak{h}^*} \), but the two definitions agree. Indeed, let \( V_\chi \cong D(\mathfrak{h})^W.v^j_\chi = D(\mathfrak{h})^W/L \) be as above. Then,

\[
N_{\chi} = D(\mathfrak{g}).(I \otimes A(\mathfrak{g})v^j_\chi), \text{ where } I \text{ is the canonical generator of } D(\mathfrak{g})/D(\mathfrak{g})\tau(\mathfrak{g}).
\]

From \( \delta(E_\chi) = E_\chi - \nu \), we get that \( (E_\chi - (b(\chi) - \nu)).(I \otimes A(\mathfrak{g})v^j_\chi) = 0 \). It follows (as above) that \( J \) is stable under the natural \( \mathbb{C}^* \)-action on \( D(\mathfrak{g}) \). Hence, \( F_\mathfrak{g}(J) = \kappa(J) \) and we have \( N_{\mathfrak{h}^*} = N_{\mathfrak{h}^*}^{F_\mathfrak{g}} \).

We can define the category \( C(M) \) similar to \( C(N) \). We clearly have \( M \in C(M) \) if, and only if, \( N = M^{F_\mathfrak{g}} \in C(N) \). Moreover, by (1.3), this is equivalent to saying that \( M \) is a \( G \)-equivariant finitely generated \( D(\mathfrak{g}) \)-module such that \( M = D(\mathfrak{g})M^G \) and \( \text{Supp } M \subset N(\mathfrak{g}) \). This is also equivalent to: \( N \) is a \( G \)-equivariant finitely generated \( D(\mathfrak{g}) \)-module such that \( N = D(\mathfrak{g})N^G \) and \( N \) is \( S^- \)-finite (meaning that each \( v \in N \) is killed by a power of \( S_+(\mathfrak{g})^G \)).

Recall that \( N_{\chi}^{\mathfrak{g}} \cong V_\chi \) through the identification of \( A(\mathfrak{g}) \) with \( D(\mathfrak{h})^W \).
Lemma 1.3. One has $\mathcal{M}_x^G \sim (V^\chi)^{F_\chi}$.  

Proof. Write $\mathcal{N}_x = \mathcal{D}(g)/J$. Then, $\mathcal{M}_x = \mathcal{D}(g)/F(g)(J)$ and $\mathcal{M}_x^G = \mathcal{D}(g)^G/F(g)^G$. By Lemma 1.3, $\delta(F(g)(J^G)) = F(g)(\delta(J^G))$, therefore $\mathcal{M}_x^G \sim (\mathcal{D}(h)^W/F(h)(\delta(J^G)))$. Since $V^\chi \cong (\mathcal{D}(h)^W/F(h)(\delta(J^G)))$, the lemma follows. \hfill \Box 

Let $g_0$ be a real form of $g$ with adjoint group $G_0 \subset G$. There exists a natural action of $\mathcal{D}(g)$ on $\mathcal{D}(g_0)$ defined by 

$$\langle \partial(v), T, f \rangle = \langle T, -\partial(v), f \rangle, \quad \langle \xi, T, f \rangle = \langle T, \xi, f \rangle$$ 

for all $T \in \mathcal{Db}(g_0)$, $f \in C_c^\infty(g_0)$, $v \in g$, $\xi \in g^*$. This induces a structure of $\mathcal{D}(g)^G$-module on $\mathcal{Db}(g_0)^G$. From $\mathcal{T}; \mathcal{Db}(g_0)^G_0 = 0$, we obtain a natural $A(g)$-module structure on $\mathcal{Db}(g_0)^G_0$.  

Fix a basis $\{u_1, \ldots, u_n\}$ of $g_0$ such that $\kappa(u_j, u_k) = \pm \delta_{jk}$ and denote by $dy$ the Lebesgue measure associated to this choice. Let $S(g_0)$ be the Schwartz space on $g_0$. Define, as in [13, Appendix 1], the Fourier transform of $f \in S(g_0)$ by 

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{g_0} f(y) e^{-in(y,x)} dy.$$ 

Let $T$ be a tempered distribution on $g_0$. The Fourier transform of $T$ is defined by 

$$\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle$$ 

for all $f \in C_c^\infty(g_0)$. Then we have 

$$(1.4) \quad \forall D \in \mathcal{D}(g), \quad \forall T \in \mathcal{Db}(g_0), \quad \hat{D}T = F_g(D) \hat{T}.$$ 

Recall [2] that $T \in \mathcal{Db}(g_0)$ is said to be homogeneous of degree $d$ if, for all $f \in C_c^\infty(g_0)$, $t \in \mathbb{R}^+$, $(T, f_t) = t^d (T, f)$, where $f_t(v) = t^{-n} f(t^{-1}v)$. Then, a homogeneous distribution of degree $d$ is tempered and satisfies $\text{E}_g T = dT$. We will need the following well-known result:  

Lemma 1.4. Let $T \in \mathcal{Db}(g_0)$ be tempered and set $M = \mathcal{D}(g) T$. Then $M^{F_\chi} \cong \mathcal{D}(g) \hat{T}$.  

Proof. By [13], we have $\text{ann}_\mathcal{D}(g)(\hat{T}) = F_g^{-1}(\text{ann}_\mathcal{D}(g)(T))$. Hence the result. \hfill \Box 

Let $N(g_0)$ be the set of nilpotent elements of $g_0$. Define $\mathcal{D}(g)$-submodules of $\mathcal{Db}(g_0)$ by 

$$\mathcal{Db}(g_0)_{nil} = \{ \Theta \in \mathcal{Db}(g_0) : \text{Supp} \Theta \subset N(g_0) \},$$ 

$$\mathcal{Db}(g_0)_S = \{ T \in \mathcal{Db}(g_0) : \exists k \in \mathbb{N}, \text{S}_+(g)^k T = 0 \}.$$ 

The elements of $\mathcal{Db}(g_0)_S$ are called $S$-finite. Observe that $\mathcal{Db}(g_0)^{G_0}_{nil}$ and $\mathcal{Db}(g_0)^{G_0}_S$ are $\mathcal{D}(g)^G$-modules. The next theorem is a consequence of the results proved in [18].  

Theorem 1.5. (1) $\mathcal{Db}(g_0)^{G_0}_{nil} = \{ \Theta \in \mathcal{Db}(g_0)^{G_0} : \mathcal{D}(g) \Theta \in \mathcal{C}(\mathcal{M}) \}$.

(2) $\mathcal{Db}(g_0)^{G_0}_S = \{ T \in \mathcal{Db}(g_0)^{G_0} : \mathcal{D}(g) T \in \mathcal{C}(\mathcal{N}) \}$.

(3) $\Theta \in \mathcal{Db}(g_0)^{G_0}_{nil} \iff \hat{\Theta} \in \mathcal{Db}(g_0)^{G_0}_S$.  

Proof. (1) follows from [18, Theorem 6.1], since $\mathcal{D}(g) \Theta \in \mathcal{C}(\mathcal{M})$ is equivalent to $\mathcal{D}(g)^G \Theta \cong \bigoplus_{\chi \in W} m_{\chi} \mathcal{M}_\chi$.  

(2) and (3) are consequences of (1) and Lemma 1.4. \hfill \Box
Remark 1.6. Let $T \in \mathcal{D}b(\mathfrak{g}_0)^{G_0}_{S_+}$. Recall that by the Harish-Chandra regularity theorem, $T$ is given by

$$\langle T, f \rangle = \int_{\mathfrak{g}_0} F_T(y)f(y)dy$$

for some analytic function $F_T$ on $\mathfrak{g}_0'$, locally integrable on $\mathfrak{g}_0$.

2. The distributions $\Theta_{u, \Gamma}$ and $T_{p, \Gamma}$

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, with adjoint group $G_0$, $\mathfrak{h}_0$ a Cartan subalgebra and let $H_0$ be the associated Cartan subgroup. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_0$ and adopt the notation of \[1\] Denote by $W(\mathfrak{h}_0)$ the real Weyl group, i.e. $W(\mathfrak{h}_0) = N_{G_0}(\mathfrak{h}_0)/Z_{G_0}(\mathfrak{h}_0)$. Define

$$\Delta_R = \{ \alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset \mathbb{R} \} \quad \text{(the real roots)}$$

$$\Delta_I = \{ \alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset i\mathbb{R} \} \quad \text{(the imaginary roots)}$$

A root which is neither real nor imaginary is called complex. Let $\Delta_I^+$ be a positive system of roots in $\Delta_I$ and set $\pi_I = \prod_{\alpha \in \Delta_I^+} \alpha$. Then each $w \in W(\mathfrak{h}_0)$ permutes the imaginary roots and one can define a character of $W(\mathfrak{h}_0)$, the imaginary signature, by

$$\varepsilon_I : W(\mathfrak{h}_0) \to \{ \pm 1 \}, \quad w.\pi_I = \varepsilon_I(w)\pi_I.$$ 

If $V$ is a $W(\mathfrak{h}_0)$-module we denote by $V^{\varepsilon_I}$ the isotypic component of type $\varepsilon_I$ in $V$.

In the sequel, we adopt the notation of \[5\] with the minor difference that we use $e^{-i\alpha(x,y)}$ in the definition of the Fourier transform.

Let $h \in \mathfrak{h}_0'$ and $f \in C_c^\infty(\mathfrak{g}_0)$. Define \[5\] \[3.1\] the distribution $\mu_{G_0,h}$ by

$$\langle \mu_{G_0,h}, f \rangle = \det \mathrm{ad}_{\mathfrak{g}_0/\mathfrak{h}_0}(h)^{1/2} \int_{G_0/H_0} f(\dot{g},h)d\dot{g}.$$ 

Then one defines the function $J_{\mathfrak{g}_0}(f)$, or simply $J(f)$, on $\mathfrak{h}_0'$ by

$$J_{\mathfrak{g}_0}(f) = \{ h \mapsto \langle \mu_{G_0,h}, f \rangle \}.$$ 

Set $\mathfrak{h}_0^{\text{reg}} = \{ h \in \mathfrak{h}_0 : \pi_I(h) \neq 0 \}$ and fix a connected component $\Gamma$ of $\mathfrak{h}_0^{\text{reg}}$. Let $u \in S(\mathfrak{h})$; Harish-Chandra has shown, see \[17\] \[8.1, p. 123\], that one can define a tempered $G_0$-invariant distribution on $\mathfrak{g}_0$ by

$$\forall f \in C_c^\infty(\mathfrak{g}_0), \quad \langle \Theta_{u, \Gamma}, f \rangle = \lim_{h \in \mathfrak{h}_0^{\text{reg}}, \Gamma} \langle \partial(u), J(f) \rangle(h).$$ 

Furthermore $\Theta_{u, \Gamma} \in \mathcal{D}b(\mathfrak{g}_0)^{G_0}_{S_+}$ and, when $u \in S^b(\mathfrak{h})$, $\Theta_{u, \Gamma}$ is homogeneous of degree $-b - \nu - \ell$.

Now let $p \in S(\mathfrak{h}^*)$ and define $T \in \mathcal{D}b(\mathfrak{g}_0)^{G_0}_{S_+}$ by

$$T_{p, \Gamma} = \Theta_{F_p, \Gamma} = \{ f \mapsto \lim_{h \in \mathfrak{h}_0^{\text{reg}}, \Gamma} \langle \partial(F_p(h)), J(f) \rangle(h) \}.$$ 

Then, $T_{p, \Gamma}$ is tempered and is homogeneous of degree $b - \nu$ when $p \in S^b(\mathfrak{h}^*)$.

Lemma 2.1. (1) Let $\varphi \in S(\mathfrak{g}^*)^G$. Then, $\varphi T_{p, \Gamma} = T_{\varphi(p), \Gamma}$.

(2) Let $q \in S(\mathfrak{g})^G$. Then, $\partial(q)T_{p, \Gamma} = T_{\partial(q)(p), \Gamma}$. 

Proof. Set \( u = F_h(p) \), \( \phi = \delta(\varphi) \in S(h) \) and \( s = \delta(q) \in S(h) \). Let \( f \in C_0^\infty(g_0) \).

(1) By definition, see \([22]\), \( \langle \varphi T_{p,\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u).J(\varphi f)](h) \). But, \([17]\) Lemma 3.2.7, p. 38], \( (1.4) \) and Lemma \( 1.2 \) imply that \( J(\varphi f) = \partial(F_h(\phi)).J(\hat{f}) \). Hence,

\[
\langle \varphi T_{p,\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u)\partial(F_h(\phi)).J(\hat{f})](h) = \lim_{h \to 0} [\partial(F_h(\phi))).J(\hat{f})](h)
\]

as desired.

(2) By \( 1.4 \), \( \partial(q).T_{p,\Gamma} \) is the Fourier transform of \( F_h^{-1}(q)\Theta_{u,\Gamma} \), hence

\[
\langle \partial(q).T_{p,\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u).J(F_h^{-1}(q)\hat{f})](h).
\]

Set \( g = J(\hat{f}) \). From \([17]\) Lemma 3.2.7, p. 38] and Lemma \( 1.2 \) we obtain that \( J(F_h^{-1}(q)\hat{f}) = F_h^{-1}(s)g \). Therefore

\[
\langle \partial(q).T_{p,\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u).(F_h^{-1}(s)g)](h).
\]

Recall (see \([11]\) that we have chosen a coordinate system \( \{ x_j; c_j \}_{1 \leq j \leq l} \). With standard notation, we write \( x^\alpha = \prod_{k=1}^l x_k^{\alpha_k}, e^\mu = \prod_{k=1}^l e_k^{\mu_k} \) and

\[
p = \sum_{\alpha \in \mathbb{N}^l} p_\alpha x^\alpha, \quad s = \sum_{\mu \in \mathbb{N}^l} s_\mu e^\mu.
\]

Set \( \partial^\mu = \prod_j \partial(c_j)^{\mu_j} \); thus \( \partial(s) = \sum_{\mu \in \mathbb{N}^l} s_\mu \partial^\mu \). Order \( \mathbb{N}^l \) by saying that \( \mu \leq \alpha \) if \( \mu_j \leq \alpha_j \) for all \( j \). Set \( \alpha! = \prod_j \alpha_j! \) and \( \binom{\alpha}{\mu} = \prod_j \binom{\alpha_j}{\mu_j} \), when \( \mu \leq \alpha \). Then,

\[
\partial^\mu(x^\alpha) = \begin{cases} 0 & \text{if } \mu \not\leq \alpha, \\ \binom{\alpha}{(\alpha-\mu)} x^{\alpha-\mu} & \text{if } \mu \leq \alpha. 
\end{cases}
\]

Now we have \( u = F_h(p) = \sum_{\alpha} p_\alpha i^{\alpha_1} \partial^\alpha \) and \( F_h^{-1}(s) = \sum_{\mu} q_\mu i^{-|\mu|} x^\mu \). Therefore, using the Leibniz formula, we get that

\[
\partial(u).J(F_h^{-1}(s)g) = \sum_{\alpha} p_\alpha i^{\alpha_1} \partial^\alpha (F_h^{-1}(s)g) = \sum_{\alpha} \sum_{\mu \leq \alpha} \sum_{\beta \leq \alpha} p_\alpha s_\mu i^{\alpha_1 - |\mu|} \binom{\alpha}{\beta} \partial^\beta(x^\mu) \partial^{\alpha-\beta}(g).
\]

But \( \lim_{h \to 0} \partial^\beta(x^\mu)(h) = 0 \) unless \( \beta = \mu \), hence

\[
\lim_{h \to 0} [\partial(u).J(F_h^{-1}(s)g)](h) = \sum_{\alpha} \sum_{\mu \leq \alpha} p_\alpha s_\mu i^{\alpha_1 - |\mu|} \binom{\alpha}{\mu} \mu! \lim_{h \to 0} [\partial^{\alpha-\mu}(g)](h).
\]

On the other hand, we have

\[
\langle T_{\partial(u),p,\Gamma}, f \rangle = \lim_{h \to 0} [\partial(F_h(\partial(s).p)).J(\hat{f})](h).
\]

Since \( \partial(s).p = \sum_{\alpha} \sum_{\mu \leq \alpha} \binom{\alpha}{(\alpha-\mu)} s_\mu p_\alpha x^{\alpha-\mu} \), we obtain that

\[
\langle T_{\partial(u),p,\Gamma}, f \rangle = \sum_{\alpha} \sum_{\mu \leq \alpha} \binom{\alpha}{(\alpha-\mu)} s_\mu p_\alpha i^{\alpha_1 - |\mu|} \lim_{h \to 0} [\partial^{\alpha-\mu}(g)](h).
\]

This proves the desired equality. \( \square \)
Theorem 2.2. Let \( p \in S(\mathfrak{h}^*) \) and \( D \in D(\mathfrak{g})^G \). Then, \( D.T_{p,\Gamma} = T\delta(D).p,\Gamma \).

Proof. Since \( T_{p,\Gamma} \) is \( G_0 \)-invariant, we have \( I. T_{p,\Gamma} = 0 \). Let \( P \in C(S(\mathfrak{g})^G, S(\mathfrak{h}^*)^G) \); by Lemma 2.1 and an obvious induction, we obtain that \( P.T_{p,\Gamma} = T\delta(P).p,\Gamma \). The theorem then follows from Lemma 1.1. \( \square \)

Recall, see Remark 1.6 that \( \hat{\Theta}_{u,\Gamma} \in Db(\mathfrak{g}_0)^G_{S^+} \) is determined by a locally integrable function on \( \mathfrak{g}_0 \). We still denote this function by \( \hat{\Theta}_{u,\Gamma} \).

Lemma 2.3. ([5 Lemme 6.1.2]) There exists \( \xi \in \mathcal{C}^* \), such that

\[
\alpha_{\Delta^+}(h) | \det \text{ad}_{\mathfrak{h}_0/\mathfrak{h}_0}(h) | \hat{\Theta}_{F_k(p),\Gamma}(h) = c_{\xi} p(h)
\]

for all \( p \in S(\mathfrak{h}^*)^{\xi} \) and \( h \in \mathfrak{h}_0^{\text{reg}} \). \( \square \)

Remark. In the notation of the lemma, if \( u = F_e(p) \), the function \( \hat{u}(ih) \) of [5] is replaced here by \( p(h) \) since we are using \( e^{-i\kappa(x,y)} \) in the definition of the Fourier transform.

Theorem 2.4. Let \( p \in S(\mathfrak{h}^*)^{\xi} \). There exists a bijective map

\[
\rho : D(\mathfrak{g})^G T_{p,\Gamma} \rightarrow D(\mathfrak{h})^W.p, \quad \rho(D.T_{p,\Gamma}) = \delta(D).p
\]

which, through \( \delta \), yields an isomorphism

\[
\rho : A(\mathfrak{g}).T_{p,\Gamma} \simeq D(\mathfrak{h})^W.p.
\]

Proof. We first need to show that \( \rho \) is well defined. Let \( D \in D(\mathfrak{g})^G \); by Theorem 2.2 we have

(\dag)

\[
D.T_{p,\Gamma} = T\delta(D).p,\Gamma = \hat{\Theta}_{F_k(\nu),\Gamma}.
\]

Suppose that \( D.T_{p,\Gamma} = 0 \). Then, the analytic function associated to \( T\delta(D),p,\Gamma \in Db(\mathfrak{g}_0)^G_{S^+} \) vanishes on \( \mathfrak{h}_0^{\text{reg}} \). Notice that, since \( \delta(D) \) is \( W \)-invariant, \( \delta(D).p \in S(\mathfrak{h}^*)^{\xi} \). Therefore Lemma 2.3 gives \( \delta(D).p = 0 \) on \( \mathfrak{h}_0^{\text{reg}} \). Thus \( \delta(D).p = 0 \) on \( \mathfrak{h} \) and \( \rho \) is well defined.

Now, it follows easily from (\dag) that \( \rho \) is a linear bijection. Since \( I.T_{p,\Gamma} = 0 \), the last assertion is clear. \( \square \)

Recall that we denote by \( V_\chi \subset \mathcal{H}^{\Delta(\chi)}(\mathfrak{h}^*) \) a simple \( W \)-module in the class of \( \chi \in W^\perp \).

Corollary 2.5. Let \( p \in S(\mathfrak{h}^*)^{\xi} \) such that \( CW.p \) is simple. Then there exists \( \chi \in W^\perp \) such that \( V_\chi^{\xi} \neq 0 \). We have

1. \( D(\mathfrak{g}).T_{p,\Gamma} \simeq N_\chi \) and \( D(\mathfrak{g})^G T_{p,\Gamma} \simeq V_\chi^{\xi} \);
2. \( D(\mathfrak{g}).\Theta_{F_k(p),\Gamma} \simeq M_\chi \) and \( D(\mathfrak{g})^G \Theta_{F_k(p),\Gamma} \simeq (V_\chi^{\xi})^{-1} \).

Proof. The first assertion follows from \( \mathcal{H}(\mathfrak{h}^*) \simeq CW \). Then, 1 and 2 are consequences of \( V_\chi^{\xi} \simeq D(\mathfrak{h})^W.p \), Lemma 1.3 and Theorem 2.4. \( \square \)

Remark 2.6. Let \( \chi \in W^\perp \) be such that \( V_\chi^{\xi} \neq 0 \). It follows obviously from the previous corollary that

\[
N_\chi \simeq D(\mathfrak{g}).T_{p,\Gamma}, \quad M_\chi \simeq D(\mathfrak{g}).\Theta_{u,\Gamma},
\]

where \( 0 \neq p \in V_\chi^{\xi} \subset \mathcal{H}^{\Delta(\chi)}(\mathfrak{h}^*)^{\xi} \) and \( u = F_e(p) \in \mathcal{H}^{\Delta(\chi)}(\mathfrak{h})^{\xi} \).
3. The decomposition of $\text{Db}(\mathfrak{g}_0)_{S_w}^G$ and $\text{Db}(\mathfrak{g}_0)_{nil}^G$

Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and let $[\mathfrak{h}_1], \ldots, [\mathfrak{h}_r]$ be the conjugacy classes of Cartan subalgebras in $\mathfrak{g}_0$. For each $j = 1, \ldots, r$ we denote by

$$\mathfrak{h}_{j,\mathbb{C}} = \mathfrak{h}_j \otimes \mathbb{C}, \quad W_j = W(\mathfrak{g}, \mathfrak{h}_{j,\mathbb{C}}), \quad \Delta_{\mathfrak{h}_{j,\mathbb{C}}}^+$$

a set of positive imaginary roots, $\epsilon_{I,j} : W(\mathfrak{h}_j) = W(G_0, \mathfrak{h}_j) \to \{ \pm 1 \}$ the imaginary signature associated to $\mathfrak{h}_j$. For each $j$ we fix a connected component $\Gamma_j$ of $\mathfrak{h}_{j,\mathbb{R}}^{\text{reg}}$. The results of [12] then apply to $\mathfrak{h}_0 = \mathfrak{h}_j$, $\Gamma = \Gamma_j$ etc.

**Remark 3.1.** Recall that the $\mathfrak{h}_{j,\mathbb{C}}$ are $G$-conjugate. Therefore, if $1 \leq j, k \leq r$, the algebras $D(\mathfrak{h}_{j,\mathbb{C}})^W$ and $D(\mathfrak{h}_{k,\mathbb{C}})^W$ are naturally isomorphic. Denote this isomorphism by $\gamma_{jk}$ and let $\delta_j$ be the Harish-Chandra isomorphism from $A(\mathfrak{g})$ onto $D(\mathfrak{h}_{j,\mathbb{C}})^W$. One can check that $\delta_k = \gamma_{jk} \circ \delta_j$. Therefore, we can choose an “abstract” Cartan subalgebra $\mathfrak{h}$ and identify $\delta_j$ with the homomorphism $\delta : D(\mathfrak{g})^W \to D(\mathfrak{h})^W$, where $W = W(G, \mathfrak{h})$. Then, if $\chi \in W^\ast$, we have an irreducible $W$-module $V_\chi \subset \mathcal{H}(\chi)(\mathfrak{h}^\ast)$ and a simple $D(\mathfrak{h})^W$-module $V_\chi$.

For each $\chi \in W^\ast$, choose a simple $W$-module $V_{\chi,j} \subset \mathcal{H}(\chi)(\mathfrak{h}_{j,\mathbb{C}})$, $V_{\chi,j} \cong V_\chi$. Write $V_{\chi,j} = V_{\chi,j}^{\epsilon_{I,j}} \oplus E_{\chi,j}$ with $E_{\chi,j}$ stable under $W(\mathfrak{h}_j)$. Let $\{ v_{\chi,j}^k \}_{1 \leq k \leq d(\chi)}$ be a basis of $V_{\chi,j}$ such that

$$V_{\chi,j}^{\epsilon_{I,j}} = \bigoplus_{k=1}^{n_j(\chi)} \mathbb{C} v_{\chi,j}^k, \quad E_{\chi,j} = \bigoplus_{k=n_j(\chi)+1}^{d(\chi)} \mathbb{C} v_{\chi,j}^k$$

(hence $n_j(\chi) = \dim V_{\chi,j}^{\epsilon_{I,j}}$).

**Lemma 3.2.** The $D(\mathfrak{h}_{j,\mathbb{C}})^W$-module $S(\mathfrak{h}_{j,\mathbb{C}})^{\epsilon_{I,j}}$ decomposes as

$$S(\mathfrak{h}_{j,\mathbb{C}})^{\epsilon_{I,j}} = \bigoplus_{\chi \in W^\ast} \bigoplus_{k=1}^{n_j(\chi)} D(\mathfrak{h}_{j,\mathbb{C}})^W . v_{\chi,j}^k$$

with $D(\mathfrak{h}_{j,\mathbb{C}})^W . v_{\chi,j}^k \cong V_\chi$.

**Proof.** Clearly, we can drop the index $j$ and write $\mathfrak{h}_0 = \mathfrak{h}_j$, $\mathfrak{h} = \mathfrak{h}_{j,\mathbb{C}}$, $\epsilon_{I,j} = \epsilon_{I,j}^\mathfrak{h}$ etc. Since $D(\mathfrak{h})^W . v_{\chi,j}^k \subset S(\mathfrak{h}^\ast)^{\epsilon_{I,j}}$ for $1 \leq k \leq n(\chi) = \dim V_{\chi}^{\epsilon_{I,j}}$, one has

$$S(\mathfrak{h}^\ast)^{\epsilon_{I,j}} \subset \bigoplus_{\chi \in W^\ast} \bigoplus_{k=1}^{n(\chi)} D(\mathfrak{h})^W . v_{\chi,j}^k.$$
Proof. (2) follows from the proof of [5, Proposition 6.1.1].

**Corollary 3.6.** We have
\[ \text{Db}(\mathfrak{g}_0)_{\text{nil}} \cong \bigoplus_{\chi \in W^*} m\chi \mathcal{M}_\chi \]
\[ \text{Db}(\mathfrak{g}_0)_{\text{red}} \cong \bigoplus_{\chi \in W^*} m\chi \mathcal{M}_\chi \]

where \( m\chi = \sum_{j=1}^r \text{dim} V_{\chi}^{e_j} \).
Remark 3.7. Let $\chi \in W^\circ$. It is not always possible to “realize” the modules $N_\chi$ and $M_\chi$ as $D(g).T$ for some $T \in Db(g_0)$, where $g_0$ is a real form of $g$. By the previous results, this statement is equivalent to the existence of a Cartan subalgebra $h_j \subset g_0$ such that $V^{x_j \iota_j} \neq 0$. D. Renard has observed that, using the results of W. Rossmann [15], this can be translated to a question about centralizers of nilpotent elements. Fix a real form $g_\mathbb{R}$ of $g$ with adjoint group $G_\mathbb{R}$. If $x \in g_\mathbb{R}$ is nilpotent one defines a subgroup of the component group $A(G.x)$ (see [14] for notation) by

$$A(G_\mathbb{R}.x) = G_\mathbb{R}^\circ / G_\mathbb{R}^\circ \cap (G^x)^0.$$  

Recall that $\chi \in W^\circ$ can be written $\sigma(O, \psi)$ via the Springer correspondence, where $O \subset g$ is a nilpotent orbit and $\psi : A(O) \to GL(E)$ is an irreducible representation. Then, by [15] Corollary 3.2 & Theorem 3.3, there exists a Cartan subalgebra $O$ such that $V^{x_j \iota_j} \neq 0$ if, and only if, there exists a nilpotent element $x \in g_\mathbb{R}$ such that $O = G.x$ and $E^{A(G_\mathbb{R}.x)} \neq 0$.

Let $g = sp(\ell, \mathbb{C})$ and let $\phi \in W^\circ$ be the long sign character, i.e. $V_\phi = \mathbb{C}_{\pi_1}$ where $\pi_1$ is the product of the long roots. Then, see [6] §13.3, $\phi = \sigma(O, \psi)$ where $O = G.x$ is the subregular nilpotent orbit with partition $[2\ell - 2, 2]$ and $\psi$ is the non-trivial character of $A(O) \equiv \{ \pm 1 \}$. The real forms of $g$ are $sp(\ell, \mathbb{R})$ and the $sp(p, q)$, $p+q = \ell$. Assume now that $\ell \geq 3$. By the classification of nilpotent orbits in $sp(p, q)$, see [7] Theorem 9.2.5, we know that $O \cap sp(p, q) = \emptyset$. Hence, by Rossmann’s results, $V^{x_j \iota_j} = 0$ for each Cartan subalgebra $h_j \subset sp(p, q)$. On the other hand, if $G_\mathbb{R}$ is the adjoint group of $sp(\ell, \mathbb{R})$, one can show that $A(G_\mathbb{R}.x) = A(G.x)$. Thus, with the above notation, $E^{A(G_\mathbb{R}.x)} = 0$ and it follows that $V^{x_j \iota_j} = 0$ for each Cartan subalgebra $h_j \subset sp(\ell, \mathbb{R})$. For instance, when $g = sp(3, \mathbb{R})$ there are six conjugacy classes of Cartan subalgebras and one can directly verify (without using [15]) that $V^{x_j \iota_j} = 0$ for $j = 1, \ldots, 6$. We thank D. Renard for showing this computation to us.

Let $x \in N(g_0)$ and denote by $\beta_x$ the Liouville (Kostant-Kirillov) measure on $G_0.x$. By [14] one can define $\Theta_x \in Db(g_0)^{G_0}_{nil}$ by $\langle \Theta_x, f \rangle = \int_{G_0.x} f \, d\beta_x$ for all $f \in C_c^\infty(g_0)$. Set $O = G.x$. Then, see [9], [10] or [18]: $\Theta_x$ is homogeneous of degree $\lambda_\Theta = \frac{1}{2} \dim O - \dim g$ and satisfies

$$D(g).\Theta_x \cong M_{\chi_\Theta}$$

for some $\chi_\Theta \in W^\circ$ such that $\lambda_\Theta = \nu - n - \chi_\Theta(\Theta)$.  

Corollary 3.8. There exists $j \in \{ 1, \ldots, r \}$ and $u \in F_h^{-1}(V_{\chi_\Theta,j})^{x_j \iota_j}$ such that

$$D(g)^{G_j}.\Theta_x \cong D(g)^{G_j}.\Theta_{a_j,r_j}.$$  

Proof. Since $D(g)^{G_j}.\Theta_x \cong M_{\chi_\Theta}^{G_j}$ is a simple submodule of $Db(g_0)^{G_0}_{nil}$, the claim follows from Corollary 3.3. 

Remark 3.9. It is proved in [11], see also [5], that $\Theta_x$ can be written as $\sum_{j=1}^r \Theta_{a_j,r_j}$ with $a_j \in H_{\lambda(x_\Theta)}(h_j, c)^{x_j \iota_j}$ for all $j$. It is easily seen that we may assume $\mathbb{C}W_{a_j} \cong V_{\chi_\Theta}$ for all $j$ such that $a_j \neq 0$. W. Rossmann [15] has given conditions to ensure that $\Theta_x = \Theta_{a_j,r_j}$ for some $j$. 


4. Example: The Complex Case

We assume in this section that \( g_0 = g_1^\mathbb{C} \) is a complex semisimple Lie algebra, \( g_1 \), viewed as a real Lie algebra. Then, \( g \) can be identified with \( g_1 \times g_1 \) and \( g_0 \) with the diagonal \( \{(a, a) \in g_1 \times g_1 \} \). Let \( h_1 \) be a Cartan subalgebra of \( g_1 \). Recall the following well-known facts, see [17] or [18]:

- \( h_0 = \{(a, a) : a \in h_1 \} \) is a Cartan subalgebra of \( h_0 \) and \( h = h_0 \otimes \mathbb{R} \mathbb{C} = h_1 \times h_1 \);
- \( W(g, h) = W_1 \times W_1 \), where \( W_1 = W(g_1, h_1) \), and \( W(h_0) = \{(w, w) \in W \} \) is isomorphic to \( W_1 \);
- there is a unique conjugacy class \([h_0]\) of Cartan subalgebras and \( h_0 \) is connected;
- the roots in \( \Delta(g, h) \) are complex and, therefore, \( \varepsilon_I = 1 \);
- the irreducible representations of \( W \) are of the form \( \chi = \phi \otimes \mu, \phi, \mu \in W_1^\mathbb{C} \);
- one has \( \phi = \phi^* \) for all \( \phi \in W_1^\mathbb{C} \), where \( \phi^* \) is the dual representation.

Observe that \( D(g) = D(g_1) \boxtimes D(g_1) \) and \( D(g)^G = D(g_1)^{G_1} \boxtimes D(g_1)^{G_1} \).

**Lemma 4.1.** Let \( \chi \in W^\mathbb{C} \). Then, the simple \( D(g) \)-module \( M_\chi \) is of the form \( M_\phi \boxtimes M_\mu \) for some \( \phi, \mu \in W_1^\mathbb{C} \).

**Proof.** The claim follows easily from the definition of the category \( \mathcal{C}(M) \) and the decomposition of the \( W \)-module \( S(h^*) = S(h_1^+) \boxtimes S(h_1^+) \).

**Corollary 4.2.** ([18] Theorem 6.11) We have
\[
\text{Db}(g_0^G)_{nil} \cong \bigoplus_{\phi \in W_1^\mathbb{C}} M^{G_1}_\phi \boxtimes M^{G_1}_\phi
\]
as a \( D(g) \)-module.

**Proof.** Let \( \chi = \phi \otimes \mu \in W^\mathbb{C} \). Then, \( V^{\chi_I} = (V_\phi \boxtimes V_\mu)^{W_1} \neq 0 \) if, and only if, \( \phi = \mu \) and therefore \( n(\chi) = 1 \). The assertion now follows from Corollary 3.3.

Recall the following general results from [18]. Since the module \( M_\chi \) is irreducible and \( G \)-equivariant, its support is the closure of a nilpotent orbit \( O = G.x \).

Furthermore, if \( i : O \hookrightarrow g \) is the inclusion, \( M_\chi \) is uniquely determined by its \( (D, G) \)-module inverse image \( L_\chi := i^! M_\chi \). The \( D_O \)-module \( L_\chi \) is an irreducible integrable connection associated to an irreducible representation \( \psi \) of the component group \( A(O) := G^*/(G^*)^0 \) where \( (G^*)^0 \) is the connected component of the centralizer \( G^\mathbb{C} \). Therefore, since \( \chi \) is uniquely determined by \( O \) and \( \psi \), we set \( \chi = \sigma(O, \psi) \).

In our situation, i.e. in the complex case, we have \( O = O_1^1 \times O_1^2 \) with \( O_1^j \) nilpotent orbits in \( g_1 \) for \( j = 1, 2 \). Then, \( \chi = \sigma(O, \psi) = \phi_1 \boxtimes \phi_2, L_\chi = L_{\phi_1} \boxtimes L_{\phi_2}, \phi_j = \sigma(O_1^j, \psi_j), \psi = \psi_1 \boxtimes \psi_2 \). Note that \( b(\chi) = b(\phi_1) + b(\phi_2) \) and \( \lambda_O = \lambda_{O_1^1} + \lambda_{O_1^2} \).

Let \( x \in N(g_0) \); set \( x = (x_1, x_1), x_1 \in N(g_1), O_1 = G_1.x_1, O = G.x = O_1 \times O_1 \). The inclusion \( i : O \hookrightarrow g \) is equal to \( i_1 \times i_1 \), where \( i_1 : O_1 \hookrightarrow g_1 \). By (3.1) and Corollary 4.2 there exist \( \chi \in W^{\mathbb{C}}, \chi \in W_1^{\mathbb{C}} \) such that \( \chi = \chi_1 \otimes \chi_1 \) and \( D(g_1)^{\Theta_x} \cong M_{\chi_1} \boxtimes M_{\chi_1} \).

It is known (Harish-Chandra) that \( \Theta_x = \Theta_u.h_0^0 \) for some \( u \in S(h_1) \boxtimes S(h_1) \). The following result has been proved by various authors; see [2, 3] (when \( O_1 \) is “special”), [8], [9], [10].

**Theorem 4.3.** One has \( \chi_1 = \sigma(O_1, \text{triv}) \), and there exists \( p \in (V_{\chi_1} \boxtimes V_{\chi_1})^{W_1} \) such that \( \Theta_x = \Theta_{F_p(p), h_0^0} \).
Proof. Recall from [9] or [10] that $\chi = \chi_1 \boxtimes \chi_1 = \sigma(O, \text{triv})$. This means that
\[
\mathcal{L}_\chi = \mathcal{L}_{\chi_1} \boxtimes \mathcal{L}_{\chi_1} = O_O = O_{\chi_1} \boxtimes O_{\chi_1},
\]
(where we denote by $O_X$ the structural sheaf of an algebraic variety $X$). This yields $\mathcal{L}_{\chi_1} = O_{\chi_1}$ and $\chi_1 = \sigma(O_1, \text{triv})$.

Set $T_x = \Theta_x$; then $D(g)T_x = N_{\chi_1} \otimes N_{\chi_1}$ (see Lemma [4]). Since $S_+(g^*)G, \Theta_x = 0$ we have $S_+(g^*)G, T_x = 0$. It follows from Proposition [3](2) that we can write $T_x = T_{p, b_0}$ for some $p \in (\mathcal{H}(b_0^*) \boxtimes \mathcal{H}(b_0))^{W_1}$ or, equivalently, $\Theta_x = \Theta_{F(p), b_0}$. Now, by Theorem [2] $D(h)^W, p = V^{\chi_1} \boxtimes V^{\chi_1}$ and therefore $CW,p \cong V^{\chi_1} \boxtimes V^{\chi_1}$. Moreover, $T_x = T_{p, b_0}$ is homogeneous of degree $b(\chi_0) - 2\nu = 2b(\chi_1) - 2\nu = \deg p - 2\nu$. Thus $\deg p = 2b(\chi_1)$ and, by definition of $V_{\chi_1}, p \in (V^{\chi_1} \boxtimes V^{\chi_1})^W_1$.

\[ \square \]

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