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METRIC PROPERTIES OF THE GROUP OF AREA PRESERVING DIFFEOMORPHISMS

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Abstract. Area preserving diffeomorphisms of the 2-disk which are identity near the boundary form a group $\mathcal{D}_2$ which can be equipped, using the L^2-norm on its Lie algebra, with a right invariant metric. With this metric the diameter of $\mathcal{D}_2$ is infinite. In this paper we show that $\mathcal{D}_2$ contains quasi-isometric embeddings of any finitely generated free group and any finitely generated abelian free group.

1. Introduction

Let $\mathcal{D}_2$ be the group of smooth area preserving diffeomorphisms of the 2-disk which are identity near the boundary of the disk . A tangent vector to $\mathcal{D}_2$ at a point ϕ is a divergence free vector field X_ϕ (see [1] and [5]) whose L^2-norm is defined by

$$\|X_\phi\|_2 = (\int_{\mathbf{D}^2} \|X_\phi(x)\|^2 dx)^{1/2},$$

where $\|.\|$ stands for the standard Euclidean norm. Consider a path $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting two maps ϕ_0 and ϕ_1. The length of the path $\{\phi_t\}$ is given by the formula

$$l_2(\{\phi_t\}) = \int_0^1 \|\frac{d\phi_t}{dt}\|_2 dt.$$

Any two maps in $\mathcal{D}_2$ being connected by a path with finite length l_2, induces a distance on $\mathcal{D}_2$ defined by $d_2(\phi_0,\phi_1) = \inf l_2(\{\phi_t\})$, where the infimum is taken over all paths joining ϕ_0 to ϕ_1. Notice that this distance is right invariant:

$$d_2(\phi_0\circ\phi, \phi_1\circ\phi) = d_2(\phi_0,\phi_1).$$

Equipped with this distance, $\mathcal{D}_2$ has an infinite diameter (see [13], [6] and [1]). The group of area preserving diffeomorphisms is important in hydrodynamics; it is the configuration space of an incompressible 2-dimensional fluid (see [1]). The Euler-Lagrange equation associated to the metric d_2 corresponds to the equation of the motion of a perfect incompressible flow (Euler equation).

Finitely generated groups, i.e. groups generated with a finite number of generators obeying a finite number of relations, are another standard class of groups equipped with a right invariant metric. Let $\mathcal{G}$ be such a group, and $\mathcal{S}$ a system of generators of $\mathcal{G}$. The *length* $l_{\mathcal{S}}(g)$ of an element g in $\mathcal{G}$ is the minimal number of

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generators in $\mathcal{S}$ and their inverses needed to write g. It defines a right invariant natural metric on $\mathcal{G}$:

$$d_{\mathcal{S}}(g_1, g_2) = l_{\mathcal{S}}(g_1 g_2^{-1}).$$

The free group $\mathbf{F}_n$ and the abelian free group $\mathbf{Z}^n$ with n generators are classical examples of finitely generated groups.

Let $(\mathcal{G}_1, d_1)$ and $(\mathcal{G}_2, d_2)$ be two metric groups, a morphism $m : \mathcal{G}_1 \to \mathcal{G}_2$ is a *quasi-expanding embedding* if:

- m is injective;
- there exists a pair of positive constants $k, k' > 0$ such that for any pair g_1, g_1' in $\mathcal{G}_1$:

$$d_2(m(g_1), m(g_1')) \geq k d_1(g_1, g_1') - k'.$$

An isomorphism m between two metric groups is a *quasi-isometry* if m and its inverse are quasi-expanding embeddings. A *quasi-isometric embedding* is a morphism which is a quasi-isometry onto its image. We refer the reader to [10] for more details and results concerning quasi-isometries.

In the case of a finitely generated group $\mathcal{G}$ equipped with 2 systems of generators $\mathcal{S}$ and $\mathcal{S}'$, we have for any g in $\mathcal{G}$:

$$\frac{1}{k(\mathcal{S}', \mathcal{S})} l_{\mathcal{S}}(g) \leq l_{\mathcal{S}'}(g) \leq k(\mathcal{S}, \mathcal{S}') l_{\mathcal{S}}(g), \tag{1}$$

where $k(\mathcal{S}, \mathcal{S}')$ stands for the maximum over all the generators s in $\mathcal{S}$ of the length $l_{\mathcal{S}'}(s)$. Thus the identity map is a quasi-isometric isomorphism from $(\mathcal{G}, d_{\mathcal{S}})$ to $(\mathcal{G}, d_{\mathcal{S}'})$ where $d_{\mathcal{S}}$ and $d_{\mathcal{S}'}$ stand respectively for the right invariant metrics induced by $\mathcal{S}$ and $\mathcal{S}'$. It follows that for a morphism of metric groups with values in (or taking values in) a finitely generated group $(\mathcal{G}, d_{\mathcal{S}})$ (where d_S is the right invariant metric induced by a system of generators $\mathcal{S}$) the property of being a quasi-isometric embedding does not depend on the set of generators.

This note is an attempt to understand the anatomy of the group $\mathcal{D}_2$ seen as a metric space. Since closed balls are not compact in $\mathcal{D}_2$, a direct approach using growth of balls seems pointless. Our point of view here is to show the complexity of $\mathcal{D}_2$ by showing that very different finitely generated groups can be embedded by quasi-isometric morphisms in $\mathcal{D}_2$. We prove the following result:

Theorem 1. *Any finitely generated free group and any finitely generated abelian free group is quasi-isometrically embedded in $\mathcal{D}_2$.*

The keys for this theorem rely on the following two facts that we will prove and make more explicit in the sequel:

- any finitely generated free group and any finitely generated abelian free group can be mapped by a quasi-isometric embedding in a braid group;
- the braid subgroups obtained this way can be in turn mapped by a quasi-isometric embedding in the group $\mathcal{D}_2$.

Finally, we conclude with three remarks related to Theorem 1 and its possible generalizations.

2. The free groups and the abelian free groups

For any integer $n > 0$ we denote by $\mathbf{F}_n$ (resp. $\mathbf{Z}^n$) the free group (resp. the abelian free group) with n generators $e_1, \dots, e_n$ and d_n the corresponding right invariant metric. For any pair of integers $2 \leq r \leq s$, there exists a quasi-isometric embedding from the (abelian) free group with r generators to the (abelian) free group with s generators. In the case of the free groups the converse is also true, more precisely we have the following standard result:

Lemma 1. *For any $n \geq 2$, there exists a morphism from the free group with n generators $\mathbf{F}_n$ to the free group with 2 generators $\mathbf{F}_2$ which is a quasi-isometric embedding.*

Proof. For a given n, one can choose an integer r large enough, and a system of n elements in $\mathbf{F}_2$, $g_1, \ldots, g_n$, such that

- these words can be written using only the generators e_1 and e_2 of $\mathbf{F}_2$ (and not their inverses;
- they have a same length equal to $10r$;
- for any pair i, j in $1, \ldots, n$ the word $g_i g_j^{-1}$ has length at least $18r$.

It follows that the morphism $\mu : \mathbf{F}_n \to \mathbf{F}_2$ which associates to each generator e_i of $\mathbf{F}_n$ the element g_i satisfies

$$8rd_n(f_1, f_2) \leq d_2(\mu(f_1), \mu(f_2)) \leq 10rd_n(f_1, f_2),$$

for any pair of elements f_1, f_2 in $\mathbf{F}_n$.

The second inequality shows that μ is an injective morphism; combined with the first one, we get that it is a quasi-isometric embedding. □

3. The braid group and its subgroups

The *Artin Braid group* $\mathbf{B}_n$ is a group given by the set of generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i,$$

for all i, j in $\{1, \ldots, n-1\}$ with $|i-j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all i in $\{1, \ldots, n-2\}$. An element of this group is called *a braid*.

A geometrical way to represent braids (see for instance [3] and [12]) consists in fixing the $2n$ points $P_i = (i/(n-1) - 1/2, 0, 1)$ and $Q_i = (i/(n-1) - 1/2, 0, 0)$ for $i = 0, \ldots, n-1$ in the solid cylinder $\mathbf{D}^2 \times [0, 1]$. A braid β can be seen as the isotopy class of a system of n non-intersecting arcs, joining each point P_i to a point $Q_{\tau_\beta(i)}$, where τ_β is a permutation on $\{0, \ldots, n-1\}$, and such that the intersection of any of these arcs with any disk $\mathbf{D}^2 \times \{t\}$, t in $[0, 1]$, consists in a unique point. With this representation, it is easy to understand what the group law, the generators and the relations mean.

There exists a classical surjective representation ρ of the braid group $\mathbf{B}_3$ in the group $\mathbf{PSL}(2, \mathbf{Z})$ which is defined by

$$\rho(\sigma_1) = s_1 \quad \text{and} \quad \rho(\sigma_2) = s_2,$$

where s_1 and s_2 are respectively the representatives in $\mathbf{PSL}(2, \mathbf{Z})$ of the following two elements in $\mathbf{SL}(2, \mathbf{Z})$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is a representation since the only relation in $\mathbf{B}_3$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

is preserved by ρ:

$$s_1s_2s_1 = s_2s_1s_2.$$

It is surjective since $\mathbf{PSL}(2,\mathbf{Z})$ is generated by the two elements s_1 and s_2. The two elements in $\mathbf{PSL}(2,\mathbf{Z})$,

$$a = s_1s_2 \text{ and } b = s_1s_2s_1,$$

satisfy $a^3 = b^2 = Id$ and generate $\mathbf{PSL}(2,\mathbf{Z})$ which actually is isomorphic to the free product of $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$:

$$\mathbf{PSL}(2,\mathbf{Z}) = \langle a, b;\, a^3, b^2\rangle.$$

The generators s_1 and s_2 can be written using a and b:

$$s_1 = a^{-1}b \text{ and } s_2 = ba^{-1}.$$

In the particular case of the group $\mathbf{PSL}(2,\mathbf{Z})$ equipped with the two systems of generators $\{s_1, s_2\}$ and $\{a, b\}$, equation (1) reads

$$\frac{1}{3}l_{\{s_1,s_2\}} \leq l_{\{a,b\}} \leq 2l_{\{s_1,s_2\}}. \tag{2}$$

The following proposition is a classical result. For the sake of completeness, we give it below followed by a short proof.

Proposition 1. *The morphism m from the free group $\mathbf{F}_2$ to $\mathbf{PSL}(2,\mathbf{Z})$ defined by*

$$m(e_1) = s_1^2 = a^{-1}ba^{-1}b \text{ and } m(e_2) = s_2^2 = ba^{-1}ba^{-1},$$

is a quasi-isometric embedding.

Proof. Consider a word f in $\mathbf{F}_2$, written in a reduced form (i.e. with the smallest possible length) and let f^n be the word in $\mathbf{F}_2$ with length $n < l_{\{e_1,e_2\}}(f)$ which coincides with f truncated after its n^{th} symbol. It is easy to check that

$$l_{\{a,b\}}(m(f^{n+1})) \geq l_{\{a,b\}}(m(f^n)) + 1.$$

It follows that

$$l_{\{a,b\}}(m(f)) \geq l_{\{e_1,e_2\}}(f) - 1 + l_{\{a,b\}}(m(f^1)) = l_{\{e_1,e_2\}}(f) + 3 \geq l_{\{e_1,e_2\}}(f). \tag{3}$$

On the other hand, for any element f in $\mathbf{F}_2$ we have

$$l_{\{a,b\}}(m(f)) \leq 4l_{\{s_1^2,s_2^2\}}(m(f)) = 4l_{\{e_1,e_2\}}(f).$$

This series of inequalities show that the morphism m is a quasi-isometric embedding. □

This quasi-isometric embedding can be easily visualized on the Caley graph of the group $\mathbf{PSL}(2,\mathbf{Z})$ (see this graph in [10]).

Corollary 1. *The morphism $\bar{m}$ from the free group $\mathbf{F}_2$ to $\mathbf{B}_3$ defined by*

$$\bar{m}(e_1) = \sigma_1^2 \text{ and } \bar{m}(e_2) = \sigma_2^2,$$

is a quasi-isometric embedding.

Proof. On one hand, we obviously have

$$l_{\{\sigma_1,\sigma_2\}}(\bar{m}(f)) \,\leq\, 2l_{\{e_1,e_2\}}(f).$$

On the other hand, we get from the relation

$$m \,=\, \rho \circ \bar{m}$$

that for any element f in $\mathbf{F}_2$,

$$l_{\{s_1,s_2\}}(m(f)) \,\leq\, l_{\{\sigma_1,\sigma_2\}}(\bar{m}(f)).$$

Combining Equations (2) and (3) we get

$$\frac{1}{2}l_{\{e_1,e_2\}}(f) \,\leq\, \frac{1}{2}l_{\{a,b\}}(m(f)) \,\leq\, l_{\{s_1,s_2\}}(m(f)).$$

Thus,

$$\frac{1}{2}l_{\{e_1,e_2\}}(f) \,\leq\, l_{\{\sigma_1,\sigma_2\}}(\bar{m}(f)).$$

These inequalities prove the corollary. □

Remark 1. The embedding of the free group $\mathbf{F}_2$ in the braid group $\mathbf{B}_3$ maps $\mathbf{F}_2$ onto the subgroup of $\mathbf{B}_3$ which consists in all equivalence classes of systems of 3 arcs, the first one and the last one being vertical and the second one turning around the other two. This interpretation allows us to identify $\bar{m}(\mathbf{F}_2)$, hence $\mathbf{F}_2$, with the fundamental group of the 2-punctured disk.

From Lemma 1 and Corollary 1 we get

Corollary 2. *Any finitely generated free group can be embedded in $\mathbf{B}_3$ by a quasi-isometric embedding.*

Consider now embeddings of the abelian free groups in the braid groups.

Proposition 2. *For any $n \geq 1$, the morphism $\tilde{m}_n$ from the group $\mathbf{Z}^n$ to the braid group $\mathbf{B}_{2n}$ defined for $i = 1, \ldots, n$ by*

$$\tilde{m}_n(e_i) \,=\, (\sigma_{2i-1})^2,$$

is a quasi-isometric embedding.[1]

Proof. Since two generators σ_i and σ_j of the braid group $\mathbf{B}_n$ commute when then are not consecutive ($|i-j| \neq 1$), the morphism $\tilde{m}_n$ is well defined. Any element z in $\tilde{m}_n(\mathbf{Z}^n)$ can be written in a unique way as the product:

$$z \,=\, \sigma_1^{\alpha_1(z)}\sigma_3^{\alpha_3(z)}\sigma_5^{\alpha_5(z)}\ldots\sigma_{2n-1}^{\alpha_{2n-1}(z)},$$

where the $\alpha_j(z)$'s are even integers.

For $i = 0, \ldots, n-1$, the map $z \to \alpha_i(z)$ is a morphism. By identifying $\mathbf{B}_2$ with $\mathbf{Z}$, this morphism corresponds to the forgetful map: $\tau_i : \mathbf{B}_{2n} \to \mathbf{B}_2$ which consists in forgetting all the strands of a representative of an element in $\mathbf{B}_n$ but the $(2i-1)^{th}$ and $(2i)^{th}$ ones. This yields the equality:

$$l_{\{\sigma_1,\ldots,\sigma_{2n-1}\}}(z) \,=\, \sum_{i=0}^{i=n-1} |\alpha_i(z)|.$$

[1]It is only for technical reasons that will be clear in the next section that we define $\tilde{m}_n(e_i) = (\sigma_{2i-1})^2$ and not $\tilde{m}_n(e_i) = (\sigma_{2i-1})$.

Thus for any f in $\mathbf{Z}^n$,

$$l_{\{\sigma_1,\dots,\sigma_{2n-1}\}}(\tilde{m}_n(f)) \,=\, 2l_{\{e_1,\dots,e_n\}}(z).$$

□

4. Area preserving diffeomorphisms

4.1. **A lower bound for the distance between two maps.** We recall in this paragraph some results that are developed in [9].

Let us fix again an element ϕ in $\mathcal{D}_2$ and an isotopy $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting identity to ϕ. To any pair of distinct points x, y in $\mathbf{D}^2$ and to every $t \in [0,1]$, we associate the unit vector

$$u(t,x,y) \,=\, \frac{\phi_t(y) \,-\, \phi_t(x)}{\|\phi_t(y) \,-\, \phi_t(x)\|}.$$

Lemma 2 ([9])**.** *The integral*

$$\mathcal{G}(\{\phi_t\}) \,=\, \frac{1}{2\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2} \|\frac{du}{dt}(t,x,y)\|dxdydt$$

is well defined.

Proof. An easy calculation yields

$$\|\frac{du}{dt}(t,x,y)\| \,=\, \frac{\|(\dot{\phi}_t(y) \,-\, \dot{\phi}_t(x)) \wedge (\phi_t(y) \,-\, \phi_t(x))\|}{\|\phi_t(y) \,-\, \phi_t(x)\|^2},$$

where $\wedge$ is the wedge product and $\dot{\phi}_t = \frac{\partial\phi_t}{\partial t}$. It follows that

$$\|\frac{du}{dt}(t,x,y)\| \,\le\, \frac{2}{\|\phi_t(y) \,-\, \phi_t(x)\|} \sup_{(t,x)\in[0,1]\times\mathbf{D}^2} \|\dot{\phi}_t(x)\|.$$

The quantity $\sup_{(t,x)\in[0,1]\times\mathbf{D}^2} \|\dot{\phi}_t(x)\|$ is bounded. Hence it is enough to show that since ϕ_t is area preserving for every t in $[0,1]$,

$$\int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{dxdy}{\|\phi_t(y) \,-\, \phi_t(x)\|} \,=\, \int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{dxdy}{\|y \,-\, x\|}$$

and that this last integral converges. □

We now introduce the quantity

$$\mathcal{G}(\phi) \,=\, \inf \mathcal{G}(\{\phi_t\}),$$

where the infimum is taken over all isotopies in $\mathcal{D}_2$ joining identity to ϕ.

Lemma 3 ([9])**.** *There exists a constant $K > 0$ such that for any map ϕ in $\mathcal{D}_2$*

$$\mathcal{G}(\phi) \,\le\, Kl_2(\phi).$$

Proof. The quantity $\mathcal{G}(\{\phi_t\})$ satisfies

$$\mathcal{G}(\{\phi_t\}) \,\le\, \frac{1}{\pi}\int_0^1\int_{\mathbf{D}^2\times\mathbf{D}^2} \frac{\|\dot{\phi}_t(x)\|}{\|\phi_t(y) \,-\, \phi_t(x)\|}dxdydt.$$

The Cauchy-Schwarz inequality gives

$$G(\{\phi_t\}) \,\le\, \frac{1}{\pi}\int_0^1 \|\dot{\phi}_t\|_2\|I_t\|_2 dt,$$

where

$$I_t(x) \;=\; \int_{\mathbf{D}^2} \frac{dy}{\|\phi_t(y) \,-\, \phi_t(x)\|}.$$

For every t in $[0,1]$ the map ϕ_t is area preserving; consequently, the L_2-norm of I_t satisfies

$$\|I_t\|_2 \;=\; (\int_{\mathbf{D}^2} (\int_{\mathbf{D}^2} \frac{dy}{\|x \,-\, y\|})^2 dx)^{1/2} \;<\; +\infty.$$

Thus, for $K = \frac{1}{\pi}\|I_t\|_2$ we get

$$\mathcal{G}(\{\phi_t\}) \;\leq\; Kl_2(\{\phi_t\}).$$

It follows that

$$\mathcal{G}(\phi) \;\leq\; Kl_2(\phi).$$

□

The quantity $\mathcal{G}(\phi)$ can be interpreted as follows. For an isotopy $\{\phi_t\}_{t\in[0,1]}$ connecting identity to ϕ and a pair of distinct points x, y in $\mathbf{D}^2$, we consider the map

$$\begin{array}{rcl} u_{x,y} : [0,1] & \to & \mathbf{S}^1 \\ t & \mapsto & u(t,x,y). \end{array}$$

The change of variables induced by the map $u_{x,y}$ leads to the equality

$$\int_0^1 \|\frac{du_{x,y}}{dt}(t)\|dt \;=\; \int_{\mathbf{S}^1} \sharp\{{u_{x,y}}^{-1}(\omega)\}d\omega,$$

where $\sharp$ stands for cardinality. By integrating we get

$$\mathcal{G}(\{\phi_t\}) \;=\; \frac{1}{2\pi}\int_{\mathbf{D}^2\times\mathbf{D}^2}\int_{\mathbf{S}^1} \sharp\{{u_{x,y}}^{-1}(\omega)\}d\omega dxdy.$$

Consider the 2 arcs $\gamma_x : t \mapsto (\phi_t(x),t)$ and $\gamma_y : t \mapsto (\phi_t(y),t)$ in the cylinder $\mathbf{D}^2 \times [0,1]$ and choose a direction ω in $\mathbf{S}^1$ tangent to $\mathbf{D}^2$. When projected in the direction of ω onto a plane orthogonal to ω, γ_y overcrosses $\sharp\{{u_{x,y}}^{-1}(\omega)\}$ times γ_x. The integral

$$Cr_{\{\phi_t\}}(x,y) \;=\; \frac{1}{2\pi}\int_{\mathbf{S}^1} \sharp\{{u_{x,y}}^{-1}(\omega)\}d\omega$$

is the averaged number of times the arc γ_y overcrosses γ_x over all directions ω tangent to $\mathbf{D}^2$. $\mathcal{G}(\{\phi_t\})$ is then the spatial average $\int_{\mathbf{D}^2\times\mathbf{D}^2} Cr_{\{\phi_t\}}(x,y)dxdy$.

Changing the isotopy $\{\phi_t\}$ may reduce the averaged quantity of overcrossings $\mathcal{G}(\{\phi_t\})$; the infimum is given by $\mathcal{G}(\phi)$.

4.2. **A subgroup of $\mathcal{D}_2$.** For r small enough, consider, for $i = 0,\ldots,n-1$, the n disks D_i in $\mathbf{D}^2$, with radius r and centered at $P_i \;=\; (i/(n-1) - 1/2, 0)$. We define the subgroup $\mathcal{B}_n$ which consists in these maps whose restriction to the disks D_i is identity for $i = 0,\ldots,n-1$. There is a natural morphism μ_n from the group $\mathcal{B}_n$ to the braid group $\mathbf{B}_n$ which is defined as follows:

- choose a map ϕ in $\mathcal{B}_n$ and an isotopy $\{\phi_t\}_{t\in[0,1]}$ from identity to ϕ;

- a representative of the braid $\mu_n(\phi)$ consists in the system of arcs in $\mathbf{D}^2 \times [0,1]$:

$$\bigcup_{i=0}^{i=n-1} \{(\phi_t(P_i), t), t \in [0,1]\}.$$

Lemma 4. *Let ϕ be a map in $\mathcal{B}_n$. There exists a quantity $C(r) \geq 0$ which depends only on r and goes to zero with r such that*

$$l_2(\phi) \geq \frac{1}{K} l_{\{\sigma_1,\ldots,\sigma_{n-1}\}}(\mu_n(\phi))(1 - C(r)) \times (area\, D_0)^2.$$

Proof. Let $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ be an isotopy connecting identity to ϕ. The integral $\mathcal{G}(\{\phi_t\})$ reads

$$\mathcal{G}(\{\phi_t\}) = \int_{\mathbf{D}^2\times\mathbf{D}^2} Cr_{\{\phi_t\}}(x,y)dxdy,$$

and consequently

$$\mathcal{G}(\{\phi_t\}) \geq \sum_{i\neq j} \int_{D_i\times D_j} Cr_{\{\phi_t\}}(x,y)dxdy = \int_{D_0\times D_0} (\sum_{i\neq j} Cr_{\{\phi_t\}}(T_i(x), T_j(y)))dxdy,$$

where T_i stands for the translation which maps D_0 onto D_i.

For $(x,y,\omega) \in D_0 \times D_0 \times \mathbf{S}^1$, let $\mathcal{U}_\phi(x,y,\omega)$ be the minimum over all isotopies in $\mathcal{D}_2$ connecting identity to ϕ of the quantity

$$\sum_{i\neq j} \sharp\{u^{-1}_{T_i(x),T_j(y)}(\omega)\},$$

We have

$$\mathcal{G}(\phi) \geq \frac{1}{2\pi} \int_{\mathcal{S}_1} \int_{D_0\times D_0} \mathcal{U}_\phi(x,y,\omega)d\omega dxdy.$$

Let $M(r)$ be the set of ω's for which the orthogonal projections (in the direction ω) of the disks D_i overlap. We denote by $2\pi C(r)$ the measure of the set $M(r)$. One can easily check that $C(r)$ goes to zero with r. It is also clear that for ω in $\mathcal{S}^1 \setminus M(r)$,

$$\mathcal{U}_\phi(x,y,\omega) \geq l_{\{\sigma_1,\ldots,\sigma_{n-1}\}}(\mu_n(\phi)).$$

By integrating over $\mathcal{S}^1 \setminus M(r)$ we get

$$\mathcal{G}(\phi) \geq l_{\{\sigma_1,\ldots,\sigma_{n-1}\}}(\mu_n(\phi))(1 - C(r)) \times (area\, D_0)^2.$$

□

5. Proof of Theorem 1

Consider in $\mathcal{B}_3$ two maps Θ_1 and Θ_2 such that

$$\mu_3(\Theta_1) = \sigma_1^2 \text{ and } \mu_3(\Theta_2) = \sigma_2^2.$$

The morphism $\pi : \mathbf{F}_2 \to \mathcal{D}_2$ defined by

$$\pi(e_1) = \Theta_1 \text{ and } \pi(e_2) = \Theta_2,$$

is injective since $\mu_3 \circ \pi = \bar{m}$ is injective. For any f in $\mathbf{F}_2$, it also satisfies

$$\begin{aligned} l_2(\pi(f)) &\geq \frac{1}{K} l_{\{\sigma_1,\sigma_2\}}\mu_3(\pi(f))(1 - C(r)) \times (area\, D_0)^2 \\ &\geq \frac{1}{2K} l_{\{e_1,e_2\}}(f)(1 - C(r)) \times (area\, D_0)^2; \end{aligned}$$

the left inequality coming from Lemma 4, the right one from the second inequality in the proof of Corollary 1.

On the other hand, for

$$f = e_{\epsilon_1}^{\alpha_1} e_{\epsilon_2}^{\alpha_2} \dots e_{\epsilon_n}^{\alpha_n},$$

where $\epsilon_i = 1$ or 2, we have

$$l_2(\pi(f)) \leq \sum_{i=1}^{n} |\alpha_i| l_2(\pi(e_{\epsilon_i})) \leq l_{e_1,e_2}(f) \times \max_{i \in \{1,2\}} l_2(\pi(e_i)).$$

Using Lemma 1, we also get a quasi-isometric embedding of $\mathbf{F}_n$ in $\mathcal{D}_2$ for any $n \geq 2$. This proves Theorem 1 for finitely generated free groups.

Consider now n maps $\Omega_1, \dots, \Omega_n$ in $\mathcal{B}_{2n}$ such that

- $\mu_{2n}(\Omega_i) = \sigma_{2i-1}^2$ for $i = 1, \dots, n$;
- the maps Ω_i are different from identity on disjoint supports, thus they commute.

Since the maps Ω_i commute, the morphism $\tilde{\pi} : \mathbf{Z}^n \to \mathcal{D}_2$ defined for $i = 1, \dots, n$, by

$$\tilde{\pi}(e_i) = \Omega_i$$

is well defined. It is injective since $\mu_{2n} \circ \tilde{\pi} = \tilde{m}_n$ is injective. For any f in $\mathbf{Z}^n$, it also satisfies

$$\begin{aligned} l_2(\tilde{\pi}(f)) &\geq \frac{1}{K} l_{\{\sigma_1,\dots,\sigma_{2n-1}\}} \mu_{2n}(\tilde{\pi}(f))(1 - C(r)) \times (area\, D_0)^2 \\ &= \frac{2}{K} l_{\{e_1,\dots,e_n\}}(f)(1 - C(r)) \times (area\, D_0)^2; \end{aligned}$$

the left inequality coming from Lemma 4, the right one from the last inequality in the proof of Proposition 2.

On the other hand, given

$$f = e_1^{\alpha_1} e_2^{\alpha_2} \dots e_n^{\alpha_n},$$

we have

$$l_2(\tilde{\pi}(f)) \leq \sum_{i=1}^{n} |\alpha_i| l_2(\tilde{\pi}(e_i)) \leq l_{e_1,\dots,e_n}(f) \times \max_{i \in \{1,\dots,n\}} l_2(\tilde{\pi}(e_i)).$$

Thus for any $n \geq 1$, we get a quasi-isometric embedding of $\mathbf{Z}^n$ in $\mathcal{D}_2$. This achieves the proof of Theorem 1.

6. Final remarks

6.1. **Random walks on $\mathcal{D}_2$.** Let G be a group equipped with a right invariant metric d. Consider the Borel σ-field of G, $\mathcal{B}(G)$, and let $\mu : \mathcal{B}(G) \mapsto [0,1]$ be a Borel probability measure on G. We can associate to μ the random walk on $\mathcal{D}_2$ $X = \{X^n\}_{n \in \mathbf{Z}^+}$ defined by

$$X^0 = Id,$$

$$X^n = g_n . g_{n-1} \dots g_1 \quad \text{for } n \geq 1,$$

where $\{g_i\}_{i \in \mathbf{Z}^+}$ is a sequence of independent random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and identically distributed according to μ.

For any random variable $X : \Omega \mapsto \mathbf{R}$ and $p \geq 1$ we let $||X||_p = (\int_\Omega |X|^p dP)^{1/p}$. $L^p(P)$ is the set of random variables X for which $||X||_p < \infty$. For $X \in L^1(P)$ we let $E(X) = \int_\Omega X dP$ denote the mathematical expectation of X.

Proposition 6.1. *Suppose* $E(d(X^1, Id)) = \int_{\mathbf{R}} d(g, Id)\mu(dg) < \infty$, *and let*

$$\gamma_1 = \inf_{n \in \mathbf{Z}^+} \frac{1}{n} E(d(X^n, Id)).$$

Then

$$\lim_{n\to\infty} \frac{l_2(d(X^n, Id))}{n} = \gamma_1$$

P-almost surely.

Proof. The proof is standard. We can always suppose that $(\Omega, \mathcal{F}, P)$ is the canonical space where $\Omega = G^{\mathbf{Z}^+}, \mathcal{F} = \mathcal{B}(G^{\mathbf{Z}^+})$ and $P = \mu^{\otimes \mathbf{Z}^+}$ is the product measure. Let $\Theta : \Omega \mapsto \Omega$ denote the shift defined by $\Theta(\mathbf{g})_i = g_{i+1}$ where $\mathbf{g} = \{g_i\}_{i \in \mathbf{Z}^+}$ and let $\tau_n(\mathbf{g}) = d(g_n.g_{n-1} \dots g_1, Id)$. The right invariance of d implies that

$$\begin{aligned}\tau_{n+m}(\mathbf{g}) &= d(g_{n+m}.\dots g_1, Id)\\ &= d(g_{n+m} \dots g_{n+1}, (g_n.\dots g_1)^{-1})\\ &\leq \tau_n(\mathbf{g}) + \tau_m(\Theta^n(\mathbf{g})).\end{aligned}$$

Hence Proposition 6.1 follows from Kingman's subadditive Ergodic Theorem [11]. □

The quantity γ_1 is called *the linear escape rate* of the random walk.

Examples.

- Consider the free group $\mathbf{F}_d$ with generators $\mathcal{S} = \{e_1, \dots, e_d\}$ and equipped with the right invariant metric induced by the word length. Charge the generators and their inverse with an equidistributed measure ($\mu(e_i) = \mu(e_i^{-1}) = \frac{1}{d}$) and consider the random walk on this group which consists in starting form an element chosen at random in $\mathcal{S} \cup \mathcal{S}^{-1}$ and mutiplying at each step on the left by an element chosen at random in $\mathcal{S} \cup \mathcal{S}^{-1}$. It is easy to check that the linear rate of escape in this particular case satifies
$$\gamma_1 = 1 - \frac{1}{d}.$$
- For a similar random walk on the abelian free group $\mathbf{Z}^d$ it is also known that the linear escape rate vanishes ($\gamma_1 = 0$).
- From Theorem 1 we get that for any positive integer d, there exist d maps $\Theta_1, \dots, \Theta_d$ in $\mathcal{D}_2$ such that the group generated by $\Theta_1, \dots, \Theta_d$ is quasi-isometric to the free group. It implies that the associated random walk on this group has a strictly positive linear escape rate.
- From Theorem 1 we also get that for any positive integer d, there exist d maps $\Omega_1, \dots, \Omega_d$ in $\mathcal{D}_2$ such that the group generated by $\Omega_1, \dots, \Omega_d$ is quasi-isometric to the abelian free group $\mathbf{Z}^d$. It implies that the associated random walk on this group has a linear escape rate which vanishes.

Question 1. For any $d > 1$ and for any generic choice of maps $\phi_1, \dots, \phi_d$ in $\mathcal{D}_2$, consider the subgroup $\mathcal{D}_2(\phi_1, \dots, \phi_d)$ of $\mathcal{D}_2$ generated by $\{\phi_1, \dots, \phi_d\}$. Is this subgroup a quasi-isometric embedding of the free group $\mathbf{F}_d$? More generally, does the random walk which gives an equal probability to each generator and their inverse, have a non-zero linear rate of escape?

6.2. **Embeddings in the kernel of the Calabi morphism.** Consider a map ϕ in $\mathcal{D}_2$ and a 1-form α on $\mathbf{D}^2$ which is a primitive of the area 2-form. Since ϕ is area preserving, the form $\phi^\star\alpha - \alpha$ is closed and vanishes near the boundary of $\mathbf{D}^2$. Thus, there exists a unique function $H(\phi,\alpha) : \mathbf{D}^2 \to \mathbf{R}$, which vanishes near the boundary of $\mathbf{D}^2$ and such that

$$dH(\phi,\alpha) \,=\, \phi^\star\alpha \,-\, \alpha.$$

The Calabi invariant [4] of ϕ is defined by

$$\mathcal{C}(\phi) \,=\, \int_{\mathbf{D}^2} H(\phi,\alpha).$$

It is easy to verify that this integral does not depend on the choice of the primitive α and that the Calabi invariant is actually a morphism from $\mathcal{D}_2$ to $\mathbf{R}$:

$$\mathcal{C}(\phi_0 \circ \phi_1) \,=\, \mathcal{C}(\phi_0) \,+\, \mathcal{C}(\phi_1).$$

A. Banyaga [2] proved that the kernel of this morphism, $ker\,\mathcal{C}$, is a simple group and Y. Eliashberg and T. Ratiu [6] showed that its subgroup has an infinite diameter (see also [9]).

There is a second definition of the Calabi invariant which is due to A. Fathi [7] (see also [8]) and which can be seen as an estimate in terms of braiding of pairs of orbits. Consider an isotopy $\{\phi_t\}_{t\in[0,1]}$ in $\mathcal{D}_2$ connecting identity to ϕ. The map

$$Ang_\phi : \mathbf{D}^2 \times \mathbf{D}^2 \setminus \Delta \to \mathbf{R},$$

(where Δ stands for the diagonal) which associates to any pair of points $x \neq y$ in $\mathbf{D}^2$ the angular variation of the vector $\overrightarrow{\phi_t(x)\phi_t(y)}$ when t goes from 0 to 1, does not depend on the choice of the isotopy and is bounded where it is defined. The Calabi invariant is the integral of this function, i.e.:

$$\mathcal{C}(\phi) \,=\, \int_{\mathbf{D}^2\times\mathbf{D}^2} Ang_\phi(x,y)dxdy.$$

With this second definition, $Ker\,\mathcal{C}$ is the subgroup of area preserving maps whose angular variation of a pair of points vanishes in average. Using this second geometric interpretation, Theorem 1 can be made more precise:

Theorem 2. *Any finitely generated free group and any finitely generated abelian free group is quasi-isometrically embedded in* $Ker\,\mathcal{C}$.

We only give here the main lines of the proof and insist on its differences with the proof of Theorem 1.

Consider the morphism $m' : \mathbf{F}_2 \to \mathbf{B}_5$ which maps the generator e_1 on $\sigma_1^2\sigma_3^{-2}$ and e_2 on $\sigma_2^2\sigma_4^{-2}$. Using the same technics as the ones we used in the proof of Theorem 1 we get that m' is a quasi-isometric embedding. It is possible to find two maps in $\mathcal{B}_5$, Θ_1' and Θ_2' such that

- $\mu_6(\Theta_1') = \sigma_1^2\sigma_3^{-2}$ and $\mu_6(\Theta_2') = \sigma_2^2\sigma_4^{-2}$;
- Θ_1' and Θ_2' are in $Ker\,\mathcal{C}$.

Similarly to the proof of Theorem 1, we prove that the morphism $\pi' : \mathbf{F}_2 \to Ker\,\mathcal{C}$ defined by $\pi'(e_1) = \Theta_1'$ and $\pi'(e_2) = \Theta_2'$ is a quasi-isometric embedding. Using Corollary 1, we get quasi-isometric embeddings of any finitely generated free group in $Ker\,\mathcal{C}$. A similar construction works for finitely generated abelian free groups.

6.3. **Symplectomorphisms in higher dimensions.** Let $n > 0$ and consider the group $\mathcal{D}_{2n}$ of diffeomorphisms of the unit ball in the $2n$-dimensional space B^{2n} which preserve the standard symplectic 2-form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$, and are identity near the boundary ∂B^{2n}. This group, like $\mathcal{D}_2$, can be equipped with a right invariant metric. With this metric, $\mathcal{D}_{2n}$ also has an infinite diameter [6].

Question 2. Does a quasi-isometric embedding of any finitely generated free group and any finitely generated abelian free group in $\mathcal{D}_{2n}$ exist for all $n > 0$?

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