TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 11, Pages 4447–4460
S 0002-9947(01)02766-0
Article electronically published on May 22, 2001

SPHERICAL CLASSES AND THE LAMBDA ALGEBRA

NGUYỄN H. V. HƯNG

ABSTRACT. Let $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$ be Singer's invariant-theoretic model of the dual of the lambda algebra with $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, where $\mathcal{A}$ denotes the mod 2 Steenrod algebra. We prove that the inclusion of the Dickson algebra, D_k, into $\Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism

$$\varphi_k^* : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong H_k(\Gamma^\wedge)\,.$$

The Lannes–Zarati homomorphisms themselves, φ_k, correspond to an associated graded of the Hurewicz map

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)\,.$$

Based on this result, we discuss some algebraic versions of the classical conjecture on spherical classes, which states that *Only Hopf invariant one and Kervaire invariant one classes are detected by the Hurewicz homomorphism.* One of these algebraic conjectures predicts that every Dickson element, i.e. element in D_k, of positive degree represents the homology class 0 in $Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$.

We also show that φ_k^* factors through $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$, where $\partial_k : \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ denotes the differential of $\Gamma^\wedge$. Therefore, the problem of determining $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$ should be of interest.

1. Introduction and statement of results

Let Q_0S^0 be the basepoint component of $QS^0 = \lim_n \Omega^n S^n$. It is a classical unsolved problem to compute the image of the Hurewicz homomorphism

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)\,.$$

Here and throughout the paper, homology and cohomology are taken with coefficients in $\mathbb{F}_2$, the field of two elements. The long-standing conjecture on spherical classes reads as follows.

Conjecture 1.1. The Hopf invariant one and the Kervaire invariant one classes are the only elements in $H_*(Q_0S^0)$ detected by the Hurewicz homomorphism. (See Curtis [5], Snaith and Tornehave [22] and Wellington [23] for a discussion.)

An algebraic version of this problem goes as follows. Let $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ be the polynomial algebra on k generators $x_1, \dots, x_k$, each of degree 1. Let the

Received by the editors February 4, 1999 and, in revised form, November 4, 1999.
2000 *Mathematics Subject Classification.* Primary 55P47, 55Q45, 55S10, 55T15.
Key words and phrases. Spherical classes, loop spaces, Adams spectral sequences, Steenrod algebra, lambda algebra, invariant theory, Dickson algebra.
The research was supported in part by the National Research Project, No. 1.4.2.

general linear group $GL_k = GL(k, \mathbb{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ both act on P_k in the usual way. The Dickson algebra of k variables, D_k, is the algebra of invariants

$$D_k := \mathbb{F}_2[x_1, \dots , x_k]^{GL_k} .$$

As the action of $\mathcal{A}$ and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathcal{A}$. In [14], Lannes and Zarati construct homomorphisms

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^* ,$$

which correspond to an associated graded of the Hurewicz map. The proof of this assertion is unpublished, but it is sketched by Lannes [12] and by Goerss [7]. The Hopf invariant one and the Kervaire invariant one classes are respectively represented by certain permanent cycles in $Ext_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $Ext_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, on which φ_1 and φ_2 are non-zero (see Adams [1], Browder [4], Lannes–Zarati [14]). Therefore, we are led to the following conjecture.

Conjecture 1.2. $\varphi_k = 0$ in any positive stem i for $k > 2$.

The present paper follows a series of our works ([8], [10], [11]) on this conjecture. To state our main result, we need to summarize Singer's invariant-theoretic description of the lambda algebra [20]. According to Dickson [6], one has

$$D_k \cong \mathbb{F}_2[Q_{k,k-1}, ..., Q_{k,0}],$$

where $Q_{k,i}$ denotes the Dickson invariant of degree $2^k - 2^i$. Singer sets $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defines $\Gamma_k^\wedge$ to be a certain "not too large" submodule of Γ_k. He also equips $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$ with a differential $\partial : \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ and a coproduct. Then, he shows that the differential coalgebra $\Gamma^\wedge$ is dual to the lambda algebra of the six authors of [3]. Thus, $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. (Originally, Singer uses the notation Γ_k^+ to denote $\Gamma_k^\wedge$. However, by D_k^+, $\mathcal{A}^+$ we always mean the submodules of D_k and $\mathcal{A}$ respectively consisting of all elements of positive degrees, so Singer's notation Γ_k^+ would cause confusion in this paper. Therefore, we prefer the notation $\Gamma_k^\wedge$.)

The main result of this paper is the following theorem, which has been conjectured in our paper [10, Conjecture 5.3].

Theorem 3.9. *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i \to Tor_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

An immediate consequence of this theorem is the equivalence between Conjecture 1.2 and the following one.

Conjecture 1.3. If $q \in D_k^+$, then $[q] = 0$ in $Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$.

This has been established for $k = 3$ in [10, Theorem 4.8], while Conjecture 1.2 has been proved for $k = 3$ in [8, Corollary 3.5] .

From the view point of this conjecture, it seems to us that Singer's model of the dual of the lambda algebra, $\Gamma^\wedge$, is somehow more natural than the lambda algebra itself.

The canonical $\mathcal{A}$-action on D_k is extended to an $\mathcal{A}$-action on $\Gamma_k^\wedge$. This action commutes with ∂_k (see [20]), so it determines an $\mathcal{A}$-action on $Ker\partial_k$, the submodule of all cycles in $\Gamma_k^\wedge$. We also prove

Proposition 4.1. *φ_k^* factors through $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$ as shown in the commutative diagram*

$$\begin{tikzcd}
\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \arrow[rr, "\varphi_k^*"] \arrow[dr, "\overline{i}"'] & & Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \\
& \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k \,, \arrow[ur, "\overline{p}"'] &
\end{tikzcd}$$

where $\overline{i}$ is induced by the inclusion $D_k \subset Ker\partial_k$, and $\overline{p}$ is an epimorphism induced by the canonical projection $p : Ker\partial_k \to H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.

From this result, the problem of determining $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$ would be of interest.

The paper is divided into 4 sections.

In Section 2 we recollect some materials on invariant theory, particularly on Singer's invariant-theoretic description of the lambda algebra and the Lannes–Zarati homomorphism. Section 3 is devoted to prove Theorem 3.9. Finally, Section 4 is a discussion on factoring φ_k^*.

The main results of this paper were announced in [9].

The author would like to thank Haynes Miller for introducing him to Stewart Priddy's work [18] on exploiting an explicit homotopy equivalence between the bar resolution of $\mathbb{F}_2$ over $\mathcal{A}$ and the dual of the lambda algebra. He also thanks the referee for helpful suggestions, which led to improving the exposition of the paper.

2. Recollections on modular invariant theory

We start this section by sketching briefly Singer's invariant-theoretic description of the lambda algebra.

Let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular $k \times k$-matrices with 1 on the main diagonal. The T_k-invariant ring, $M_k = P_k^{T_k}$, is called the Mùi algebra. In [17], Mùi shows that

$$P_k^{T_k} = \mathbb{F}_2[V_1, ..., V_k],$$

where

$$V_i = \prod_{\lambda_j \in \mathbb{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{i-1} x_{i-1} + x_i).$$

Then, the Dickson invariant $Q_{k,i}$ can inductively be defined by

$$Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i},$$

where, by convention, $Q_{k,k} = 1$ and $Q_{k,i} = 0$ for $i < 0$.

Let $S(k) \subset P_k$ be the multiplicative subset generated by all the non-zero linear forms in P_k. Let Φ_k be the localization, $\Phi_k = (P_k)_{S(k)}$. Using the results of Dickson [6] and Mùi [17], Singer notes in [20] that

$$\Delta_k := (\Phi_k)^{T_k} = \mathbb{F}_2[V_1^{\pm 1}, ..., V_k^{\pm 1}],$$

$$\Gamma_k := (\Phi_k)^{GL_k} = \mathbb{F}_2[Q_{k,k-1}, ..., Q_{k,1}, Q_{k,0}^{\pm 1}].$$

Further, he sets

$$v_1 = V_1, \quad v_k = V_k / V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtains

$$\Delta_k = \mathbb{F}_2[v_1^{\pm 1}, ..., v_k^{\pm 1}],$$

with $\deg v_i = 1$ for every i.

Singer defines $\Gamma_k^\wedge$ to be the submodule of $\Gamma_k = D_k[Q_{k,0}^{-1}]$ spanned by all monomials $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$ with $i_{k-1}, ..., i_1 \geq 0, i_0 \in \mathbb{Z}$, and $i_0 + \deg \gamma \geq 0$. He also shows in [20] that the homomorphism

$$\partial_k : \mathbb{F}_2[v_1^{\pm 1}, ..., v_k^{\pm 1}] \to \mathbb{F}_2[v_1^{\pm 1}, ..., v_{k-1}^{\pm 1}],$$

$$\partial_k(v_1^{j_1} \cdots v_k^{j_k}) := \begin{cases} v_1^{j_1} \cdots v_{k-1}^{j_{k-1}}, & \text{if } j_k = -1, \\ 0, & \text{otherwise,} \end{cases}$$

maps $\Gamma_k^\wedge$ to $\Gamma_{k-1}^\wedge$. Moreover, it is a differential on $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$. This module is bigraded by putting $\mathrm{bideg}(v_1^{j_1} \cdots v_k^{j_k}) = (k, k + \sum j_i)$.

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [3]. It is also bigraded by putting (as in [19, p. 90]) $\mathrm{bideg}(\lambda_i) = (1, 1+i)$. Singer proves in [20] that $\Gamma^\wedge$ is a differential bigraded coalgebra, which is dual to the differential bigraded lambda algebra Λ via the isomorphisms

$$\begin{array}{ccc} \Gamma_k^\wedge & \to & \Lambda_k^* \\ v_1^{j_1} \cdots v_k^{j_k} & \mapsto & (\lambda_{j_1} \cdots \lambda_{j_k})^*. \end{array}$$

Here the duality $*$ is taken with respect to the basis of admissible monomials of Λ. As a consequence, one gets an isomorphism of bigraded coalgebras

$$H_*(\Gamma^\wedge) \cong Tor_*^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

In the remaining part of this section, we recall the definition of the Lannes–Zarati homomorphism.

Let $P_1 = \mathbb{F}_2[x]$ with $|x| = 1$. Let $\hat{P} \subset \mathbb{F}_2[x, x^{-1}]$ be the submodule spanned by all powers x^i with $i \geq -1$. The canonical $\mathcal{A}$-action on P_1 is extended to an $\mathcal{A}$-action on $\mathbb{F}_2[x, x^{-1}]$ (see Adams [2], Wilkerson [24]). Then $\hat{P}$ is an $\mathcal{A}$-submodule of $\mathbb{F}_2[x, x^{-1}]$. One has a short-exact sequence of $\mathcal{A}$-modules

2.1. $$0 \to P_1 \xrightarrow{\iota} \hat{P} \xrightarrow{\pi} \Sigma^{-1}\mathbb{F}_2 \to 0\ ,$$

where ι is the inclusion and π is given by $\pi(x^i) = 0$ if $i \neq -1$ and $\pi(x^{-1}) = 1$. Let e_1 be the corresponding element in $Ext_{\mathcal{A}}^1(\Sigma^{-1}\mathbb{F}_2, P_1)$.

Definition 2.2 (Singer [21]). (i) $e_k = \underbrace{e_1 \otimes \cdots \otimes e_1}_{k \text{ times}} \in Ext_{\mathcal{A}}^k(\Sigma^{-k}\mathbb{F}_2, P_k)$.

(ii) $e_k(M) = e_k \otimes M \in Ext_{\mathcal{A}}^k(\Sigma^{-k}M, P_k \otimes M)$, for M a left $\mathcal{A}$-module. Here M also means the identity map of M.

Following Lannes–Zarati [14], the destabilization of M is defined by

$$\mathcal{D}M = M/EM,$$

where $EM := \mathrm{Span}\{Sq^i x |\, i > \deg x, x \in M\}$. They show that the functor associating M to $\mathcal{D}M$ is a right exact functor. Then they define $\mathcal{D}_k$ to be the kth left derived functor of $\mathcal{D}$. So one gets

$$\mathcal{D}_k(M) = H_k(\mathcal{D}F_*(M)),$$

where $F_*(M)$ is an $\mathcal{A}$-free (or $\mathcal{A}$-projective) resolution of M.

The cap-product with $e_k(M)$ gives rise to the homomorphism

$$\begin{array}{rcl} e_k(M): \quad \mathcal{D}_k(\Sigma^{-k}M) & \to & \mathcal{D}_0(P_k \otimes M) \equiv P_k \otimes M \\ e_k(M)(z) & = & e_k(M) \cap z\,. \end{array}$$

Since $\mathbb{F}_2$ is an unstable $\mathcal{A}$-module, one gets

Theorem 2.3 (Lannes–Zarati [14])**.** *Let $D_k \subset P_k$ be the Dickson algebra of k variables. Then $\alpha_k := e_k(\Sigma\mathbb{F}_2) : \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \to \Sigma D_k$ is an isomorphism of internal degree 0.*

By definition of the functor $\mathcal{D}$, one has a natural homomorphism, $\mathcal{D}(M) \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} M$. Then it induces a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{D}F_k(M) \arrow[r] \arrow[d, "i_k"] & \mathcal{D}F_{k-1}(M) \arrow[r] \arrow[d, "i_{k-1}"] & \cdots \\ \cdots \arrow[r] & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} F_k(M) \arrow[r] & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} F_{k-1}(M) \arrow[r] & \cdots\,. \end{tikzcd}$$

Here the horizontal arrows are induced from the differential in $F_*(M)$, and

$$i_k[Z] = [1 \underset{\mathcal{A}}{\otimes} Z]$$

for $Z \in F_k(M)$. Passing to homology, one gets a homomorphism

$$\textbf{2.4.} \qquad \begin{array}{rcl} i_k: \quad \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(M) & \to & Tor_k^{\mathcal{A}}(\mathbb{F}_2, M) \\ 1 \underset{\mathcal{A}}{\otimes} [Z] & \mapsto & [1 \underset{\mathcal{A}}{\otimes} Z]\,. \end{array}$$

Taking $M = \Sigma^{1-k}\mathbb{F}_2$, one obtains a homomorphism

$$i_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2)\,.$$

Note that the suspension $\Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \Sigma D_k$ and the desuspension

$$\Sigma^{-1} : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2) \xrightarrow{\cong} Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2)$$

are isomorphisms of internal degree 1 and (-1), respectively. This leads one to

Definition 2.5 (Lannes–Zarati [14])**.** The homomorphism φ_k of internal degree 0 is the dual of

$$\varphi_k^* = \Sigma^{-1} i_k (1 \underset{\mathcal{A}}{\otimes} \alpha_k^{-1}) \Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2)\,.$$

Remark 2.6. In Theorem 3.9 we also denote by φ_k^* the composite of the above φ_k^* with the suspension isomorphism $\Sigma^k : Tor_{k,i}^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \xrightarrow{\cong} Tor_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.

We need to relate $\alpha_k = e_k(\Sigma\mathbb{F}_2)$ with connecting homomorphisms.

Suppose $f \in Ext_{\mathcal{A}}^1(M_3, M_1)$ is represented by the short-exact sequence of left $\mathcal{A}$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. Let $\Delta(f) : \mathcal{D}_s(M_3) \to \mathcal{D}_{s-1}(M_1)$ be the connecting homomorphism associated with this short-exact sequence. Then one easily verifies

$$\Delta(f)(z) = f \cap z$$

for any $z \in \mathcal{D}_s(M_3)$.

One has

2.7. $e_k(\Sigma\mathbb{F}_2) = (e_1(\Sigma\mathbb{F}_2)\otimes P_{k-1})\circ\cdots\circ(e_1(\Sigma^{3-k}\mathbb{F}_2)\otimes P_1)\circ e_1(\Sigma^{2-k}\mathbb{F}_2)$.

Therefore, one gets

2.8. $\alpha_k = \Delta(e_1(\Sigma\mathbb{F}_2)\otimes P_{k-1})\circ\cdots\circ\Delta(e_1(\Sigma^{3-k}\mathbb{F}_2)\otimes P_1)\circ\Delta e_1(\Sigma^{2-k}\mathbb{F}_2)$.

(See Singer [21, p. 498].)

This formula will be useful to construct a chain-level representation of α_k.

3. A chain-level representation of the Lannes–Zarati homomorphism

Suppose again M is a left graded $\mathcal{A}$-module. Let $B_*(M)$ be the bar resolution of M over $\mathcal{A}$. Recall that

$$B_k(M) = \mathcal{A}\otimes\underbrace{I\otimes\cdots\otimes I}_{k\text{ times}}\otimes M \qquad (k\geq 0),$$

where I denotes the augmentation ideal of $\mathcal{A}$ and the tensor products are taken over $\mathbb{F}_2$. The module $B_*(M)=\bigoplus_k B_k(M)$ is bigraded by assigning an element $a_0\otimes a_1\otimes\cdots\otimes a_k\otimes x$ with homological degree k and internal degree $\sum_{i=0}^k(\deg a_i)+\deg x$.

The differential $d_k : B_k(M)\to B_{k-1}(M)$ is defined by

$$\begin{aligned} d_k(a_0\otimes a_1\otimes\cdots\otimes a_k\otimes x) \quad &= \quad a_0a_1\otimes\cdots\otimes a_k\otimes x + a_0\otimes a_1a_2\otimes\cdots\otimes a_k\otimes x \\ &\qquad +\cdots+a_0\otimes a_1\otimes\cdots\otimes a_kx. \end{aligned}$$

So d_k preserves internal degree and lowers homological degree by 1.

The action of $\mathcal{A}$ on $B_k(M)$ is given by

$$a(a_0\otimes a_1\otimes\cdots\otimes a_k\otimes x) = aa_0\otimes a_1\otimes\cdots\otimes a_k\otimes x,$$

for $a\in\mathcal{A}$.

Suppose additionally that N is a right graded $\mathcal{A}$-module. As the bar resolution is an $\mathcal{A}$-free resolution, by definition one has

$$Tor_k^{\mathcal{A}}(N,M) := H_k(N\underset{\mathcal{A}}{\otimes}B_*(M)).$$

Since $D_k\subset\mathbb{F}_2[v_1,...,v_k]$, every element $q\in D_k$ has an unique expansion

$$q=\sum_{(j_1,\ldots,j_k)} v_1^{j_1}\cdots v_k^{j_k},$$

where $j_1,...,j_k$ are non-negative. We associate with $q\in D_k$ the following element of internal degree $\sum_{i=1}^k j_i+1$:

Definition 3.1.

$$\tilde{q}=\sum_{(j_1,\ldots,j_k)} Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes\Sigma^{1-k}1\in B_{k-1}(\Sigma^{1-k}\mathbb{F}_2).$$

Lemma 3.2. *If $q\in D_k$, then*

$$\tilde{q}\in EB_{k-1}(\Sigma^{1-k}\mathbb{F}_2) := \operatorname{Span}\{Sq^ix|\, i>\deg x, x\in B_{k-1}(\Sigma^{1-k}\mathbb{F}_2)\}.$$

Proof. From the definition of the $\mathcal{A}$-action on the bar resolution, one has

$$Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1 = Sq^{j_1+1}(1 \otimes Sq^{j_2+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1).$$

Hence, it suffices to show that

$$j_1 + 1 > (j_2 + 1) + \cdots + (j_k + 1) + (1 - k) = j_2 + \cdots + j_k,$$

for every term in the expansion of $\tilde{q}$.

Recall that $V_i = v_1^{2^{i-2}} v_2^{2^{i-3}} \cdots v_{i-1} v_i$. So, one easily verifies that every element $v \in M_k = \mathbb{F}_2[V_1, ..., V_k]$ is a sum of monomials $v_1^{j_1} \cdots v_k^{j_k}$, which satisfy the condition

$$j_1 \geq j_2 + \cdots + j_k.$$

The lemma follows from the fact that the Dickson algebra D_k is a subalgebra of the Mùi algebra M_k. □

Lemma 3.3. *$\tilde{q}$ is a cycle in the chain complex $EB_*(\Sigma^{1-k}\mathbb{F}_2)$, for every $q \in D_k$.*

This is a consequence of the following lemma, which is actually an exposition of the Adem relations.

Lemma 3.4. *The homomorphism*

$$\begin{aligned} \pi_{k,p} : \Delta_k &\to \mathcal{A}^{k-1} = \mathcal{A} \otimes \cdots \otimes \mathcal{A} \qquad (k-1 \quad \textit{times}) \\ v_1^{j_1} \cdots v_p^{j_p} v_{p+1}^{j_{p+1}} \cdots v_k^{j_k} &\mapsto Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_p+1} Sq^{j_{p+1}+1} \otimes \cdots \otimes Sq^{j_k+1} \end{aligned}$$

vanishes on $\Gamma_k \subset \Delta_k$, for $1 \leq p < k$.

Proof. Consider the diagonal $\psi : \Delta_k \to \Delta_{p-1} \otimes \Delta_2 \otimes \Delta_{k-p-1}$ defined by

$$\psi(v_i) = \begin{cases} v_i \otimes 1 \otimes 1, & i < p, \\ 1 \otimes v_{i-p+1} \otimes 1, & p \leq i \leq p+1, \\ 1 \otimes 1 \otimes v_{i-p-1}, & p+1 < i. \end{cases}$$

From Proposition 2.1 of Singer [20], one gets

$$\psi(\Gamma_k) \subset \Gamma_{p-1} \otimes \Gamma_2 \otimes \Gamma_{k-p-1}.$$

Define the homomorphism $\omega_t : \Gamma_t \to \mathcal{A}^t$ by

$$\omega_t(v_1^{j_1} \cdots v_t^{j_t}) = Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_t+1}.$$

Then one has

$$\pi_{k,p} = (\omega_{p-1} \otimes \pi_{2,1} \otimes \omega_{k-p-1})\psi.$$

By Proposition 3.1 of Singer [20], the Adem relations yield

$$\pi_{2,1}(\Gamma_2) = 0.$$

Hence, $\pi_{k,p}(\Gamma_k) = 0$ for $1 \leq p < k$. The lemma is proved. □

Proof of Lemma 3.3. First, we note that $Sq^{j_k+1}(\Sigma^{1-k}1) = 0$ for any $j_k \geq 0$. Then, by definition of the differential in the bar resolution, we get

$$d_{k-1}(\tilde{q}) = \sum_{p=1}^{k-1} (\pi_{k,p} \otimes id_{\Sigma^{1-k}\mathbb{F}_2})(q \otimes \Sigma^{1-k}1).$$

Since $q \in D_k \subset \Gamma_k$, Lemma 3.4 yields $\pi_{k,p}(q) = 0$. Thus $d_{k-1}(\tilde{q}) = 0$. The lemma is proved. □

For the convenience of the latter use, we define $\tilde{\pi}_{k,p}$ as follows:

$$\tilde{\pi}_{k,p}(Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}) = Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_p+1}Sq^{j_{p+1}+1}\otimes\cdots\otimes Sq^{j_k+1}$$

for $1\le p<k$.

Suppose as before that

$$q=\sum_{J=(j_1,\ldots,j_k)} v_1^{j_1}\cdots v_k^{j_k}\in D_k.$$

For a fixed $(k-s)$-index $(j_{s+1},...,j_k)$, we define $J(j_{s+1},...,j_k)$ to be the set of all s-indices $(j_1,...,j_s)$'s such that $(j_1,...,j_s,j_{s+1},...,j_k)$ occurs as a k-index in the above sum.

The following lemma is a slight generalization of Lemma 3.4.

Lemma 3.5. *If $q=\sum_J v_1^{j_1}\cdots v_k^{j_k}\in D_k$, then*

$$\tilde{\pi}_{s,p}\big(\sum_{J(j_{s+1},\ldots,j_k)} Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_s+1}\big)=0$$

for $1\le p<s\le k$.

Proof. Let us consider the diagonal $\psi_2:\Delta_k\to\Delta_s\otimes\Delta_{k-s}$ given by

$$\psi_2(v_i)=\begin{cases} v_i\otimes 1, & 1\le i\le s,\\ 1\otimes v_{i-s}, & s<i\le k.\end{cases}$$

According to Proposition 2.1 of Singer [20], $\psi(\Gamma_k)\subset\Gamma_s\otimes\Gamma_{k-s}$. Since $q\in D_k\subset\Gamma_k$, it implies $\sum_{J(j_{s+1},\ldots,j_k)} v_1^{j_1}\cdots v_s^{j_s}\in\Gamma_s$. Then, by Lemma 3.4, we have

$$\tilde{\pi}_{s,p}\big(\sum_{J(j_{s+1},\ldots,j_k)} Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_s+1}\big)=\pi_{s,p}\big(\sum_{J(j_{s+1},\ldots,j_k)} v_1^{j_1}\cdots v_s^{j_s}\big)=0.$$

The lemma is proved. □

By definition of the destabilization functor $\mathcal{D}$, for any left $\mathcal{A}$-module M, one has an exact sequence of chain complexes

$$0\to EB_*(M)\xrightarrow{i_E} B_*(M)\xrightarrow{j_{\mathcal{D}}}\mathcal{D}B_*(M)\to 0,$$

in which the bar resolution $B_*(M)$ is exact. Hence, by use of the induced long exact sequence, the connecting homomorphism is an isomorphism

$$\partial_*:\mathcal{D}_k(M):=H_k(\mathcal{D}B_*(M))\xrightarrow{\cong}H_{k-1}(EB_*(M)).$$

Take $M=\Sigma^{1-k}\mathbb{F}_2$. The following lemma deals with the connecting isomorphism

$$\partial_*:\mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2):=H_k(\mathcal{D}B_*(\Sigma^{1-k}\mathbb{F}_2))\xrightarrow{\cong}H_{k-1}(EB_*(\Sigma^{1-k}\mathbb{F}_2)).$$

Let $[\tilde{q}]$ be the homology class of the cycle $\tilde{q}$ in

$$\mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2)\cong H_{k-1}(EB_*(\Sigma^{1-k}\mathbb{F}_2)).$$

Lemma 3.6. *If $q\in D_k$, then*

$$\partial_*[1\otimes\tilde{q}]=[\tilde{q}].$$

Proof. Suppose $q=\sum_J v_1^{j_1}\cdots v_k^{j_k}$. The element $\sum_J 1\otimes Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes\Sigma^{1-k}1\in B_k(\Sigma^{1-k}\mathbb{F}_2)$ is a lifting over $j_{\mathcal{D}}$ of its class modulo $EB_k(\Sigma^{1-k}\mathbb{F}_2)$ in $\mathcal{D}B_k(\Sigma^{1-k}\mathbb{F}_2)$. Let d denote the differential in $B_*(\Sigma^{1-k}\mathbb{F}_2)$, we get

$$\begin{aligned} d(\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1) \\ = \sum_J 1 \cdot Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1 \\ + \sum_{p=1}^{k-1} 1 \otimes \tilde{\pi}_{k,p}(\sum_J Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1}) \otimes \Sigma^{1-k}1 \\ + \sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1}\Sigma^{1-k}1. \end{aligned}$$

By Lemma 3.4

$$\tilde{\pi}_{k,p}(\sum_J Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1}) = \pi_{k,p}(q) = 0.$$

On the other hand, $Sq^{j_k+1}(\Sigma^{1-k}1) = 0$ for any $j_k \geq 0$. Therefore, we obtain

$$\begin{aligned} d(\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1) &= \sum_J Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1 \\ &= i_E(\tilde{q}). \end{aligned}$$

By definition of the connecting homomorphism, we have

$$\partial_*[1 \otimes \tilde{q}] = [\tilde{q}].$$

The lemma is proved. $\square$

The following theorem deals with the isomorphism $\alpha_k : \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \to \Sigma D_k$ treated in Theorem 2.3.

Theorem 3.7. *If $q \in D_k$, then*

$$\alpha_k[\tilde{q}] = \Sigma q.$$

Proof. We compute α_k by means of the following formula

$$\begin{aligned} \alpha_k &= \Delta(e_1(\Sigma\mathbb{F}_2) \otimes P_{k-1}) \circ \cdots \circ \Delta(e_1(\Sigma^{3-k}\mathbb{F}_2) \otimes P_1) \circ \Delta e_1(\Sigma^{2-k}\mathbb{F}_2) \\ &= \delta_k \cdots \delta_2\delta_1. \end{aligned}$$

Here δ_s stands for $\Delta(e_1(\Sigma^{1-k+s}\mathbb{F}_2) \otimes P_{s-1})$, for brevity.

Consider the short exact sequence representing $e_1(\Sigma^{2-k}\mathbb{F}_2)$:

$$0 \to \Sigma^{2-k}P_1 \xrightarrow{\iota} \Sigma^{2-k}\hat{P} \xrightarrow{\pi} \Sigma^{1-k}\mathbb{F}_2 \to 0.$$

Then the connecting homomorphism induced by this exact sequence is nothing but

$$\delta_1 : H_{k-1}(EB_*(\Sigma^{1-k}\mathbb{F}_2)) \to H_{k-2}(EB_*(\Sigma^{2-k}P_1)).$$

A lifting of $\tilde{q} = \sum_J Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1$ over π is

$$\sum_J Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{2-k}x_k^{-1} \in EB_*(\Sigma^{2-k}\hat{P}),$$

where we are writing $P_1 = \mathbb{F}_2[x_k], \hat{P} = \mathrm{Span}\{x_k^i | \, i \geq -1\}$. The boundary of this element in $EB_*(\Sigma^{2-k}\hat{P})$ is pulled back under ι to a cycle in $EB_*(\Sigma^{2-k}P_1)$, which represents $\delta_1[\tilde{q}]$. That means

$$\begin{aligned}
\delta_1[\tilde{q}] &= [d(\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes\Sigma^{2-k}x_k^{-1})] \\
&= \big[\sum_{p=1}^{k-1}\tilde{\pi}_{k,p}(\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1})\otimes\Sigma^{2-k}x_k^{-1} \\
&\quad +\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1}\otimes Sq^{j_k+1}(\Sigma^{2-k}x_k^{-1})\big] \\
&= \big[\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1}\otimes\Sigma^{2-k}Sq^{j_k+1}(x_k^{-1})\big],
\end{aligned}$$

where the last equality follows from Lemma 3.4. Indeed,

$$\tilde{\pi}_{k,p}(\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}) = \pi_{k,p}(q) = 0.$$

Similarly, $\delta_2 : H_{k-2}(EB_*(\Sigma^{2-k}P_1)) \to H_{k-3}(EB_*(\Sigma^{3-k}P_2))$ is the connecting homomorphism induced by the short exact sequence representing $e_1(\Sigma^{3-k}\mathbb{F}_2)\otimes P_1$:

$$0\to\Sigma^{3-k}P_2 \overset{\iota\otimes P_1}{\to} \Sigma^{3-k}(\hat{P}\otimes P_1) \overset{\pi\otimes P_1}{\to} \Sigma^{2-k}P_1\to 0.$$

Here we are writing $P_1=\mathbb{F}_2[x_k], P_2=\mathbb{F}_2[x_{k-1},x_k], \hat{P}=\mathrm{Span}\{x_{k-1}^i | i\geq -1\}$. A lifting of $Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1}\otimes\Sigma^{2-k}Sq^{j_k+1}(x_k^{-1})$ over $\pi\otimes P_1$ is

$$Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1}\otimes\Sigma^{3-k}x_{k-1}^{-1}Sq^{j_k+1}(x_k^{-1}).$$

Therefore, by an argument similar to the one given above, we get

$$\begin{aligned}
\delta_2\delta_1[\tilde{q}] &= [d\big(\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1}\otimes\Sigma^{3-k}x_{k-1}^{-1}Sq^{j_k+1}(x_k^{-1})\big)] \\
&= \big[\sum_{p=1}^{k-2}\sum_J\tilde{\pi}_{k-1,p}(Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-1}+1})\otimes\Sigma^{3-k}x_{k-1}^{-1}Sq^{j_k+1}(x_k^{-1}) \\
&\quad +\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-2}+1}\otimes Sq^{j_{k-1}+1}(\Sigma^{3-k}x_{k-1}^{-1}Sq^{j_k+1}(x_k^{-1}))\big] \\
&= \big[\sum_J Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_{k-2}+1}\otimes\Sigma^{3-k}Sq^{j_{k-1}+1}(x_{k-1}^{-1}Sq^{j_k+1}(x_k^{-1}))\big] \\
&\qquad\qquad \text{(by Lemma 3.5).}
\end{aligned}$$

Repeating the above argument, we then have

$$\begin{aligned}
\alpha_k[\tilde{q}] &= \delta_k\cdots\delta_1[\tilde{q}] \\
&= [\sum_J\big(\Sigma Sq^{j_1+1}(x_1^{-1}Sq^{j_2+1}(x_2^{-1}\cdots Sq^{j_k+1}(x_k^{-1})\cdots))\big)].
\end{aligned}$$

By Theorem 3.2 of our paper [10], we get

$$[\sum_J\big(\Sigma Sq^{j_1+1}(x_1^{-1}Sq^{j_2+1}(x_2^{-1}\cdots Sq^{j_k+1}(x_k^{-1})\cdots))\big)] = [\Sigma q] = \Sigma q.$$

The theorem is proved. □

This theorem has an immediate consequence as follows.

Corollary 3.8. *The homomorphism $D_k \to EB_{k-1}(\Sigma^{1-k}\mathbb{F}_2), q \mapsto \tilde{q}$ is a chain-level representation of the homomorphism*

$$(1 \underset{\mathcal{A}}{\otimes} \alpha_k^{-1})\Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2).$$

Theorem 3.9. *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i \to Tor^{\mathcal{A}}_{k,k+i}(\mathbb{F}_2, \mathbb{F}_2).$$

Proof. Suppose again that

$$q = \sum_{J=(j_1,\ldots,j_k)} v_1^{j_1} \cdots v_k^{j_k} \in D_k.$$

By Corollary 3.8 and Lemma 3.6, we have

$$\begin{array}{rccl} (1 \underset{\mathcal{A}}{\otimes} \alpha_k^{-1})\Sigma : & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k & \to & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \\ & & & \\ & [q] & \mapsto & [\tilde{q}] \overset{\partial_*}{\equiv} [1 \otimes \tilde{q}]. \end{array}$$

From the definition of i_k (see 2.4), we get

$$\begin{array}{rrcl} i_k : & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} H_k(\mathcal{D}B_*(\Sigma^{1-k}\mathbb{F}_2)) & \to & Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2) \\ & [1 \otimes \tilde{q}] & \mapsto & [1 \otimes \tilde{q}]. \end{array}$$

Let us consider the desuspension

$$\Sigma^{-1} : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2) \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2),$$

which sends $[\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1]$ to $[\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{-k}1]$. Then the map

$$\varphi_k^* = \Sigma^{-1} i_k (1 \underset{\mathcal{A}}{\otimes} \alpha_k^{-1})\Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2)$$

is given by

$$\varphi_k^*[q] = [\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{-k}1].$$

The canonical isomorphism

$$\Sigma^k : Tor^{\mathcal{A}}_{k,i}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \to Tor^{\mathcal{A}}_{k,k+i}(\mathbb{F}_2, \mathbb{F}_2)$$

is defined by the chain-level version

$$\Sigma^k(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes \Sigma^{-k}1) = a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes 1.$$

By ambiguity of notation, the composite $\Sigma^k \varphi_k^*$ is also denoted by φ_k^* (see Remark 2.6). Hence

$$\begin{array}{rcl} \varphi_k^* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i & \to & Tor^{\mathcal{A}}_{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \\ [q] & \mapsto & [\sum_J 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes 1]. \end{array}$$

In [18], Priddy constructs the Koszul complex $\overline{K}_*(\mathcal{A})$, a subcomplex of $B_*(\mathbb{F}_2)$, which is isomorphic to the dual of the lambda algebra. More precisely, it is defined as follows. Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [3]. (See Singer [20, p. 687] for a precise definition of Λ.) Then, according to Priddy [18, §7], $\overline{K}_*(\mathcal{A})$ is the image of the monomorphism

$$\begin{array}{rcl}\Lambda^* & \to & B_*(\mathbb{F}_2)\\ (\lambda_{j_1}\cdots\lambda_{j_k})^* & \mapsto & 1\otimes Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes 1,\end{array}$$

which is a homotopy equivalence. Here Λ^* denotes the dual of Λ and the duality $*$ is taken with respect to the basis of admissible monomials of Λ. Combining it with Singer's isomorphism

$$\begin{array}{rcl}\Gamma^\wedge & \to & \Lambda^*\\ v_1^{j_1}\cdots v_k^{j_k} & \mapsto & (\lambda_{j_1}\cdots\lambda_{j_k})^*,\end{array}$$

we get the following homotopy equivalence

$$\begin{array}{rcl}\Gamma^\wedge & \to & B_*(\mathbb{F}_2)\\ v_1^{j_1}\cdots v_k^{j_k} & \mapsto & 1\otimes Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes 1.\end{array}$$

As a consequence, for any $q\in D_k$, we obtain

$$\begin{aligned}\varphi_k^*[q] &= [\sum_J 1\otimes Sq^{j_1+1}\otimes\cdots\otimes Sq^{j_k+1}\otimes 1]\\ &= [\sum_J v_1^{j_1}\cdots v_k^{j_k}]\\ &= [q].\end{aligned}$$

It means that the inclusion $D_k\subset\Gamma_k^\wedge$ is a chain-level representation of φ_k^*. The theorem is completely proved. $\square$

Corollary 3.10. *Conjecture 1.2 is equivalent to Conjecture 1.3.*

This follows immediately from Theorem 3.9.

We have proved Conjecture 1.2 for $k=3$ in [8] and Conjecture 1.3 for $k=3$ in [10].

4. Factoring the Lannes–Zarati homomorphism

The purpose of this section is to prove the following proposition.

Proposition 4.1. *φ_k^* factors through $\mathbb{F}_2\underset{\mathcal{A}}{\otimes}Ker\partial_k$ as shown in the commutative diagram:*

$$\begin{tikzcd}\mathbb{F}_2\underset{\mathcal{A}}{\otimes}D_k \arrow[rr, "\varphi_k^*"] \arrow[dr, "\overline{i}"'] & & Tor_k^{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)\\ & \mathbb{F}_2\underset{\mathcal{A}}{\otimes}Ker\partial_k\,, \arrow[ur, "\overline{p}"'] & \end{tikzcd}$$

where $\overline{i}$ is induced by the inclusion $D_k\subset Ker\partial_k$, and $\overline{p}$ is an epimorphism induced by the canonical projection $p: Ker\partial_k\to H_k(\Gamma^\wedge)\cong Tor_k^{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$.

Proof. The canonical projection

$$p: Ker\partial_k\to Tor_k^{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)=Ker\partial_k/Im\partial_{k+1}$$

sends x to $[x]=x+Im\partial_{k+1}$.

By Theorem 5.15 of Singer [20], the action of $\mathcal{A}$ on $Ker\partial_k$ induces a trivial action of $\mathcal{A}$ upon $Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. Therefore, p induces the epimorphism

$$\begin{aligned} \overline{p} : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k &\to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \\ [x] &\mapsto [x]. \end{aligned}$$

For any $q \in D_k$, we have

$$\overline{p} \cdot \overline{\overline{i}}[q] = \overline{p}[q] = [q] = \varphi_k^*[q].$$

So, we get $\varphi_k^* = \overline{p} \cdot \overline{\overline{i}}$. The proposition is proved. $\square$

In [10], we have stated the following conjecture.

Conjecture 4.2. $D_k^+ \subset \mathcal{A}^+ \cdot Ker\partial_k$ for $k > 2$.

Obviously, this is stronger than Conjectures 1.2 and 1.3 and equivalent to the following one.

Conjecture 4.3. The homomorphism $\overline{\overline{i}} : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$, induced by the inclusion $i : D_k \to Ker\partial_k$, is trivial for $k > 2$.

Based on the above discussion, we believe the following problem is something of interest.

Problem 4.4. Determine $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} Ker\partial_k$.

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Department of Mathematics, Vietnam National University, Hanoi, 334 Nguyên Trãi Street, Hanoi, Vietnam

E-mail address: nhvhung@hotmail.com