IN Variant Distributions Supported On
The Nilpotent Cone of a Semisimple Lie Algebra

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Abstract. Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \) and \( D(g) \) be the algebra of differential operators with polynomial coefficients on \( g \). If \( g_0 \) is a real form of \( g \), we give the decomposition of the semisimple \( D(g)^G \)-module of invariant distributions on \( g_0 \) supported on the nilpotent cone.

0. Introduction

Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \). Choose a Cartan subalgebra \( h \) of \( g \) and let \( W \) be the associated Weyl group. Denote by \( W_h \) the set of isomorphism classes of irreducible \( W \)-modules and by \( H(h) \) the graded vector space of \( W \)-harmonic polynomials on \( h \). For \( \chi \in W \), set
\[
b(\chi) = \inf \{ j \in \mathbb{N} : |H^j(h^*) : \chi| \neq 0 \}
\]
and choose a \( W \)-submodule \( V_\chi \subset H^{b(\chi)}(h^*) \) in the class of \( \chi \). Denote by \( d(\chi) \) the dimension of \( V_\chi \).

Let \( S(g^*) \) be the algebra of polynomial functions on \( g \) and \( D(g) \) be the algebra of differential operators on \( g \), with coefficients in \( S(g^*) \). The group \( G \) acts on \( S(g^*) \), via the adjoint action, and hence has an induced action on \( S(g^*) \), \( S(g) \) and \( D(g) \). Denote the differential of this action by \( \tau : g \rightarrow D(g) \). Let \( S(g)^G \) and \( S(g^*)^G \) be the set of invariant elements without constant term. Recall that \( N(g) \), the nilpotent cone of \( g \), is the variety of zeroes of the ideal \( S(g^*) \).

Let \( g_0 \) be a real form of \( g \) with adjoint group \( G_0 \subset G \). Denote by \( Db(g_0) \) the \( D(g) \)-module of distributions on \( g_0 \). Then, the subspace of invariant distributions \( Db(g_0)^G = \{ T \in Db(g_0) : \tau(g).T = 0 \} \) is a \( D(g)^G \)-module, containing the submodule of invariant distributions supported on the nilpotent cone
\[
Db(g_0)^G_{nil} = \{ \Theta \in Db(g_0)^G : \text{Supp} \Theta \subset N(g_0) \}
\]
where \( N(g_0) = N(g) \cap g_0 \) is the nilpotent cone of \( g_0 \). The structure of \( Db(g_0)^G_{nil} \) as a vector space is well understood, see, for example, [1][5]. Let \( [h_1], \ldots, [h_r] \) be the conjugacy classes of Cartan subalgebras of \( g_0 \). For each \( j \), let \( \varepsilon_{i,j} : W(h_j) \rightarrow \{ \pm 1 \} \) be

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the imaginary signature of the real Weyl group \(W(\mathfrak{h}_j)\). Then [5 Proposition 6.1.1] there exists a vector space isomorphism

\[
\bigoplus_{j=1}^{r} S(\mathfrak{h}_j, \mathbb{C})_{\varepsilon_{I,j}} \simeq \text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}}
\]

where \(S(\mathfrak{h}_j, \mathbb{C})_{\varepsilon_{I,j}}\) is the isotypic component of type \(\varepsilon_{I,j}\) in the \(W(\mathfrak{h}_j)\)-module \(S(\mathfrak{h}_j, \mathbb{C})\).

One aim of this note is to give a complete description of the \(D(\mathfrak{g})^G\)-module \(\text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}}\). This description is given in terms of the simple summands of the equivariant holonomic \(D(\mathfrak{g})\)-module

\[
\mathcal{M} = D(\mathfrak{g})/(D(\mathfrak{g})\tau(\mathfrak{g}) + D(\mathfrak{g})S_+(\mathfrak{g}^*)^G).
\]

By [9], [13] or [13], it is known that we have a decomposition

\[
\mathcal{M} = \bigoplus_{\chi \in W^\infty} d(\chi) \mathcal{M}_\chi
\]

where the \(\mathcal{M}_\chi\) are pairwise non-isomorphic simple \(D(\mathfrak{g})\)-modules. Moreover, the support (in \(\mathfrak{g}\)) of \(\mathcal{M}_\chi\) is the closure of a nilpotent orbit and \(\mathcal{M}_\chi^G\) is a simple \(D(\mathfrak{g})^G\)-module. Then we have, see Corollary 4.4

**Theorem A.** The \(D(\mathfrak{g})^G\)-module \(\text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}}\) decomposes as

\[
\text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \cong \bigoplus_{\chi \in W^\infty} m_\chi \mathcal{M}_\chi^G
\]

where \(m_\chi = \sum_{j=1}^{r} \dim V^{\varepsilon_{I,j}}\).

This theorem is proved by combining the isomorphism (*) and the properties, established in [13] [11] [13], of the Harish-Chandra homomorphism

\[
\delta : D(\mathfrak{g})^G \rightarrow D(\mathfrak{h})^W.
\]

In the particular case where \(\mathfrak{g}_0\) is a complex Lie algebra \(\mathfrak{g}_1\) (viewed as a real Lie algebra), Theorem A was proved by N. Wallach [18]. In this case, \(\mathfrak{g} \simeq \mathfrak{g}_1 \times \mathfrak{g}_1\), \(W \simeq W_1 \times W_1\) where \(W_1\) is the Weyl group of \(\mathfrak{g}_1\). Then, each \(\mathcal{M}_\chi\) occurring in the decomposition of \(\text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}}\) is of the form \(\mathcal{M}_\phi \boxtimes \mathcal{M}_\phi\) with \(\chi = \phi \boxtimes \phi, \phi \in W_1^\infty\), and one has \(m_\chi = 1\). Hence \(\text{Db}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \cong \bigoplus_{\phi \in W_1^\infty} \mathcal{M}_\phi^G \boxtimes \mathcal{M}_\phi^G\) as a \(D(\mathfrak{g})^G\)-module.

The next corollary is an easy consequence of Theorem A

**Corollary B.** Let \(\chi \in W^\infty\), then, \(\mathcal{M}_\chi \cong D(\mathfrak{g}).\Theta\) for some \(\Theta \in \text{Db}(\mathfrak{g}_0)\) if, and only if, \(V^{\varepsilon_{I,j}} \neq 0\) for some \(j \in \{1, \ldots, r\}\).

In Remark 3.1 we apply this result to give examples of modules \(\mathcal{M}_\chi\) which cannot be generated by a distribution on any real form of \(\mathfrak{g}\).

1. **Preliminary results**

We retain the notation of the introduction. Denote by \(\Delta\) the root system of \(\mathfrak{h}\) in \(\mathfrak{g}\) and fix a system \(\Delta^+\) of positive roots. Set \(n = \dim \mathfrak{g}, \ell = \dim \mathfrak{h}\) and \(\nu = \# \Delta^+, \) hence \(n = 2\nu + \ell\). Let \(\pi\) be the product of positive roots and recall that \(x \in \mathfrak{g}\) is called generic if \(\pi(x) \neq 0\). If \(a \subset \mathfrak{g}\), we denote by \(a'\) the set of generic elements in \(a\).

For \(q \in S(\mathfrak{g})\), let \(\partial(q) \in D(\mathfrak{g})\) be the corresponding differential operator with constant coefficients. Let \(\{e_i\}_{1 \leq i \leq n}\) be an orthonormal basis of \(\mathfrak{g}\) with respect to the Killing form \(\kappa\) such that \(\{e_i\}_{1 \leq i \leq \ell}\) is a basis of \(\mathfrak{h}\). Denote by \(x_i \in S(\mathfrak{g}^*)\),
1 \leq i \leq n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\mathfrak{g}$ and $\mathfrak{h}$ by $E_\mathfrak{g} = \sum_{i=1}^\ell x_i \partial_i$ and $E_\mathfrak{h} = \sum_{i=1}^\ell x_i \partial_i$.

We now give some notation and results from [11][12][13][18]. Recall first that the algebra homomorphism, defined by Harish-Chandra,

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$$

extends the Chevalley isomorphisms $S(\mathfrak{g})^G \cong S(\mathfrak{h})^W$ and $S(\mathfrak{g}^*)^G \cong S(\mathfrak{h}^*)^W$. The map $\delta$ is surjective and its kernel is $\mathcal{I} = (\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$. This enables one to identify, through $\delta$, modules over $A(\mathfrak{g}) := \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$ with $\mathcal{D}(\mathfrak{h})^W$-modules.

**Lemma 1.1.** Let $D \in \mathcal{D}(\mathfrak{g})^G$. Then $D = P + Q$ with $P \in \mathbb{C}\langle S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G \rangle$ and $Q \in \mathcal{I}$.

**Proof.** By [11], we know that $\mathcal{D}(\mathfrak{h})^W = \mathbb{C}\langle S(\mathfrak{h})^W, S(\mathfrak{h}^*)^W \rangle$. The lemma is therefore a consequence of the properties of $\delta$ previously recalled. \( \square \)

Recall that the $(\mathcal{D}(\mathfrak{h})^W, W)$-module $S(\mathfrak{h}^*)$ decomposes as

$$S(\mathfrak{h}^*) \cong \bigoplus_{x \in W^\circ} V_x \otimes_{\mathbb{C}} V_x$$

where $V_x = \text{Hom}_{W}(V_x, S(\mathfrak{h}^*))$ is a simple $(\mathcal{D}(\mathfrak{h})^W)$-module. Let $\{v_1^\alpha, \ldots, v_\lambda^\alpha \}$ be a basis of $V_x$, then $V_x \cong \mathcal{D}(\mathfrak{h})^W.v_\lambda^\alpha$ for all $j$ and [11] implies that

$$S(\mathfrak{h}^*) = \bigoplus_{x \in W^\circ} \bigoplus_{j=1}^{d(\lambda)} \mathcal{D}(\mathfrak{h})^W.v_\lambda^j.$$

Now, set $\mathcal{N} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} S(\mathfrak{h}^*)$ and $\mathcal{N}_x = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} V_x$. We have

$$\mathcal{N} = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G)$$

and, using [11],

$$\mathcal{N} = \bigoplus_{x \in W^\circ} \mathcal{N}_x \otimes_{\mathbb{C}} V_x.$$

Then each $\mathcal{N}_x$ is a simple (holonomic) $\mathcal{D}(\mathfrak{g})$-module [13] and, therefore, $\mathcal{N}$ is a semisimple $\mathcal{D}(\mathfrak{g})$-module (see also [9]). Let $\mathbb{C}(\mathcal{N})$ be the full subcategory of finitely generated $\mathcal{D}(\mathfrak{g})$-modules of the form $\bigoplus_{x \in W^\circ} m_x \mathcal{N}_x$, $m_x \in \mathbb{N}$. From [13] we know that the category $\mathbb{C}(\mathcal{N})$ is equivalent to the category $W$-mod (of finite dimensional $W$-modules) via the functor

$$\text{Sol} : \mathbb{C}(\mathcal{N}) \longrightarrow W\text{-mod}, \quad \text{Sol}(N) = \text{Hom}_{\mathcal{D}(\mathfrak{h})^W}(N^G, S(\mathfrak{h}^*))$$

where $W$ acts on $\text{Sol}(N)$ through its natural action on $S(\mathfrak{h}^*)$.

The Killing form $\kappa$ induces a $G$-isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and an algebra automorphism $\kappa$ of $\mathcal{D}(\mathfrak{g})$, defined by $\kappa(\partial(v)) = \kappa(v, -)$, $\kappa(\kappa(v, -)) = -\partial(v)$, for all $v \in \mathfrak{g}$. Hence, in coordinates, $\kappa(\partial_j) = x_j$, $\kappa(x_j) = -\partial_j$. Set $i = \sqrt{-1} \in \mathbb{C}$ and denote by $i$ the automorphism of $\mathcal{D}(\mathfrak{g})$ given by $i(\partial_j) = -i\partial_j$, $i(x_j) = ix_j$. Define then the “Fourier transformation” $F_\mathfrak{g} \in \text{Aut} \mathcal{D}(\mathfrak{g})$ by $F_\mathfrak{g} = i \circ \kappa = \kappa \circ i^{-1}$; thus $F_\mathfrak{g}(x_j) = i\partial_j$, $F_\mathfrak{g}(\partial_j) = ix_j$. One easily checks that $\kappa(\tau(x)) = F_\mathfrak{g}(\tau(x)) = \tau(x)$
for all $x \in \mathfrak{g}$; moreover, $\kappa$ and $F_\Psi$ are $G$-equivariant. Similarly, since $\kappa$ is non-degenerate and $W$-invariant on $\mathfrak{h}$, one can define $W$-equivariant automorphisms $\kappa$ and $F_\Psi = \mathbf{i} \circ \kappa$ in $\text{Aut} \, D(\mathfrak{h})$.

**Lemma 1.2.** One has $\delta \circ F_\Psi = F_\Psi \circ \delta$.

**Proof.** A direct computation shows that $\delta(F_{\Psi}(P)) = F_{\Psi}(\delta(P))$ when $P$ belongs to $S(\mathfrak{g})^G$ or $S(\mathfrak{g}^*)^G$. Since $\delta$ is a homomorphism, it follows that $\delta(F_{\Psi}(P)) = F_{\Psi}(\delta(P))$ for all $P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G)$. Now, let $D \in D(\mathfrak{g})^G$ and write $D = P + Q$ as in Lemma 1.1. Then, since $F_{\Psi}(I) = I$, we have $\delta(F_{\Psi}(D)) = \delta(F_{\Psi}(P)) = F_{\Psi}(\delta(P)) = F_{\Psi}(\delta(D))$. \hfill \Box

Recall that $\mathcal{H}(\mathfrak{h}^*)$ is the vector space of $W$-harmonic polynomials on $\mathfrak{h}$. Hence

$$\mathcal{H}(\mathfrak{h}^*) = \{ f \in S(\mathfrak{h}^*) : \partial(q).f = 0 \text{ for all } q \in S_+(\mathfrak{h})^W \}$$

and, as a $W$-module, $\mathcal{H}(\mathfrak{h}^*)$ identifies with the regular representation of $W$. The vector space $\mathcal{H}(\mathfrak{h}^*)$ is a graded subspace of $S(\mathfrak{h}^*)$ and we set $\mathcal{H}(\mathfrak{h}^*) = S(\mathfrak{h}^*) \cap \mathcal{H}(\mathfrak{h}^*)$, $0 \leq j \leq \nu$. Define the harmonic elements of $S(\mathfrak{h})$ by $\mathcal{H}(\mathfrak{h}) = F_{\Psi}(\mathcal{H}(\mathfrak{h}^*)) = \bigoplus_{j=0}^\nu \mathcal{H}_j(\mathfrak{h})$. (We could as well have set $\mathcal{H}(\mathfrak{h}) = \kappa(\mathcal{H}(\mathfrak{h}^*))$, since $\mathcal{H}(\mathfrak{h}^*)$ is stable under $\mathbf{i}$.)

Since $V_\chi \subset \mathcal{H}(\mathfrak{h}^*)$, we have $(E_\chi - b(\chi)).v^{\chi} = 0$. For all $d \in L := \text{ann}_{D(\mathfrak{h})^W}(v^{\chi})$, we have $(E_\chi - b(\chi), d) = (E_\chi, d) \in L$. It follows that $L = \bigoplus_{k \in \mathbb{Z}} L \cap D^k(\mathfrak{h})^W$, where $D^k(\mathfrak{h}) = \{ d \in D(\mathfrak{h}) : [E_\chi, d] = kd \}$. Equivalently, $L$ is stable under the $\mathbb{C}^*$-action on $D(\mathfrak{h})$ given by $f \mapsto \lambda f$, $\partial(v) \mapsto \lambda^{-1} \partial(v)$, $f \in \mathfrak{h}^*$, $v \in \mathfrak{h}$. In particular, we see that $F_{\Psi}(L) = \kappa(L)$.

Let $R$ be a ring and $\alpha \in \text{Aut}(R)$. If $M$ is an $R$-module, we define the $R$-module $M^\alpha$ to be the abelian group $M$ with action of $a \in R$ on $x \in M$ given by $a x = \alpha(a)x$. This applies to the modules $N$, $N_{\chi}$ and the automorphism $\alpha = F_{\Psi}^{-1}$. Define

$$M = N_{F_{\Psi}^{-1}}^{\chi'}, \quad M_{\chi} = N_{\chi'}^{F_{\Psi}^{-1}}.$$ 

Thus, from 1.2 and 1.3, we obtain

$$M = D(\mathfrak{g})/(D(\mathfrak{g})\tau(\mathfrak{g}) + D(\mathfrak{g})S_+^{\nu}(\mathfrak{g}^*)^G) \cong \bigoplus_{\chi \in W^\nu} M_{\chi} \otimes_{\mathbb{C}} V_{\chi}.$$ 

**Remark.** In 1.3 one defines $M_{\chi}$ to be $N_{\chi}^{\nu^{-1}}$, but the two definitions agree. Indeed, let $V_{\chi} \cong D(\mathfrak{h})^W \cdot v_{\chi}^J = D(\mathfrak{h})^W/L$ be as above. Then,

$$N_{\chi} \cong D(\mathfrak{g})/J, \quad J \cong D(\mathfrak{g})\tau(\mathfrak{g}) + D(\mathfrak{g})S_+^{\nu}(\mathfrak{g}^*)^G + D(\mathfrak{g})\delta^{-1}(L).$$

Write $N_{\chi} = D(\mathfrak{g}).(I \otimes_{A(\mathfrak{g})} v_{\chi}^J)$, where $I$ is the canonical generator of $D(\mathfrak{g})/D(\mathfrak{g})\tau(\mathfrak{g})$. From $\delta(E_\chi) = E_\chi - \nu$, we get that $(E_\chi - (b(\chi) - \nu))(I \otimes_{A(\mathfrak{g})} v_{\chi}^J) = 0$. It follows (as above) that $J$ is stable under the natural $\mathbb{C}^*$-action on $D(\mathfrak{g})$. Hence, $F_{\Psi}(J) = \kappa(J)$ and we have $N_{\chi}^{\nu^{-1}} = N_{F_{\Psi}^{-1}}^{\chi'}$.

We can define the category $C(M)$ similar to $C(N)$. We clearly have $M \in C(M)$ if, and only if, $N = M^{F_{\Psi}} \in C(N)$. Moreover, by 1.3, this is equivalent to saying that $M$ is a $G$-equivariant finitely generated $D(\mathfrak{g})$-module such that $M = D(\mathfrak{g})M^G$ and $\text{Supp} \, M \subset \text{N}(\mathfrak{g})$. This is also equivalent to: $N$ is a $G$-equivariant finitely generated $D(\mathfrak{g})$-module such that $N = D(\mathfrak{g})N^G$ and $N$ is $S_+$-finite (meaning that each $v \in N$ is killed by a power of $S_+(\mathfrak{g})^G$).

Recall that $N_{\chi}^G \cong V_{\chi}$ through the identification of $A(\mathfrak{g})$ with $D(\mathfrak{h})^W$. 


Lemma 1.3. One has $M_x^G \sim \langle V^x \rangle F_{\theta}^{-1}$.

Proof. Write $N_x = D(g)/J$. Then, $M_x = D(g)/F_{\theta}(J)$ and $M_x^G = D(g)^G/F_{\theta}(J^G)$. By Lemma 1.2, $\delta(D(g)(J^G)) = F_{\theta}(\delta(J^G))$, therefore $M_x^G \sim \langle D(h) \rangle F_{\theta}(\delta(J^G))$. Since $V^x \cong D(h)W/\langle D(h) \rangle$, the lemma follows.

Let $g_0$ be a real form of $g$ with adjoint group $G_0 \subset G$. There exists a natural action of $D(g)$ on $Db(g_0)$ defined by

$$\langle \partial(v).T, f \rangle = \langle T, -\partial(v).f \rangle, \quad \langle \xi, T, f \rangle = \langle T, \xi f \rangle$$

for all $T \in Db(g_0), f \in C_c^\infty(g_0), v \in g^*, \xi \in g^*$. This induces a structure of $D(g)^G$-module on $Db(g_0)^G$. From $T. Db(g_0)^G = 0$, we obtain a natural $A(g)$-module structure on $Db(g_0)^G$.

Fix a basis $\{u_1, \ldots, u_n\}$ of $g_0$ such that $\kappa(u_j, u_k) = \pm \delta_{jk}$ and denote by $dy$ the Lebesgue measure associated to this choice. Let $S(g_0)$ be the Schwartz space on $g_0$. Define, as in [13] Appendix 1, the Fourier transform of $f \in S(g_0)$ by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{g_0} f(y)e^{-i\kappa(y,x)}dy.$$ 

Let $T$ be a tempered distribution on $g_0$. The Fourier transform of $T$ is defined by

$$\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle$$

for $f \in C_c^\infty(g_0)$. Then we have

(1.4) $\forall D \in D(g), \forall T \in Db(g_0), \quad \hat{D}T = F_{\theta}(D)\hat{T}.$

Recall [2] that $T \in Db(g_0)$ is said to be homogeneous of degree $d$ if, for all $f \in C_c^\infty(g_0), t \in \mathbb{R}^*, \langle T, f_t \rangle = t^d \langle T, f \rangle$, where $f_t(v) = t^{-n}f(t^{-1}v)$. Then, a homogeneous distribution of degree $d$ is tempered and satisfies $E_{\theta}. T = dT$. We will need the following well-known result:

Lemma 1.4. Let $T \in Db(g_0)$ be tempered and set $M = D(g).T$. Then $M^{F_{\theta}} \cong D(g)\hat{T}.$

Proof. By [13] we have $\text{ann}_D(g)(\hat{T}) = F_{\theta}^{-1}(\text{ann}_D(g)(T))$. Hence the result.

Let $N(g_0)$ be the set of nilpotent elements of $g_0$. Define $D(g)$-submodules of $Db(g_0)$ by

$$Db(g_0)_{nil} = \{\Theta \in Db(g_0) : \text{Supp } \Theta \subset N(g_0)\},$$

$$Db(g_0)_{S_+} = \{T \in Db(g_0) : \exists k \in \mathbb{N}, S_+(g)^k.T = 0\}.$$ 

The elements of $Db(g_0)_{S_+}$ are called $S_+$-finite. Observe that $Db(g_0)^{G_0}_{nil}$ and $Db(g_0)^{G_0}_{S_+}$ are $D(g)^G$-modules. The next theorem is a consequence of the results proved in [13].

Theorem 1.5. (1) $Db(g_0)^{G_0}_{nil} = \{\Theta \in Db(g_0)^{G_0} : D(g).\Theta \in C(M)\}.$

(2) $Db(g_0)^{G_0}_{S_+} = \{T \in Db(g_0)^{G_0} : D(g).T \in C(N)\}.$

(3) $\Theta \in Db(g_0)^{G_0}_{nil} \iff \hat{\Theta} \in Db(g_0)^{G_0}_{S_+}.$

Proof. (1) follows from [13] Theorem 6.1, since $D(g).\Theta \in C(M)$ is equivalent to $D(g)^G.\Theta \cong \bigoplus_{\chi \in \mathbb{W}^+} m_{\chi}M_x^G.$

(2) and (3) are consequences of (1) and Lemma 1.4.
**Remark 1.6.** Let $T \in \mathbb{D}b(g_0)^{G_0}_{S_+}$. Recall that by the Harish-Chandra regularity theorem, $T$ is given by

$$
\langle T, f \rangle = \int_{g_0} F_T(y)f(y)dy
$$

for some analytic function $F_T$ on $g_0^*$, locally integrable on $g_0$.

2. **The distributions $\Theta_{u,\Gamma}$ and $T_{p,\Gamma}$**

Let $g_0$ be a real form of $g$, with adjoint group $G_0$, $h_0$ a Cartan subalgebra and let $H_0$ be the associated Cartan subgroup. Set $h = \mathbb{C} \otimes \mathbb{R} h_0$ and adopt the notation of [1]. Denote by $W(h_0)$ the real Weyl group, i.e. $W(h_0) = N_{G_0}(h_0)/Z_{G_0}(h_0)$. Define

$$
\Delta_R = \{ \alpha \in \Delta : \alpha(h_0) \subset \mathbb{R} \} \quad \text{(the real roots)},
$$

$$
\Delta_I = \{ \alpha \in \Delta : \alpha(h_0) \subset i\mathbb{R} \} \quad \text{(the imaginary roots)}.
$$

A root which is neither real nor imaginary is called complex. Let $\Delta_R^+$ be a positive system of roots in $\Delta_I$ and set $\pi_I = \prod_{\alpha \in \Delta_R^+} \alpha$. Then each $w \in W(h_0)$ permutes the imaginary roots and one can define a character of $W(h_0)$, the imaginary signature, by

$$
\varepsilon_I : W(h_0) \to \{ \pm 1 \}, \quad w.\pi_I = \varepsilon_I(w)\pi_I.
$$

If $V$ is a $W(h_0)$-module we denote by $V^{\varepsilon_I}$ the isotypic component of type $\varepsilon_I$ in $V$.

In the sequel, we adopt the notation of [3] with the minor difference that we use $e^{-i\alpha(x,y)}$ in the definition of the Fourier transform.

Let $h \in h_0'$ and $f \in C^\infty_c(g_0)$. Define [3, §3.1] the distribution $\mu_{G_0,h}$ by

$$
\langle \mu_{G_0,h}, f \rangle = | \det \text{ad}_{g_0/h_0}(h) |^{\frac{1}{2}} \int_{G_0/H_0} f(g,h)dg.
$$

Then one defines the function $J_{g_0}(f)$, or simply $J(f)$, on $h_0'$ by

$$
J_{g_0}(f) = \{ h \mapsto \langle \mu_{G_0,h}, f \rangle \}.
$$

Set $h_0^{reg} = \{ h \in h_0 : \pi_I(h) \neq 0 \}$ and fix a connected component $\Gamma$ of $h_0^{reg}$. Let $u \in S(h)$; Harish-Chandra has shown, see [17, §8.1, p. 123], that one can define a tempered $G_0$-invariant distribution on $g_0$ by

$$
(2.1) \quad \forall f \in C^\infty_c(g_0), \quad \langle \Theta_{u,\Gamma}, f \rangle = \lim_{h \in \Gamma} [ \partial(h).J(f)](h).
$$

Furthermore $\Theta_{u,\Gamma} \in \mathbb{D}b(g_0)^{G_0}_{S_+}$ and, when $u \in S^b(h)$, $\Theta_{u,\Gamma}$ is homogeneous of degree $-b - \nu - \ell$.

Now let $p \in S(h^*)$ and define $T \in \mathbb{D}b(g_0)^{G_0}_{S_+}$ by

$$
(2.2) \quad T_{p,\Gamma} = \hat{\Theta}_{F_p(p),\Gamma} = \{ f \mapsto \lim_{h \in \Gamma} [ \partial(F_h(p)).J(f)](h) \}.
$$

Then, $T_{p,\Gamma}$ is tempered and is homogeneous of degree $b - \nu$ when $p \in S^b(h^*)$.

**Lemma 2.1.** (1) Let $\varphi \in S(g)^G$. Then, $\varphi T_{p,\Gamma} = T_{\delta(\varphi)p,\Gamma}$.

(2) Let $q \in S(g)^G$. Then, $\partial(q).T_{p,\Gamma} = T_{\partial(\delta(q))p,\Gamma}$. 

Proof. Set \( u = F_b(p), \phi = \delta(\varphi) \in S(h)^W \) and \( s = \delta(q) \in S(h)^W \). Let \( f \in C_c^\infty(g_0) \).

1. By definition, see (2.2), \( \langle \varphi_{T_{\Gamma}, f} \rangle = \lim_{h \to 0} [\partial(u).J(\varphi f)](h) \). But, \( J \) Lemma 3.2.7, p. 38, (1.4) and Lemma 1.2 imply that \( J(\varphi f) = \partial(F_b(\phi)).J(\hat{f}) \).

Hence,
\[
\langle \varphi_{T_{\Gamma}, f} \rangle = \lim_{h \to 0} [\partial(u).\partial(F_b(\phi)).J(\hat{f})](h) = \lim_{h \to 0} [\partial(F_b(\phi))].J(\hat{f})(h)
\]

as desired.

2. By (1.4), \( \partial(h).T_{\Gamma} \) is the Fourier transform of \( F_b^{-1}(q)\Theta_{\Gamma} \), hence
\[
\langle \partial(h).T_{\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u).J(F_b^{-1}(q)\hat{f})](h).
\]

Set \( g = J(\hat{f}) \). From \( J \) Lemma 3.2.7, p. 38 and Lemma 1.2 we obtain that \( J(F_b^{-1}(q)\hat{f}) = F_b^{-1}(s)g \). Therefore
\[
\langle \partial(h).T_{\Gamma}, f \rangle = \lim_{h \to 0} [\partial(u).J(F_b^{-1}(s)g)](h).
\]

Recall (see \( J \)) that we have chosen a coordinate system \( \{x_j, e_j\}_{1 \leq j \leq \ell} \). With standard notation, we write \( x^\alpha = \prod_{k=1}^\ell x_k^{\alpha_k}, e^\mu = \prod_{k=1}^\ell e_k^{\mu_k} \) and
\[
p = \sum_{\alpha \in \mathbb{N}^\ell} p_\alpha x^\alpha, \quad s = \sum_{\mu \in \mathbb{N}^\ell} s_\mu e^\mu.
\]

Set \( \partial^\mu = \prod_j \partial(e_j)^{\mu_j} \); thus \( \partial(s) = \sum_{\mu \in \mathbb{N}^\ell} s_\mu \partial^\mu \). Order \( \mathbb{N}^\ell \) by saying that \( \mu \leq \alpha \) if \( \mu_j \leq \alpha_j \) for all \( j \). Set \( \alpha! = \prod_j \alpha_j! \) and \( \binom{\alpha}{\mu} = \prod_j \binom{\alpha_j}{\mu_j} \), when \( \mu \leq \alpha \). Then,
\[
\partial^\mu(x^\alpha) = \begin{cases} 0 & \text{if } \mu \not\leq \alpha, \\ \frac{\alpha!}{(\alpha-\mu)!} x^{\alpha-\mu} & \text{if } \mu \leq \alpha. \end{cases}
\]

Now we have \( u = F_b(p) = \sum_\alpha p_\alpha x^\alpha \partial^\alpha \) and \( F_b^{-1}(s) = \sum_\mu s_\mu \partial^\mu x^\mu \). Therefore, using the Leibniz formula, we get that
\[
\partial(u).J(F_b^{-1}(s)g) = \sum_\alpha p_\alpha x^\alpha \partial^\alpha (F_b^{-1}(s)g)
\]
\[
= \sum_\alpha \sum_\beta \sum_{\mu \leq \alpha} p_\alpha s_\mu \partial^{\alpha-\beta} \left( x^\mu \partial^\beta (F_b^{-1}(s)g) \right).
\]

But \( \lim_{h \to 0} \partial^\beta (x^\mu)(h) = 0 \) unless \( \beta = \mu \), hence
\[
\lim_{h \to 0} [\partial(u).J(F_b^{-1}(s)g)](h) = \sum_\alpha \sum_\mu \sum_{\beta \leq \alpha} p_\alpha s_\mu \partial^{\alpha-\beta} \left( x^\mu \partial^\beta (F_b^{-1}(s)g) \right) \mu! \lim_{h \to 0} [\partial^{\alpha-\beta}](g)(h).
\]

On the other hand, we have
\[
\langle T_{\partial(s), p, \Gamma}, f \rangle = \lim_{h \to 0} [\partial(F_b(\partial(s), p)).J(\hat{f})](h).
\]

Since \( \partial(s) = \sum_\alpha \sum_{\mu \leq \alpha} \frac{\alpha!}{(\alpha-\mu)!} s_\mu p_\alpha x^\alpha \partial^\mu \), we obtain that
\[
\langle T_{\partial(s), p, \Gamma}, f \rangle = \sum_\alpha \sum_{\mu \leq \alpha} \frac{\alpha!}{(\alpha-\mu)!} s_\mu p_\alpha x^\alpha \partial^{\alpha-\beta} \left( x^\mu \partial^\beta (F_b^{-1}(s)g) \right) \mu! \lim_{h \to 0} [\partial^{\alpha-\beta}](g)(h).
\]

This proves the desired equality.
\( \square \)
Theorem 2.2. Let \( p \in S(\mathfrak{h}^*) \) and \( D \in \mathcal{D}(g)^G \). Then, \( D.T_{p,\Gamma} = T_{\delta(D),p,\Gamma} \).

Proof. Since \( T_{p,\Gamma} \) is \( G_0 \)-invariant, we have \( I.T_{p,\Gamma} = 0 \). Let \( P \in \mathbb{C}(S(g)^G, S(\mathfrak{g}^*)^G) \); by Lemma 2.4 and an obvious induction, we obtain that \( P.T_{p,\Gamma} = T_{\delta(P),p,\Gamma} \). The theorem then follows from Lemma 1.1.

Recall, see Remark 1.6 that \( \tilde{\Theta}_{u,\Gamma} \in \text{Db}(\mathfrak{g}_0)^{G_0}_{S_u^+} \) is determined by a locally integrable function on \( \mathfrak{g}_0 \). We still denote this function by \( \tilde{\Theta}_{u,\Gamma} \).

Lemma 2.3. ([5 Lemme 6.1.2]) There exists \( c_T \in \mathbb{C}^* \), such that

\[
\alpha_{\Delta}^T(h) | \det \text{ad}_{b_0/b_0}(h) | \tilde{\Theta}_{F(h),\Gamma}(h) = c_T p(h)
\]

for all \( p \in S(\mathfrak{h}^*)^{\varepsilon I} \) and \( h \in \mathfrak{h}_0^{\text{reg}} \).

Proof. The first assertion follows from Corollary 2.5.

Theorem 2.4. Let \( p \in S(\mathfrak{h}^*)^{\varepsilon I} \). There exists a bijective map

\[
\rho : \mathcal{D}(g)^G.T_{p,\Gamma} \rightarrow \mathcal{D}(g)^W.p, \quad \rho(D.T_{p,\Gamma}) = \delta(D).p
\]

which, through \( \delta \), yields an isomorphism

\[
\rho : A(g).T_{p,\Gamma} \cong \mathcal{D}(g)^W.p.
\]

Proof. We first need to show that \( \rho \) is well defined. Let \( D \in \mathcal{D}(g)^G \); by Theorem 2.2 we have

\[
(\dagger) \quad D.T_{p,\Gamma} = T_{\delta(D),p,\Gamma} = \tilde{\Theta}_{F(h),\delta(D),p,\Gamma}.
\]

Suppose that \( D.T_{p,\Gamma} = 0 \). Then, the analytic function associated to \( T_{\delta(D),p,\Gamma} \in \text{Db}(\mathfrak{g}_0)^{G_0}_{S_u^+} \) vanishes on \( \mathfrak{h}_0^{\text{reg}} \). Notice that, since \( \delta(D) \) is \( W \)-invariant, \( \delta(D).p \in S(\mathfrak{h}^*)^{\varepsilon I} \). Therefore Lemma 2.3 gives \( \delta(D).p = 0 \) on \( \mathfrak{h}_0^{\text{reg}} \). Thus \( \delta(D).p = 0 \) on \( \mathfrak{h} \) and \( \rho \) is well defined.

Now, it follows easily from (\dagger) that \( \rho \) is a linear bijection. Since \( I.T_{p,\Gamma} = 0 \), the last assertion is clear.

Recall that we denote by \( V_\chi \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*) \) a simple \( W \)-module in the class of \( \chi \in W^- \).

Corollary 2.5. Let \( p \in S(\mathfrak{h}^*)^{\varepsilon I} \) such that \( CW.p \) is simple. Then there exists \( \chi \in W^- \) such that \( V_\chi^{\varepsilon I} \neq 0 \). We have

1. \( \mathcal{D}(g).T_{p,\Gamma} \cong N_\chi \) and \( \mathcal{D}(g)^G.T_{p,\Gamma} \cong V_\chi \);
2. \( \mathcal{D}(g).\Theta_{F(h),p,\Gamma} \cong M_\chi \) and \( \mathcal{D}(g)^G.\Theta_{F(h),p,\Gamma} \cong (V_\chi)^{F(h)^{-1}} \).

Proof. The first assertion follows from \( \mathcal{H}(\mathfrak{h}^*) \cong CW \). Then, 1 and 2 are consequences of \( V_\chi \cong \mathcal{D}(g)^W.p \), Lemma 1.3 and Theorem 2.4.

Remark 2.6. Let \( \chi \in W^- \) be such that \( V_\chi^{\varepsilon I} \neq 0 \). It follows obviously from the previous corollary that

\[
N_\chi \cong \mathcal{D}(g).T_{p,\Gamma}, \quad M_\chi \cong \mathcal{D}(g).\Theta_{u,\Gamma}
\]

where \( 0 \neq p \in V_\chi^{\varepsilon I} \subset \mathcal{H}^{b(\chi)}(\mathfrak{h}^*)^{\varepsilon I} \) and \( u = F(h) \in \mathcal{H}^{b(\chi)}(\mathfrak{h})^{\varepsilon I} \).
3. THE DECOMPOSITION OF $\text{Db}(g_0)^{G_0}_{\mathfrak{sl}_k}$ AND $\text{Db}(g_0)^{G_0}_{\text{mil}}$

Fix a real form $g_0$ of $g$ and let $[h_1], \ldots, [h_r]$ be the conjugacy classes of Cartan subalgebras in $g_0$. For each $j = 1, \ldots, r$ we denote by

$$h_{j,C} = h_j \otimes \mathbb{C}, \quad W_j = W(g_0, h_{j,C}), \quad \Delta^+_j, \ \text{a set of positive imaginary roots},$$

$$\varepsilon_{I,j} : W(h_j) = W(g_0, h_j) \to \{ \pm 1 \} \text{ the imaginary signature associated to } h_j.$$ 

For each $j$ we fix a connected component $\Gamma_j$ of $h_j^{\text{reg}}$. The results of \cite{2} then apply to $h_0 = h_j, \Gamma = \Gamma_j$ etc.

**Remark 3.1.** Recall that the $h_j, \mathbb{C}$ are $G$-conjugate. Therefore, if $1 \leq j, k \leq r$, the algebras $D(h_j, \mathbb{C})^{W_j}$ and $D(h_k, \mathbb{C})^{W_k}$ are naturally isomorphic. Denote this isomorphism by $\gamma_{jk}$ and let $\delta_j$ be the Harish-Chandra isomorphism from $A(g)$ onto $D(h_j, \mathbb{C})^{W_j}$. One can check that $\delta_k = \gamma_{jk} \circ \delta_j$. Therefore, we can choose an “abstract” Cartan subalgebra $h$ and identify $\delta_j$ with the homomorphism $\delta : D(g)^{W} \to D(h)^{W}$, where $W = W^\gamma(G, h)$. Then, if $\chi \in W^\gamma$, we have an irreducible $W$-module $V_{\chi} \subset H^{\beta(\chi)}(h^*)$ and a simple $D(h)^{W}$-module $V_{\chi}$.

For each $\chi \in W^\gamma$, choose a simple $W$-module $V_{\chi,j} \subset H^{\beta(\chi)}(h_{j,C}^*), \ V_{\chi,j} \cong V_{\chi}$. Write $V_{\chi,j} = V_{\chi,j}^{\varepsilon_{I,j}} \otimes E_{\chi,j}$ with $E_{\chi,j}$ stable under $W(h_j)$. Let \{ $e_{\chi,j}^k$ \}$_{1 \leq k \leq d(\chi)}$ be a basis of $V_{\chi,j}$ such that

$$V_{\chi,j}^{\varepsilon_{I,j}} = \bigoplus_{k=1}^{d(\chi)} \mathbb{C} e_{\chi,j}^k, \quad E_{\chi,j} = \bigoplus_{k=n(\chi)+1}^{d(\chi)} \mathbb{C} e_{\chi,j}^k$$

(hence $n_j(\chi) = \dim V_{\chi,j}^{\varepsilon_{I,j}}$).

**Lemma 3.2.** The $D(h_j, \mathbb{C})^{W_j}$-module $S(h_j^*, \chi)^{\varepsilon_{I,j}}$ decomposes as

$$S(h_j^*, \chi)^{\varepsilon_{I,j}} = \bigoplus_{\chi \in W^\gamma} \bigoplus_{k=1}^{n_j(\chi)} D(h_j, \mathbb{C})^{W_j} . e_{\chi,j}^k$$

with $D(h_j, \mathbb{C})^{W_j} . e_{\chi,j}^k \cong V_{\chi}^k$.

**Proof.** Clearly, we can drop the index $j$ and write $h_0 = h_j, \ h = h_{j,C}, \ v^k = e_{\chi,j}^k$ etc. Since $D(h)^{W_j} . e_{\chi}^k \subset S(h^*)^{\varepsilon_{I}}$ for $1 \leq k \leq n(\chi) = \dim V_{\chi}^{\varepsilon_{I}}$, one has

$$S(h^*)^{\varepsilon_{I}} \supset \bigoplus_{\chi \in W^\gamma} \bigoplus_{k=1}^{n(\chi)} D(h)^{W_j} . e_{\chi}^k.$$ 

Recall from \cite{3} that $S(h^*) = \bigoplus_{\chi} S(h^*)[\chi]$ with $S(h^*)[\chi] = \bigoplus_{k=1}^{d(\chi)} D(h)^{W} . e_{\chi}^k$. Write $S(h^*)[\chi] = E_1 \oplus E_2$, where $E_1 = \bigoplus_{k=1}^{d(\chi)} D(h)^{W} . e_{\chi}^k$ and $E_2 = \bigoplus_{k=n(\chi)+1}^{d(\chi)} D(h)^{W} . e_{\chi}^k$. Notice that $E_1, E_2$ are stable under $W(h_0)$ and that we have $S(h^*)[\chi]^{\varepsilon_{I}} = E_1 \oplus E_2^{\varepsilon_{I}}$.

We now show that $E_2^{\varepsilon_{I}} = 0$. This will prove that

$$S(h^*)^{\varepsilon_{I}} = \bigoplus_{\chi \in W^\gamma} \bigoplus_{k=1}^{n(\chi)} D(h)^{W} . e_{\chi}^k.$$ 

Let $D \in D(h)^{W}$ and $v \in V_{\chi}$. Notice first that if $D.v \neq 0$, the operator $D$ yields an isomorphism of $W$-modules $V_{\chi} \cong D.V_{\chi}$. Therefore, if $V_{\chi} = \bigoplus_k S_k$ with an
$S_k$ irreducible $W(h_0)$-module, we get that $D.V_\chi = \bigoplus_k D.S_k$, $D.S_k \cong S_k$. It follows that if $v \in E_\chi$ (the $W(h_0)$-stable complement of $V_\chi^{e_i}$), then $D.v \in D.E_\chi$ with $D.E_\chi \cap S(h_*^{e_i}) = 0$. Let $p = \sum_{k=n(\chi)+1}^{\Delta(\chi)} D_k v_\chi^k \in E_2$. Then, $CW(h_0).p \subset \sum_{k>n(\chi)} CW(h_0).D_k v_\chi^k$ and, by the previous remarks, $(CW(h_0).D_k v_\chi^k)^{e_i} = 0$. Thus $(CW(h_0).p)^{e_i} = 0$, which shows that $E_2^{e_i} = 0$.

Recall the following result:

**Proposition 3.3** ([5 Proposition 6.1.1]). (1) The linear map

$$T : \bigoplus_{j=1}^r S(h_{j(C)}^{e_{i,j}}) \rightarrow Db(g_0)^{Go}_{S_+}, \quad T(p_1, \ldots , p_r) = \sum_{j=1}^r T_{p_j, \Gamma_j}$$

is an isomorphism of vector spaces.

(2) The map $T$ induces an isomorphism:

$$\bigoplus_{j=1}^r H(h_{j(C)}^{e_{i,j}}) \cong \{ T \in Db(g)^{Go}_{S_+} : S_+(g)^G . T = 0 \}.$$  

**Proof.** (2) follows from the proof of [5 Proposition 6.1.1].

**Theorem 3.4.** Set $T(h_j) = \sum_{p \in S(h_{j(C)}^{e_{i,j}}) \subset \mathbb{C} T_\Gamma_\Gamma}$. Then we have the following decomposition of $D(g)^G$-modules:

$$Db(g_0)^{Go}_{S_+} = \bigoplus_{j=1}^r T(h_j)$$

with

$$T(h_j) = \bigoplus_{\chi \in W} \bigoplus_{k=1}^{n_\chi(\chi)} D(g)^G . T_{\chi(k), \Gamma_j}$$

and $D(g)^G . T_{\chi(k), \Gamma_j} \cong \mathcal{N}_\chi^G$.

**Proof.** The decomposition of $T(h_j)$, as a $D(g)^G$-module, is consequence of Theorem 3.3. Lemma 3.2 (using the isomorphism $\delta : A(g) \cong D(h_{j(C)}^{W_j})$) and Proposition 3.3. The decomposition of $Db(g_0)^{Go}_{S_+}$ follows from Proposition 3.3.

Using the Fourier transform, we obtain the following:

**Corollary 3.5.** The $D(g)^G$-module $Db(g_0)^{Go}_{nil}$ decomposes as

$$Db(g_0)^{Go}_{nil} = \bigoplus_{j=1}^r \bigoplus_{\chi \in W} \bigoplus_{k=1}^{n_\chi(\chi)} D(g)^G . \Theta F_{\chi}^{-1}(v_{\chi(k),j}), \Gamma_j \cong \mathcal{N}_\chi^G.$$  

The next corollary follows from Theorem 3.4 and Corollary 3.5.

**Corollary 3.6.** We have

$$Db(g_0)^{Go}_{S_+} \cong \bigoplus_{\chi \in W} m_\chi \mathcal{N}_\chi^G, \quad Db(g_0)^{Go}_{nil} \cong \bigoplus_{\chi \in W} m_\chi \mathcal{N}_\chi^G,$$

where $m_\chi = \sum_{j=1}^r \dim V_\chi^{e_{i,j}}$.  

\[ \square \]
Remark 3.7. Let $\chi \in W^\omega$. It is not always possible to “realize” the modules $N_\chi$ and $M_\chi$ as $D(g).T$ for some $T \in Db(g_0)$, where $g_0$ is a real form of $g$. By the previous results, this statement is equivalent to the existence of a Cartan subalgebra $h_j \subset g_0$ such that $V^{\chi_{\phi}}_{\phi} \neq 0$. D. Renard has observed that, using the results of W. Rossmann [15], this can be translated to a question about centralizers of nilpotent elements. Fix a real form $g_\mathbb{R}$ of $g$ with adjoint group $G_\mathbb{R}$. If $x \in g_\mathbb{R}$ is nilpotent one defines a subgroup of the component group $A(G.x)$ (see [4] for notation) by

$$A(G_\mathbb{R}.x) = G_\mathbb{R}^x / G_\mathbb{R}^x \cap (G^x)^0.$$ 

Recall that $\chi \in W^\omega$ can be written $\sigma(O, \psi)$ via the Springer correspondence, where $O \subset g$ is a nilpotent orbit and $\psi : A(O) \to GL(E)$ is an irreducible representation. Then, by [15] Corollary 3.2 & Theorem 3.3, there exists a Cartan subalgebra $H_0 \subset g_\mathbb{R}$ such that $V^{\chi_{\phi}}_{\phi} \neq 0$ if, and only if, there exists a nilpotent element $x \in g_\mathbb{R}$ such that $O = G.x$ and $E^{A(G_\mathbb{R}.x)} \neq 0$.

Let $g = \mathfrak{sp}(\ell, \mathbb{C})$ and let $\phi \in W^\omega$ be the long sign character, i.e. $V_\phi = \mathbb{C}\pi_l$ where $\pi_l$ is the product of the long roots. Then, see [5] §13.3, $\phi = \sigma(O, \psi)$ where $O = G.x$ is the subgroup of nilpotent orbit with partition $[2\ell - 2, 2]$ and $\psi$ is the non-trivial character of $A(O) \equiv \{ \pm 1 \}$. The real forms of $g$ are $\mathfrak{sp}(\ell, \mathbb{R})$ and the $\mathfrak{sp}(p, q)$, $p+q = \ell$. Assume now that $\ell \geq 3$. By the classification of nilpotent orbits in $\mathfrak{sp}(p, q)$, see [7] Theorem 9.2.5, we know that $O^{\phi} \cap \mathfrak{sp}(p, q) = \emptyset$. Hence, by Rossmann’s results, $V^{\chi_{\phi}}_{\phi} \neq 0$ for each Cartan subalgebra $h_j \subset \mathfrak{sp}(p, q)$. On the other hand, if $G_\mathbb{R}$ is the adjoint group of $\mathfrak{sp}(\ell, \mathbb{R})$, one can show that $A(G_\mathbb{R}.x) = A(G.x)$. Thus, with the above notation, $E^{A(G_\mathbb{R}.x)} = 0$ and it follows that $V^{\chi_{\phi}}_{\phi} = 0$ for each Cartan subalgebra $h_j \subset \mathfrak{sp}(\ell, \mathbb{R})$. For instance, when $g = \mathfrak{sp}(3, \mathbb{R})$ there are six conjugacy classes of Cartan subalgebras and one can directly verify (without using [15]) that $V^{\chi_{\phi}}_{\phi} = 0$ for $j = 1, \ldots, 6$. We thank D. Renard for showing this computation to us. 

Let $x \in N(g_0)$ and denote by $\beta_x$ the Liouville (Kostant-Kirillov) measure on $G_0.x$. By [14] one can define $\Theta_x \in Db(g_0)^{G_0}_{nil}$ by $\langle \Theta_x, f \rangle = \int_{g_0.x} f d\beta_x$ for all $f \in C_c^\infty(g_0)$. Set $O = G.x$. Then, see [9], [10] or [18]; $\Theta_x$ is homogeneous of degree $\lambda_\Theta = 1/2 \dim O - \dim g$ and satisfies

$$(3.1) \quad D(g).\Theta_x \cong M_{\chi_\Theta}$$

for some $\chi_\Theta \in W^\omega$ such that $\lambda_\Theta = \nu - n - b(\chi_\Theta)$.

Corollary 3.8. There exists $j \in \{1, \ldots, r\}$ and $u \in F_{h_j}^{-1}(V_{\chi_{\Theta}})^{\chi_{\phi}}$ such that $D(g)^{G_j}.\Theta_x \cong D(g)^{G_j}.\Theta_{u, r_j}$.

Proof. Since $D(g)^{G_j}.\Theta_x \cong M^{G_j}_{\chi_{\Theta}}$ is a simple submodule of $Db(g_0)^{G_0}_{nil}$, the claim follows from Corollary 3.3. 

Remark 3.9. It is proved in [11], see also [3], that $\Theta_x$ can be written as $\sum_{j=1}^r \Theta_{a_j, r_j}$ with $a_j \in H^{\chi_{\Theta}}(h_j, C)^{\chi_{\phi}}$. It is easily seen that we may assume $CW.a_j \cong V_{\chi_\Theta}$ for all $j$ such that $a_j \neq 0$. W. Rossmann [15] has given conditions to ensure that $\Theta_x = \Theta_{a_j, r_j}$ for some $j$. 


4. Example: the complex case

We assume in this section that \( g_0 = g_1^{\text{triv}} \) is a complex semisimple Lie algebra, \( g_1 \), viewed as a real Lie algebra. Then, \( g \) can be identified with \( g_1 \times g_1 \) and \( g_0 \) with the diagonal \( \{ (a,a) : a \in g_1 \times g_1 \} \). Let \( h_0 \) be a Cartan subalgebra of \( g_1 \). Recall the following well-known facts, see [17] or [18]:

- \( h_0 = \{ (a,a) : a \in h_1 \} \) is a Cartan subalgebra of \( h_0 \) and \( h = h_0 \otimes \mathbb{R} \subseteq h_1 \times h_1 \);
- \( W(g,h) = W_1 \times W_1 \), where \( W_1 = W(g_1,h_1) \), and \( W(h_0) = \{ (w,w) \in W \} \) is isomorphic to \( W_1 \);
- there is a unique conjugacy class \( [h_0] \) of Cartan subalgebras and \( h_0' \) is connected;
- the irreducible representations of \( W \) are of the form \( \chi = \phi \otimes \mu, \phi, \mu \in W_1^* \);
- one has \( \phi = \phi^* \) for all \( \phi \in W_1^* \), where \( \phi^* \) is the dual representation.

Observe that \( D(g) = D(g_1) \boxtimes D(g_1) \) and \( D(g)^G = D(g_1)^{G_1} \boxtimes D(g_1)^{G_1} \).

Lemma 4.1. Let \( \chi \in W^* \). Then, the simple \( D(g) \)-module \( M_\chi \) is of the form \( M_{\phi} \boxtimes M_{\mu} \) for some \( \phi, \mu \in W_1^* \).

Proof. The claim follows easily from the definition of the category \( \mathcal{C}(M) \) and the decomposition of the \( W \)-module \( S(h^*) = S(h_1^*) \boxtimes S(h_1^*) \).

Corollary 4.2. [18 Theorem 6.11] We have

\[
\text{Db}(g_0^{G_0})_{\text{nil}} \cong \bigoplus_{\phi \in W_1^*} M_\phi^{G_1} \boxtimes M_\phi^{G_1}
\]

as a \( D(g)^G \)-module.

Proof. Let \( \chi = \phi \otimes \mu \in W^* \). Then, \( V_\chi^{G_1} = (V_\phi \otimes V_\mu)^{W_1} \neq 0 \) if, and only if, \( \phi = \mu \) and therefore \( n(\chi) = 1 \). The assertion now follows from Corollary 3.3.

Recall the following general results from [13]. Since the module \( M_\chi \) is irreducible and \( G \)-equivariant, its support is the closure of a nilpotent orbit \( O = G.x \). Furthermore, if \( i : O \hookrightarrow g \) is the inclusion, \( M_\chi \) is uniquely determined by its \( (D,O) \)-module inverse image \( \mathcal{L}_\chi := i^* M_\chi \). The \( D_O \)-module \( \mathcal{L}_\chi \) is an irreducible integrable connection associated to an irreducible representation \( \psi \) of the component group \( A(O) := G^*/(G^*)^0 \) (where \( (G^*)^0 \) is the connected component of the centralizer \( G^* \)). Therefore, since \( \chi \) is uniquely determined by \( O \) and \( \psi \), we set \( \chi = \sigma(O, \psi) \).

In our situation, i.e., in the complex case, we have \( O = O_1^j \times O_2^j \) with \( O_1^j \) nilpotent orbits in \( g_1 \) for \( j = 1,2 \). Then, \( \chi = \sigma(O, \psi) = \phi_1 \otimes \phi_2, \mathcal{L}_\chi = L_{\phi_1} \boxtimes L_{\phi_2}, \phi_j = \sigma(O^j, \psi_j), \psi = \psi_1 \otimes \psi_2. \)

Note that \( b(\chi) = b(\phi_1) + b(\phi_2) \) and \( \lambda_O = \lambda_{O^1} + \lambda_{O^2}. \)

Let \( x \in N(g_0) \); set \( x = (x_1, x_1), x_1 \in N(g_1), O_1 = G_1.x_1, O = G.x = O_1 \times O_1. \)

The inclusion \( i : O \hookrightarrow g \) is equal to \( i_1 \times i_1, \) where \( i_1 : O_1 \hookrightarrow g_1 \). By (3.1) and Corollary 4.2 there exist \( \chi \in W^*, \chi_1 \in W_1 \) such that \( \chi = \chi_1 \otimes \chi_2 \) and \( D(g) \cdot \Theta_x \cong M_{\chi_1} \boxtimes M_{\chi_2}. \)

It is known (Harish-Chandra) that \( \Theta_x = \Theta_{u,h_0} \) for some \( u \in S(h_1) \otimes S(h_1). \)

The following result has been proved by various authors; see [2, 3] (when \( O_1 \) is “special”), [3, 9, 16].

Theorem 4.3. One has \( \chi_1 = \sigma(O_1, \text{triv}) \), and there exists \( p \in (V_{\chi_1} \otimes V_{\chi_1})^{W_1} \) such that \( \Theta_x = \Theta_{F_p(p),h_0^p}. \)
Proof. Recall from [9] or [10] that χ = χ1 ⊗ χ1 = σ(𝒪, triv). This means that

\[ \mathcal{L}_X = \mathcal{L}_{\chi_1} \otimes \mathcal{L}_{\chi_1} = \mathcal{O}_X = \mathcal{O}_{\chi_1} \otimes \mathcal{O}_{\chi_1} \]

(where we denote by \( \mathcal{O}_X \) the structural sheaf of an algebraic variety \( X \)). This yields \( \mathcal{L}_{\chi_1} = \mathcal{O}_{\chi_1} \) and \( \chi_1 = \sigma(\mathcal{O}_1, \text{triv}) \).

Set \( T_x = \bar{\Theta}_z \); then \( \mathcal{D}(g)T_x = N_x \otimes N_{\chi_1} \) (see Lemma 1.4). Since \( S_+(g^*)G, \Theta_x = 0 \) we have \( S_+(g)G, T_x = 0 \). It follows from Proposition 3.3(2) that we can write \( T_x = T_{p, b_x} \) for some \( p \in (\mathcal{H}(b_x) \otimes \mathcal{H}(b_x))^{W} \) or, equivalently, \( \Theta_x = \bar{\Theta}_{F_p(p), b_x} \). Now, by Theorem 3.4 \( \mathcal{D}(b)^W, p = V_{\chi_1} \otimes V_{\chi_1} \) and therefore \( \mathcal{C}W, p \equiv V_{\chi_1} \otimes V_{\chi_1} \). Moreover, \( T_x = T_{p, b_x} \) is homogeneous of degree \( b(\chi_0) - 2\nu = 2b(\chi_1) - 2\nu = \deg p - 2\nu \). Thus \( \deg p = 2b(\chi_1) \) and, by definition of \( V_{\chi_1}, p \in (V_{\chi_1} \otimes V_{\chi_1})^W \).

\[ \square \]

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