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REPRESENTATION THEORY AND ADHM-CONSTRUCTION ON QUATERNION SYMMETRIC SPACES

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ABSTRACT. We determine all irreducible homogeneous bundles with anti-self-dual canonical connections on compact quaternion symmetric spaces. To deform the canonical connections, we give a relation between the representation theory and the theory of monads on the twistor space. The moduli spaces are described via the Bott-Borel-Weil Thereom. The Horrocks bundle is also generalized to higher-dimensional projective spaces.

1. INTRODUCTION

It is known that the concept of anti-self-duality exists on a quaternion-Kähler manifold ([M-S] and [Ni]), and is interpreted as holomorphicity on the Salamon twistor space. In short, constructing a bundle with an anti-self-dual connection on a quaternion-Kähler manifold is equivalent to finding a holomorphic vector bundle with reality condition on its twistor space. This relation has as its origin the well-known equivalence used in the classification problem of anti-self-dual bundles over half-conformally-flat 4-manifolds. Atiyah-Drinfeld-Hitchin-Manin succeeded in classifying instantons on the 4-dimensional sphere, which is the so called ADHM-construction ([A]). In a similar way, instantons on the complex projective plane were classified by Buchdahl ([B]). To classify the corresponding holomorphic bundles, they used Horrocks' monad methods (see for example [O-S-S]) on the Penrose twistor space. This monad method also has the advantage of describing the moduli space, in particular, of 1-instantons.

The 4-dimensional sphere S^4 and the projective plane $\mathbb{P}^2$ are the only compact positive 1-dimensional quaternion-Kähler manifolds with twistor spaces with positive Einstein-Kähler metrics ([H]). Salamon showed that every higher-dimensional quaternion-Kähler manifold with positive scalar curvature has a twistor space with a positive Einstein-Kähler metric ([S]). *All* known examples of complete quaternion-Kähler manifolds with positive scalar curvature are symmetric spaces. Wolf showed that there is a one-to-one correspondence between simple Lie algebras and compact quaternion symmetric spaces ([W]). From this point of view, the quaternion projective space $\mathbb{H}P^n$ and the 2-plane complex Grassmannian $Gr_2(\mathbb{C}^{n+2})$ are considered as direct generalizations of S^4 and $\mathbb{P}^2$, corresponding to the respective Lie algebras C_{n+1} and A_{n+1}.

It is easy to find anti-self-dual bundles on $\mathbb{H}P^n$ and $Gr_2(\mathbb{C}^{n+2})$. In particular, Mamone Capria and Salamon gave examples of k-instantons on $\mathbb{H}P^n$ ([M-S]), which

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are also of interest for algebraic geometers. The pull-back bundle of a *k-instanton* is called a *mathematical instanton bundle with quantum number k* ([O-S]), and there are many excellent results about them (for example, see [S-T], [A-O] and [O-T]). The classification of k-instantons is an important problem. The author and Kametani established vanishing theorems for k-instantons ([Na-2] and [K-N]) which, combined with Beilinson's spectral sequence, yield the classification of k-instantons by monads, which is a direct generalization of the ADHM-classification. In particular, a monad for 1-instantons is

$$\mathcal{O}(-1) \longrightarrow V \longrightarrow \mathcal{O}(1),$$

on $\mathbb{P}^{2n+1}$, where V is a trivial bundle of rank $2n+2$. A similar argument can be applied on $Gr_2(\mathbb{C}^{n+2})$, and we can consider a natural extension of a k-instanton on $\mathbb{P}^2$. It can be shown that every k-instanton is expressed as the cohomology of an appropriate monad ([N-N1] and [N-N2]). As in the case of $\mathbb{H}P^n$, a monad for 1-instantons is described as

$$\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \longrightarrow V \longrightarrow \mathcal{O}(1,0) \oplus \mathcal{O}(0,1),$$

on the twistor space of $Gr_2(\mathbb{C}^{n+2})$, where V is a trivial bundle of rank $2n+4$. Aside from these direct generalizations, Mamone Capria and Salamon ([M-S]) showed that the well-known Horrocks bundle ([Ho]) on $\mathbb{P}^5$ can be obtained as the pull-back of an anti-self-dual bundle on $\mathbb{H}P^2$. As another example, they mentioned a homogeneous anti-self-dual bundle over $G_2/SO(4)$, which will be treated in this paper. These are the only known concrete examples of anti-self-dual bundles on higher-dimensional quaternion-Kähler manifolds.

In the present paper, we determine irreducible homogeneous bundles with anti-self-dual canonical connections on compact quaternion symmetric spaces (Theorem 3.4). This result assures us that *every* compact quaternion symmetric space has an anti-self-dual bundle (Corollary 3.5). Moreover, making use of monads, we will deform these connections. The usual method to derive monads is to use vanishing theorems and spectral sequences (for example, see [O-S-S]). Although there exist general vanishing theorems for anti-self-dual bundles ([Na-2] and [N-N1]), these are insufficient to make the spectral sequence converge rapidly in higher dimensions ([M-S], [K-N] and [N-N2]). Therefore it is natural to seek out a new way to obtain monads. In fact, we give a relationship between representation theory and monad constructions (Theorem 4.2 and Lemma 4.3), that systematically yields monads for which the cohomology bundles are the pull-backs of anti-self-dual bundles (Theorem 4.5). These monads include the above two examples on $\mathbb{H}P^n$ and $Gr_2(\mathbb{C}^{n+2})$ (see Example 4.6), and the cohomology bundles are considered as generalizations of 1-instantons to the other quaternion symmetric spaces. Applying the Bott-Borel-Weil (BBW) theorem, we describe moduli spaces which are the sets consisting of isomorphism classes of cohomology bundles of the monads. (We refer to [K] for the BBW-theorem. In [K], the proof of the BBW-theorem relies on the Peter-Weyl theorem, and so we can explicitly obtain an identification between the cohomology groups and the representation spaces.) The moduli space is identified with either (1) an open ball in $\mathbb{R}^N$ (Theorem 5.10, *e.g.* the moduli of 1-instantons on $\mathbb{H}P^n$ [D-O] and [M-S]) or (2) an open cone over a complex projective space (Theorem 6.9, *e.g.* the moduli of 1-instantons on $Gr_2(\mathbb{C}^{n+2})$ [N-N3]). These moduli may be regarded as the space parametrizing deformations of canonical connections. As a by-product, we give a description of the moduli space of *nullcorrelation bundles* on $\mathbb{P}^{2n+1}$ from the group theoretical point of view (Theorem 5.5). (The moduli of nullcorrelation bundles have already obtained by Spindler [Sp].) The Horrocks bundle is also generalized to higher-dimensional projective spaces (Theorem 7.1), and we obtain analogues of the Horrocks bundle that are pull-backs of anti-self-dual bundles over higher-dimensional quaternion projective spaces.

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2. Preliminaries

Let M be a connected quaternion-Kähler manifold with non-zero scalar curvature, and let Z be the twistor space of M ([S]).

2.1. Self-dual and anti-self-dual connections. The vector bundle $\wedge^2 T^*M$ has the following holonomy invariant decomposition:

$$\wedge^2 T^*M = S^2\mathbb{H} \oplus S^2\mathbb{E} \oplus (S^2\mathbb{H} \oplus S^2\mathbb{E})^{\perp}, \tag{2.1}$$

where $\mathbb{H}$ and $\mathbb{E}$ are vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. For example, $\mathbb{H}$ is a tautological quaternion line bundle when the base space is a quaternion projective space $\mathbb{H}P^n$.

Definition 2.1.1. An $\omega \in \Omega^2 T^*M$ is called a *self-dual* (resp. *anti-self-dual*) form if

$$\omega \in \Gamma(S^2\mathbb{H}) \quad (\text{resp. } \Gamma(S^2\mathbb{E})).$$

We note that the above definition coincides with that of self-duality or anti-self-duality on a 4-dimensional oriented Riemannian manifold in the case $n = 1$. We shall investigate metric connections on a complex vector bundle E equipped with a hermitian metric h throughout this paper.

Definition 2.1.2. A connection is called (*anti-*) *self-dual* if its curvature 2-form is an (anti-) self-dual form.

Remark. As we mentioned in the introduction, Mamone Capria and Salamon, and also Nitta, have defined these (anti-) self-dual connections independently ([M-S] and [Ni]). On the other hand, Galicki and Poon [G-P] call self-dual and anti-self-dual connections c_1-self-dual and c_2-self-dual connections, respectively.

Theorem 2.1.3 ([M-S], [G-P] and [Ni])**.** *Self-dual and anti-self-dual connections are Yang-Mills connections.*

Remark. Moreover, if M is compact, self-dual and anti-self-dual connections minimize the Yang-Mills functional ([M-S] and [G-P]). It is known that we have an essentially unique non-flat self-dual connection over a simply connected quaternion-Kähler manifold whose dimension is greater than or equal to 8 ([Na-1]).

It is known that on the twistor space Z, the pull-back of an anti-self-dual bundle has a holomorphic structure with the induced connection ([M-S] and [Ni]). The twistor space has the real structure σ which is induced by the quaternion structure

([S]). As in the 4-dimensional case, we have the well-known Ward correspondence (see, for example [A-H-S]);

Ward correspondence. *There is a one-to-one correspondence between anti-self-dual bundles with unitary structures on a quaternion-Kähler manifold M and holomorphic vector bundles E on the twistor space such that*

1. *the restricted bundle $E|_{\mathbb{P}^1_x}$ to the twistor fibre $\mathbb{P}^1_x$ is trivial for every $x \in M$, and*
2. *there is an isomorphism $\tau : E \longrightarrow \sigma^*\overline{E}^*$ with $(\sigma^*\bar{\tau})^* = \tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbb{P}^1_x}$ for all $x \in M$.*

2.2. Quaternion symmetric spaces. We give a quick review of the relation between simple Lie algebras and compact symmetric spaces, mostly in order to fix our notation (for details in 2.2, see [W]).

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex simple Lie algebra and B the Killing form on $\mathfrak{g}^{\mathbb{C}}$. We fix a Cartan subalgebra and choose simple roots. The highest root is denoted by θ. Then the Lie subalgebra $\mathfrak{sp}(1)$ in the compact real form $\mathfrak{g}$ is generated by the highest root vector. We define a Lie subalgebra $\mathfrak{k}$ in $\mathfrak{g}$ as the centralizer of $\mathfrak{sp}(1)$. For brevity, $\mathfrak{sp}(1) \oplus \mathfrak{k}$ is denoted by $\mathfrak{k}_4$. Let G be the associated simply connected compact Lie group with $\mathfrak{g}$ as Lie algebra. The subgroup of G corresponding to $\mathfrak{k}_4$ is denoted by K_4. Then G/K_4 is a compact quaternion symmetric space. Conversely, a compact quaternion symmetric space is necessarily obtained in this way.

Next we describe the twistor space of G/K_4. We take a Lie subalgebra $\mathfrak{u}(1)$ in $\mathfrak{sp}(1)$ which consists of constant multiples of h_θ in the Cartan subalgebra which corresponds to θ under the identification by the Killing form B. Let $\mathfrak{k}_Z$ be a Lie subalgebra $\mathfrak{u}(1) \oplus \mathfrak{k}$ and K_Z the corresponding Lie subgroup in G. The homogeneous space G/K_Z is the twistor space of G/K_4. Since the twistor space G/K_Z is a compact simply connected homogeneous Kähler manifold, we can also express the twistor space using a complex simply connected Lie group $G^{\mathbb{C}}$ which has $\mathfrak{g}^{\mathbb{C}}$ as Lie algebra. Then the twistor space is denoted by $G^{\mathbb{C}}/P$, where P is the corresponding parabolic subgroup of $G^{\mathbb{C}}$.

Finally we take homogeneous holomorphic bundles on $G^{\mathbb{C}}/P$ into account. Let I be the set of integral weights for $\mathfrak{g}^{\mathbb{C}}$. For an integral weight λ in I, $E(\lambda)$ is defined as the irreducible representation space of $\mathfrak{g}^{\mathbb{C}}$ with $-\lambda$ as an extremal weight. (Hence, if λ is dominant, $E(\lambda)$ is the dual of the irreducible representation space with the highest weight λ.) When we consider a representation space for the parabolic subalgebra $\mathfrak{p}$ whose exponential group in $G^{\mathbb{C}}$ is P, we denote by $E_{\mathfrak{p}}(\lambda)$ the irreducible representation space of $\mathfrak{p}$ with $-\lambda$ as an extremal weight. With this notation, the homogeneous holomorphic bundle $G^{\mathbb{C}} \times_P E_{\mathfrak{p}}(\lambda)$ is denoted by $\mathcal{O}_{\mathfrak{p}}(\lambda)$. (Since we consider only integral weights for $\mathfrak{g}^{\mathbb{C}}$, the parabolic subgroup P also acts on $E_{\mathfrak{p}}(\lambda)$.)

Remark. Our notation for representation spaces and holomorphic homogeneous bundles follows that of [B-E].

Remark. As mentioned after Definition 2.1.1, we treat complex vector bundles. Hence, we take account of only *complex* representations in this paper.

Example 2.2.1. We take a Lie algebra of type C_{n+1}. Then the homogeneous space $G^{\mathbb{C}}/P$ is a complex projective space $\mathbb{P}^{2n+1}$. Let ϖ_1 be one of the fundamental weights. (We number the fundamental weights as in Bourbaki ([Bo]) throughout this paper.) The vector bundle $\mathcal{O}_{\mathfrak{p}}(\varpi_1)$ is nothing but the hyperplane section bundle $\mathcal{O}(1)$. In general, for an integer k, $\mathcal{O}_{\mathfrak{p}}(k\varpi_1)$ is a line bundle $\mathcal{O}(k)$ on $\mathbb{P}^{2n+1}$.

Remark. We do not distinguish between holomorphic vector bundles and locally free sheaves on the twistor space.

3. Homogeneous anti-self-dual bundles

We introduce a function f which plays a key role in this paper.

Definition 3.1. We define a function $f : I \longrightarrow \mathbb{Z}$ by

$$f(\lambda) = B(\lambda, \theta^\vee),$$

for an integral weight λ, where $\theta^\vee$ is the co-root of the highest root θ ($\theta^\vee = 2\theta/B(\theta,\theta)$).

Remark. For each complex simple Lie algebra, we write down the function f explicitly below:

$$A_n : f(\sum_{i=1}^{n} p_i\varpi_i) = \sum p_i, \quad B_n : f(\sum_{i=1}^{n} p_i\varpi_i) = p_1 + 2\sum p_i + p_n,$$

$$C_n : f(\sum_{i=1}^{n} p_i\varpi_i) = \sum p_i, \quad D_n : f(\sum_{i=1}^{n} p_i\varpi_i) = p_1 + 2\sum p_i + p_{n-1} + p_n,$$

$$E_6 : f(\sum_{i=1}^{6} p_i\varpi_i) = p_1 + 2p_2 + 2p_3 + 3p_4 + 2p_5 + p_6,$$

$$E_7 : f(\sum_{i=1}^{7} p_i\varpi_i) = 2p_1 + 2p_2 + 3p_3 + 4p_4 + 3p_5 + 2p_6 + p_7,$$

$$E_8 : f(\sum_{i=1}^{8} p_i\varpi_i) = 2p_1 + 3p_2 + 4p_3 + 6p_4 + 5p_5 + 4p_6 + 3p_7 + 2p_8,$$

$$F_4 : f(\sum_{i=1}^{4} p_i\varpi_i) = 2p_1 + 3p_2 + 2p_3 + p_4, \quad G_2 : f(p_1\varpi_1 + p_2\varpi_2) = p_1 + 2p_2,$$

where the ϖ_i are the fundamental weights and each p_i is an integer.

The function $f : I \longrightarrow \mathbb{Z}$ has the following two properties.

Lemma 3.2. *For any integral weight λ, denote by $\Pi\,(E_{\mathfrak{p}}(\lambda))$ the set of all weights of $E_{\mathfrak{p}}(\lambda)$. If μ_1 and μ_2 are in $\Pi\,(E_{\mathfrak{p}}(\lambda))$, then $f(\mu_1) = f(\mu_2)$.*

Proof. Since $E_{\mathfrak{p}}(\lambda)$ is an irreducible representation space of $\mathfrak{p}$, we need only take account of the reductive part $\mathfrak{k}_Z^{\mathbb{C}}$ of the Levi decomposition of $\mathfrak{p}$, where $\mathfrak{k}_Z^{\mathbb{C}}$ is the complexification of $\mathfrak{k}_Z$. From the definition of $\mathfrak{p}$, if α is a root of the semisimple part of $\mathfrak{k}_Z^{\mathbb{C}}$, we have $B(\alpha, \theta) = 0$. So the structure theory of the representation of $\mathfrak{p}$ yields that $f(\lambda) = f(\mu_1)$ and $f(\lambda) = f(\mu_2)$. □

Lemma 3.3. *If a homogeneous holomorphic bundle $\mathcal{O}_{\mathfrak{p}}(\lambda)$ is restricted to the twistor fibre $\mathbb{P}^1_x$, then we have the identification on $\mathbb{P}^1_x$:*

$$\mathcal{O}_{\mathfrak{p}}(\lambda)|_{\mathbb{P}^1_x} \cong \mathcal{O}\left(f(\lambda)\right)^{\oplus rank\, \mathcal{O}_{\mathfrak{p}}(\lambda)}.$$

Proof. By homogeneity, it suffices to consider the origin o of $G^{\mathbb{C}}/P$, which corresponds to the unit element in $G^{\mathbb{C}}$. The theorem of Grothendieck (see, for example, [O-S-S, Theorem 2.1.1, p. 22] implies that the restricted bundle $\mathcal{O}_{\mathfrak{p}}(\lambda)|_{\mathbb{P}^1_o}$ is decomposed into a direct sum of line bundles. To find the first Chern classes of these line bundles, we may inspect the action of $h_{\theta^\vee}$ in $\mathfrak{u}(1)$ on $E_{\mathfrak{p}}(\lambda)$, where $h_{\theta^\vee}$ is the dual element of $\theta^\vee$ under the Killing form B. To do this, we compute the value of $\lambda(h_{\theta^\vee})$. This value is nothing but $f(\lambda)$. Then, by Lemma 3.2, $h_{\theta^\vee}$ acts on $E_{\mathfrak{p}}(\lambda)$ as scalar multiplication by $-f(\lambda)$. □

Theorem 3.4. *Let E be an irreducible homogeneous bundle over G/K_4 whose canonical connection is anti-self-dual. Then, there exists an integral weight λ with $f(\lambda) = 0$ such that $\mathcal{O}_{\mathfrak{p}}(\lambda)$ is the pull-back bundle of E on the twistor space G/K_Z. Conversely, if an integral weight λ satisfies $f(\lambda) = 0$, an irreducible homogeneous holomorphic bundle $\mathcal{O}_{\mathfrak{p}}(\lambda)$ on G/K_Z is the pull-back of an anti-self-dual homogeneous bundle on G/K_4.*

Proof. Since K_Z is a subgroup of K_4, the pull-back bundle F of E is also a homogeneous bundle on G/K_Z. The anti-self-duality of the canonical connection of E implies that F has a holomorphic structure with respect to the induced canonical connection. Therefore we may express F as $\bigoplus_i \mathcal{O}_{\mathfrak{p}}(\lambda_i)$ using integral weights λ_i. Then we must have $f(\lambda_i) = 0$ by Lemma 3.3, because F is the pull-back bundle. As in the proof of Lemma 3.3, this means that the action of $h_{\theta^\vee}$ is trivial. Hence, the classical representation theory of $\mathfrak{sp}(1)$ implies that $\mathfrak{sp}(1)$ in the Lie algebra $\mathfrak{k}_4$ also acts trivially on $E_{\mathfrak{p}}(\lambda_i)$. Consequently, by irreducibility of E, $i = 1$ and so F is $\mathcal{O}_{\mathfrak{p}}(\lambda)$ for an appropriate integral weight λ with $f(\lambda) = 0$.

Conversely, the assumption about λ yields that $h_{\theta^\vee}$ acts trivially on $E_{\mathfrak{p}}(\lambda)$. So we get the irreducible homogeneous bundle E over G/K_4 with a connection whose pull-back is $\mathcal{O}_{\mathfrak{p}}(\lambda)$. Since the pull-back of E is holomorphic with the induced connection, the connection of E is anti-self-dual. □

Combined with the classification of compact quaternion symmetric spaces by Wolf ([W]), Theorem 3.4 implies

Corollary 3.5. *Every compact quaternion symmetric space has an anti-self-dual bundle.*

4. Representation theory and monad construction

Let W be the Weyl group of $\mathfrak{g}^{\mathbb{C}}$, and w^0 the longest element of W.

Lemma 4.1. *If λ in I is dominant, then the image of $f : \Pi(E(\lambda)) \longrightarrow \mathbb{Z}$ is $\{-f(\lambda), -f(\lambda)+1, \cdots, -1, 0, 1, \cdots, f(\lambda)-1, f(\lambda)\}$.*

Proof. From the hypothesis, $-\lambda$ is the lowest weight of $E(\lambda)$ and $w^0(-\lambda)$ is the highest weight. Since the Weyl group preserves the Killing form and the inverse element of w^0 is itself, we have $f(-w^0\lambda) = B(-w^0\lambda, \theta^\vee) = B(-\lambda, w^0\theta^\vee) = B(\lambda, \theta^\vee) = f(\lambda)$. On the other hand, if α is a simple root of $\mathfrak{g}^{\mathbb{C}}$, a direct computation shows that $f(\alpha) = 0$ or 1. Consequently, a familiar argument about root strings implies the desired result. □

From now on, we focus attention on an integral dominant weight λ which satisfies $f(\lambda) = 1$.

Theorem 4.2. *For an integral dominant weight λ, the following two conditions are equivalent:*

1. $f(\lambda) = 1$.
2. *There exist two $\mathfrak{p}$-equivariant maps*

$$i : E_{\mathfrak{p}}(w^0\lambda) \longrightarrow E(\lambda) \quad \textit{and} \quad \pi : E(\lambda) \longrightarrow E_{\mathfrak{p}}(\lambda),$$

where i is injective, π is surjective and $\pi \circ i = 0$. Moreover, $Ker\,\pi/Im\,i$ is isomorphic to $\bigoplus_i E_{\mathfrak{p}}(\mu_i)$ as $\mathfrak{p}$-modules for some integral weights μ_i such that $f(\mu_i) = 0$.

Proof. For an integral dominant weight λ such that $f(\lambda) = 1$, we restrict an action on $E(\lambda)$ of $\mathfrak{g}$ to the subalgebra $\mathfrak{p}$. As in the proof of Lemma 4.1, $-\lambda$ is the lowest weight and $w^0(-\lambda)$ is the highest weight of $E(\lambda)$. On the other hand, an irreducible representation of $\mathfrak{p}$ corresponds to an irreducible representation of the reductive part $\mathfrak{k}_Z^{\mathbb{C}}$ of $\mathfrak{p}$. From these, we have two $\mathfrak{p}$-equivariant maps

$$i : E_{\mathfrak{p}}(w^0\lambda) \longrightarrow E(\lambda) \quad \text{and} \quad \pi : E(\lambda) \longrightarrow E_{\mathfrak{p}}(\lambda),$$

where i is injective and π is surjective. Using Lemma 3.2, we have the composite $\pi \circ i = 0$.

Then $\operatorname{Ker}\pi$ and $\operatorname{Coker} i$ are $\mathfrak{p}$-modules, because i and π are $\mathfrak{p}$-equivariant, and so $\operatorname{Ker}\pi/\operatorname{Im} i$ is also a $\mathfrak{p}$-module.

First, instead of $\mathfrak{p}$-modules, we regard $\operatorname{Ker}\pi/\operatorname{Im} i$ and $E(\lambda)$ as $\mathfrak{k}_Z$-modules. Since $\operatorname{Ker}\pi/\operatorname{Im} i$ and $E(\lambda)$ are finite dimensional, they are completely reducible as $\mathfrak{k}_Z$-modules. Now $E(\lambda)$ is assumed to be decomposed into $E_{\mathfrak{k}_Z}(w^0\lambda) \oplus \bigoplus_i E_{\mathfrak{k}_Z}(\mu_i) \oplus E_{\mathfrak{k}_Z}(\lambda)$. ($E_{\mathfrak{k}_Z}(\lambda)$ is as usual the irreducible representation space of $\mathfrak{k}_Z$ with $-\lambda$ as an extremal weight.) An argument about root strings for the highest weight $w^0(-\lambda)$ and the proof of Lemma 3.2 show that a weight vector in $E(\lambda)$ whose weight is mapped into 1 by f necessarily belongs to $E_{\mathfrak{k}_Z}(w^0\lambda)$. In the same way, each weight vector whose weight is mapped into -1 by f necessarily belongs to $E_{\mathfrak{k}_Z}(\lambda)$. Consequently, $\operatorname{Ker}\pi/\operatorname{Im} i$ is identified with $\bigoplus_i E_{\mathfrak{k}_Z}(\mu_i)$ such that $f(\mu_i) = 0$ as a $\mathfrak{k}_Z$-module. If X is a root vector in the nilpotent part of $\mathfrak{p}$ with a root α, we have $f(\alpha) > 0$ from the construction of Wolf (see §2.2). Hence, using Lemma 3.2, we see that the nilpotent part of $\mathfrak{p}$ acts trivially on $\operatorname{Ker}\pi/\operatorname{Im} i$. Therefore, as a $\mathfrak{p}$-module, $\operatorname{Ker}\pi/\operatorname{Im} i$ is isomorphic to $\bigoplus_i E_{\mathfrak{k}_Z}(\mu_i) = \bigoplus_i E_{\mathfrak{p}}(\mu_i)$.

Conversely, assume condition (2). Then, $E(\lambda)$ has weights μ such that $f(\mu) = f(\lambda)$, $f(w^0\lambda)$ or 0. Since λ is dominant, $f(\lambda)$ is positive. If $f(\lambda) \geqq 2$, $E(\lambda)$ has a weight ν such that $f(\nu) = 1$ by Lemma 4.1. Lemma 3.2 implies that ν is not a weight of $E_{\mathfrak{p}}(\lambda)$ or $E_{\mathfrak{p}}(w^0\lambda)$. Then $\operatorname{Ker}\pi/\operatorname{Im} i$ also has ν as weight. Using Lemma 3.2 again, we see that this is a contradiction. $\square$

Remark. From the remark after Definition 3.1, the following are integral dominant weights μ satisfying $f(\mu) = 1$ for each complex simple Lie algebra:

$$\begin{aligned} &A_n : \varpi_1, \cdots, \varpi_n, \quad B_n : \varpi_1, \varpi_n, \quad C_n : \varpi_1, \cdots, \varpi_n, \quad D_n : \varpi_1, \varpi_{n-1}, \varpi_n, \\ &E_6 : \varpi_1, \varpi_6, \quad E_7 : \varpi_7, \quad E_8 : \text{nothing}, \quad F_4 : \varpi_4, \quad G_2 : \varpi_1, \end{aligned}$$

where ϖ_i is the i-th fundamental weight. (As stated in Example 2.2.1, the numbering of the fundamental weights is adopted from Bourbaki ([Bo]).) For each ϖ_i

such that $f(\varpi_i) = 1$, the irreducible $\mathfrak{k}_Z$-decomposition of $E(\varpi_i)$ is as follows:

$A_n : E(\varpi_i) \cong E_{\mathfrak{k}_Z}(\varpi_i) \oplus E_{\mathfrak{k}_Z}(-\varpi_1 + \varpi_{i+1}) \oplus E_{\mathfrak{k}_Z}(\varpi_{i-1} - \varpi_n) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_i)$.

$B_n : E(\varpi_1) \cong E_{\mathfrak{k}_Z}(\varpi_1) \oplus E_{\mathfrak{k}_Z}(-\varpi_2 + \varpi_3) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_1)$

(where $E(\varpi_1) \cong E_{\mathfrak{k}_Z}(\varpi_1) \oplus E_{\mathfrak{k}_Z}(-\varpi_2 + 2\varpi_3) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_1)$, if $n = 3$).

$E(\varpi_n) \cong E_{\mathfrak{k}_Z}(\varpi_n) \oplus E_{\mathfrak{k}_Z}(\varpi_1 - \varpi_2 + \varpi_n) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_n)$.

$C_n : E(\varpi_i) \cong E_{\mathfrak{k}_Z}(\varpi_i) \oplus E_{\mathfrak{k}_Z}(-\varpi_1 + \varpi_{i-1}) \oplus E_{\mathfrak{k}_Z}(-\varpi_1 + \varpi_{i+1}) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_i)$.

$D_n : E(\varpi_1) \cong E_{\mathfrak{k}_Z}(\varpi_1) \oplus E_{\mathfrak{k}_Z}(-\varpi_2 + \varpi_3) \oplus E_{\mathfrak{k}_Z}(-\varpi_1)$.

$E(\varpi_{n-1}) \cong E_{\mathfrak{k}_Z}(\varpi_{n-1}) \oplus E_{\mathfrak{k}_Z}(\varpi_1 - \varpi_2 + \varpi_n) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_{n-1})$.

$E(\varpi_n) \cong E_{\mathfrak{k}_Z}(\varpi_n) \oplus E_{\mathfrak{k}_Z}(\varpi_1 - \varpi_2 + \varpi_{n-1}) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_n)$.

$E_6 : E(\varpi_1) \cong E_{\mathfrak{k}_Z}(\varpi_1) \oplus E_{\mathfrak{k}_Z}(-\varpi_2 + \varpi_5) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_1)$.

$E(\varpi_6) \cong E_{\mathfrak{k}_Z}(\varpi_6) \oplus E_{\mathfrak{k}_Z}(-\varpi_2 + \varpi_3) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_6)$.

$E_7 : E(\varpi_7) \cong E_{\mathfrak{k}_Z}(\varpi_7) \oplus E_{\mathfrak{k}_Z}(-\varpi_1 + \varpi_2) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_7)$.

$F_4 : E(\varpi_4) \cong E_{\mathfrak{k}_Z}(\varpi_4) \oplus E_{\mathfrak{k}_Z}(-\varpi_1 + \varpi_3) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_4)$.

$G_2 : E(\varpi_1) \cong E_{\mathfrak{k}_Z}(\varpi_1) \oplus E_{\mathfrak{k}_Z}(2\varpi_1 - \varpi_2) \oplus E_{\mathfrak{k}_Z}(w^0\varpi_1)$.

In general, if λ is an integral dominant weight, we have in a similar way

$$E_{\mathfrak{p}}(w^0\lambda) \xrightarrow{i} E(\lambda) \xrightarrow{\pi} E_{\mathfrak{p}}(\lambda),$$

where i and π are a $\mathfrak{p}$-equivariant monomorphism and epimorphism respectively, and $\pi \circ i = 0$. Then we call this complex a *monad of representations*.

Lemma 4.3. *A monad of representations for an integral dominant λ induces a monad of holomorphic vector bundles on $G^{\mathbb{C}}/P$.*

Proof. With the above notation, we can define

$$\alpha_0 : \mathcal{O}_{\mathfrak{p}}(w^0\lambda) \longrightarrow G^{\mathbb{C}}/P \times E(\lambda) \quad \text{and} \quad \beta_0 : G^{\mathbb{C}}/P \times E(\lambda) \longrightarrow \mathcal{O}_{\mathfrak{p}}(\lambda),$$

such that

$$\alpha_0\big([g, e]\big) = \big([g], gi(e)\big) \quad \text{and} \quad \beta_0\big(([g], u)\big) = \big[g, \pi(g^{-1}u)\big],$$

where g, e, u are elements of $G^{\mathbb{C}}, E_{\mathfrak{p}}(w^0\lambda), E(\lambda)$, respectively. (For brevity, a trivial bundle $G^{\mathbb{C}}/P \times E(\lambda)$ is also denoted simply by $E(\lambda)$.) The properties of i and π imply that

$$\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \xrightarrow{\alpha_0} E(\lambda) \xrightarrow{\beta_0} \mathcal{O}_{\mathfrak{p}}(\lambda)$$

is a monad of vector bundles. □

Definition 4.4. A monad of vector bundles on $G^{\mathbb{C}}/P$ obtained in Lemma 4.3 is called *the standard monad induced by λ*.

Remark. The word "standard " in the definition means that the monomorphism α_0 and the epimorphism β_0 are specified. In the fifth and sixth sections, we take a *not* standard induced monad into account. In other words, we deform bundle homomorphisms to obtain various anti-self-dual connections.

Theorem 4.5. *For an integral dominant weight λ, the following two conditions are equivalent:*

1. $f(\lambda) = 1$.
2. *The cohomology bundle of the standard monad induced by λ is the pull-back of an anti-self-dual bundle on G/K_4.*

Proof. The cohomology bundle of the standard monad is also homogeneous. Then, Theorems 3.4 and 4.2 (and the argument in the proof of the latter) imply the desired equivalence. □

Example 4.6. We give two examples.

1. The fundamental weight ϖ_1 of C_{n+1} induces

$$\mathcal{O}_{\mathfrak{p}}(w^0\varpi_1) \xrightarrow{\alpha_0} E(\varpi_1) \xrightarrow{\beta_0} \mathcal{O}_{\mathfrak{p}}(\varpi_1).$$

2. The fundamental weights ϖ_1 and ϖ_{n+1} of A_{n+1} induce

$$\mathcal{O}_{\mathfrak{p}}(w^0\varpi_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\varpi_{n+1}) \xrightarrow{\alpha_0} E(\varpi_1) \oplus E(\varpi_{n+1}) \xrightarrow{\beta_0} \mathcal{O}_{\mathfrak{p}}(\varpi_1) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_{n+1}).$$

These are the monads in the introduction whose cohomology bundle are the pull-back of 1-instantons on $\mathbb{H}P^n$ and $Gr_2(\mathbb{C}^{n+2})$. The reason for taking a direct sum in (2) is explained in the next section.

Remark. Even if an integral dominant weight λ satisfies $f(\lambda) \geqq 2$, it is possible to construct a monad whose cohomology bundle is the pull-back of an anti-self-dual bundle. In this case, using Lemmas 3.2 and 4.1, we need at least $f(\lambda)$ monads to obtain the pull-back of an anti-self-dual bundle, because representations of K_Z whose weights, say μ's, satisfy $f(\mu) > 0$ must be expelled from the representation $E(\lambda)$. For example, making use of a weight $2\varpi_1$ of C_3, we obtain

$$\mathcal{O}_{\mathfrak{p}}(w^0\varpi_2) \longrightarrow F \longrightarrow \mathcal{O}_{\mathfrak{p}}(\varpi_2),$$

where F is the cohomology bundle of the standard monad induced by $2\varpi_1$. If we denote by $\mathbb{E}$ the homogeneous nullcorrelation bundle on $\mathbb{P}^5$, the above monad is

$$\mathbb{E}(-1) \longrightarrow F \longrightarrow \mathbb{E}(1),$$

and the cohomology bundle is $S^2\mathbb{E} \oplus \mathcal{O}$, where $S^2\mathbb{E}$ is the symmetric tensor bundle of $\mathbb{E}$.

5. Moduli spaces I

Our reason for the use of monad methods is to obtain deformations of anti-self-dual connections or holomorphic structures. Even if we deform the bundle homomorphisms in a monad, it could happen that the holomorphic structure of the cohomology bundles does not change. In the case of a monad induced by an integral dominant weight, there is a possibility that all the cohomology bundles are homogeneous. Such an example is the monad induced by ϖ_1 of A_n: $\mathcal{O}_{\mathfrak{p}}(w^0\varpi_1) \xrightarrow{\alpha_0} E(\varpi_1) \xrightarrow{\beta_0} \mathcal{O}_{\mathfrak{p}}(\varpi_1)$, and the cohomology bundle is $\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2)$. In fact, the Bott-Borel-Weil (BBW) theorem implies that $H^1(Z, \mathrm{End}\,(\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2))) = 0$, where Z is the twistor space and $\mathrm{End}\,(\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2))$ is the endomorphism bundle of $\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2)$. However, in the same way, we see that

$$H^1(Z, \mathrm{End}\,(\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_{n-1} - \varpi_n)))$$

does not vanish, where $\mathcal{O}_{\mathfrak{p}}(\varpi_{n-1} - \varpi_n)$ is the cohomology bundle of the standard monad induced by ϖ_n. This gives one reason for taking a direct sum in Example 4.6(2). Hence, we consider only induced monads for which the endomorphism bundle of the standard cohomology bundle (*i.e.* $\operatorname{Ker}\beta_0/\operatorname{Im}\alpha_0$) has non-vanishing first cohomology. Since we already have good examples for A_n and C_n, we consider an integral weight λ for the Lie algebras B, D, E, F and G satisfying $f(\lambda) = 1$. The BBW theorem yields

$$\begin{aligned}
B_n :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3))) = 0,\\
&(\text{if}\, n = 3, H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + 2\varpi_3))) = 0),\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_n))) \cong E(\varpi_1),\\
D_n :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3))) = 0,\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_n))) = 0,\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_{n-1}))) = 0,\\
E_6 :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_5))) = 0,\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3))) = 0,\\
E_7 :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_2))) = 0,\\
F_4 :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_3))) \cong E(\varpi_4),\\
G_2 :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(2\varpi_1 - \varpi_2))) \cong E(\varpi_1).
\end{aligned}$$

Therefore, to find interesting monads in the case of D_n and E_6, we take a direct sum in a similar way to Example 4.6 (2). Indeed, from the BBW theorem we have

$$\begin{aligned}
D_n :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_n) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_{n-1}))) \cong 2E(\varpi_1),\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_n)))\\
&\qquad \cong \begin{cases} 2E(\varpi_n), & n \text{ even},\\ E(\varpi_{n-1}) \oplus E(\varpi_n), & n \text{ odd},\end{cases}\\
&H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_1 - \varpi_2 + \varpi_{n-1})))\\
&\qquad \cong \begin{cases} 2E(\varpi_{n-1}), & n \text{ even},\\ E(\varpi_{n-1}) \oplus E(\varpi_n), & n \text{ odd},\end{cases}\\
E_6 :& H^1(\operatorname{End}(\mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_5) \oplus \mathcal{O}_{\mathfrak{p}}(-\varpi_2 + \varpi_3))) \cong E(\varpi_1) \oplus E(\varpi_6).
\end{aligned}$$

In this way, we consider the monad induced by

$$\begin{aligned}
&A_n : \varpi_1 \oplus \varpi_n, \quad B_n : \varpi_n, \quad C_n : \varpi_1,\\
&D_n : \varpi_{n-1} \oplus \varpi_n,\, \varpi_1 \oplus \varpi_{n-1},\, \varpi_1 \oplus \varpi_n,\\
&E_6 : \varpi_1 \oplus \varpi_6, \quad F_4 : \varpi_4, \quad G_2 : \varpi_1,
\end{aligned}$$

where the monad induced by $\varpi_i \oplus \varpi_j$ means the one which is a direct sum of the monads induced by ϖ_i and ϖ_j. (For A_n and C_n, these monads induce 1-instantons and so the moduli are already known ([D-O], [M-S], [K-N] and [N-N3]).) We call these monad M_A, M_B, M_C, M_{D_1}, M_{D_2}, M_{D_3}, M_E, M_F and M_G, respectively. Moreover, the monads M_B, M_C, M_F, M_G are said to be of *simple type*, and the monads M_A, M_{D_1}, M_{D_2}, M_{D_3}, M_E of *direct sum type*.

First of all, we take the monads of simple type M_B, M_C, M_F and M_G into account in this section. In general, these monads are expressed as

$$M_s \,:\, \mathcal{O}_{\mathfrak{p}}(w^0\lambda) \xrightarrow{\alpha} E(\lambda) \xrightarrow{\beta} \mathcal{O}_{\mathfrak{p}}(\lambda),$$

where λ is an integral dominant weight satisfying $f(\lambda) = 1$.

First, the homomorphisms α and β are explicitly described.

Lemma 5.1. *For homomorphisms α and β in M_s, there exist linear endomorphisms A and B of $E(\lambda)$ such that*

$$\alpha\big([g,e]\big) = \big([g], Agi(e)\big), \quad \beta\big(([g],u)\big) = \big[g, \pi(g^{-1}Bu)\big],$$

where g, e and u are elements of G, $E_{\mathfrak{p}}(w^0\lambda)$ and $E(\lambda)$, respectively.

Proof. A homomorphism α is an element of $H^0(\mathrm{End}\,(\mathcal{O}_{\mathfrak{p}}(w^0\lambda), E(\lambda))) \cong E(\lambda) \otimes H^0(\mathcal{O}_{\mathfrak{p}}(w^0\lambda)^*)$. The BBW theorem implies $H^0(\mathcal{O}_{\mathfrak{p}}(w^0\lambda)^*) \cong E(-w^0\lambda) \cong E(-\lambda)$. Consequently, α can be regarded as an element of $\mathrm{End}\,(E(\lambda))$. We can explicitly write down this identification (*i.e.* the BBW theorem) by the method of Kostant [K]. As for a homomorphism β, we obtain $H^0(\mathrm{End}\,(E(\lambda), \mathcal{O}_{\mathfrak{p}}(\lambda))) \cong E(\lambda)^* \otimes H^0(\mathcal{O}_{\mathfrak{p}}(\lambda)) \cong E(\lambda)^* \otimes E(\lambda)$. □

Remark. From now on, we assume that G is a *compact* Lie group.

Since M_s is a monad, α is injective, β is surjective and $\beta \circ \alpha = 0$. Making use of Lemma 5.1, these are equivalent to the conditions that A and B are automorphisms of $E(\lambda)$ and $\pi \circ g^{-1}BAg \circ i : E_{\mathfrak{p}}(w^0\lambda) \longrightarrow E_{\mathfrak{p}}(\lambda)$ is a 0-map for all g in G. The representation space $\mathrm{End}\,(E(\lambda))$ of G is assumed to be decomposed into $\mathrm{End}\,(E(\lambda)) \cong \bigoplus_j E(\nu_j)$, where $E(\nu_j)$ is an irreducible representation space of G and ν_j is a dominant weight. Hence, from our assumption, ν_j satisfies $0 \leqq f(\nu_j) \leqq 2$. According to this decomposition, BA is expressed as $BA = \sum_j BA_j$.

Proposition 5.2. *With this notation, $\beta \circ \alpha = 0$ if and only if $BA_j = 0$ for every j such that $f(\nu_j) = 2$.*

Proof. For each weight η of $\mathrm{End}\,(E(\lambda))$ and μ of $E(\lambda)$, we put U_η and e_μ as the corresponding (non-zero) weight vectors, respectively. It is known that if $U_\eta e_\mu$ is a non-zero vector in $E(\lambda)$, then $\eta+\mu$ is a weight of $E(\lambda)$. Let $\{e_\mu\}$ be a weight basis of $E(\lambda)$ and h_λ a G-invariant hermitian inner product on $E(\lambda)$. The hermitian inner product h_λ induces a G-invariant hermitian inner product h_{End} on $\mathrm{End}\,(E(\lambda))$ and so h_{ν_j} on $E(\nu_j)$. Then we can regard the decomposition $\mathrm{End}\,(E(\lambda)) \cong \bigoplus_j E(\nu_j)$ as orthogonal.

Now $f(\nu_j)$ is assumed to equal 2, and so we can take a weight vector U_{η_j} in $E(\nu_j)$ such that $f(\eta_j) = -2$. When $U_{\eta_j}e_\mu$ is non-zero, we see that $f(\eta_j + \mu) = -1, 0$ or 1 from Lemma 4.1. Since $f(\mu) = -1, 0$ or 1, $f(\mu)$ must be 1 and $f(\eta_j + \mu) = -1$. If $U_{\eta_j}e_\mu = 0$ for all e_μ such that $f(\mu) = 1$, we obtain $U_{\eta_j} = 0$, and this is a contradiction. Consequently, there exists a weight μ of $E(\lambda)$ such that $f(\mu) = 1$ and $U_{\eta_j}e_\mu$ is non-zero. Then, it follows from Theorem 4.2 and the definition of h_{End} that $\pi \circ g^{-1}BAg \circ i : E_{\mathfrak{p}}(w^0\lambda) \longrightarrow E_{\mathfrak{p}}(\lambda)$ is a 0-map for all g in G if and only if

$$h_{\mathrm{End}}(g^{-1}BAg,\, U_\eta) = h_{\mathrm{End}}(BA,\, g \cdot U_\eta) = 0$$

for all g in G and all weights η of $\mathrm{End}\,(E(\lambda))$ such that $f(\eta) = -2$. Since a weight η satisfying $f(\eta) = -2$ is necessarily a weight of $E(\nu_j)$ such that $f(\nu_j) = 2$ by Lemma 4.1, the above condition is equivalent to

$$h_{\nu_j}(BA_j,\, g \cdot U_{\eta_j}) = 0$$

for all g in G, all j satisfying $f(\nu_j) = 2$ and all weights η_j of $E(\nu_j)$ such that $f(\eta_j) = -2$. Then the irreducibility of $E(\nu_j)$ yields the desired result.

It is clear from our argument using the function f that the converse holds. $\square$

For brevity, the conditions

1. A and B are automorphisms of $E(\lambda)$,
2. $BA_j = 0$ for all j such that $f(\nu_j) = 2$

are called *monad conditions*.

In general, isomorphism classes of monads do not correspond to isomorphism classes of the cohomology bundles. However, Okonek, Schneider and Spindler gave a sufficient condition for the existence of a one-to-one correspondence between them ([O-S-S], pp. 276-281). In short,

Lemma 5.3 ([O-S-S]). *Let $A \longrightarrow B \longrightarrow C$ be a monad. If*

$$H^0(B^* \otimes A) = H^1(B^* \otimes A) = 0,$$
$$H^0(C^* \otimes B) = H^1(C^* \otimes B) = 0,$$
$$H^1(C^* \otimes A) = H^2(C^* \otimes A) = 0,$$

then there exists a bijection between isomorphism classes of monads and isomorphism classes of the cohomology bundles.

From the BBW theorem, a direct computation shows that the monads M_B, M_C, M_F and M_G satisfy the above Okonek-Schneider-Spindler condition. (In our case, note that the highest weights of $B^* \otimes A$, $C^* \otimes B$ and $C^* \otimes A$ are singular weights in the sense of [B-E, p. 39].) We therefore describe the isomorphism classes of monads.

Proposition 5.4. *Monads M_{s_1} and M_{s_2} are isomorphic to each other (in other words, the following diagram is commutative):*

$$\begin{array}{ccccc} M_{s_1} : \mathcal{O}_{\mathfrak{p}}(w^0\lambda) & \xrightarrow{\alpha_1} & E(\lambda) & \xrightarrow{\beta_1} & \mathcal{O}_{\mathfrak{p}}(\lambda) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle C} & & \big\downarrow{\scriptstyle q} \\ M_{s_2} : \mathcal{O}_{\mathfrak{p}}(w^0\lambda) & \xrightarrow[\alpha_2]{} & E(\lambda) & \xrightarrow[\beta_2]{} & \mathcal{O}_{\mathfrak{p}}(\lambda), \end{array}$$

where C is an automorphism of $E(\lambda)$, p and q are automorphisms of $\mathcal{O}_{\mathfrak{p}}(w^0\lambda)$ and $\mathcal{O}_{\mathfrak{p}}(\lambda)$ respectively) if and only if there exists a non-zero constant c such that $B_2A_2 = cB_1A_1$ under the identification $\alpha_i \sim A_i$ and $\beta_i \sim B_i$ of Lemma 5.1.

Proof. Since $\mathcal{O}_{\mathfrak{p}}(w^0\lambda)$ and $\mathcal{O}_{\mathfrak{p}}(\lambda)$ are Einstein-Hermitian bundles over a compact Kähler manifold, they are simple bundles. Consequently, p and q can be regarded as non-zero constants. The commutative diagram shows that

$$([g],\, CA_1ge) = ([g],\, pA_2ge) \quad \text{and} \quad \left[g,\, q\pi(g^{-1}B_1u)\right] = \left[g,\, \pi(g^{-1}B_2Cu)\right],$$

and the irreducibility of $E(\lambda)$ implies that $CA_1 = pA_2$ and $qB_1 = B_2C$. Hence $B_2A_2 = p^{-1}qB_1A_1$.

Conversely, if $B_2A_2 = cB_1A_1$, we may put $C = A_2A_1^{-1} = cB_1B_2^{-1}$, $p = 1$ and $q = c$. $\square$

We refer to the set consisting of the isomorphism classes of the cohomology bundles as the *complex moduli space*. From the monad conditions, the Okonek-Sneider-Spindler condition and Proposition 5.4, we obtain

Theorem 5.5. *Let $\mathcal{M}_B^{\mathbb{C}}$, $\mathcal{M}_C^{\mathbb{C}}$, $\mathcal{M}_F^{\mathbb{C}}$, $\mathcal{M}_G^{\mathbb{C}}$ be complex moduli spaces of the cohomology bundles of monads M_B, M_C, M_F, M_G respectively. Then we have identifications*

$$\begin{aligned}
\mathcal{M}_B^{\mathbb{C}} &\cong \{[C] \in \mathbb{P}(E(\varpi_1) \oplus \mathbb{C}) \,|\, \det C \neq 0\},\\
\mathcal{M}_C^{\mathbb{C}} &\cong \{[C] \in \mathbb{P}(E(\varpi_2) \oplus \mathbb{C}) \,|\, \det C \neq 0\},\\
\mathcal{M}_F^{\mathbb{C}} &\cong \{[C] \in \mathbb{P}(E(\varpi_4) \oplus \mathbb{C}) \,|\, \det C \neq 0\},\\
\mathcal{M}_G^{\mathbb{C}} &\cong \{[C] \in \mathbb{P}(E(\varpi_1) \oplus \mathbb{C}) \,|\, \det C \neq 0\},
\end{aligned}$$

where $\det C \neq 0$ means that C is an automorphism of $E(\lambda)$, and $E(\varpi_i) \oplus \mathbb{C}$ is the subspace of $\operatorname{End}(E(\lambda))$ *which does not have weight vectors whose weights are mapped into* 2 *by f.*

Remark. The cohomology bundle of M_C is called the *nullcorrelation bundle* on $\mathbb{P}^{2n-1}$. The moduli space $\mathcal{M}_C^{\mathbb{C}}$ of nullcorrelation bundles has already been described by Spindler [Sp].

To obtain the moduli of anti-self-dual connections, we must additionally consider the reality condition (recall the Ward correspondence in §2.1).

First, we decompose $E(\lambda)$ into $E_{\mathfrak{k}_Z}(w^0\lambda) \oplus E_0 \oplus E_{\mathfrak{k}_Z}(\lambda)$, which is an orthogonal decomposition with respect to the G-invariant hermitian inner product h_λ. Relative to this, we denote the natural inclusions and orthogonal projections by

$$\begin{aligned}
i_{w^0\lambda} : E_{\mathfrak{k}_Z}(w^0\lambda) \longrightarrow E(\lambda), \quad \pi_{w^0\lambda} : E(\lambda) \longrightarrow E_{\mathfrak{k}_Z}(w^0\lambda),\\
i_\lambda : E_{\mathfrak{k}_Z}(\lambda) \longrightarrow E(\lambda), \quad \pi_\lambda : E(\lambda) \longrightarrow E_{\mathfrak{k}_Z}(\lambda).
\end{aligned}$$

Proposition 5.6. *Let E be the cohomology bundle of the monad M_s, and $\mathbb{P}_x$ the twistor fibre. The restricted bundle $E|_{\mathbb{P}_x}$ to the twistor fibre is trivial for each x in G/K_4 if and only if the map $\pi_{w^0\lambda} \circ g^{-1}BAg \circ i_{w^0\lambda} : E_{\mathfrak{k}_Z}(w^0\lambda) \longrightarrow E_{\mathfrak{k}_Z}(w^0\lambda)$ is an isomorphism for each g in G.*

Proof. This will be a slight modification of [O-S-S, Lemma 4.2.3, p. 325].

From Grothendieck's theorem ([O-S-S, Theorem 2.1.1, p. 22]) and the condition $c_1(E) = 0$, $E|_{\mathbb{P}_x}$ is trivial if and only if for an arbitrary non-zero section s of $E|_{\mathbb{P}_x}$ we have $s(z) \neq 0$ for all z in $\mathbb{P}_x$. On the other hand, combined with BBW, a display of the monad M_s restricted to $\mathbb{P}_x$ implies that $H^0(\mathbb{P}_x, E|_{\mathbb{P}_x}) \cong H^0(\mathbb{P}_x, \operatorname{Ker}\beta|_{\mathbb{P}^x})$ and there exists an injection $H^0(\mathbb{P}_x, \operatorname{Ker}\beta|_{\mathbb{P}^x}) \longrightarrow E(\lambda)$. For convenience, we denote $\alpha([g, \cdot])$ by $\alpha_{[g]} : \mathcal{O}_{\mathfrak{p}}(w^0\lambda)|_{[g]} \longrightarrow E(\lambda)$ and $\beta([g], \cdot)$ by $\beta_{[g]} : E(\lambda) \longrightarrow \mathcal{O}_{\mathfrak{p}}(\lambda)|_{[g]}$ for g in G.

If $E|_{\mathbb{P}_x}$ is trivial, the composite $H^0(\mathbb{P}_x, E|_{\mathbb{P}_x}) \cong H^0(\mathbb{P}_x, \operatorname{Ker}\beta|_{\mathbb{P}^x}) \longrightarrow E(\lambda)$ gives us a subspace E_x of $E(\lambda)$ such that $E|_{\mathbb{P}_x} \cong \mathbb{P}_x \times E_x$. Thus,

$$\bigcap_{[g]\in\mathbb{P}_x} \operatorname{Ker}\beta_{[g]} = E_x \quad \text{and} \quad \bigcup_{[g]\in\mathbb{P}_x} \operatorname{Im}\alpha_{[g]} \cap E_x = \{0\}.$$

We claim that if $[g_1]$, $[g_2]$ are different points in $\mathbb{P}_x$, then $\operatorname{Im}\alpha_{[g_1]} \cap \operatorname{Im}\alpha_{[g_2]} = \{0\}$. Let $Sp(1)$ be the subgroup of G whose Lie algebra is $\mathfrak{sp}(1)$ in §2.2. We assume that there exist non-zero vectors e_1, e_2 in $E_{\mathfrak{k}_Z}(w^0\lambda)$ and s in $Sp(1)$ such that

$\alpha([g_1, e_1]) = \alpha([g_1 s, e_2])$ or, equivalently, $Ag_1 se_2 = Ag_1 e_1$, and so $se_2 = e_1$. Making use of Theorem 4.2 and the hypothesis $f(\lambda) = 1$, we can regard $E(\lambda) = \mathbb{H}^m \oplus \mathbb{C}^r$ as an $Sp(1)$-module, where $\mathbb{H}^m = E_{\mathfrak{k}_Z}(w^0\lambda) \oplus E_{\mathfrak{k}_Z}(\lambda)$ and $\mathbb{C}^r = E_0$. ($\mathbb{H}$ is the standard representation and $\mathbb{C}$ is a trivial representation of $Sp(1)$.) Classical representation theory of $Sp(1)$ shows that e_1 and e_2 are in the same $\mathbb{H}$ and s is an element of $U(1)$, thereby proving our claim. Then if $[g_1] \neq [g_2]$ in $\mathbb{P}_x$, we have $E(\lambda) \cong \operatorname{Im}\alpha_{[g_1]} \oplus \operatorname{Im}\alpha_{[g_2]} \oplus E_x$. The monad condition implies that $\operatorname{Ker}\beta_{[g_2]} \cong \operatorname{Im}\alpha_{[g_2]} \oplus E_x$. Consequently, $\beta_{[g_2]} \circ \alpha_{[g_1]}$ is an isomorphism.

Conversely, we assume that the restricted bundle $E|_{\mathbb{P}_x}$ is not trivial, and so there exist a non-zero section s of $E|_{\mathbb{P}_x}$ and an element g in G such that $[g_1]$ is in $\mathbb{P}_x$ and $s([g_1]) = 0$. Since $\alpha(s)$ is an section of $\operatorname{Ker}\beta$, there exists a unique element u in $E(\lambda)$ such that $\alpha(s) = ([g], u)$ for all $[g]$ in $\mathbb{P}_x$ and $\beta\left(([g], u)\right) = 0$. On the other hand, the hypothesis $s([g_1]) = 0$ implies that $\alpha(s)([g_1])$ is contained in $\operatorname{Im}\alpha_{[g_1]}$, and so there exists a non-zero element e in $E_{\mathfrak{k}_Z}(w^0\lambda)$ such that $\alpha_{[g_1]}e = u$. Then we have $\beta_{[g]} \circ \alpha_{[g_1]}e = 0$.

Now we know that $E|_{\mathbb{P}_x}$ is trivial if and only if $\beta_{[g_2]} \circ \alpha_{[g_1]} : E_{\mathfrak{k}_Z}(w^0\lambda) \longrightarrow E_{\mathfrak{k}_Z}(\lambda)$ is an isomorphism for $[g_1] \neq [g_2]$ in $\mathbb{P}_x$. The latter condition is equivalent to $\pi_\lambda \circ s^{-1}g^{-1}BAg \circ i_{w^0\lambda} : E_{\mathfrak{k}_Z}(w^0\lambda) \longrightarrow E_{\mathfrak{k}_Z}(\lambda)$ being an isomorphism for every $[g]$ in $\mathbb{P}_x$ and for every s in $Sp(1)$ which does not belong to the subgroup $U(1)$. From the representation theory of $Sp(1)$ and the decomposition of $E(\lambda)$ as an $Sp(1)$-module, such an s and $Sp(1)$-invariant hermitian inner product give an isomorphism between $E_{\mathfrak{k}_Z}(w^0\lambda)$ and $E_{\mathfrak{k}_Z}(\lambda)$. Hence the above condition is equivalent to the condition that $\pi_{w^0\lambda} \circ g^{-1}BAg \circ i_{w^0\lambda} : E_{\mathfrak{k}_Z}(w^0\lambda) \longrightarrow E_{\mathfrak{k}_Z}(w^0\lambda)$ is an isomorphism for every $[g]$ in $\mathbb{P}_x$. □

Let σ be the real structure on the twistor space [S].

Lemma 5.7. *If λ is an integral dominant weight such that $f(\lambda) = 1$, then $\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ is isomorphic to $\mathcal{O}_{\mathfrak{p}}(w^0\lambda)$ as a holomorphic bundle.*

Proof. Let j be an element in $Sp(1)$ which determines the real structure σ. A familiar argument about the representation theory of $Sp(1)$ and Theorem 4.2 imply that $j \cdot E_{\mathfrak{k}_Z}(\lambda) = E_{\mathfrak{k}_Z}(w^0\lambda)$, where $E_{\mathfrak{k}_Z}(\lambda)$ and $E_{\mathfrak{k}_Z}(w^0\lambda)$ are subspaces of $E(\lambda)$ and so $j \cdot E_{\mathfrak{k}_Z}(\lambda)$ is also a subspace of $E(\lambda)$. On the other hand, making use of the G-invariant hermitian inner product h_λ, we have an (anti-holomorphic) bundle homomorphism $\mathcal{O}_{\mathfrak{p}}(\lambda) \longrightarrow \overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ given by $[g,\, e] \longmapsto [g,\, h_\lambda(e, \cdot)]$ (e is an element of $E_{\mathfrak{k}_Z}(\lambda)$). Now, from the definition of the pull-back bundle, we can express $\sigma^*\mathcal{O}_{\mathfrak{p}}(\lambda)$ as $\left\{([g]_Z,\, [gj, e])\right\}$, where $[g]_Z$ is a point in the twistor space represented by g in G, and e is a vector in $E_{\mathfrak{k}_Z}(\lambda)$. Thus, we may define a holomorphic bundle homomorphism $\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^* \longrightarrow \mathcal{O}_{\mathfrak{p}}(w^0\lambda)$ by

$$([g]_Z,\, [gj, h(e, \cdot)]) \longmapsto [g,\, \pi_{w^0\lambda} \circ j \circ i_\lambda e].$$

□

Remark. Even if an integral dominant weight λ satisfies $f(\lambda) \geqq 2$, Lemma 4.1 and an argument about root strings yields $j \cdot E_{\mathfrak{k}_Z}(\lambda) = E_{\mathfrak{k}_Z}(w^0\lambda)$ and so $\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^* \cong \mathcal{O}_{\mathfrak{p}}(w^0\lambda)$.

Proposition 5.8. *Let E be the cohomology bundle of the monad M_s. Moreover, the restricted bundle $E|_{\mathbb{P}_x}$ is assumed to be trivial for every x in G/K_4. Then, there is an isomorphism $\tau : E \longrightarrow \sigma^*\overline{E}^*$ with $(\sigma^*\bar{\tau})^* = \tau$ which induces a positive* *definite hermitian form on sections of $E|_{\mathbb{P}^1_x}$ for every $x \in M$ if and only if there exist a hermitian inner product on $E(\lambda)$ and isomorphisms $\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ and $\mathcal{O}_{\mathfrak{p}}(\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*$.*

Proof. By hypothesis, $\sigma^*\overline{E}^*$ is the cohomology bundle of the monad

$$\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^* \xrightarrow{\sigma^*\overline{\beta}^*} \sigma^*\overline{E(\lambda)}^* \xrightarrow{\sigma^*\overline{\alpha}^*} \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*.$$

We can verify the Okonek-Schneider-Spindler condition ([O-S-S, Lemma 4.1.3, p. 276]) using Lemma 5.7 and the BBW theorem. Then we obtain that there is an isomorphism $\tau : E \longrightarrow \sigma^*\overline{E}^*$ if and only if there exist isomorphisms $E(\lambda) \cong \sigma^*\overline{E(\lambda)}^*$, $\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ and $\mathcal{O}_{\mathfrak{p}}(\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*$.

Next we take the condition imposed on τ into account. Since $E|_{\mathbb{P}^1_x}$ is trivial, in the notation in the proof of Proposition 5.6, this condition yields that E_x has a positive hermitian inner product. Consequently the irreducibility of $E(\lambda)$ implies that the above isomorphism $E(\lambda) \cong \sigma^*\overline{E(\lambda)}^*$ induces a hermitian inner product on $E(\lambda)$.

Conversely, a hermitian inner product on $E(\lambda)$ induces an isomorphism $E(\lambda) \cong \sigma^*\overline{E(\lambda)}^*$. Combined with isomorphisms $\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ and $\mathcal{O}_{\mathfrak{p}}(\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*$, this induces the desired $\tau : E \longrightarrow \sigma^*\overline{E}^*$ under the hypothesis that $E|_{\mathbb{P}_x}$ is trivial. □

Therefore, to describe the moduli space, we fix the G-invariant hermitian inner product h_λ on $E(\lambda)$ and isomorphisms $\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ and $\mathcal{O}_{\mathfrak{p}}(\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*$ defined in Lemma 5.7.

Proposition 5.9. *Relative to the fixed isomorphisms $E(\lambda) \cong \sigma^*\overline{E(\lambda)}^*$, $\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^*$ and $\mathcal{O}_{\mathfrak{p}}(\lambda) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*$, the following two conditions are equivalent*:

1. *There exists a commutative diagram*

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \arrow[r, "\alpha"] \arrow[d] & E(\lambda) \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_{\mathfrak{p}}(\lambda) \arrow[d] \\
\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda)}^* \arrow[r, "\sigma^*\overline{\beta}^*"'] & \sigma^*\overline{E(\lambda)}^* \arrow[r, "\sigma^*\overline{\alpha}^*"'] & \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda)}^*.
\end{tikzcd}$$

2. *For all u and v in $E(\lambda)$, $h_\lambda(Au, v) = h_\lambda(u, Bv)$.*

Proof. If the diagram is commutative, we see from $\sigma^*\overline{\beta}^* = h_\lambda(\alpha, \ )$ that for all e in $E_{\mathfrak{k}_Z}(\lambda)$ and all g in G, $h_\lambda(e, -jg^{-1}B\cdot) = h_\lambda(Agje, \cdot)$. Since h_λ is G-invariant, we have $h_\lambda(gje, Bu) = h_\lambda(Agje,\ u)$ for all u in $E(\lambda)$, e in $E_{\mathfrak{k}_Z}(\lambda)$ and g in G. The irreducibility of $E(\lambda)$ yields that $h_\lambda(Au, v) = h_\lambda(u, Bv)$. Now it is clear that (2) implies (1). □

We denote by A^* the adjoint operator of A with respect to h_λ. If A is an automorphism of $E(\lambda)$, the restricted endomorphism $g^{-1}A^*Ag$ to an arbitrary subspace of $E(\lambda)$ is also an automorphism. Hence, we call the relation $B = A^*$ the *reality condition for the monad M_s*.

Theorem 5.10. *Let $\mathcal{M}_B$, $\mathcal{M}_C$, $\mathcal{M}_F$, $\mathcal{M}_G$ be the moduli spaces of anti-self-dual bundles induced by the monads M_B, M_C, M_F, M_G respectively. Then we have* *identifications such that*

$$\mathcal{M}_B \cong B^{\dim E(\varpi_1)}, \qquad \mathcal{M}_C \cong B^{\dim E(\varpi_2)},$$
$$\mathcal{M}_F \cong B^{\dim E(\varpi_4)}, \qquad \mathcal{M}_G \cong B^{\dim E(\varpi_1)},$$

where B^k is a k-dimensional open ball.

Proof. The cohomology bundle of the monad M_s is the pull-back of an anti-self-dual bundle on G/K_4 if and only if the monad M_s satisfies the monad and reality conditions. In the notation in Lemma 5.1, these conditions are expressed in the following way:

1. A and B are automorphisms of $E(\lambda)$.
2. $BA_j = 0$ for all j such that $f(\nu_j) = 2$.
3. $B = A^*$.

(Propositions 5.2, 5.6, 5.8, 5.9). If we put $C = A^*A$, C is in the real subspace $\mathrm{End}\,(E(\lambda))^{\mathbb{R}}$ of $\mathrm{End}\,(E(\lambda))$ with respect to the standard real structure. This real structure is compatible with the G-decomposition $\mathrm{End}\,(E(\lambda))^{\mathbb{R}} \cong \bigoplus_j E(\nu_j)^{\mathbb{R}}$. By this and Theorem 5.5, C is an element of $E(\varpi_i)^{\mathbb{R}} \oplus \mathbb{R}$ for an appropriate i. Since A is an automorphism, C is positive with respect to h_λ.

On the other hand, if two monads M_{s_1} and M_{s_2} satisfy the reality conditions, Proposition 5.4 implies that M_{s_1} is isomorphic to M_{s_2} if and only if there exists a non-zero real constant c such that $A_2^*A_2 = cA_1^*A_1$. Thus the moduli space $\mathcal{M}_s$ of anti-self-dual connections induced by the monad M_s is described by

$$\mathcal{M}_s \cong \left\{C \in E(\varpi_i)^{\mathbb{R}} \oplus \mathbb{R} \subset \mathrm{End}\,(E(\lambda)) \,|\, f(\varpi_i) = 1,\ C \text{ is positive}\right\}/\sim$$

where $C_1 \sim C_2$ means that there exists a non-zero real constant c such that $C_2 = c\,C_1$.

Now a positive endomorphism C is assumed to be in $E(\varpi_i)^{\mathbb{R}}$. Then we have $\mathrm{trace}\, C = h_{\mathrm{End}}(C, I) = 0$, where h_{End} is the hermitian inner product on $\mathrm{End}\,(E(\lambda))$ induced by h_λ. This is a contradiction, and so we have $C = aI + D$, where a is a non-zero real constant and D is an element of $E(\varpi_i)^{\mathbb{R}}$. Making use of the $\mathbb{R}^*$-action, we may express C as $I + D$. Since C is positive, we get $h_\lambda(Du,\, u) > -|u|^2$ for all u in $E(\lambda)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the eigenvalues of D. These eigenvalues are real, because D is also a self-adjoint operator. Hence the condition imposed on D implies that $\alpha_i > -1$ for all $i = 1, \cdots, n$. Consequently, we obtain

$$\mathcal{M}_s \cong \left\{D \in E(\varpi_i)^{\mathbb{R}} \subset \mathrm{End}\,(E(\lambda)) \,|\, f(\varpi_i) = 1 \text{ and } \alpha_i > -1\right\}.$$

We induce a G-invariant inner product on $E(\varpi_i)^{\mathbb{R}}$ using h_{End}. From the definition, we have $|D|^2 = \mathrm{trace}\, D^2 = \sum_j \alpha_j^2$, and so $\mathcal{M}_s$ contains the unit ball. Since $\sum_j \alpha_j = \mathrm{trace}\, D = h_{\mathrm{End}}(D,\, I) = 0$, there exists a negative α_j if $D \neq 0$. We may define α_1 as the smallest eigenvalue of D. Then the smallest eigenvalue of $c\,D$ is $c\,\alpha_1$, where c is a positive number. Hence $\mathcal{M}_s$ is a star-shaped open set centered at the origin. This completes the proof. □

Remark 1. We have not discussed the completeness of these families. In the case of $\mathcal{M}_C$, an argument using Beilinson's spectral sequence ([O-S-S, Theorem 3.1.3, p. 240 or Theorem 3.1.4, p. 245]) and the vanishing theorem ([Na-2]) shows that this moduli space is complete ([K-N]; see also [M-S]). The "center" of the family represents a homogeneous bundle, and so we can compute the first cohomology of the endomorphism bundle of the pull-back; this computation has already done (at the beginning in this section). Combining this with the reality condition, we see that we can expect our families to be complete.

Remark 2. The study of degenerations of anti-self-dual connections in these families is of independent interest. Our theory now enables explicit computations to be carried out, though this topic will be treated elsewhere. (See also [N-N3].)

6. Moduli spaces II

The monads of direct sum type $M_A, M_{D_1}, M_{D_2}, M_{D_3}, M_E$ are considered in this section. We express these monads as

$$M_d \,:\, \mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \xrightarrow{\alpha} E(\lambda_1) \oplus E(\lambda_2) \xrightarrow{\beta} \mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2),$$

where λ_1 and λ_2 are integral dominant weights satisfying $f(\lambda_1) = f(\lambda_2) = 1$.

Making use of the BBW theorem, we have

Lemma 6.1. *For homomorphisms α and β in M_d, there exist linear maps A_1, B_1 : $E(\lambda_1) \longrightarrow E(\lambda_1)$, A_2, B_2 : $E(\lambda_2) \longrightarrow E(\lambda_1)$, A_3, B_3 : $E(\lambda_1) \longrightarrow E(\lambda_2)$ and A_4, $B_4 : E(\lambda_2) \longrightarrow E(\lambda_2)$ such that*

$$\begin{aligned}\alpha\big([g,\,(e_1,e_2)]\big) \;&=\; \Big([g],\,\big(A_1gi_1(e_1)+A_2gi_2(e_2),\,A_3gi_1(e_1)+A_4gi_2(e_2)\big)\Big),\\ \beta\big([g],\,(u_1,u_2)\big) \;&=\; \Big[g,\,\big\{\pi_1g^{-1}(B_1(u_1)+B_2(u_2)),\,\pi_2g^{-1}(B_3(u_1)+B_4(u_2))\big\}\Big],\end{aligned}$$

where g, e_1, e_2, u_1 and u_2 are elements of G, $E_{\mathfrak{p}}(w^0\lambda_1)$, $E_{\mathfrak{p}}(w^0\lambda_2)$, $E(\lambda_1)$ and $E(\lambda_2)$, respectively.

Remark. For brevity, we define linear endomorphisms A and B of $E(\lambda_1) \oplus E(\lambda_2)$ as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

When no confusion can arise, we express α and β as

$$\alpha\big([g,\,(e_1,e_2)]\big) \;=\; \big([g],\,Ag\circ i(e_1,\,e_2)\big), \quad \beta\big([g],\,(u_1,u_2)\big) \;=\; \big[g,\,\pi\circ g^{-1}B(u_1,\,u_2)\big].$$

In contrast to monads of simple type, $E(\lambda_1) \oplus E(\lambda_2)$ is not irreducible. Therefore, if A is an isomorphism, then α is injective, and if B is an isomorphism, then β is surjective. However, the converse does not hold in general.

First, we consider the condition $\beta \circ \alpha = 0$, because we can also apply Proposition 5.2 to this case.

Proposition 6.2. *For the monads $M_A, M_{D_1}, M_{D_2}, M_{D_3}, M_E$, we have $\beta \circ \alpha = 0$ if and only if*

$$BA = \begin{pmatrix} c_1\, I & Q \\ R & c_2\, I \end{pmatrix},$$

where c_1 and c_2 are constants and

$$\begin{aligned} &M_A;\, Q \in E(\varpi_2), \quad R \in E(\varpi_{n-1}),\\ &M_{D_1};\, Q \text{ and } R \in E(\varpi_1)\\ &M_{D_2};\, Q \in E(\varpi_n)\,(n \text{ even}),\;\; E(\varpi_{n-1})\,(n \text{ odd}), \quad R \in E(\varpi_n),\\ &M_{D_3};\, Q \in E(\varpi_{n-1})\,(n \text{ even}),\;\; E(\varpi_n)\,(n \text{ odd}), \quad R \in E(\varpi_{n-1}),\\ &M_E;\, Q \in E(\varpi_6), \quad R \in E(\varpi_1).\end{aligned}$$

Next, we take the Ward condition (1) and (2) (§2.2.2) into account. Though we cannot use irreducibility, we have the following easy lemma.

Lemma 6.3. *Let V_0 be the subspace of $E(\lambda_1) \oplus E(\lambda_2)$ which is spanned by all elements represented by $g(e_1, e_2)$ for g in G, e_1 in $E_{\mathfrak{p}}(w^0\lambda_1)$ and e_2 in $E_{\mathfrak{p}}(w^0\lambda_2)$. Then V_0 coincides with $E(\lambda_1) \oplus E(\lambda_2)$.*

Proof. The irreducibility of $E(\lambda_1)$ and $E(\lambda_2)$ implies that V_0 contains $E(\lambda_1)$ and $E(\lambda_2)$. □

In this case, Lemma 5.7 still holds. If we use Lemma 6.3 instead of the irreducibility, we can obtain an analogue of Proposition 5.8.

Proposition 6.4. *Let E be the cohomology bundle of the monad M_d. Moreover, assume that the restricted bundle $E|_{\mathbb{P}_x}$ is trivial for every x in G/K_4. Then, there is an isomorphism $\tau : E \longrightarrow \sigma^*\overline{E}^*$ with $(\sigma^*\bar{\tau})^* = \tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbb{P}^1_x}$ for every $x \in M$ if and only if there exist a hermitian inner product on $E(\lambda_1) \oplus E(\lambda_2)$ and isomorphisms*

$$\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_2)}^*$$

and

$$\mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_2)}^*.$$

Consequently, we fix G-invariant hermitian inner products h_{λ_1} and h_{λ_2} on $E(\lambda_1)$ and $E(\lambda_2)$, respectively, and the isomorphisms

$$\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_2)}^*$$

and

$$\mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_2)}^*$$

defined in Lemma 5.7. We denote by $h_{\lambda_1+\lambda_2}$ the direct sum inner product on $E(\lambda_1) \oplus E(\lambda_2)$. Making use of Lemma 6.3 again, we have

Proposition 6.5. *Relative to the fixed isomorphisms*

$$E(\lambda_1) \oplus E(\lambda_2) \cong \sigma^*\overline{E(\lambda_1) \oplus E(\lambda_2)}^*,$$
$$\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_2)}^*$$

and

$$\mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2) \cong \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_2)}^*,$$

the following two conditions are equivalent:

1. *There exists a commutative diagram:*

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \arrow[r, "\alpha"] \arrow[d] & E(\lambda_1) \oplus E(\lambda_2) \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2) \arrow[d] \\
\sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(\lambda_2)}^* \arrow[r, "\sigma^*\overline{\beta}^*"'] & \sigma^*\overline{E(\lambda) \oplus E(\lambda_2)}^* \arrow[r, "\sigma^*\overline{\alpha}^*"'] & \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1)}^* \oplus \sigma^*\overline{\mathcal{O}_{\mathfrak{p}}(w^0\lambda_2)}^*.
\end{tikzcd}$$

2. *For all u and v in $E(\lambda_1) \oplus E(\lambda_2)$, $h_{\lambda_1+\lambda_2}(Au, v) = h_{\lambda_1+\lambda_2}(u, Bv)$.*

From now on, we only consider the case in which $B = A^*$, where A^* is the adjoint operator of A with respect to $h_{\lambda_1+\lambda_2}$.

Proposition 6.6. *Let E be the cohomology bundle of the monad M_d satisfying $B = A^*$, and $\mathbb{P}_x$ the twistor fibre. The restriction $E|_{\mathbb{P}_x}$ is trivial for every x in G/K_4 if and only if A is an automorphism of $E(\lambda_1) \oplus E(\lambda_2)$.*

Proof. If $E|_{\mathbb{P}_x}$ is trivial, there exists a subspace E_x of $E(\lambda)$ such that

$$\bigcap_{[g]\in\mathbb{P}_x} \operatorname{Ker}\beta_{[g]} = E_x \quad \text{and} \quad \bigcup_{[g]\in\mathbb{P}_x} \operatorname{Im}\alpha_{[g]} \cap E_x = \{0\}.$$

Let $Sp(1)$ be the subgroup of G whose Lie algebra is $\mathfrak{sp}(1)$ in §2.2, and let $[g]$ be a point in $\mathbb{P}_x$. Since $B = A^*$, we have

$$E(\lambda_1) \oplus E(\lambda_2) \;=\; E_x \oplus \bigcup_{s\in Sp(1)} Ags\big(E_{\mathfrak{p}}(w^0\lambda_1) \oplus E_{\mathfrak{p}}(w^0\lambda_2)\big).$$

Representation theory of $Sp(1)$ and a dimension count implies that for all g in G, there exists an s in $Sp(1)$ such that

$$Ag\big(E_{\mathfrak{p}}(w^0\lambda_1) \oplus E_{\mathfrak{p}}(w^0\lambda_2)\big) \,\cap\, Ags\big(E_{\mathfrak{p}}(w^0\lambda_1) \oplus E_{\mathfrak{p}}(w^0\lambda_2)\big) = \{0\}.$$

This means that the restricted map

$$g^{-1}A^*Ag : E_{\mathfrak{p}}(w^0\lambda_1) \oplus E_{\mathfrak{p}}(w^0\lambda_2) \longrightarrow E_{\mathfrak{p}}(w^0\lambda_1) \oplus E_{\mathfrak{p}}(w^0\lambda_2)$$

is an isomorphism for every g in G (see the proof of Proposition 5.6). Then Lemma 6.3 yields that A is an automorphism.

For a proof of the converse, we can also apply the same argument as in the proof of Proposition 5.6. □

On the other hand, the condition $B = A^*$ results in a slight modification of Proposition 6.2.

Proposition 6.7. *For the monads $M_A, M_{D_1}, M_{D_2}, M_{D_3}, M_E$ satisfying $B = A^*$, we have $\beta\circ\alpha = 0$ if and only if*

$$A^*A = \begin{pmatrix} c_1\, I & R^* \\ R & c_2\, I \end{pmatrix},$$

where c_1 and c_2 are non-negative real constants and

$$\begin{aligned} &M_A;\, R \in E(\varpi_{n-1}), \qquad M_{D_1}; R \in E(\varpi_1) \qquad M_{D_2};\, R \in E(\varpi_n), \\ &M_{D_3};\, R \in E(\varpi_{n-1}), \qquad M_E;\, R \in E(\varpi_1). \end{aligned}$$

Finally, we verify the Okonek-Sneider-Spindler condition (Lemma 5.3) using the BBW theorem. As a result, there exists a one-to-one correspondence between isomorphism classes of monads and isomorphisms of the cohomology bundles. We also describe the isomorphism classes of monads in this case.

Proposition 6.8. *Suppose that A_1, B_1, A_2, B_2 in the monads M_{d_1} and M_{d_2} are automorphisms of $E(\lambda_1) \oplus E(\lambda_2)$. The monads M_{d_1} and M_{d_2} are isomorphic to each other* (*in other words, the following diagram is commutative*:

$$\begin{tikzcd} M_{s_1} : \mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \arrow[r, "\alpha_1"] \arrow[d, "p"'] & E(\lambda_1) \oplus E(\lambda_2) \arrow[r, "\beta_1"] \arrow[d, "C"] & \mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2) \arrow[d, "q"] \\ M_{s_2} : \mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2) \arrow[r, "\alpha_2"'] & E(\lambda_1) \oplus E(\lambda_2) \arrow[r, "\beta_2"'] & \mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2), \end{tikzcd}$$

where C is an automorphism of $E(\lambda_1) \oplus E(\lambda_2)$, and p, q are automorphisms of $\mathcal{O}_{\mathfrak{p}}(w^0\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\lambda_2)$, $\mathcal{O}_{\mathfrak{p}}(\lambda_1) \oplus \mathcal{O}_{\mathfrak{p}}(\lambda_2)$ respectively) if and only if there exist non-zero constants a, b, c, d such that

$$B_2 A_2 \begin{pmatrix} a\,I & O \\ O & b\,I \end{pmatrix} = \begin{pmatrix} c\,I & O \\ O & d\,I \end{pmatrix} B_1 A_1.$$

Proof. First, from the BBW theorem, we can express p and q as $p = (a\,I,\, b\,I)$ and $q = (c\,I,\, d\,I)$, where a, b, c and d are non-zero constants. Then the commutative diagram shows that

$$\begin{aligned} CA_1(ge_1,\, ge_2) \;&=\; A_2(ae_1,\, be_2), \\ c\pi_1 \circ g^{-1} B_1(u_1,\, u_2) \;&=\; \pi_1 \circ g^{-1} B_2 C(u_1,\, u_2), \\ d\pi_2 \circ g^{-1} B_1(u_1,\, u_2) \;&=\; \pi_2 \circ g^{-1} B_2 C(u_1,\, u_2), \end{aligned}$$

where e_1, e_2, u_1 and u_2 are elements of $E_{\mathfrak{p}}(w^0\lambda_1)$, $E_{\mathfrak{p}}(w^0\lambda_2)$, $E(\lambda_1)$ and $E(\lambda_2)$ respectively. These equations are equivalent to

$$CA_1 \;=\; A_2 \begin{pmatrix} a\,I & O \\ O & b\,I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c\,I & O \\ O & d\,I \end{pmatrix} B_1 \;=\; B_2 C.$$

Consequently we have

$$B_2 A_2 \begin{pmatrix} a\,I & O \\ O & b\,I \end{pmatrix} = \begin{pmatrix} c\,I & O \\ O & d\,I \end{pmatrix} B_1 A_1.$$

It is now clear that the converse holds. □

Theorem 6.9. *Let $\mathcal{M}_A$, $\mathcal{M}_{D_1}$, $\mathcal{M}_{D_2}$, $\mathcal{M}_{D_3}$, $\mathcal{M}_E$ be the moduli spaces of anti-self-dual bundles induced by the monads M_A, M_{D_1}, M_{D_2}, M_{D_3}, M_E respectively. Then $\mathcal{M}_A$, $\mathcal{M}_{D_1}$, $\mathcal{M}_{D_2}$, $\mathcal{M}_{D_3}$, $\mathcal{M}_E$ are identified with open cones over the respective complex projective spaces $\mathbb{P}(E(\varpi_{n-1}))$, $\mathbb{P}(E(\varpi_1))$, $\mathbb{P}(E(\varpi_n))$, $\mathbb{P}(E(\varpi_{n-1}))$, $\mathbb{P}(E(\varpi_1))$.*

Proof. The cohomology bundle of M_d is the pull-back of an anti-self-dual bundle on G/K_4 if and only if the monad satisfies

1. A and B are automorphisms of $E(\lambda_1) \oplus E(\lambda_2)$.
2. $$A^*A \;=\; \begin{pmatrix} c_1\,I & R^* \\ R & c_2\,I \end{pmatrix},$$
   where c_1 and c_2 are non-negative real constants and R is an element of $E(\varpi_i)$, and
3. $B \;=\; A^*$.

(See Propositions 6.5, 6.6, 6.7.) Here, if c_1 is zero, it is easy to show that A is not an automorphism. Hence c_1 is a positive constant and c_2 is also positive.

On the other hand, in this situation, Proposition 6.8 implies that A_1 and A_2 induce isomorphic monads if and only if there exist non-zero constants a, b, c, d such that

$$A_2^* A_2 \begin{pmatrix} a\,I & O \\ O & b\,I \end{pmatrix} = \begin{pmatrix} c\,I & O \\ O & d\,I \end{pmatrix} A_1^* A_1.$$

We now set

$$A_1^* A_1 \;=\; \begin{pmatrix} c_1\,I & R_1^* \\ R_1 & c_2\,I \end{pmatrix} \quad \text{and} \quad A_2^* A_2 \;=\; \begin{pmatrix} c_3\,I & R_2^* \\ R_2 & c_4\,I \end{pmatrix}.$$

Then we obtain that $c_3 = c_1 a^{-1} c$, $c_4 = c_2 b^{-1} d$, $R_2 = a^{-1} d R_1$ and $R_2^* = b^{-1} c R_1^*$. Since $A_2^* A_2$ is a positive self-adjoint operator, we have that $a^{-1}c$ and $b^{-1}d$ are positive and $\overline{a^{-1}d} = b^{-1}c$. Consequently, we may put $c_i = 1$ without loss of generality. Then our equivalence relation implies that there exists $\alpha \in S^1$ such that $R_2 = \alpha R_1$. Moreover, an easy argument in linear algebra implies that if all the eigenvalues μ_p of R^*R are less than or equal to 1, there exist linear maps $A_1 : E(\lambda_1) \longrightarrow E(\lambda_1)$, $A_2 : E(\lambda_2) \longrightarrow E(\lambda_1)$, $A_3 : E(\lambda_1) \longrightarrow E(\lambda_2)$ and $A_4 : E(\lambda_2) \longrightarrow E(\lambda_2)$ such that

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \text{and} \quad A^*A = \begin{pmatrix} I & R^* \\ R & I \end{pmatrix}.$$

Finally, with the above notation, A^*A has 1 or $1 \pm \sqrt{\mu_p}$ as eigenvalues. The positivity of A^*A yields that μ_ps must be less than 1. Thus the moduli space $\mathcal{M}_d$ of anti-self-dual connections induced by the monad M_d is described as follows:

$$\mathcal{M}_d \cong \Big\{ R \in E(\varpi_i) \subset \operatorname{Hom}(E(\lambda_2), E(\lambda_1)) \,|\, f(\varpi_i) = 1, \\ \text{all the eigenvalues of } R^*R \text{ are less than } 1 \Big\} / \sim$$

where $R_1 \sim R_2$ means that there exists a constant $\alpha \in S^1$ such that $R_2 = \alpha R_1$. This set is identified with an open cone over the projective space $\mathbb{P}(E(\varpi_i))$. □

7. Generalized Horrocks bundles

In this section, we give an example of a non-homogeneous anti-self-dual bundle on $\mathbb{H}P^n$ when $n \geqq 2$. This example is inspired by Mamone Capria and Salamon [M-S].

Theorem 7.1. *On $\mathbb{P}^{2n-1}$ $(n \geqq 3)$, we have a monad of the following type*:

$$\mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_n) \longrightarrow \mathcal{O}(1),$$

and the cohomology bundle of this monad is the pull-back of an anti-self-dual bundle on $\mathbb{H}P^{n-1}$. In the case $n = 3$, this cohomology bundle is the well-known Horrocks bundle on $\mathbb{P}^5$ [Ho].

Proof. Consider the standard monad induced by ϖ_{n-1} for the Lie algebra C_n:

$$\mathcal{O}_{\mathfrak{p}}(w^0 \varpi_{n-1}) \longrightarrow E_{C_n}(\varpi_{n-1}) \longrightarrow \mathcal{O}_{\mathfrak{p}}(\varpi_{n-1}).$$

From the remark after Theorem 4.2, the cohomology bundle of this standard monad equals $\mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_{n-2}) \oplus \mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_n)$.

We now make use of the standard embedding of $SL(n, \mathbb{C})$ into $Sp(n, \mathbb{C})$. As an A_{n-1}-module, $E_{C_n}(\varpi_{n-1})$ contains the direct sum of irreducible representations $E_{A_{n-1}}(\varpi_1) \oplus E_{A_{n-1}}(\varpi_{n-1})$. This subspace can be regarded as $E_{C_n}(\varpi_1)$, and so we have a monad

$$\mathcal{O}(-1) \longrightarrow E_{C_n}(\varpi_{n-1}) \longrightarrow \mathcal{O}(1).$$

Now endow these homogeneous bundles with hermitian metrics induced by an $Sp(n)$-invariant hermitian inner product on $E_{C_n}(\varpi_{n-1})$. From the two monads, we have two orthogonal projections

$$p_1' : E_{C_n}(\varpi_{n-1}) \longrightarrow \mathcal{O}_{\mathfrak{p}}(w^o \varpi_{n-1}) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_{n-1}),$$
$$p_2 : E_{C_n}(\varpi_{n-1}) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1).$$

It is well known that the unitary connections on $\operatorname{Ker} p_1'$ and $\operatorname{Ker} p_2$ are compatible with the holomorphic structures of the corresponding cohomology bundles (see for example [D-K, Lemma 3.1.20, p. 82]). Moreover, we have another orthogonal projection:

$$p_1 : E_{C_n}(\varpi_{n-1}) \longrightarrow \mathcal{O}_{\mathfrak{p}}(w^o\varpi_{n-1}) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_{n-1}) \oplus \mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_{n-2}).$$

It is easy to show that the restricted map p_2 to $\operatorname{Ker} p_1$ is also surjective. Hence we have a monad

$$\mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_n) \longrightarrow \mathcal{O}(1).$$

Finally, our orthogonal projection

$$\pi : \mathcal{O}_{\mathfrak{p}}(-\varpi_1 + \varpi_n) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

can be regarded as a bundle homomorphism on $\mathbb{H}P^{n-1}$. Consequently, $\operatorname{Ker} \pi$ can also be regarded as a bundle on $\mathbb{H}P^{n-1}$. The pull-back has a holomorphic structure with respect to the induced connection, and so the connection of this bundle is anti-self-dual. $\square$

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