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UNBOUNDED COMPONENTS OF THE SINGULAR SET OF THE DISTANCE FUNCTION IN $\mathbb{R}^n$

PIERMARCO CANNARSA AND ROBERTO PEIRONE

Abstract. Given a closed set $F \subseteq \mathbb{R}^n$, the set Σ_F of all points at which the metric projection onto F is multi-valued is nonempty if and only if F is nonconvex. The authors analyze such a set, characterizing the unbounded connected components of Σ_F. For F compact, the existence of an asymptote for any unbounded component of Σ_F is obtained.

1. Introduction

Motzkin's Theorem in convex analysis states that a nonempty closed set $F \subseteq \mathbb{R}^n$ is convex if and only if the projection onto F of any point $x \in \mathbb{R}^n$, $\mathrm{proj}_F(x)$, is a singleton. This is one of the simplest examples that show how properties of proj_F can be used to describe properties of F.

In this paper we are interested in the set of all points $x \in \mathbb{R}^n$ at which proj_F is multivalued, hereafter denoted by Σ_F. As is well-known, an equivalent definition of Σ_F can be given in terms of the euclidean distance function d_F—another well-known object. In fact, the set on which proj_F is multivalued coincides with the set of all points $x \in \mathbb{R}^n \setminus F$ at which d_F fails to be differentiable, see, e.g., [5, p. 62]. We refer to the latter set as the *singular set* of d_F. The above characterization of Σ_F as the singular set of d_F is very useful as it provides simple proofs of rather fine measure theoretic estimates. For example, since d_F is Lipschitz continuous, Rademacher's Theorem implies that Σ_F has Lebesgue measure zero. It is noteworthy that this result was proved by Erdös [4] by a completely different method.

The local structure of the singular set has been investigated by several authors in view of its importance for best approximation theory. For instance, Bartke and Berens [3] proved that any non-isolated singular point of d_F is the initial point of a Lipschitz singular arc—a result formerly obtained by Pauc [8] for $n = 2$. This analysis was later extended to Hilbert spaces by Westphal and Frerking [10], see also [9].

The first object of our analysis is the global structure of the connected components of Σ_F. In particular, we are interested in classifying the connected components of Σ_F according to their boundedness properties. We observe that, for a simply connected compact set F that is not convex, the unboundedness of Σ_F is ensured by another classical result of Motzkin's [7]. Our approach uses the one-to-one correspondence between the connected components of the set $\overline{\mathrm{co}}\, F \setminus F$ and the

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components of Σ_F that was introduced by Veselý in [9]. Denoting by $\mathcal{M}$ such a correspondence, in Theorem 3.1 we show that a component C of Σ_F is bounded if and only if $\mathcal{M}^{-1}(C)$ is bounded and is also a component of the set $\mathbb{R}^n \setminus F$. In particular, whenever F fails to contain the topological boundary of $\overline{\mathrm{co}}\, F$, the existence of an unbounded connected component of Σ_F follows.

Once the unbounded components of Σ_F have been characterized, we turn our attention to the asymptotic behaviour of such components. More precisely, for any $n \geq 2$ and for F compact we show that an unbounded component C of Σ_F always possesses an asymptotic half-line, in the sense that $d_C(x_0 + tv) \to 0$ as $t \to +\infty$ for some $x_0, v \in\mathbb{R}^n$ with $||v|| = 1$. Also, for $n = 2$, we prove that the above convergence to the asymptote takes a stronger form. In fact, in this case, we construct a continuous arc, γ, contained in C and asymptotic to the half-line $x_0 + tv$, and we show that γ coincides—up to reparametrization—with the unique absolutely continuous solution of

$$\begin{cases} \xi'(s) \in D^+ d_F(\xi(s))\,, & \text{a.e. } s \geq 0, \\ \xi(0) = \overline{x} \end{cases} \tag{1.1}$$

(here $\overline{x}$ is a fixed point of C). Notice that inclusion (1.1) has already been used in the literature to study the local structure of Σ_F (see [3] and [1]).

This paper is organized as follows. In section 2 we recall basic notions on the distance function and on the singular set of the metric projection. Section 3 is devoted to the characterization of the unbounded connected components of Σ_F. In section 4 we prove the existence of an asymptote for general n, and then we improve this result in section 5 for $n = 2$.

2. Notation and Preliminary Results

For any nonempty subset A of $\mathbb{R}^n$, we denote by $\mathcal{P}(A)$ the family of all subsets of A, and by $\mathcal{C}(A)$ the set of all connected components of A.

We denote by $x \cdot y$ the scalar product between two vectors $x, y \in\mathbb{R}^n$, and by $||x||$ the usual euclidean norm of x. The canonical basis of $\mathbb{R}^n$ will be denoted by $e_1, \dots, e_n$ and we shall set, as usual, $x_k = x \cdot e_k$. Given $x, y \in \mathbb{R}^n$ we denote by $[x, y]$ the line segment

$$[x, y] = \{tx + (1 - t)y \;:\; t \in [0, 1]\}\,.$$

For any $x \in\mathbb{R}^n$ and any $r > 0$ we set

$$B(x, r) = \{y \in \mathbb{R}^n : ||y - x|| < r\}$$

and we abbreviate $B(r) = B(0, r)$.

In this paper, F stands for a nonempty closed subset of $\mathbb{R}^n$. Moreover, we denote by $\mathrm{co}\, F$ the convex hull of F and by $\overline{\mathrm{co}}\, F$ the closure of $\mathrm{co}\, F$.

The distance function from a closed set F, d_F, is defined as

$$d_F(x) = \min_{y \in F} ||x - y|| \qquad \forall x \in \mathbb{R}^n.$$

For any $\eta > 0$ we set $F_\eta = \{x \in \mathbb{R}^n \;:\; d_F(x) < \eta\}$. The set of closest points in F to a given point $x \in \mathbb{R}^n$ will be denoted by

$$\mathrm{proj}_F(x) = \{y \in F \;:\; d_F(x) = \|x - y\|\}\,.$$

We shall also refer to $\mathrm{proj}_F(x)$ as the (euclidean) projection of x onto F. The following is an elementary property of the distance function that we need for the sequel. We give a proof for the reader's convenience.

Lemma 2.1. *Let $\xi\colon]a,b[\to\mathbb{R}^n$ be a continuous arc such that $\xi(t)\notin F$ for any $t\in]a,b[$, and define $f(t)=d_F(\xi(t))$. If ξ and f are both differentiable at some point $\overline{t}\in]a,b[$, then*

$$f'(\overline{t})=\xi'(\overline{t})\cdot\frac{\xi(\overline{t})-x}{||\xi(\overline{t})-x||}\qquad\forall x\in proj_F(\xi(\overline{t}))\,.$$

Proof. Let $x\in\mathrm{proj}_F(\xi(\overline{t}))$. Observing that $f(t)\le||\xi(t)-x||$ for any $t\in]a,b[$ and $f(\overline{t})=||\xi(\overline{t})-x||$, we obtain

$$\frac{f(t)-f(\overline{t})}{t-\overline{t}}\le\frac{||\xi(t)-x||-||\xi(\overline{t})-x||}{t-\overline{t}}\qquad\forall t>\overline{t}\,.$$

Hence, taking the limit as $t\downarrow\overline{t}$,

$$f'(\overline{t})\le\xi'(\overline{t})\cdot\frac{\xi(\overline{t})-x}{||\xi(\overline{t})-x||}\,.$$

The opposite inequality is easily obtained arguing similarly for $t<\overline{t}$. □

We denote by Σ_F the singular set of d_F, that is the set of all points $x\in\mathbb{R}^n\setminus F$ at which d_F fails to be differentiable. As observed in the Introduction, Σ_F has Lebesgue measure zero. More is actually true. In fact, the representation formula

$$d_F^2(x)-\|x\|^2=\min_{y\in F}\left(\|x-y\|^2-\|x\|^2\right)=\min_{y\in F}\left(\|y\|^2-2x\cdot y\right)$$

implies that $x\mapsto d_F^2(x)-\|x\|^2$ is concave. So, by known results for singular sets of concave functions (see, e.g., [11] and [2]), Σ_F can be covered by countably many Lipschitz hypersurfaces. In particular, the Hausdorff dimension of Σ_F is at most $N-1$, a result that was originally conjectured by Erdös in [4] (see also [8]).

We now recall a known relation between the family of all connected components of the singular set Σ_F and the family of the connected components of $\overline{\mathrm{co}}\,F\setminus F$. For this purpose, let us consider the map $\mathcal{M}:\mathcal{C}(\overline{\mathrm{co}}F\setminus F)\to\mathcal{P}(\Sigma_F)$ defined by

$$\mathcal{M}(S)=\Sigma_F\cap\mathrm{proj}^{-1}_{\overline{\mathrm{co}}F}(S)\qquad\forall S\in\mathcal{C}(\overline{\mathrm{co}}F\setminus F).$$

The following result was proved by Veselý in [9].

Lemma 2.2. *$\mathcal{M}$ is an injective map with range equal to $\mathcal{C}(\Sigma_F)$.*

For a given point $x\notin F$, consider a ball $B(x,r)\subseteq\mathbb{R}^n\setminus F$. Clearly, the family

$$\mathcal{B}(x,r):=\{B(y,R)\ :\ B(x,r)\subseteq B(y,R)\subseteq\mathbb{R}^n\setminus F\}$$

is non-empty. If $B(\widehat{x},\widehat{r})$ is an element of $\mathcal{B}(x,r)$ having maximal radius, then we say that $B(\widehat{x},\widehat{r})$ is a *maximal ball* containing $B(x,r)$. Obviously, the radius $\widehat{r}=\widehat{r}(x,r)$ of a maximal ball is a well-defined function of (x,r). On the other hand, the center of a maximal ball need not be unique. Nevertheless, the norm of the center of any maximal ball satisfies the lower bound

$$\|\widehat{x}\|\ge\|\widehat{x}-x_0\|-\|x_0\|\ge\widehat{r}-\|x_0\| \tag{2.1}$$

where x_0 is a fixed element of F.

The next lemma is a usual tool for proving Motzkin's Theorem [6]. Among other things, it ensures the existence of a maximal ball.

Lemma 2.3. *Let $x \in \overline{\mathrm{co}}\, F \setminus F$, and let $r > 0$ be such that $\overline{B(x,r)} \subseteq \mathbb{R}^n \setminus F$. Then, the following results hold.*

1. *There exists a maximal ball containing $B(x,r)$.*
2. *If $B(\widehat{x},\widehat{r})$ is a maximal ball containing $B(x,r)$, then $\partial B(\widehat{x},\widehat{r}) \cap F$ contains at least two different points, and so $\widehat{x} \in \Sigma_F$. If, in addition, $x \in \partial(\overline{\mathrm{co}}\, F) \setminus F$, then*

$$\lim_{r \downarrow 0} \|\widehat{x}\| = \infty\,. \tag{2.2}$$

Proof. We imitate the proof of Motzkin's Theorem given in [5]. First, let us prove that

$$K := \{(y,R) \in \mathbb{R}^n \times]0,\infty[\ :\ B(y,R) \in \mathcal{B}(x,r)\}$$

is compact. For this purpose, we observe that, for any $(y,R) \in \mathbb{R}^n \times \mathbb{R}$,

$$(y,R) \in K \qquad \Longleftrightarrow \qquad \|y-x\| \le R-r \quad \text{and} \quad d_F(y) \ge R\,. \tag{2.3}$$

In order to show that K is bounded, we note that there exist a point $z \in (\mathrm{co}\, F \setminus F) \cap B(x,r)$ and a positive number ε such that $B(z,\varepsilon) \subseteq B(x,r)$. Also, for a suitable choice of points $z_1, \dots, z_{n+1} \in F$, we have that $z \in \mathrm{co}\{z_1, \dots, z_{n+1}\}$. Set

$$A = \max\{\|z - z_i\|\ :\ 1 \le i \le n+1\}.$$

Let $(y,R) \in K$. Then

$$\|y - z\| \le R - \varepsilon \tag{2.4}$$

and

$$(y-z)\cdot(z_i - z) \ge 0 \tag{2.5}$$

for some $i \in \{1, \dots, n+1\}$. Indeed, (2.5) follows observing that $0 = (y-z)\cdot(z-z)$ is a convex combination of $\{(y-z)\cdot(z-z_i)\ :\ 1 \le i \le n+1\}$.

Now, by (2.4) and (2.5),

$$(R-\varepsilon)^2 \ge \|z-y\|^2 \ge \|y-z_i\|^2 - \|z_i - z\|^2 \ge R^2 - A^2.$$

So, $R \le \frac{\varepsilon^2 + A^2}{2\varepsilon}$. Hence, (2.3) implies that K is bounded and closed, thus compact. Therefore, a maximal ball does exist in $\mathcal{B}(x,r)$.

For the proof of the fact that $\partial B(\widehat{x},\widehat{r}) \cap F$ contains at least two different points the reader is referred to [5, p. 62].

Finally, let $x \in \partial(\overline{\mathrm{co}}\, F) \setminus F$ and fix $r_0 > 0$ so that $\overline{B(x,r_0)} \subseteq \mathbb{R}^n \setminus F$. Let θ be any unit vector in the normal cone to $\overline{\mathrm{co}}\, F$ at x, that is

$$\|\theta\| = 1 \qquad \text{and} \qquad (y-x)\cdot\theta \le 0 \quad \forall y \in \overline{\mathrm{co}} F\,. \tag{2.6}$$

We will show that

$$0 < r < r_0 \quad \Longrightarrow \quad \widehat{r} \ge \frac{r^2 + r_0^2}{2r}\,. \tag{2.7}$$

This, in view of (2.1), will imply the conclusion (2.2). To prove (2.7), let us define $B_R := B(x + (R-r)\theta, R)$ for any $R > r$, and observe that $B(x,r) \subseteq B_R$. Then, it suffices to check that

$$r < R \le \frac{r^2 + r_0^2}{2r} \quad \Longrightarrow \quad B_R \subseteq \mathbb{R}^n \setminus F\,. \tag{2.8}$$

Aiming at (2.8), let $z \in B_R \cap \overline{\text{co}}\, F$. Then, $\|z - x - (R - r)\theta\|^2 < R^2$. So, owing to (2.6),

$$\|z - x\|^2 < 2rR - r^2 \leq r_0^2$$

provided that R is chosen as in (2.8). Thus, $z \in B(x, r_0) \subseteq \mathbb{R}^n \setminus F$ and the proof is complete. □

Remark 2.4. We note that a simple consequence of (2.2) above is that Σ_F is unbounded whenever $\partial(\overline{\text{co}}\, F) \nsubseteq F$. This result was originally proved by Motzkin in [7] for F compact.

3. Connected Components of Σ_F

The main result of this section is the following characterization of the bounded connected components of Σ_F in terms of the one-to-one correspondence $\mathcal{M}$ defined in section 2.

Theorem 3.1. *Let $S \in \mathcal{C}(\overline{\text{co}}\, F \setminus F)$. Then $\mathcal{M}(S)$ is bounded if and only if S is bounded and $S \in \mathcal{C}(\mathbb{R}^n \setminus F)$.*

For the proof of Theorem 3.1 we need two simple lemmas.

Lemma 3.2. *Any two points $y \in S$ and $x \in \mathcal{M}(S)$ belong to the same connected component of $\mathbb{R}^n \setminus F$.*

Proof. First, by the very definition of $\mathcal{M}$, we have that $z := \text{proj}_{\overline{\text{co}}\, F}(x) \in S$. Moreover, $z \notin F$ as x is singular for d_F. Thus, $[x, z]$ is contained in $\mathbb{R}^n \setminus F$ and so x and z lie in the same connected component of $\mathbb{R}^n \setminus F$. Since $y, z \in S$ and S is a connected component of $\overline{\text{co}}\, F \setminus F$, the conclusion follows. □

We recall that the notion of maximal ball was introduced just before the statement of Lemma 2.3.

Lemma 3.3. *Let $y \in S$ and suppose $\overline{B(y, r)} \subseteq \mathbb{R}^n \setminus F$. If $B(\widehat{y}, \widehat{r})$ is a maximal ball containing $B(y, r)$, then $\widehat{y} \in \mathcal{M}(S)$.*

Proof. We have to show that $z := \text{proj}_{\overline{\text{co}}\, F}(\widehat{y})$ lies in S. For this purpose let us note that $z \in B(\widehat{y}, \widehat{r})$, as any closed maximal ball touches F at two points at least. Hence,

$$y, z \in B(\widehat{y}, \widehat{r}) \cap \overline{\text{co}}\, F\,.$$

Since $B(\widehat{y}, \widehat{r}) \cap \overline{\text{co}}\, F$ is a convex set contained in $\overline{\text{co}} F \setminus F$, we conclude that $z \in S$ as required. □

The following is a refinement of the result described in Remark 2.4.

Corollary 3.4. *If $\partial(\overline{\text{co}}\, F) \nsubseteq F$, then at least one connected component of Σ_F is unbounded.*

Proof. Take $y \in \partial(\overline{\text{co}}\, F) \setminus F$ and let S be the connected component of $\overline{\text{co}}\, F \setminus F$ containing y. By (2.2), $\mathcal{M}(S)$ is unbounded. □

We are now ready to prove the announced characterization.

Proof of Theorem 3.1. The proof of the "if" part is easy. Indeed, let S be a bounded connected component of $\mathbb{R}^n \setminus F$. Then, by Lemma 3.2, $\mathcal{M}(S) \subseteq S$.

The "only if" part will be proved by contradiction. So, suppose S is not a bounded component of $\mathbb{R}^n \setminus F$. We will then show that $\mathcal{M}(S)$ is unbounded. Aiming at this, let us distinguish two cases.

Case 1: S is an unbounded component of $\mathbb{R}^n \setminus F$. Let $z \in F$ be fixed. For $y \in S$, let $\overline{B(y,r)} \subseteq \mathbb{R}^n \setminus F$. If $B(\widehat{y},\widehat{r})$ is a maximal ball containing $B(y,r)$, then $\widehat{y} \in \mathcal{M}(S)$ by Lemma 3.3. Also, $||y-\widehat{y}|| \leq ||z-\widehat{y}||$ as $z \notin B(\widehat{y},\widehat{r})$. Thus,

$$||\widehat{y}|| \geq ||y|| - ||\widehat{y}-y|| \geq ||y|| - ||\widehat{y}-z|| \geq ||y|| - ||\widehat{y}|| - ||z||.$$

Therefore, $||\widehat{y}|| \geq \frac{1}{2}(||y|| - ||z||)$. Since S is unbounded, $||y||$ can be taken arbitrarily large. So, $\mathcal{M}(S)$ is unbounded.

Case 2: S is not a component of $\mathbb{R}^n \setminus F$. Let $\widetilde{S}$ be the component of $\mathbb{R}^n \setminus F$ containing S and fix $x_0 \in S$. Clearly, $\widetilde{S} \not\subseteq \overline{\mathrm{co}}\, F$ and so there exists a point $x_1 \in \widetilde{S} \setminus \overline{\mathrm{co}}\, F$. Let $\gamma : [0,1] \to \widetilde{S}$ be a continuous arc such that $\gamma(0) = x_0, \gamma(1) = x_1$ and define

$$\overline{t} = \inf\{t \in [0,1] \ : \ \gamma(t) \notin \overline{\mathrm{co}}\, F\}\,.$$

Then, $\gamma(\overline{t}) \in \overline{\mathrm{co}}\, F$ as $\overline{\mathrm{co}}\, F$ is closed. Furthermore, $\gamma(\overline{t}) \in \partial(\overline{\mathrm{co}}\, F)$ as a sequence $t_j \downarrow \overline{t}$ such that $\gamma(t_j) \notin \overline{\mathrm{co}}\, F$ does exist. On the other hand, $\gamma(t) \in \overline{\mathrm{co}}\, F$ for every $t \in [0,\overline{t}]$. Therefore, $\gamma(\overline{t})$ and x_0 belong to the same component of $\overline{\mathrm{co}}\, F \setminus F$. In conclusion, $\gamma(\overline{t}) \in S \in \mathcal{C}(\overline{\mathrm{co}}\, F \setminus F)$ and the proof can be easily completed applying Lemma 3.3 and (2.2). □

Remark 3.5. We note that the above proof (Case 2) also shows the following result: if $S \in \mathcal{C}(\overline{\mathrm{co}}\, F \setminus F)$ is not a connected component of $\mathbb{R}^n \setminus F$, then $S \cap \partial(\overline{\mathrm{co}}\, F)$ is nonempty.

Corollary 3.6. *Given* $G \in \mathcal{C}(\mathbb{R}^n \setminus F)$*, let*

$$\mathcal{R}(G) = \big\{\mathcal{M}(S) : S \in \mathcal{C}(\overline{\mathrm{co}}\, F \setminus F), S \subseteq G\big\}.$$

If G is bounded, then $\mathcal{R}(G)$ has precisely one element, which is bounded. If, on the contrary, G is unbounded, then all elements of $\mathcal{R}(G)$ are unbounded.

Proof. If G is bounded, we have that $G \subseteq \overline{\mathrm{co}}\, F$ for, by the Hahn-Banach Theorem, any $x \notin \overline{\mathrm{co}}\, F$ lies in an unbounded component of $\mathbb{R}^n \setminus F$. So, G is also a connected component of $\overline{\mathrm{co}}\, F \setminus F$. If, on the contrary, G is unbounded, then no set S in the definition of $\mathcal{R}(G)$ can be a bounded component of $\mathbb{R}^n \setminus F$. The conclusion follows directly from Theorem 3.1. □

In particular, we obtain the following.

Corollary 3.7. *All connected components of Σ_F are unbounded if and only if all connected components of $\mathbb{R}^n \setminus F$ are unbounded.*

We note that, in the above corollary, the singular set Σ_F may well be empty.

4. Asymptotic Behaviour of Σ_F

In this section we are interested in the behaviour at infinity of the unbounded components of the singular set of the distance. Hereafter, we shall assume that $n \geq 2$, the case $n = 1$ being trivial.

Theorem 4.1. *Let F be compact and let C be an unbounded connected component of Σ_F. Then there exists a half line $\{x_0 + tv \ : \ t \geq 0\}$, where $x_0, v \in\mathbb{R}^n$ and $\|v\| = 1$, such that $d_C(x_0 + tv) \to 0$ as $t \to +\infty$.*

The proof of the above theorem relies on the following simple property of compact sets.

Lemma 4.2. *Let $K \subseteq\mathbb{R}^n$ be a compact set satisfying*

$$x_n \leq 0 \qquad \forall x \in K$$

and

$$K_0 := \{x \in K \ : \ x_n = 0\} \neq \emptyset\,.$$

Then, for any $R > 0$,

$$\sup_{\|x\|\leq R,\, x_n=0} \ \sup_{y\in proj_K(x+te_n)} d_{K_0}(y) \to 0 \qquad as \quad t \to +\infty\,.$$

Proof. The conclusion will follow if we show that for any $\varepsilon > 0$ there exists $\alpha_\varepsilon > 0$ such that

$$y \in K,\ d_{K_0}(y) \geq \varepsilon \qquad \Rightarrow \qquad y_n \leq -\alpha_\varepsilon \tag{4.1}$$

and that, for any $R > 0$,

$$\sup_{\|x\|\leq R,\, x_n=0} \ \sup_{y\in \mathrm{proj}_K(x+te_n)} |y_n| \to 0 \tag{4.2}$$

as $t \to +\infty$. Also, it suffices prove (4.2) since (4.1) is an immediate consequence of the compactness of K.

Let $q \in K_0$ be fixed. Then, for any $R > 0$, any $x \in\mathbb{R}^n$ satisfying $\|x\| \leq R,\ x_n = 0$, and any $y \in \mathrm{proj}_K(x + te_n)$ we have that

$$|t - y_n| \leq \|x + te_n - y\| \leq \|x + te_n - q\| \leq \sqrt{\|x - q\|^2 + t^2} \leq \sqrt{(R + \|q\|)^2 + t^2}\,.$$

Therefore,

$$t - \sqrt{(R + \|q\|)^2 + t^2} \leq y_n \leq 0$$

and (4.2) follows as $t \to +\infty$. □

We are now ready to prove the existence of an asymptote.

Proof of Theorem 4.1. Let $S \in \mathcal{C}(\overline{\mathrm{co}}\, F \setminus F)$ be such that $C = \mathcal{M}(S)$. Since C is unbounded, Theorem 3.1 implies that either S is unbounded or $S \notin \mathcal{C}(\mathbb{R}^n \setminus F)$. On the other hand, $\overline{\mathrm{co}}\, F \setminus F$ is bounded as F is compact, and this shows that S cannot be a component of $\mathbb{R}^n \setminus F$.

Now, in view of Remark 3.5 there exists a point $\overline{x} \in S$ such that $\overline{x} \in \partial(\overline{\mathrm{co}}\, F) \setminus F$. Let Π be a support hyperplane to $\overline{\mathrm{co}}\, F$ through $\overline{x}$. Possibly changing coordinates, we can and do assume that

$$\overline{x} \in \Pi = \{x \in \mathbb{R}^n \ : \ x_n = 0\}$$

and

$$\overline{\mathrm{co}}\, F \subseteq \{x \in \mathbb{R}^n \ : \ x_n \leq 0\}.$$

Let us set

$$A := F \cap \Pi \qquad \text{and} \qquad B := (\overline{\mathrm{co}}\, F) \cap \Pi.$$

Notice that B is the closed convex hull of A. Indeed, the inclusion $\overline{\mathrm{co}}\, A \subseteq B$ is trivial. Conversely, let $x \in B$. Then, $x_n = 0$ and there exist $x^{(1)}, \ldots, x^{(k)} \in F$ and $\lambda_1, \ldots, \lambda_k > 0$ such that

$$x = \sum_{i=1}^{k} \lambda_i x^{(i)}\,, \qquad \sum_{i=1}^{k} \lambda_i = 1\,.$$

Hence, $x_n^{(i)} = 0$ for any $1 \le i \le k$, and so $x^{(1)}, \ldots, x^{(k)} \in A$.

Next, we observe that $\overline{x} \in B \setminus A$ by construction. Applying Lemma 2.2 to the set $A \subseteq \Pi \simeq \mathbb{R}^{n-1}$, we find a point $\overline{z} \in \Pi \cap \Sigma_A$ whose projection onto B lies in the same connected component of $B \setminus A$ as $\overline{x}$. Without loss of generality we will assume that $\overline{z} = 0$.

Let $R > 0$ be such that $B(R) \subseteq \mathbb{R}^n \setminus F$. Then, $d_A(0) \ge R$. We want to show that for any $0 < r < R$ there exists $T(r) > 0$ such that, for any $t \ge T(r)$, the translated disc $\big(B(r) \cap \Pi\big) + te_n$ contains a singular point $x(t) \in \Sigma_F$. We will split our reasoning into four steps.

Step 1: we claim that, for any $0 < r < R$, there exists a number $\sigma_r > 0$ such that

$$d_A(x) < r - \sigma_r + d_A(0) \qquad \forall x \in \Pi \cap \partial B(r)\,. \tag{4.3}$$

Indeed, let $p, p' \in A$ ($p \neq p'$) be such that

$$\|p\| = d_A(0) = \|p'\|\,.$$

For any $x \in \Pi \cap \partial B(r)$ we have that either $0 \notin [x, p]$ or $0 \notin [x, p']$. Assume that $0 \notin [x, p]$. Then,

$$d_A(x) \le \|p - x\| < \|p\| + \|x\| = d_A(0) + r\,.$$

A compactness argument yields (4.3).

Step 2: for $0 < r < R$ and $t \ge 0$ let us define

$$\alpha_r(x) = \mathrm{proj}_{\Pi \cap \overline{B(r)}}(x), \qquad x \in \mathbb{R}^n,$$

and

$$\phi_t(x) = \mathrm{proj}_F(x + te_n), \qquad x \in \Pi \cap \overline{B(r)}\,.$$

Notice that α_r is, in fact, single-valued and continuous. Moreover,

$$\alpha_r = \alpha_r \circ \mathrm{proj}_\Pi\,.$$

Map ϕ_t, on the contrary, is set-valued in general. Let us denote by Φ the set of all $t \ge 0$ such that ϕ_t is single-valued. Then, the product map

$$\beta_{r,t} := \alpha_r \circ \phi_t : \Pi \cap \overline{B(r)} \to \Pi \cap \overline{B(r)}$$

is well-defined and continuous for any $0 < r < R$ and $t \in \Phi$.

Step 3: let $r \in]0, R[$ be fixed and set $\rho := \min\{R - r, \sigma_r\}$, with σ_r given by Step 1. Then, by Lemma 4.2, there exists $T(r) > 0$ such that

$$d_A(\phi_t(x)) < \rho \qquad \forall x \in \Pi \cap \overline{B(r)} \tag{4.4}$$

for every $t \in \Phi$ satisfying $t \ge T(r)$. We claim that, for such values of t,

$$\beta_{r,t}(x) \in \partial B(r) \cap \Pi \qquad \forall x \in \Pi \cap \overline{B(r)} \tag{4.5}$$

and

$$\beta_{r,t}(x) \neq -x \qquad \forall x \in \Pi \cap \partial B(r)\,. \tag{4.6}$$

To prove our first claim, let $x \in \Pi \cap \overline{B(r)}$ be fixed. In view of (4.4) we have that, for some $y \in A$, $\|y - \phi_t(x)\| < \rho$. Set

$$x(t) := \mathrm{proj}_\Pi(\phi_t(x)) \,. \tag{4.7}$$

Then, $\|y - x(t)\| \leq \|y - \phi_t(x)\| < \rho$, and so

$$\|x(t)\| > \|y\| - \rho \geq R - \rho \geq r \,. \tag{4.8}$$

Inclusion (4.5) easily follows.

We now turn to the proof of (4.6). Suppose that a point x exists on the sphere $\Pi \cap \partial B(r)$ such that $\beta_{r,t}(x) = -x$. We will show that this gives a contradiction. In fact, since $x(t)$ is a negative multiple of x, using (4.8) we have that

$$\|x - x(t)\| = r + \|x(t)\| \geq r + \|y\| - \rho \geq r + d_A(0) - \sigma_r > 0 \,.$$

Therefore, recalling (4.7),

$$\|x + te_n - \phi_t(x)\| \geq \sqrt{\|x - x(t)\|^2 + t^2} \geq \sqrt{(r + d_A(0) - \sigma_r)^2 + t^2} \,. \tag{4.9}$$

On the other hand, owing to (4.3) we can find a point $y' \in A$ such that

$$\|x + te_n - y'\| < \sqrt{(r + d_A(0) - \sigma_r)^2 + t^2} \,. \tag{4.10}$$

Since (4.9) and (4.10) contradict the definition of ϕ_t, the proof of (4.6) is complete.

Step 4: we claim that, if $t \geq T(r)$, then $t \notin \Phi$. In other words, we will show that the projection map ϕ_t must be multi-valued at some point for sufficiently large t. For suppose not. Then, on account of (4.5) and (4.6), the continuous map $-\beta_{r,t}$ would have no fixed points in $\Pi \cap \overline{B(r)}$, contrary to Brouwer's Theorem. Hence, our claim follows.

From Steps 1 to 4 above we conclude that, for any $r > 0$, there exists $T(r) > 0$ such that, for any $t \geq T(r)$, a singular point $\xi(t) \in \Sigma_F$ exists so that $\|\xi(t) - te_n\| \leq r$. In order to complete the proof, it remains to show that $\xi(t)$ belongs to the connected component C of Σ_F, that we fixed at the beginning. This follows from our next lemma. □

Lemma 4.3. *Let $S, \overline{x}, \Pi, e_n$ be defined as in the proof of Theorem 4.1. Then there exist $r_0, T_0 > 0$ such that, for any $t \geq T_0$,*

$$\mathrm{proj}_{\overline{\mathrm{co}}\, F}(x + te_n) \in S \qquad \forall x \in \Pi \cap \overline{B(r_0)}.$$

Proof. First, without loss of generality, we will assume that

$$\overline{x} = \mathrm{proj}_B(0) \,. \tag{4.11}$$

In fact, since $\overline{x}$ and $\mathrm{proj}_B(0)$ belong to the same connected component of $B \setminus A$, they belong to the same component of $\overline{\mathrm{co}}\, F \setminus F$ as well.

Now, let $B(\overline{x}, \rho) \subseteq \mathbb{R}^n \setminus F$. Then,

$$B(\overline{x}, \rho) \cap \overline{\mathrm{co}}\, F \subseteq S \,. \tag{4.12}$$

Therefore, in view of (4.11), a compactness argument shows that

$$\|y\| \geq \|\overline{x}\| + \sigma \qquad \forall y \in B \setminus B(\overline{x}, \rho)$$

for some $\sigma > 0$. Furthermore, by a continuity argument we conclude that, for some $\eta > 0$, all points $y \in \overline{\mathrm{co}}\, F \setminus B(\overline{x}, \rho)$, with $y_n \geq -\eta$, satisfy the estimate below

$$\|\mathrm{proj}_\Pi(y)\| \geq \|\overline{x}\| + \frac{\sigma}{2} \,.$$

By another continuity argument, the last inequality can be easily refined as follows: there exists $r_0 > 0$ such that

$$\|\mathrm{proj}_\Pi(y) - x\| > \|\overline{x} - x\| \qquad \forall x \in \Pi \cap \overline{B(r_0)} \tag{4.13}$$

for any $y \in \overline{\mathrm{co}}\, F \setminus B(\overline{x}, \rho)$ satisfying $y_n \geq -\eta$.

Next, by Lemma 4.2 applied to $K = \overline{\mathrm{co}}\, F$ we deduce that, if t is sufficiently large, say $t \geq T_0$, then

$$e_n \cdot \mathrm{proj}_{\overline{\mathrm{co}}\, F}(x + te_n) \geq -\eta \qquad \forall x \in \Pi \cap \overline{B(r_0)} \tag{4.14}$$

where $\eta > 0$ is fixed as in (4.13).

To complete the proof of the lemma, in view of (4.12), it suffices to show that, for $t \geq T_0$,

$$\mathrm{proj}_{\overline{\mathrm{co}}\, F}(x + te_n) \in B(\overline{x}, \rho) \qquad \forall x \in \Pi \cap \overline{B(r_0)}\,. \tag{4.15}$$

We now proceed to check (4.15). Let $t \geq T_0$ and $x \in \Pi \cap \overline{B(r_0)}$ be fixed, and set $z := \mathrm{proj}_{\overline{\mathrm{co}}\, F}(x + te_n)$. Suppose $z \notin B(\overline{x}, \rho)$. Then, owing to (4.13) and (4.14), we have

$$\|\mathrm{proj}_\Pi(x + te_n - z)\| = \|x - \mathrm{proj}_\Pi(z)\| > \|\overline{x} - x\| = \|\mathrm{proj}_\Pi(x + te_n - \overline{x})\|\,.$$

Moreover, $|(x + te_n - z)_n| \geq t = |(x + te_n - \overline{x})_n|$. So,

$$\|x + te_n - z\| > \|x + te_n - \overline{x}\|$$

in contrast with the definition of z. □

5. Asymptotic Behaviour in Dimension Two

In this last section we shall restrict our attention to the two-dimensional case for which stronger results can be obtained. Let us recall that Theorem 4.1 ensures the existence of a half line in $\mathbb{R}^n$, asymptotic to any unbounded connected component, C, of Σ_F. However, from the proof of this theorem it actually follows that there exists $\phi : [0, +\infty[\to v^\perp$, with $\phi(t) \to 0$ as $t \to +\infty$, such that $x_0 + tv + \phi(t) \in C$. As we will see next, when $n = 2$ one can show that ϕ is continuous.

Theorem 5.1. *Let $F \subseteq \mathbb{R}^2$ be a compact set, and C be an unbounded connected component of Σ_F. Then there exists a half line $\{x_0 + tv \;:\; t \geq 0\}$, where $x_0, v \in \mathbb{R}^2$ and $\|v\| = 1$, and a continuous map $\phi : [0, +\infty[\to v^\perp$ such that*

$$x_0 + tv + \phi(t) \in C \qquad \textit{and} \qquad \phi(t) \to 0 \quad \textit{as} \quad t \to +\infty.$$

Proof. We shall use the same notation as in the proof of Theorem 4.1. Notice that, in this special case, Π is a straight line, $0 \in \Pi \cap \Sigma_A$, and $\mathrm{proj}_A(0) = \{(a, 0), (-a, 0)\}$ for some $a > 0$.

Let $0 < r < a$. By Lemma 4.2 we deduce that there exists $\tau(r) > 0$ such that, for any $t \geq \tau(r)$ and any $s \in [-r, r]$,

$$d_A(y) < a - r \qquad \forall y \in \mathrm{proj}_F(se_1 + te_2)\,.$$

Hence, for all such values of t and s,

$$\mathrm{proj}_F(se_1 + te_2) \subseteq F_1 \cup F_2 \tag{5.1}$$

where

$$F_1 := \{x \in F \;:\; x_1 \leq -r\}\,, \qquad F_2 := \{x \in F \;:\; x_1 \geq r\}\,.$$

Therefore,

$$d_{F_1}(se_1+te_2)=d_{F_2}(se_1+te_2) \qquad \Longrightarrow \qquad se_1+te_2\in\Sigma_F \tag{5.2}$$

for any $t\geq\tau(r)$ and any $s\in[-r,r]$.

Now, let us define

$$f_t(s)=d_{F_1}(se_1+te_2)-d_{F_2}(se_1+te_2) \qquad (t\geq\tau(r), -r\leq s\leq r)\,.$$

We claim that, if t is sufficiently large, say $t\geq T(r)\geq\tau(r)$,
(i) f_t is strictly increasing;
(ii) $f_t(-r)<0\,,\ f_t(r)>0$.
Indeed, let $-r\leq s_1<s_2\leq r$ and let $y\in\mathrm{proj}_{F_1}(s_2e_1+te_2)$. Then

$$d_{F_1}(s_1e_1+te_2)\leq\|s_1e_1+te_2-y\|<\|s_2e_1+te_2-y\|=d_{F_1}(s_2e_1+te_2)\,.$$

Similarly, $d_{F_2}(s_1e_1+te_2)>d_{F_2}(s_2e_1+te_2)$. So, our claim (i) follows.

Next, to check that $f_t(r)>0$, we can apply Lemma 4.2 to prove the existence of a number $T(r)\geq\tau(r)$ such that

$$d_{F_1\cap\Pi}(y)<r \qquad \forall y\in\mathrm{proj}_{F_1}(se_1+te_2) \quad \forall s\in[-r,r]$$

for every $t\geq T(r)$. Also, observe that $x_1\leq -a$ for any $x\in F_1\cap\Pi$. Thus,

$$y_1\leq r-a \qquad \forall y\in\mathrm{proj}_{F_1}(se_1+te_2) \quad \forall s\in[-r,r]\,.$$

Consequently,

$$\|re_1+te_2-y\|=\sqrt{(r-y_1)^2+(t-y_2)^2}\geq\sqrt{a^2+t^2}$$

for any $y\in\mathrm{proj}_{F_1}(re_1+te_2)$ and any $t\geq T(r)$. Therefore,

$$\begin{aligned}d_{F_1}(re_1+te_2)&\geq\sqrt{a^2+t^2}>\sqrt{(a-r)^2+t^2}\\&=\|re_1+te_2-(a,0)\|\geq d_{F_2}(re_1+te_2)\end{aligned}$$

and so $f_t(r)>0$, for any $t\geq T(r)$. Since the inequality $f_t(-r)<0$ can be obtained similarly, the proof of (ii) is complete.

In conclusion, from properties (i) and (ii) above we deduce that, for any $t\geq T(r)$, the function f_t has a unique zero in $[-r,r]$, say $s(t)$. Moreover, as one can easily check, $s(t)$ depends continuously on t. In view of (5.2), we have thus shown that for any $r\in]0,a[$ there exists $T(r)>0$ such that

$$s(t)e_1+te_2\in\Sigma_F \qquad \forall t\geq T(r)\,.$$

To construct the desired map, it suffices to fix $r=r_0\in]0,a[$ and take

$$x_0=T(r_0)e_2\,, \qquad v=e_2\,, \qquad \phi(t)=s(t+T(r_0))e_1\,.$$

The fact that $x_0+tv+\phi(t)\in C$ follows from Lemma 4.3. □

Remark 5.2. In the sequel, we will need a property that we have derived, incidentally, in the proof of Theorem 5.1, namely that $\mathrm{proj}_F(x_0+tv+\phi(t))$ tends to the set $\{(a,0),(-a,0)\}$ as $t\to+\infty$. More precisely, we claim that the distance of $(a,0)$ from $\mathrm{proj}_{F_2}(x_0+tv+\phi(t))$ tends to 0 as $t\to+\infty$, and so does the distance of $(-a,0)$ from $\mathrm{proj}_{F_1}(x_0+tv+\phi(t))$. In fact, by (5.1), for any $t\geq\tau(r), s\in[-r,r]$, and any $x\in\mathrm{proj}_F(se_1+te_2)$, we have that $|x_1|\geq r$. Also, $|x_1|\leq a$, for otherwise either $(a,0)$ or $(-a,0)$ would be closer than x to se_1+te_2. Since, in view of Lemma 4.2, the e_2-component of any $x\in\mathrm{proj}_F(se_1+te_2)$ tends to 0 as $t\to+\infty$, the claimed property follows.

Theorem 5.1 ensures that, if $n = 2$, then any unbounded connected component of Σ_F contains at least a continuous arc that propagates up to ∞. On the other hand, it is known that any point $\overline{x} \in \Sigma_F$ satisfying $\overline{x} \notin \overline{\mathrm{co}}\, F$ is the initial point of a Lipschitz singular arc $\xi(\cdot)$ for d_F, see [3] for the case of $n = 2$ and [1] for general n. Moreover, $\xi(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} \xi'(s) \in D^+ d_F(\xi(s)), & s \in [0, \sigma] \quad \text{a.e.}, \\ \xi(0) = \overline{x} \end{cases} \tag{5.3}$$

for some $\sigma > 0$. Here, $D^+ d_F(x)$ denotes the superdifferential of d_F at x, which in this case can also be represented as

$$D^+ d_F(x) = \frac{x - \mathrm{co}(\mathrm{proj}_F(x))}{d_F(x)} \qquad (x \notin F)\,.$$

Notice that, in particular, $D^+ d_F(x) \subseteq \overline{B}_1$, so that the solution ξ of (5.3) is Lipschitz continuous with constant equal to 1.

It is then natural to ask whether the two singular arcs above—i.e. the one given by Theorem 5.1 and the solution of problem (5.3)—coincide up to reparametrization. The following result is intended to answer such a question.

Theorem 5.3. *Let $F \subseteq \mathbb{R}^2$ be a compact set, let C be an unbounded connected component of Σ_F, and let*

$$\gamma(t) := x_0 + tv + \phi(t) \qquad (t \geq 0)$$

be the continuous singular arc given by Theorem 5.1. Then there exists $\overline{t} \geq 0$ such that the solution ξ of problem (5.3) with $\overline{x} = \gamma(\overline{t})$ is defined for any $s \geq 0$ and coincides with $\gamma(t), t \geq \overline{t}$, up to reparametrization.

Proof. We shall use the same notation as in the proof of Theorem 5.1. In particular, we assume, without loss of generality, that $x \cdot e_2 \leq 0$ for all $x \in F$. Moreover, having set $A = \{x \in F \,:\, x \cdot e_2 = 0\}$, we suppose that $\mathrm{proj}_A(0) = \{(a, 0), (-a, 0)\}$ for some $a > 0$. Notice that $\gamma(t) = s(t + T(r))e_1 + (t + T(r))e_2$, where $r \in]0, a[$ is fixed and $|s(t)| < a$. We also define sets F_1, F_2 as

$$F_1 := \{x \in F \,:\, x_1 \leq -r\}\,, \qquad F_2 := \{x \in F \,:\, x_1 \geq r\}\,.$$

To begin the proof we note that, in view of Remark 5.2, for sufficiently large t, say $t \geq \overline{t}$, we have that

$$\mathrm{proj}_{F_1}(\gamma(t)) \subseteq B_{\frac{a}{3}}(-a, 0) \qquad \text{and} \qquad \mathrm{proj}_{F_2}(\gamma(t)) \subseteq B_{\frac{a}{3}}(a, 0)\,. \tag{5.4}$$

Let ξ be the solution of problem (5.3) with $\overline{x} = \gamma(\overline{t})$. For technical reasons, we will assume—without loss of generality—that $\overline{t} \geq a$.

The core of the proof will consist in showing that, for some $\delta \in]0, \sigma]$,

$$d_{F_1}(\xi(s)) = d_{F_2}(\xi(s)) \qquad \forall s \in [0, \delta], \tag{5.5}$$

$$|\xi_1(s)| \leq r \qquad \forall s \in [0, \delta] \tag{5.6}$$

and

$$\xi_2'(s) \geq \frac{1}{2}, \qquad s \in [0, \delta] \quad \text{a.e.} \tag{5.7}$$

To check the above claims, we note first that the proof of Theorem 5.1 actually shows that $|\gamma_1(\overline{t})| = |\xi_1(0)| < r$ provided that δ above is properly chosen. Thus, estimate (5.6) follows.

Next, to check (5.7), we observe that, for any $s \in [0, \delta]$ satisfying the inclusion in (5.3), we have that

$$\xi'(s) = \frac{\xi(s) - w(s)}{d_F(\xi(s))}, \tag{5.8}$$

where $w(s) \in \text{co}\big(\text{proj}_F(\xi(s))\big)$. Hence,

$$\xi_2'(s) = \frac{\xi_2(s) - w_2(s)}{d_F(\xi(s))} \geq \frac{\xi_2(s)}{d_F(\xi(s))} \quad \text{a.e. in} \quad [0, \delta].$$

Recalling that, on account of (5.6), $|\xi_1(s)| \leq a$, we conclude that

$$d_F(\xi(s)) \leq \min\left\{||\xi(s) - (a, 0)||, ||\xi(s) - (-a, 0)||\right\} \leq \sqrt{\xi_2^2(s) + a^2}.$$

Hence,

$$\xi_2'(s) \geq \frac{\xi_2(s)}{\sqrt{\xi_2^2(s) + a^2}} =: g(s) \quad \text{a.e. in} \quad [0, \delta].$$

Now, $\xi_2(0) \geq \bar{t} \geq a$ as $\xi(0) = (\bar{t} + T(r))e_2 + s(\bar{t} + T(r))e_1$. So, $g(0) \geq 1/\sqrt{2}$. Since g is continuous, we conclude that (5.7) is satisfied possibly reducing $\delta > 0$.

We now turn to the proof of (5.5). Let us set $\alpha_i(s) = d_{F_i}\big(\xi(s)\big)$, for $i = 1, 2$. Then, it suffices to show that

$$\big(\alpha_1(s) - \alpha_2(s)\big)\big(\alpha_1'(s) - \alpha_2'(s)\big) \leq 0 \tag{5.9}$$

for every $s \in]0, \delta[$ at which ξ, α_1, α_2 are differentiable, and inclusion (5.3) is satisfied. Indeed, from (5.9) we deduce that the function $\big(\alpha_1(s) - \alpha_2(s)\big)^2$ has nonpositive derivative a.e. in $[0, \delta]$. Thus, it is decreasing. Since it takes the value 0 at 0, it vanishes identically in $[0, \delta]$. We now proceed to prove (5.9). Let s be as above, and suppose $\alpha_2(s) < \alpha_1(s)$. Then, thanks to (5.6), (5.7) and (5.1),

$$\text{proj}_F(\xi(s)) = \text{proj}_{F_2}(\xi(s)). \tag{5.10}$$

Now, fix $x_{s,i} \in \text{proj}_{F_i}(\xi(s))$. By Lemma 2.1 we have

$$\alpha_i'(s) = \frac{\xi'(s) \cdot (\xi(s) - x_{s,i})}{\alpha_i(s)}.$$

Hence, by (5.8),

$$\begin{aligned} \alpha_i'(s) &= \frac{(\xi(s) - w(s)) \cdot (\xi(s) - x_{s,i})}{\alpha_i(s) d_F(\xi(s))} \\ &= \frac{||\xi(s) - w(s)||^2 + \alpha_i(s)^2 - ||w(s) - x_{s,i}||^2}{2\alpha_i(s) d_F(\xi(s))}. \end{aligned} \tag{5.11}$$

By (5.4) and the upper semicontinuity of the projection on a closed set we have, possibly taking a smaller $\delta > 0$,

$$\text{proj}_{F_1}(\xi(s)) \subseteq B_{\frac{a}{3}}(-a, 0) \qquad \text{and} \qquad \text{proj}_{F_2}(\xi(s)) \subseteq B_{\frac{a}{3}}(a, 0).$$

Therefore, $||w(s) - x_{s,2}|| \leq \frac{2}{3}a$, as $x_{s,2} \in \text{proj}_{F_2}(\xi(s))$ and $w(s) \in \text{co}\big(\text{proj}_{F_2}(\xi(s))\big)$ in view of (5.10). By similar considerations, $||w(s) - x_{s,1}|| \geq \frac{4}{3}a$. Since $\alpha_1(0) = \alpha_2(0)$, as $\xi(0) \in \text{Im}\gamma$, (5.11) yields $\alpha_2'(s) > \alpha_1'(s)$ for $0 < s < \delta$, provided δ is sufficiently small. The proof of (5.9) is complete in the case $\alpha_2(s) < \alpha_1(s)$. The opposite case $\alpha_2(s) < \alpha_1(s)$, can be treated similarly. We have thus proved (5.5).

Now, let $\xi : [0, \tau[\to \mathbb{R}^2$ be the maximal solution of problem (5.3), with $\overline{x} = \gamma(\bar{t})$, satisfying (5.5), (5.6) and (5.7). We claim that $\tau = +\infty$, which yields in turn the

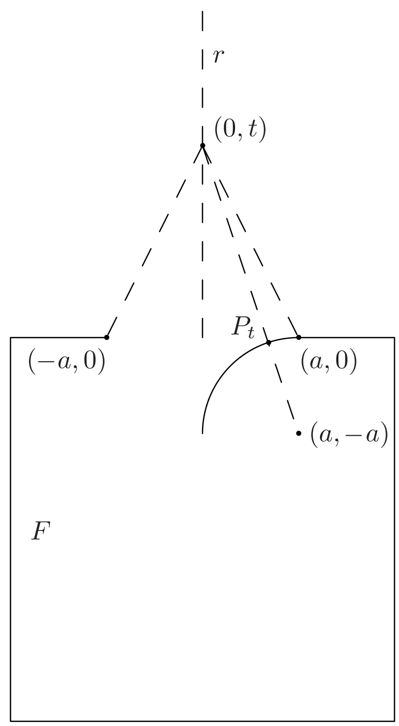

FIGURE. Here, $P_t = \mathrm{proj}_F(0, t)$

conclusion of the theorem. For suppose $\tau < +\infty$. Then, owing to (5.5) and (5.6), $\xi(\tau^-) := \lim_{t\uparrow\tau} \xi(t) \in \gamma([\overline{t}, +\infty[)$. So, we can solve problem (5.3) with $\overline{x} = \xi(\tau^-)$, obtaining a contradiction with the maximality of ξ. □

We conclude the paper with an example showing that an unbounded component of Σ_F may well possess an asymptote no point of which belongs to Σ_F.

Example 5.4. Let F be the compact subset of $\mathbb{R}^2$ pictured in the Figure, the round part of which consists of an arc of circle of radius a, centered at $(a, -a)$

From the proof of Theorem 5.1 it follows that the asymptote of the unbounded connected component of Σ_F (which in fact coincides with Σ_F itself) is given by the half-line $r = \{(0, t) \ : \ t \geq 0\}$. On the other hand, a simple geometric argument shows that $\Sigma_F \cap r = \emptyset$. Indeed, the projection onto F of any point $(0, t) \in r$ is given by the intersection P_t of F with the straight line through $(0, t)$ and $(a, -a)$.

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Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma (Italy)

E-mail address: `cannarsa@ mat.uniroma2.it`

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma (Italy)

E-mail address: `peirone@ mat.uniroma2.it`