TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 11, Pages 4637–4660
S 0002-9947(01)02874-4
Article electronically published on July 3, 2001

ISOMETRIES OF HILBERT C^*-MODULES

BARUCH SOLEL

Abstract. Let X and Y be right, full, Hilbert C^*-modules over the algebras A and B respectively and let $T : X \to Y$ be a linear surjective isometry. Then T can be extended to an isometry of the linking algebras. T then is a sum of two maps: a (bi-)module map (which is completely isometric and preserves the inner product) and a map that reverses the (bi-)module actions. If A (or B) is a factor von Neumann algebra, then every isometry $T : X \to Y$ is either a (bi-)module map or reverses the (bi-)module actions.

1. Introduction

Given a right Hilbert C^*-module X over a C^*-algebra A it is a module over A and has an A-valued inner product. One then defines the norm of X using the inner product and it makes X a Banach space. It is known that once the module structure and Banach space structure are given (for a C^*-module X) the A-valued inner product is uniquely defined. This was proved by Lance in [L1, Theorem] and, shortly afterwards, by Blecher in [B1, Theorems 3.1 and 3.2]. In fact, as Blecher showed, the inner product can be recovered from the module and Banach space structures. The results of Lance and Blecher can be stated as follows.

Theorem 1.1 ([B2], [L1])**.** *Let X_1 and X_2 be right Hilbert C^*-modules over a C^*-algebra A and let $S : X_1 \to X_2$ be a surjective isometry which is an A-module map. Then S preserves the inner product, i.e. $\langle Sx, Sy\rangle_2 = \langle x, y\rangle_1$ (where $\langle \cdot, \cdot\rangle_j$ is the inner product in X_j). Moreover, the inner product of a right Hilbert C^*-module X over A can be recovered from the norm and the module structure by*

$$\langle x, x\rangle = \sup\{r(x)^* r(x) : r \text{ is an } A\text{-module map: } X \to A, \|r\| \le 1\}$$

and

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle x + i^k y,\ x + i^k y\right\rangle \quad (i = \sqrt{-1}).$$

Another proof can be found in [F, Theorem 5]. One can modify the first part of the theorem for the case where X_1 is a C^*-module over A and X_2 is a C^*-module over B and S is a module map in the sense that there is a *-isomorphism $\alpha : A \to B$ such that $S(xa) = (Sx)\alpha(a)$. In this case S satisfies $\langle Sx, Sy\rangle_2 = \alpha(\langle x, y\rangle_1)$. (See [MS, Lemma 5.10].)

Received by the editors June 26, 2000.
2000 *Mathematics Subject Classification.* Primary 46L08.
Supported by Technion V.P.R. Fund–Steigman Research Fund, Technion V.P.R. Fund–Fund for the Promotion of Sponsored Research and the Fund for the Promotion of Research at the Technion.

In the present paper we study to what extent it is possible to recover the C^*-module structure from the Banach space structure alone. In other words, given an isometry T (linear and surjective) of a C^*-module X over A onto a C^*-module Y over B, is it a module map? i.e. can we find a *-isomorphism α of A onto B with $T(xa) = T(x)\alpha(a)$? If we can, then, as we mentioned above, we have $\langle Tx, Ty\rangle = \alpha(\langle x, y\rangle)$. Also, denoting by $y \otimes z^*$ (for $y, z \in X$) the operator on X defined by $(y \otimes z^*)(x) = y\,\langle z, x\rangle$, we have

$$\begin{aligned} T\big((y\otimes z^*)(x)\big) &= T\big(y\,\langle z,x\rangle\big) = T(y)\alpha\big(\langle z,x\rangle\big)\\ &= T(y)\,\langle T(z),T(x)\rangle = \big(T(y)\otimes T(z)^*\big)(T(x)). \end{aligned}$$

The operators of the form $y \otimes z^*,\ y, z \in X$, generate a C^*-algebra denoted by $\mathbb{K}(X)$. It is possible to show that, for T as above, $\beta(y \otimes z^*) = Ty \otimes (Tz)^*$ defines a *-isomorphism $\beta : \mathbb{K}(X) \to \mathbb{K}(Y)$. The computation above shows that $T(Kx) = \beta(K)Tx$ for $x \in X,\ K \in \mathbb{K}(X)$. The relationship we have now between T, α on β can be summarized by considering the *-algebra $\mathcal{L}(X)$ defined by

$$\mathcal{L}(X) = \begin{pmatrix} \mathbb{K}(X) & X \\ \bar{X} & A \end{pmatrix}$$

(where $\bar{X}$ and the product and involution on $\mathcal{L}(X)$ will be defined shortly) and noting that the map $\psi : \mathcal{L}(X) \to \mathcal{L}(Y)$ defined by

$$\psi\begin{pmatrix} K & x \\ \bar{y} & a \end{pmatrix} = \begin{pmatrix} \beta(K) & Tx \\ \overline{Ty} & \alpha(a) \end{pmatrix}$$

is a *-isomorphism.

Hence to say that T preserves the C^*-module structure amounts to saying that T can be extended to a *-isomorphism of $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$.

By considering the transpose map that maps the Hilbert column space H^c (a right C^*-module over $\mathbb{C}$, isometric to a Hilbert space H) onto the Hilbert row space H^r (a right Hilbert C^*-module over $K(H)$) it is clear that we don't always have such a *-isomorphism.

Our main result, Theorem 3.2, shows that, if X and Y are full, T can always be extended to an isometry of $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$.

The celebrated result of Kadison [K, Theorem 7] states that every unital isometry of unital C^*-algebras is a selfadjoint Jordan map. For von Neumann algebras we can, in fact, decompose the algebras as a direct sum of two summands. On one summand the map is a *-isomorphism and on the other it is a *-anti-isomorphism ([K, Theorem 10]). A similar result was proved also for isometries of some non-selfadjoint operator algebras ([S]). For an isometry T of self-dual C^*-modules over von Neumann algebras we find that T can be written as a sum of an isometry which is a module map (and preserves the inner product) and an isometry that is, in some sense, an anti-module-map. (For a precise statement see Corollary 2.25.) The case of (not necessarily self-dual) Hilbert C^*-modules over general C^*-algebras is similar except that the decomposition of X is done by a projection in the enveloping von Neumann algebra of $\mathcal{L}(X)$ (Theorem 3.2).

As a corollary we show that, if we assume that the isometry T is in fact a 2-isometry (i.e., the map $I \otimes T : M_2 \otimes X \to M_2 \otimes Y$ is an isometry), then T preserves the C^*-module structure (Corollary 3.4). In particular, a 2-isometry of Hilbert C^*-modules is necessarily a complete isometry.

After this work was completed it was pointed out to us by D. Blecher that M. Hamana had previously proved this last fact using different methods [Ha].

Also we show that, for a given Hilbert space H, H^c and H^r are the only Hilbert C^*-modules that are isometric to H (Corollary 3.6).

Now we turn to set some notation and recall the definitions that we need.

Definitions. (1) A right pre-Hilbert C^*-module over a C^*-algebra A is a right-module X equipped with a map $\langle \cdot,\cdot \rangle : X \times X \to A$ satisfying:

(i) $\langle x,x\rangle \geq 0,\ x \in X,$ and $\langle x,x\rangle = 0$ only if $x=0$.
(ii) $\langle x,y\rangle^* = \langle y,x\rangle,\quad y,x \in X.$
(iii) $y \longmapsto \langle x,y\rangle$ is a linear map for all $x \in X$.
(iv) $\langle x,ya\rangle = \langle x,y\rangle a,\quad x,y \in X,\ a \in A.$

(2) The norm on a pre-Hilbert C^*-module X over A is defined by $\|x\| = \| \langle x,x\rangle \|^{\frac{1}{2}}$. If X is complete with respect to this norm, then X is said to be a (right) Hilbert C^*-module over A.

(3) A Hilbert C^*-module X over A is said to be full if

$$A = \overline{\operatorname{span}}\{\langle x,y\rangle : x,y \in X\}.$$

One can define left Hilbert C^*-module similarly. X is then a left A-module and the inner product is assumed to be linear in the first entry. Also $\langle ax,y\rangle = a\langle x,y\rangle$.

Given a right Hilbert C^*-module X over A we define $\bar{X}$, the *conjugate module*, as follows. As a set we write $\bar{X} = \{\bar{x} : x \in X\}$. The linear structure is defined by $\overline{\lambda x + y} = \bar{\lambda}\bar{x} + \bar{y}$. $\bar{X}$ becomes a left A-module when we set

$$a \cdot \bar{x} = \overline{xa^*}$$

and the A-valued inner product is

$$\langle \bar{x},\bar{y}\rangle = \langle x,y\rangle .$$

This makes $\bar{X}$ a left Hilbert C^*-module over A.

From now on, unless we say otherwise, all Hilbert C^*-modules are assumed to be right modules and full.

A bounded module map $T : X \to X$ (where X is a Hilbert C^*-module) is said to be adjointable if there exists a map $T^* : X \to X$ with $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all x,y in X. The set of all adjointable maps on X is a C^*-algebra (with respect to the operator norm) and is denoted $\mathbb{B}(X)$.

Given X and Y in X we can define an adjointable operator $x \otimes y^* \in \mathbb{B}(X)$ by

$$x \otimes y^*(z) = x\langle y,z\rangle .$$

(Another notation frequently used for this operator is $\theta_{x,y}$.) The C^*-subalgebra generated by these operators will be written $\mathbb{K}(X)$. Elements of $\mathbb{K}(X)$ are sometimes referred to as “compact operators”. If H is a Hilbert space, viewed as a C^*-module over $\mathbb{C}$, then $\mathbb{K}(X) = K(H)$, the algebra of compact operators on H. In general $\mathbb{K}(X) \neq \mathbb{B}(X)$.

Given a Hilbert C^*-module X over A one can form

$$\mathcal{L}(X) = \begin{pmatrix} \mathbb{K}(X) & X \\ \bar{X} & A \end{pmatrix}.$$

Then $\mathcal{L}(X)$ is a *-algebra with product and involution defined by

$$\begin{pmatrix} K_1 & x_1 \\ \bar{y}_1 & a_1 \end{pmatrix}\begin{pmatrix} K_2 & x_2 \\ \bar{y}_2 & a_2 \end{pmatrix} = \begin{pmatrix} K_1K_2 + x_1 \otimes y_2^* & K_1x_2 + x_1a_2 \\ \overline{K_2^*y_1} + a_1 \cdot \bar{y}_2 & \langle y_1,x_2\rangle + a_1a_2 \end{pmatrix}$$

and

$$\begin{pmatrix} K & x \\ \bar{y} & a \end{pmatrix}^* = \begin{pmatrix} K^* & y \\ \bar{x} & a^* \end{pmatrix}.$$

There is also a natural action of $\mathcal{L}(X)$ on $X \oplus A$ which defines a norm on $\mathcal{L}(X)$ making it a C^*-algebra. We shall refer to $\mathcal{L}(X)$ as the *linking algebra* of X.

A (right) Hilbert C^*-module X over A is said to be *self-dual* if for every A-module map

$$f : X \to A$$

there is some $y \in X$ such that $f(x) = \langle y, x\rangle$. Suppose now that X is a self-dual Hilbert C^*-module over a von Neumann algebra M. Then X is a dual Banach space (i.e. there is a Banach space X_* such that $X = (X_*)^*$) and $\mathbb{B}(X)$ is a von Neumann algebra. (See [P, Proposition 3.8 and Proposition 3.10].) In this case we set

$$\mathcal{L}_w(X) = \begin{pmatrix} \mathbb{B}(X) & X \\ \bar{X} & M \end{pmatrix}.$$

This is then a von Neumann algebra which we call the *linking von Neumann algebra* of X. (See [B2].)

For more about Hilbert C^*-modules see [L2], [RW] and [P].

2. Isometries of self-dual modules

The main theorem in this section is the following.

Theorem 2.1. *Let M, N be von Neumann algebras and $p \in N$, $q \in N$ be projections such that each of the projections p, $I-p$, q and $I-q$ has central support equal to I. Let $T : pM(I-p) \to qN(I-q)$ be a surjective linear isometry. Then there are central projections e_1, e_2 in M, f_1, f_2 in N with $e_1+e_2 = I$ and $f_1+f_2 = I$ and there are maps*

$$\Psi : e_1Me_1 \to f_1Nf_1,$$
$$\Phi : e_2Me_2 \to f_2Nf_2$$

satisfying

(1) *Ψ is a (surjective) *-isomorphism,*
   Φ is a (surjective) *-antiisomorphism.
(2) *For $x \in pM(I-p)$,*

$$T(x) = \Psi(e_1xe_1) + \Phi(e_2xe_2).$$

(3) *$\Psi(e_1p) = f_1q$ and $\Phi(e_2p) = f_2(I-q)$.*

The proof will be divided into several lemmas and propositions. The final arguments can be found following Corollary 2.23.

Note first that both $pM(I-p)$ and $qN(I-q)$ have a structure of a JB^*-triple with $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.

Since linear isometries preserve the triple product we have the following result which is well known (see [Ka, Proposition 5.5] or [H, Theorem 4]).

Lemma 2.2. *For $x, y, z \in pM(I-p)$,*

$$T(xy^*z + zy^*x) = T(x)T(y)^*T(z) + T(z)T(y)^*T(x)$$

and, in particular,

$$T(xy^*x) = T(x)T(y)^*T(x).$$

Note that an element $u \in pM(I-p)$ is a partial isometry if and only if it is a tripotent (i.e. $\{u,u,u\} = u$); hence $T(u)$ is also a partial isometry.

From now on we assume that M, N, p, q and T are as in assumptions of Theorem 2.1.

Lemma 2.3. *Suppose $\{z_\alpha\}_{\alpha\in\Lambda}$ is an orthogonal family of central projections in M with $\sum z_\alpha = I$. Then there is an orthogonal family $\{c_\alpha\}_{\alpha\in\Lambda}$ of central projections in N with $\sum c_\alpha = I$ and such that, for every $\alpha \in \Lambda$,*

$$T\big(z_\alpha pM(I-p)z_\alpha\big) = c_\alpha qN(I-q)c_\alpha.$$

Proof. Suppose $M \subseteq B(H)$ and $N \subseteq B(K)$ for Hilbert spaces H, K. For each $\alpha \in \Lambda$ write

$$c^1_\alpha = \big[T\big(z_\alpha pM(I-p)z_\alpha\big)K\big],$$
$$c^2_\alpha = \big[T\big(z_\alpha pM(I-p)z_\alpha\big)^*K\big]$$

(where $[S]$ for a subspace $S \subseteq K$ is the projection onto the closure of S). Clearly $c^1_\alpha,\ c^2_\alpha \in N$. Also $c^1_\alpha \le q,\ c^2_\alpha \le I-q$ (in particular $c^1_\alpha c^2_\alpha = 0$).

If $u \in T\big(z_\alpha pM(I-p)z_\alpha\big)$ and $v \in T\big(z_\beta pM(I-p)z_\beta\big)$ are partial isometries and $\alpha \ne \beta$ in Λ then $T^{-1}(u), T^{-1}(v)$ are orthogonal; hence u and v are orthogonal. Since $pM(I-p)$ is spanned by its partial isometries we find that

$$T\big(z_\beta pM(I-p)z_\beta\big)^*T\big(z_\alpha pM(I-p)z_\alpha\big) = 0.$$

It follows that

$$c^1_\alpha c^1_\beta = 0, \quad \alpha \ne \beta,$$

and similarly

$$c^2_\alpha c^2_\beta = 0, \quad \alpha \ne \beta.$$

We now set $c_\alpha = c^1_\alpha + c^2_\alpha \in N$ and we find that $\{c_\alpha\}_{\alpha\in\Lambda}$ is an orthogonal family of projections in N. Now suppose $y \in qN(I-q)$ satisfies $c^1_\alpha y = 0$ for all $\alpha \in \Lambda$. Then $T(z_\alpha x)^*y = 0$ for all $x \in pM(1-p)$. In particular

$$yT\big(z_\alpha T^{-1}(y)\big)^*y = 0.$$

Applying T^{-1} we get

$$T^{-1}(y)\big(z_\alpha T^{-1}(y)\big)^*T^{-1}(y) = 0.$$

Hence $z_\alpha T^{-1}(y)\big(z_\alpha T^{-1}(y)\big)^*z_\alpha T^{-1}(y) = 0$; thus $z_\alpha T^{-1}(y) = 0$ for all $\alpha \in \Lambda$. As $\sum z_\alpha = I,\ T^{-1}(y) = 0$ and, consequently, $y = 0$.

This proves that $\sum c^1_\alpha = q$ (recall that the central support of $I-q$ is I). Similarly $\sum c^2_\alpha = I-q$. Hence $\sum c_\alpha = I$. To show that each c_α is in the center of N it suffices to show that, for $\alpha \ne \beta$

$$c_\alpha N c_\beta = 0.$$

To show that $c^2_\alpha N c^1_\beta = 0$ we fix $y \in M$, and $x_2, x_1 \in pM(1-p)$ and compute

$$\begin{aligned}
&T(z_\alpha x_1)y^*T(z_\beta x_2) + T(z_\beta x_2)y^*T(z_\alpha x_1)\\
&= T\left(z_\alpha x_1 T^{-1}((qy(I-q)))^* z_\beta x_2 + z_\beta x_2 T^{-1}\big((qy(I-q))\big)^* z_\alpha x_1\right)\\
&= T(0+0) \quad (\text{as } z_\alpha z_\beta = 0).
\end{aligned}$$

Since $c^1_\alpha T(z_\alpha x_1) = T(z_\alpha x_1)$ while $c^1_\alpha T(z_\beta x_2) = 0$, we get

$$T(z_\alpha x_1)y^* T(z_\beta x_2) = 0$$

for all $y \in N,\ x_1, x_2 \in pM(I-p)$. Hence

$$c^2_\alpha N c^1_\beta = 0.$$

Now we turn to the proof of $c^1_\alpha N c^1_\beta = 0$. We have to show that $c^1_\alpha qNq c^1_\beta = 0$ and, since qNq is the σ-weak closure of span of $\{ab^* : a, b \in qN(I-q)\}$, we need to show $c^1_\alpha ab^* c^1_\beta = 0$ for all such a, b.

Write $a = T(d)$ and $b = T(g)$ and then compute, for $x_1, x_2 \in pM(I-p)$,

$$\begin{aligned}
&T(z_\alpha x_1)^* T(d) T(g)^* T(z_\beta x_2) \\
&= T(z_\alpha x_1)^* \big[T(d)T(g)^* T(z_\beta x_2) + T(z_\beta x_2) T(g)^* T(d)\big] \\
&= T(z_\alpha x_1)^* T\big(dg^* z_\beta x_2 + z_\beta x_2 g^* d\big) \\
&= T(z_\alpha x_1)^* c^1_\alpha c^1_\beta T\big(z_\beta d g^* x_2 + z_\beta x_2 g^* d\big) = 0.
\end{aligned}$$

Hence $c^1_\alpha N c^1_\beta = 0$.

Thus $c_\alpha N c_\beta = 0$ for all $\alpha \neq \beta$; i.e., $c_\alpha \in Z(N)$. ■

Lemma 2.4. *Let $\{z_\alpha\}_{\alpha\in\Lambda}$ be an orthogonal family of central projections of M with $\sum z_\alpha = I$. Let $\{c_\alpha\}$ be as Lemma 2.3. Suppose that, for every $\alpha \in \Lambda$, Theorem 2.1 holds for $z_\alpha M, c_\alpha N$ and the restriction of T to $z_\alpha pM(I-p)z_\alpha$, in place of M, N and T. Then it holds for M, N, T.*

Proof. From the conclusion of Theorem 2.1, applied to $z_\alpha M$ and $c_\alpha N$, we get projections $e_{1,\alpha},\ e_{2,\alpha},\ f_{1,\alpha},\ f_{2,\alpha}$ and maps Ψ_α, Φ_α. Setting $e_i = \sum e_{i,\alpha},\ f_i = \sum f_{i,\alpha},\ \Psi = \sum \bigoplus \Psi_\alpha$ and $\Phi = \sum \bigoplus \Psi_\alpha$ we obtain the conclusion of the theorem for M, N. ■

Lemma 2.5. *There is an orthogonal family of central projections $\{z_\alpha : \alpha \in \Lambda\}$ in M with $\sum z_\alpha = I$ such that, for each $\alpha \in \Lambda$, either*

(1) *$z_\alpha p$ and $z_\alpha(I-p)$ are abelian projections in M or*

(2) *There is a family $\{u_i : i \in I\}$, of cardinality $|I| \geq 2$, of partial isometries in $z_\alpha pM(I-p)z_\alpha$ satisfying*

(i) *$u_i^* u_i = u_j^* u_j$ for all $i, j \in I$,*

(ii) *$u_i u_i^* u_j u_j^* = 0$ for all $i \neq j$ in I,*

(iii) *$\sum u_i u_i^* = p z_\alpha$*

or

(3) *There is a family $\{u_i : i \in I\}$, of cardinality $|I| \geq 2$, of partial isometries in $z_\alpha pM(1-p)z_\alpha$ satisfying*

(i′) *$u_i u_i^* = u_j u_j^*$ for all $i, j \in I$,*

(ii′) *$u_i^* u_i u_j^* u_j = 0$ for all $i \neq j$ in I,*

(iii′) *$\sum u_i^* u_i = z_\alpha(I-p)$.*

Proof. Since M can be written as a direct sum of algebras of different types we can deal with each type separately. Recall that for a projection $g,\ c(g)$ is its central support.

Case 1: M is of type III.

Then we can write $p = p_1 + p_2$ with $p_1 \sim p_2$. Since $c(p_1) = c(p_2) = c(p) = I = c(I-p)$ and $p_1, I-p$ are both properly infinite, $p_1 \sim I-p,\quad i = 1, 2$. Hence there are u_1, u_2 in M such that $u_i u_i^* = p_i,\ u_i^* u_i = I-p,\ i = 1, 2$, and we are done.

Case 2: M is of type I.
In this case there is an abelian projection $e_1 \in M$ with $c(e_1) = I$. Since $c(e_1) \leq c(I-p)\ (= I)$, we have $e_1 \precsim I-p$ ([KR, Proposition 6.4.6]) and, thus, there is an abelian projection $e \leq I-p$ with $c(e) = I$. It now follows [KR, Corollary 6.5.5] that there is a family $\{q_j : 1 \leq j \leq \infty\}$ of central projections with $\sum q_j = I$ and such that $q_j p$ is the sum of j equivalent abelian projections $q_j p = \sum_{i=1}^{j} q_j p_i$. As $c(q_j p_i) = c(q_j p) = q_j = c(q_j e)$ we have $q_j p_i \sim q_j e$ for all $i \leq j$. Hence for each algebra $q_j M q_j$ with $j \geq 2$ statement (2) holds. We are left to deal with the case q_1. So we assume now that p is abelian. If $I-p$ is not abelian we can use a similar argument to the one above (interchanging the roles of p and $I-p$) and get statement (3). We are left with the case where both p and $I-p$ are abelian projections and this is (1).
Case 3: M is of type II.
By splitting M using central projections we can assume that each of the projections p and $I-p$ are either finite or properly infinite.

If $I-p$ is properly infinite we can argue as in the type III case: $p = p_1 + p_2, p_1 \sim p_2 \precsim I-p$ and, thus, there are u_1, u_2 with $p_i = u_i u_i^*,\ u_1^* u_1 = u_2^* u_2 \leq I-p$; hence statement (2) holds. If p is properly infinite we reverse the roles of p and $I-p$ and get statement (3). So we assume that both p and $I-p$ are finite (thus we are in the type II_1 case). In this case we let Δ be the center-valued dimension function, defined on the projections of M with range equal to the set of all positive operators in the unit ball of the center (see [KR, § 8.4]).

For every $j \geq 2$ we can let q_j be the maximal central projection satisfying $\frac{1}{j}\, q_j \Delta(p) \leq q_j \Delta(I-p)$, and $q_0 = I - \bigvee q_j$. But, for every $j \geq 2$,

$$q_0 \Delta(I-p) \leq q_0 \, \frac{1}{j}\, \Delta(p) \leq \frac{1}{j}\, q_o.$$

Hence $q_0 \Delta(I-p) = 0$ and $\Delta(q_0(I-p)) = 0$ implying that $q_0(I-p) = 0$. But $c(q_0(I-p)) = q_0 c(I-p) = q_0$; hence $q_0 = 0$ and $I = \sum_{j=2}^{\infty} (q_j - q_{j-1})$ (setting $q_1 = 0$). Given $j \geq 2$,

$$\frac{1}{j}\, (q_j - q_{j-1}) \Delta(p) \leq (q_j - q_{j-1}) \Delta(I-p).$$

Restricting our attention to the algebra $(q_j - q_{j-1}) M (q_j - q_{j-1})$ we can write $\frac{1}{j}\, \Delta(p) \leq (I-p)$. Thus we can write p as a sum of j equivalent subprojections $p = \sum p_i$ with $p_i \precsim I-p$; hence $p_i \sim e \leq I-p$ for all $i \leq j$. This shows that, in this case, (2) holds. ■

Lemma 2.6. *If M, N, T are as in Theorem 2.1 and, in addition, p and $I-p$ are abelian projections in M, then Theorem 2.1 holds.*

Proof. Since p and $I-p$ are abelian projections, M is *-isomorphic to $M_2 \otimes \mathcal{A}$ where $\mathcal{A}$ is an abelian von Neumann algebra and M_2 is the algebra of 2×2 complex matrices. We assume now that $M = M_2 \otimes \mathcal{A}$. Write $u = e_{12} \otimes I$ (where $\{e_{ij}\}$ are the matrix units in M_2) and $v = T(u)$. Given $a \in \mathcal{A}$ we have

$$T(e_{12} \otimes a) = T\big((e_{12} \otimes I)(e_{12} \otimes a^*)^*(e_{12} \otimes I)\big) = vT(e_{12} \otimes a^*)^* v.$$

Then $qN(1-q) = T(e_{12} \otimes \mathcal{A}) = vT(e_{12} \otimes \mathcal{A})^* v$. Hence $q \leq vv^*$ and $I - q \leq v^* v$. But $v \in qN(1-q)$ so that $q = vv^*,\ I - q = v^* v$.

Now write

$$\psi_{1,1}(a) = T(e_{12} \otimes a)v^* \in qN(1-q)v^* = qNq, \quad a \in \mathcal{A}.$$

$\psi_{1,1}$ maps I into $vv^* = q$ and it is an isometry onto the von Neumann algebra qNq. By [K, Theorem 10] this map is a *-isomorphism (using the fact that $\mathcal{A}$ is abelian). Hence qNq is abelian. Similarly one sees that $I - q$ is also an abelian projection.

We now have, for $a, b, c \in \mathcal{A}$,

$$\begin{aligned} &T(e_{12} \otimes a)v^*T(e_{12} \otimes b) = T(e_{12} \otimes a)v^*T(e_{12} \otimes b)v^*v \\ &= \psi_{11}(a)\psi_{11}(b)v = \psi_{11}(ab)v = T(e_{12} \otimes ab), \end{aligned}$$

$$\begin{aligned} &T(e_{12} \otimes b)T(e_{12} \otimes c^*)^* = \psi_{11}(b)v[\psi_{11}(c^*)v]^* = \psi_{11}(b)vv^*\psi_{11}(c^*)^* \\ &= \psi_{11}(b)\psi_{11}(c) = T(e_{12} \otimes bc)v^*, \end{aligned}$$

and, using similar identities we see that the map

$$\psi : M \to N$$

defined by

$$\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} T(e_{12} \otimes a)v^* & T(e_{12} \otimes b) \\ T(e_{12} \otimes c^*)^* & v^*T(e_{12} \otimes d) \end{pmatrix}$$

is a *-isomorphism of M onto N extending T. This completes the proof of Theorem 2.1 in this case. (Here $e_1 = I,\ e_2 = 0$.) ■

From now on, in this section, we assume that condition (2) of Lemma 2.5 holds (with $z_\alpha = I$).

We fix the family $\{u_i\}$ as in Lemma 2.5 and write v_i for $T(u_i) \in qN(1-q)$. Then v_i is a partial isometry and we write

$$r_i = v_i v_i^* (\leq q),\ d_i = v_i^* v_i (\leq I - q).$$

Now fix $i \neq j$. We wish to study the relative position of v_i and v_j.

We have

$$\begin{aligned} v_i &= T(u_i u_i^* u_i) = T(u_i u_j^* u_j) = T(u_i u_j^* u_j + u_j u_j^* u_i) \\ &= v_i v_j^* v_j + v_j v_j^* v_i = v_i d_j + r_j v_i. \end{aligned}$$

It then follows that $r_j v_i d_j = 0$. Since we can interchange i and j we get

$$\begin{aligned} v_i = v_i d_j + r_j v_i \quad &\text{and} \quad r_j v_i d_j = 0, \\ v_j = v_j d_i + r_i v_j \quad &\text{and} \quad r_i v_j d_i = 0, \end{aligned}$$

and, thus,

$$r_i = v_i v_i^* = (v_i d_j + r_j v_i)(d_j v_i^* + v_i^* r_j) = v_i d_j v_i^* + r_j r_i r_j.$$

But then $r_j r_i r_j \leq r_i$ and, consequently $(I - r_i) r_j r_i r_j (I - r_i) = 0$ which implies that $r_i r_j = r_i r_j r_i$ and $r_i r_j = r_j r_i$. Similar analysis works for d_i, d_j and we find that

$$r_i r_j = r_j r_i,\ d_i d_j = d_j d_i.$$

The computation above now shows that

$$r_i = v_i d_j v_i^* + r_i r_j$$

and similarly

$$d_i = v_i^* r_j v_i + d_i d_j.$$

We summarize as follows.

Lemma 2.7. *With the notation and assumptions above, for $i \neq j$,*

(1) $v_i = v_i d_j + r_j v_i = (I - r_j) v_i d_j + r_j v_i (I - d_j)$,
(2) $v_j = v_j d_i + r_i v_j = (I - r_i) v_j d_i + r_i v_j (I - d_i)$,
(3) $r_j v_i d_j = r_i v_j d_i = 0$,
(4) $r_i r_j = r_j r_i, \; d_i d_j = d_j d_i$,
(5) $v_i d_j v_i^* = r_i (I - r_j)$

and

(6) $v_i^* r_j v_i = d_i (I - d_j)$. ■

Lemma 2.8. *With the notation and assumption above we have for every triple $\{i, j, k\}$ of different indices,*

(1) $d_i d_j (I - d_k) = 0$,

and

(2) $r_i r_j (I - r_k) = 0$.

Consequently, if we write r for $\bigvee\{r_i : i \in I\}$ and r_0 for $\bigwedge\{r_i : i \in I\}$, then $\{r_i - r_0 : i \in I\}$ is an orthogonal family of projections with sum equal to $r - r_0$. Similar statement holds for $\{d_i - d_0\}$.

Proof. For every $a \in u_i M u_i^*$ we have

$$a u_i u_j^* u_k + u_k u_j^* a u_i = 0 \quad (\text{as } \; u_j^* u_k = 0 = u_j^* u_i).$$

Thus

$$(*) \qquad T(a u_i) v_j^* v_k + v_k v_j^* T(a u_i) = 0.$$

Now set

$$a = u_i u_i^* T^{-1}\big(r_j (I - r_k) v_i\big) u_i^* \in u_i M u_i^*.$$

Then

$$\begin{aligned} T(a u_i) &= T\big(u_i u_i^* T^{-1}(r_j (I - r_k) v_i) u_i^* u_i\big) \\ &= v_i T\big(u_i T^{-1}(r_j (I - r_k) v_i)^* u_i\big)^* v_i \\ &= v_i \big[v_i (v_i^* r_j (I - r_k)) v_i\big]^* v_i \\ &= r_i r_j (I - r_k) v_i. \end{aligned}$$

Using $(*)$ we have

$$r_i r_j (I - r_k) v_i v_j^* v_k + v_k v_j^* r_i r_j (I - r_k) v_i = 0.$$

Multiplying on the left by $v_j v_k^*$ we get

$$v_j v_k^* r_i r_j (I - r_k) v_i v_j^* v_k + v_j d_k v_j^* r_i r_j (I - r_k) v_i = 0.$$

As $v_k^*(I - r_k) = 0$, the first term vanishes. From Lemma 2.7 we know that $v_j d_k v_j^* = r_j (I - r_k)$; hence $r_j (I - r_k) r_i r_j (1 - r_k) v_i = 0$.

It follows that $r_i r_j (I - r_k) = 0$. Statement (1) is proved similarly and the final statement of the lemma follows immediately. ■

Lemma 2.9. (1) *For $a \in pM(I - p)$ and a partial isometry $u \in pM(I - p)$,*

$$T(u u^* a u^* u) = v v^* T(a) v^* v$$

where $v = T(u)$.

(2) *For a partial isometry $u \in pM(I-p)$ with $T(u)=v$,*

$$T(u^*Mu^*u) = vv^*Nv^*v.$$

(3) *If $x, y \in pM(I-p)$ and $x^*y = yx^* = 0$, then*

$$T(x)^*T(y) = 0 = T(y)T(x)^*.$$

Proof. (1) $T(uu^*au^*u) = vT(ua^*u)^*v = v[vT(a)^*v]^*v = vv^*T(a)v^*v$.

(2) From (1) it follows that

$$T(uu^*Mu^*u) \subseteq vv^*Nv^*v.$$

Applying the same argument to T^{-1} we get equality.

(3) Let $x, y \in pM(I-p)$ satisfy $x^*y = yx^* = 0$. Let $x = u_1|x|$ be the polar decomposition of x and $y = u_2|y|$ be the one for y. Then $u_1^*u_2 = u_2u_1^* = 0$. If we write $v_i = T(u_i)$ then

$$0 = T(u_1u_1^*u_2 + u_2u_1^*u_1) = v_1v_1^*v_2 + v_2v_1^*v_1$$

and

$$0 = T(u_1u_2^*u_1) = v_1v_2^*v_1.$$

hence

$$\begin{aligned} v_1^*v_2 &= v_1^*v_1v_1^*v_2 + (v_1^*v_2v_1^*)v_1 \\ &= v_1^*(v_1v_1^*v_2 + v_2v_1^*v_1) = 0. \end{aligned}$$

Similarly $v_2v_1^* = 0$.

Since $x \in u_1u_1^*Mu_1^*u_1,\ y \in u_2u_2^*Mu_2^*u_2$, part (1) shows that

$$T(x) \in v_1v_1^*Nv_1^*v_1,\ T(y) \in v_2v_2^*Nv_2^*v_2.$$

As $v_2v_1^* = v_1^*v_2 = 0$,

$$T(x)^*T(y) = 0 = T(y)T(x)^*. \qquad \blacksquare$$

Lemma 2.10. *If u is a partial isometry in $pM(I-p)$, $v = T(u)$ and e is a projection satisfying $e \le vv^*$, then there is a projection $e_0 \le uu^*$ with $T^{-1}(ev) = e_0u$.*

Proof. Write $v' = ev, v'' = (vv^* - e)v$. Both are partial isometries and they satisfy $v'v''^* = v''v'^* = 0$. Hence (Lemma 2.9), the partial isometries $u' = T^{-1}(v')$ and $u'' = T^{-1}(v'')$ satisfy $0 = u'u''^* = u''u'^*$. Since $u'+u'' = T^{-1}(v'+v'') = T^{-1}(v) = u$ the conclusion follows. $\blacksquare$

Lemma 2.11. *For all $i, j \in I$, r_j commutes with the elements in r_iNr_i.*

Proof. For $i = j$ it is clear, so assume $i \ne j$. Since $r_iNd_i = T(u_iu_i^*Mu_i^*u_i)$ (Lemma 2.9 (2)),

$$r_iNr_i = r_iNd_iv_i^* = T(u_iu_i^*Mu_i^*u_i)v_i^*.$$

So fix $x = u_iu_i^*xu_i^*u_i$ and compute (using the fact that $u_ju_j^*u_iu_i^* = 0$ and $u_j^*u_j = u_i^*u_i$).

$$\begin{aligned} T(x) &= T(xu_i^*u_i) = T(xu_j^*u_j + u_ju_j^*x) = T(x)v_j^*v_j + v_jv_j^*T(x) \\ &= T(x)d_j + r_jT(x). \end{aligned}$$

Hence

$$T(x) = (I - r_j)T(x)d_j + r_jT(x)(I - d_j).$$

$$\begin{aligned} T(x)v_i^*r_j &= T(x)v_i^*r_jv_iv_i^* = T(x)d_i(I-d_j)v_i^* \\ &= r_jT(x)d_i(1-d_j)v_i^*. \end{aligned}$$

Hence, for every $y \in r_iNr_i$, $yr_j = r_jyr_j$ and the claim follows. ■

Our next objective is to show that, for $i \neq j$ and for x, y, z in u_iMu_i, $r_i(I-r_j)T(xy^*z) = r_i(I-r_j)T(x)T(y)^*T(z)$. This will be proved in Proposition 2.13.

We first consider the map

$$\varphi : u_iu_i^*Mu_iu_i^* \to v_iv_i^*Nv_iv_i^*$$

defined by

$$\varphi(u_iu_i^*xu_iu_i^*) = T(u_iu_i^*xu_i)v_i^*.$$

Since $T(u_iu_i^*Mu_i) = r_iNd_i$ (Lemma 2.9), the map φ is a surjective isometry from the von Neumann algebra $u_iu_i^*Mu_iu_i^*$ onto the von Neumann algebra r_iNr_i that is unital $\big(\varphi(u_iu_i^*) = r_i\big)$. By [K, Theorem 10] there are central projections g, h in r_iNr_i and central projections g_0, h_0 in $u_iu_i^*Mu_iu_i^*$ with $g + h = r_i$, $g_0 + h_0 = u_iu_i^*$ and such that φ, restricted to g_0Mg_0, is a *-isomorphism onto gNg and φ, restricted to h_0Mh_0, is a *-anti-isomorphism of h_0Mh_0 onto hNh.

Lemma 2.12. *With the notation above, $h(I-r_j)$ is an abelian projection in r_iNr_i.*

Proof. Since $h \in r_iNr_i$ it follows from Lemma 2.11 that $h(I-r_j)$ is a projection in r_iNr_i. Write $c = h(I-r_j)$. To show that cNc is abelian it suffices to show that one cannot find in cNc projections e_1, e_2 that are equivalent (in cNc) and orthogonal (i.e. $e_1e_2 = 0$). Assume, by negation, that there are such projections. Then there is a partial isometry $w \in cNc$ with

$$ww^* = e_2, \quad w^*w = e_1.$$

Write

$$t_1 = e_1v_i,\ t_2 = v_jv_i^*e_1v_i,\ t_3 = wv_i$$

and set $s_i = T^{-1}(t_i)$. Then t_i and s_i are partial isometries. We have $t_1^*t_1 = v_i^*e_1v_1$, $t_2^*t_2 = v_i^*e_1v_iv_j^*v_jv_i^*e_1v_i$ and $t_3^*t_3 = v_i^*w^*wv_i = v_i^*e_1v_i$. Since

$$v_i^*e_1v_i \le v_i^*cv_i \le v_i^*(I-r_j)v_i = d_id_j \le d_j,$$

(using Lemma 2.7), we have

$$t_i^*t_i = v_i^*e_1v_i, \quad i = 1, 2, 3.$$

Also, $t_1t_1^* = e_1$, $t_2t_2^* = v_jv_i^*e_1v_iv_j^* \le r_j$ and $t_3t_3^* = wv_iv_i^*w^* = e_2 \le I - r_j$. Hence $\{t_it_i^* : i = 1, 2, 3\}$ is an orthogonal set.

By Lemma 2.10, there are projections $c_1 \le u_iu_i^*$ and $c_2 \le u_ju_j^*$ with

$$s_1 = c_1u_i,\ s_2 = c_2u_j.$$

Now, by the definition of h, the map

$$\varphi(x) = T(xu_i)v_i^*, \qquad x \in h_0Mh_0,$$

is a *-anti-isomorphism. Hence

$$\begin{aligned} T(s_3 s_1^* s_1) &= \varphi(s_3 s_1^* s_1 u_i^*) v_i = \varphi\big((s_3 u_i^*)(u_i s_1^*)(s_1 u_i^*)\big) v_i \\ &= \varphi(s_1 u_i^*)\varphi(s_1 u_i^*)^* \varphi(s_3 u_i^*) v_i \\ &= \varphi(s_1 u_i^*) v_i v_i^* \varphi(s_1 u_i^*)^* \varphi(s_3 u_i^*) v_i \\ &= T(s_1) T(s_1)^* T(s_3) = t_1 t_1^* t_3 = 0. \end{aligned}$$

Hence $s_3 s_1^* = 0$.

We can use Lemma 2.8 and Lemma 2.7 for $\{t_1, t_2, t_3\}$ and T^{-1} in place of $\{v_i, v_j, v_k\}$ and T (since they also have pairwise orthogonal ranges and the same initial space). Since $s_3^* s_3 s_1^* s_1 = 0$ (by the computation above) it follows from Lemma 2.8 (1) that $s_2^* s_2 s_1^* s_1 = s_2^* s_2 s_3^* s_3 s_1^* s_1 = 0$ and, similarly, $s_2^* s_2 s_3^* s_3 = 0$. So that $\{s_i^* s_i\}$ is an orthogonal family. By Lemma 2.7 (5) (applied to the situation here) we get for $i \neq j$ in $\{1,2,3\}$, $s_i s_i^*(I - s_j s_j^*) = s_i s_j^* s_j s_i^* = 0$.

We conclude that $s_i s_i^* = s_j s_j^*$ for all i, j in $\{1,2,3\}$. But $s_1 s_1^* \leq u_i u_i^*$ (as $s_1 = c_1 u_i$) and $s_2 s_2^* \leq u_j u_j^* \leq I - u_i u_i^*$.

This is a contradiction and it completes the proof. ∎

Proposition 2.13. *For x, y, z in $u_i M u_i$ and $j \neq i$,*

$$r_i(I - r_j) T(xy^*z) = r_i(I - r_j) T(x) T(y)^* T(z).$$

Proof. Fix x, y, z in $u_i M u_i$ and write

$$r_i(I - r_j) T(xy^*z) = r_i(I - r_j) h T(xy^*z) + r_i(I - r_j) g T(xy^*z)$$

where g, h are defined above. With φ as above we have

$$(I - r_j) h T(xy^*z) = (I - r_j) h \varphi(xy^*zu_i^*) v_i.$$

Since $\varphi(xy^*zu_i^*)$ lies in $r_i N r_i$, we can use Lemma 2.11 and Lemma 2.7 to get $(I - r_j)\varphi(xy^*zu_i^*) v_i = \varphi(xy^*zu_i^*)(I - r_j) v_i = \varphi(xy^*zu_i^*) v_i v_i^* (I - r_j) v_i$. Hence, by the definition of h, $(I - r_j) h T(xy^*z) = (I - r_j) h \varphi(xy^*zu_i^*) h r_i (I - r_j) v_i = (I - r_j) h \varphi(zu_i^*) \varphi(yu_i^*)^* \varphi(xu_i^*) h r_i (I - r_j) v_i$. But since $(I - r_j) h$ is an abelian projection in $r_i N r_i$ (Lemma 2.12) we have

$$\begin{aligned} (I - r_j) h T(xy^*z) &= (I - r_j) h \varphi(xu_i^*) \varphi(yu_i^*)^* \varphi(zu_i^*) h r_i (I - r_j) v_i \\ &= (I - r_j) h T(x) v_i^* v_i T(y)^* T(z) v_i^* h r_i (I - r_j) v_i \\ &= (I - r_j) h T(x) T(y)^* T(z). \end{aligned}$$

Also, from the definition of g,

$$\begin{aligned} (I - r_j) g T(xy^*z) &= (I - r_j) g \varphi(xy^*zu_i^*) v_i \\ &= (I - r_j) g \varphi(xu_i^*) \varphi(yu_i^*)^* \varphi(zu_i^*) v_i \\ &= (I - r_j) g T(x) T(y)^* T(z). \end{aligned}$$

This completes the proof. ∎

We now turn to define a map

$$\theta : pMp \to N.$$

For if we note first that $p = \sum u_i u_i^*$ and every $x \in pMp$, $x = \sum_{i,j} u_i u_i^* x u_j u_j^*$ (σ-weakly). For every $i, j \in I$ we set

$$\theta_{ij} : u_i u_i^* M u_j u_j^* \longrightarrow N$$

by

$$\theta_{ij}(u_iu_i^*xu_ju_j^*) = (r_i - r_0)T(u_iu_i^*xu_j)v_j^* + v_j^*T(u_iu_i^*xu_j)(d_i - d_0),$$

where $r_0 = \bigwedge r_i,\ d_0 = \bigwedge d_i$.

To study the map θ defined by $\{\theta_{ij}\}$ we first write

$$\alpha_{ij}(u_iu_i^*xu_ju_j^*) = (r_i - r_0)T(u_iu_i^*xu_j)v_j^*$$

and

$$\beta_{ij}(u_iu_i^*xu_ju_j^*) = v_j^*T(u_iu_i^*xu_j)(d_i - d_0).$$

Also write, for every finite subset $F \subseteq I,\ p_F = \sum_{i\in F} u_iu_i^*,\ r_F = \sum_{i\in F} r_i,\ d_F = \sum_{i\in F} d_i$ and

$$x(F) = p_F x p_F \quad (\text{for } x \in pMp)$$

and $\alpha_F : p_FMp_F \to N$ is defined by

$$\alpha_F(p_Fxp_F) = \sum_{i,j\in F} \alpha_{ij}(u_iu_i^*xu_ju_j^*).$$

(Similarly, β_F can be defined.) We have $\alpha_F(p_FMp_F) \subseteq (r_F - r_0)N(r_F - r_0)$ and if $F_1 \subseteq F_2,\ x \in N$,

$$\alpha_{F_1}(p_{F_1}xp_{F_1}) = (r_{F_1} - r_0)\alpha_{F_2}(p_{F_2}xp_{F_2})(r_{F_1} - r_0).$$

Lemma 2.14. *Given a finite subset $F \subseteq I$, α_F is a *-homomorphism of p_FMp_F onto $(r_F - r_0)N(r_F - r_0)$ and β_F is a *-antihomomorphism of p_FMp_F onto $(d_F - d_0)N(d_F - d_0)$.*

Proof. We prove the properties of α_F. The proof for β_F is similar.

For $i, j \in I,\ u_iu_i^*Mu_j = u_iu_i^*Mu_ju_i^*u_i \subseteq u_iu_i^*Mu_i^*u_i = u_iu_i^*Mu_j^*u_j \subseteq u_iu_i^*Mu_j$. Hence $u_iu_i^*Mu_j = u_iu_i^*Mu_i^*u_i$ and $T(u_iu_i^*Mu_j) = r_iNd_i$. Hence $T(u_iu_i^*Mu_j)v_j^* = r_iNd_iv_j^*$. For $i = j$ this is r_iNr_i and, thus, $\alpha_{ii}(u_iu_i^*Mu_iu_i^*) = (r_i - r_0)Nr_i = (r_i - r_0)N(r_i - r_0)$ (as $I - r_0$ commutes with r_iNr_i by Lemma 2.11).

For $i \neq j$

$$\alpha_{ij}(u_iu_i^*Mu_ju_j^*) = (r_i - r_0)Nd_iv_j^* = (r_i - r_0)Nv_j^*(v_jd_iv_j^*) = (r_i - r_0)Nv_j^*(I - r_i).$$

As $r_j(I - r_i) = r_j - r_0$ (Lemma 2.8) we see that α_{ij} is onto $(r_i - r_0)N(r_j - r_0)$. This shows that α_F maps p_FMp_F onto $(r_F - r_0)N(r_F - r_0)$.

We now show that α_F is a *-map. For that, fix $x = u_iu_i^*xu_ju_j^*$ and consider

$$\alpha_{ij}(x) = (r_i - r_0)T(xu_j)v_j^* = (r_i - r_0)T(xu_j)v_j^*(r_j - r_0).$$

If $i = j$ we have

$$\begin{aligned} \alpha_{ii}(x) &= (r_i - r_0)T(u_iu_i^*xu_i)v_i^*(r_i - r_0) \\ &= (r_i - r_0)v_iT(x^*u_i)^*v_iv_i^*(r_i - r_0) \\ &= \left[(r_i - r_0)T(x^*u_i)v_i^*(r_i - r_0)\right]^* \\ &= \alpha_{ii}(x^*)^*. \end{aligned}$$

If $i \neq j$ we have $T(u_iu_i^*xu_j) = T(u_iu_i^*xu_j + u_ju_i^*xu_i) = v_iT(x^*u_i)^*v_j + v_jT(x^*u_i)^*v_i$. Hence

$$\alpha_{ij}(x) = (r_i - r_0)\left[v_iT(x^*u_i)^*v_j + v_jT(x^*u_i)^*v_i\right]v_j^*(r_j - r_0).$$

Since $(r_i - r_0)v_j = 0$ we have

$$\alpha_{ij}(x) = (r_i - r_0)v_i T(x^* u_i)^* (r_j - r_0) = \alpha_{ji}(x^*)^*.$$

This shows that α_F is a *-map.

Finally, we shall show that α_F is a homomorphism. For that we fix

$$x = u_i u_i^* x u_k u_k^*,\ y = u_j u_j^* y u_m u_m^*.$$

If $k \neq j$, then $xy = 0$. In this case $r_k(r_j - r_0) = 0$ and $\alpha_F(x)\alpha_F(y) = (r_i - r_0)T(xu_k)v_k^*(r_j - r_0)T(yu_m)v_m^* = 0$. So we suppose now that $k = j$ and then $xy = u_i u_i^* x u_j u_j^* y u_m u_m^*$. Hence $\alpha_F(xy) = (r_i - r_0)T(xyu_m)v_m^*$. If $i \neq j$ this is equal to

$$\begin{aligned}(r_i - r_0)T\big((xu_j)u_j^*(yu_m)\big)v_m^* &= (r_i - r_0)T\left[(xu_j)u_j^*(yu_m) + yu_m u_j^* xu_j\right]v_m^* \\ &= (r_i - r_0)\left[T(xu_j)v_j^*T(yu_m) + T(yu_m)v_j^*T(xu_j)\right]v_m^*.\end{aligned}$$

Now $T(yu_m) = T(yu_m u_j^* u_j) \in r_j N d_j$ and $(r_i - r_0)T(yu_m) = 0$. Hence

$$\alpha_F(xy) = (r_i - r_0)T(xy_j)v_j^*T(yu_m)v_m^* = \alpha_F(x)\alpha_F(y).$$

Now consider the case $i = j$. Then

$$\alpha_F(xy) = (r_i - r_0)T\big((xu_i)u_i^*(yu_m)\big)v_m^*.$$

Since $yu_m = u_i u_i^* y u_m u_i^* u_i \in u_i M u_i$ and xu_i, u_i also lie in $u_i M u_i$, we can apply Proposition 2.13 and get

$$\alpha_F(xy) = (r_i - r_0)T(xu_i)v_i^*T(yu_m)v_m^* = \alpha_F(x)\alpha_F(y). \quad \blacksquare$$

It follows from Lemma 2.14 that for each finite subset $F \subseteq I$ and each $x \in pMp$

$$\|\alpha_F(x)\| \leq \|x\|.$$

Hence, for a fixed $x \in pMp$ the net $\{\alpha_F(x) : F \subseteq I\}$ is bounded and we can find a σ-weakly convergent subset $\alpha_{F'}(x) \longrightarrow \alpha(x)$.

For every finite subset $F \subseteq I$ there is some F_0' in the subnet with $F_0' \supseteq F$. But then, for every $F' \supseteq F_0'$, $r_F \alpha_{F'}(x) r_F = \alpha_F(x)$; hence

$$r_F \alpha(x) r_F = \alpha_F(x)$$

and, consequently.

$$\alpha(x) = \sigma\text{-weak} \lim_F \alpha_F(x).$$

We can now conclude from Lemma 2.14 the following. (The statement for β is proved similarly.)

Corollary 2.15. *There is a surjective *-homomorphism*

$$\alpha : pMp \to (r - r_0)N(r - r_0)$$

and a surjective *-antihomomorphism

$$\beta : pMp \to (d - d_0)N(d - d_0)$$

such that, for all i, j,

$$r_i \alpha(x) r_j = \alpha_{ij}(x), \quad d_j \beta(x) d_i = \beta_{ij}(x).$$

Lemma 2.16. *For $x \in pM(I - p)$ and $i, j \in I$ we have $r_i(I - r_j)T(u_j u_j^* x) = 0$.*

Proof. We can assume $i \neq j$ and $x = u_j u_j^* x$. Write $x = x_1 + x_2$ where $x_1 = x u_j^* u_j$ and $x_2 = x(I - u_j^* u_j)$. Then $x_1 \in u_j A u_j$ and, thus, $T(x_1) \in v_j N v_j = r_j N d_j$. Hence $(I - r_j)T(u_j u_j^* x_1) = 0$. For x_2 we have $u_i u_i^* x_2 = x_2 u_i^* u_i = 0$ (as $u_i^* u_i = u_j^* u_j$). Hence $0 = T(u_i u_i^* x_2 + x_2 u_i^* u_i) = v_i v_i^* T(x_2) + T(x_2) v_i^* v_i = r_i T(x_2) + T(x_2) d_i$. We also have $u_i x_2^* u_i = 0$; hence $v_i T(x_2)^* v_i = 0$. Therefore

$$\begin{aligned} r_i T(x_2) &= r_i T(x_2) d_i + r_i T(x_2)(I - d_i) \\ &= r_i T(x_2) d_i + \big[r_i T(x_2) + T(x_2) d_i\big](I - d_i) = 0. \end{aligned}$$

Note that a similar argument shows that

$$T(x_2) d_i = 0. \qquad \blacksquare$$

Lemma 2.17. *For $x \in u_j u_j^* M(I - p)$, $y \in u_i u_i^* M u_i$,*

$$r_i(I - r_0)T(y)v_j^* T(x) = r_i(I - r_0)T(y u_j^* x).$$

Proof. Assume first that $i \neq j$. Then it follows from Lemma 2.16 that $r_i(I - r_0)T(x) = 0$. Also we have $x u_j^* y = 0$. Hence

$$\begin{aligned} &r_i(I - r_0)T(y u_j^* x) = r_i(I - r_j)T(y u_j^* x + x u_j^* y) \\ &= r_i(I - r_j)\big[T(y)v_j^* T(x) + T(x) v_j^* T(y)\big] \\ &= r_i(I - r_j)T(y)v_j^* T(x). \end{aligned}$$

Now consider the case $i = j$. if $x \in u_i u_i^* M u_i^* u_i$ the result follows from Proposition 2.13. So assume now that $x = u_i u_i^* x(I - u_i^* u_i)$. Then $x u_i^* y = 0$ and $r_i T(x) v_i^* = r_i T(x) d_i v_i^* = T(u_i u_i^* x u_i^* u_i) v_i^* = 0$ (where we used Lemma 2.9). Hence

$$\begin{aligned} r_i T(y u_i^* x) &= r_i T(y u_i^* x + x u_i^* y) \\ &= r_i\big[T(y) v_i^* T(x) + T(x) v_i^* T(y)\big] \\ &= r_i T(y) v_i^* T(x). \qquad \blacksquare \end{aligned}$$

Corollary 2.18. *For every i, j, k if $a = u_i u_i^* a u_j u_j^*$ and $x = u_k u_k^* x(I - p)$, then*

$$(r - r_0)T(ax) = \alpha_{ij}(a)T(x).$$

Proof. Assume first that $j \neq k$. Then $ax = 0$. Also $\alpha_{ij}(a) \in (r_i - r_0)N(r_j - r_0)$ (see the proof of Lemma 2.14) and $(r_j - r_0)T(x) = r_j(I - r_k)T(x) = 0$ by Lemma 2.16. Hence $\alpha_{ij}(a)T(x) = 0$. We now consider the case $j = k$. In this case $\alpha_{ij}(a)T(x) = (r_i - r_0)T(a u_j) v_j^* T(x)$ and Lemma 2.17 (with $y = a u_j \in u_i u_i^* M u_j^* u_j = u_i u_i^* M u_i^* u_i$) applies to give

$$\alpha_{ij}(a)T(x) = r_i(I - r_0)T(a u_j u_j^* x) = r_i(I - r_0)T(ax).$$

As $(r - r_i)T(ax) = 0$ (Lemma 2.16), we are done. $\blacksquare$

Before we conclude from the last corollary that T is a module map we need the following.

Lemma 2.19.

(1) *$\theta(= \alpha + \beta)$ is an injective map.*
(2) *α and β are σ-weakly continuous maps on pMp.*
(3) *There are projections g_1, g_2 in $Z(pMp)$ such that*
  (i) *$g_1 + g_2 = p$,*
  (ii) *$\ker \alpha = g_2 M g_2$ and $\ker \beta = g_1 M g_1$,*
  (iii) *if $c(g_i)$ is the central support of g_i in M, then $c(g_1) + c(g_2) = I$.*

Proof. (1) Recall that for every $i, j \in I$,

$$\begin{aligned}\theta(u_i u_i^* x u_j u_j^*) &= (r_i - r_0)T(u_i u_i^* x u_j)v_j^*(r_j - r_0) \\ &+ (d_j - d_0)v_j^* T(u_i u_i^* x u_j)(d_i - d_0).\end{aligned}$$

Since $\{r_i - r_0\}$ and $\{d_i - d_0\}$ are orthogonal families, it will suffice to show the injectivity of θ_{ij} $\left(= \theta|u_i u_i^* M u_j u_j^*\right)$ for all $i, j \in I$.

So fix $i, j \in I$ and $x = u_i u_i^* x u_j u_j^*$ such that

$$(r_i - r_0)T(u_i u_i^* x u_j)v_j^*(r_j - r_0) = 0 = (d_j - d_0)v_j^* T(u_i u_i^* x u_j)(d_i - d_0).$$

From Lemma 2.7 we get $v_j(d_j - d_0)v_j^* = r_0$ and $v_j^*(r_j - r_0)v_j = d_0$ and we have

$$(r_i - r_0)T(u_i u_i^* x u_j)d_0 = 0 = r_0 T(u_i u_i^* x u_j)(d_i - d_0).$$

Also, fix $k \neq i$, and compute

$$r_0 T(u_i u_i^* x u_j)d_0 = r_k T(u_i u_i^* x u_j)d_k = T(u_k u_k^* u_i u_i^* x u_j u_k^* u_k) = 0$$

(using Lemma 2.9).

Now we will show that $(r_i - r_0)T(u_i u_i^* x u_j)(d_i - d_0) = 0$. It will then follow that $T(u_i u_i^* x u_j) = 0$ (as it lies in $r_i N d_i$ by Lemma 2.9) and consequently, $u_i u_i^* x u_j = 0$ implying $x = 0$.

Fix $k \neq i$ and note that

$$(r_i - r_0)T(u_i u_i^* x u_j)(d_i - d_0) = (r_i - r_0)T(u_i u_i^* x u_j)v_i r_k v_i^* \in (r_i - r_0)N r_i r_k v_i^*.$$

But the last set is $\{0\}$ since r_k commutes with $r_i N r_i$ (Lemma 2.11) and $r_k(r_i - r_0) = 0$.

(2) β is a map onto $(d - d_0)N(d - d_0)$ and α is onto $(r - r_0)N(r - r_0)$. Since $d - d_0$ and $r - r_0$ are orthogonal projections, we can view N as acting on a Hilbert space H with two orthogonal subspaces $H_1 = (d - d_0)(H)$ and $H_2 = (r - r_0)(H)$. Let $\tau : B(H_1) \to B(H_1)$ be a transpose map, then $\alpha \oplus \tau \circ \beta$ is a *-isomorphism of pMp into $B(H)$. Thus it is σ-weakly continuous and so are its compressions to H_1 and H_2; i.e. α and β are σ-weakly continuous.

(3) Since α, β are σ-weakly continuous their kernels are σ-weakly closed ideals in pMp and the existence of projections g_1, g_2 in the center of pMp and satisfying (ii) follows. We now turn to proving (i). Since $\theta = \alpha + \beta$ is injective, $g_1 g_2 = 0$. So we write $h = p - (g_1 + g_2)$ and claim that $h = 0$. Write, for $i \in I$, $\tilde{u}_i = h u_i$ and $\tilde{v}_i = T(\tilde{u}_i)$ and note that Lemma 2.7 and Lemma 2.8 apply to $\tilde{u}_i, \tilde{v}_i$ in place of u_i, v_i (since $\{\tilde{u}_i \tilde{u}_i^*\}$ is an orthogonal family and $\tilde{u}_i^* \tilde{u}_i = \tilde{u}_j^* \tilde{u}_j$ for all i, j). We also write $\tilde{r}_i = \tilde{v}_i \tilde{v}_i^*$, $d_i = \tilde{v}_i^* \tilde{v}_i$. Using Lemma 2.10 applied to T^{-1}, we find that $\tilde{v}_i = c v_i$ for some projection $c \leq v_i v_i^* = r_i$ and thus $\tilde{r}_i \in r_i N r_i$ and, by Lemma 2.11, $\tilde{r}_i$ commutes with all r_j. In particular $\tilde{r}_i r_0 = r_0 \tilde{r}_i$, $i \in I$.

Now $r_0 \tilde{v}_i = \tilde{v}_i \tilde{v}_i r_0 \tilde{v}_i \in \tilde{r}_i N \tilde{d}_i = T(h u_i u_i^* M u_i^* u_i)$ and we can find $a = u_i u_i^* a u_i^* u_i$ such that $r_0 \tilde{v}_i = T(ha)$. Hence

$$\alpha(h a u_i^*) = (r_i - r_0)T(ha)v_i^* = (r_i - r_0)r_0 \tilde{v}_i \tilde{v}_i^* = 0.$$

But α, restricted to hMh is injective. Hence $h a u_i^* = 0$ and also $ha = 0$ and $r_0 \tilde{v}_i = T(ha) = 0$. But then $r_0 \tilde{r}_i = 0$ for all i; i.e. $\tilde{r}_i \leq r_i - r_0$ and it follows that $\tilde{r}_i \tilde{r}_j = 0$ for all $i \neq j$. Similar argument shows that $\tilde{d}_i \tilde{d}_j = 0$, $i \neq j$. For a given $i \in I$ and $j \neq i$,

$$\tilde{r}_i = \tilde{r}_i(I - \tilde{r}_j) = \tilde{v}_i \tilde{d}_j \tilde{v}_i^* = \tilde{v}_i \tilde{d}_j \tilde{d}_i \tilde{v}_i^* = 0.$$

But then $\tilde{v}_i = 0$ and, thus, $hu_i = 0$ for all $i \in I$. This shows that $h = 0$ and we are done.

It is now left to prove (iii): $c(g_1) + c(g_2) = I$. But since $g_1 + g_2 = p$ and $c(p) = I$ and $g_i \in Z(pMp)$ it is obvious. ■

Because of Lemma 2.4 it will suffice now, in order to prove Theorem 2.1 to restrict our attention to the cases $c(g_1) = I$ and $c(g_2) = I$. Since the proof is similar in these cases we now assume $c(g_1) = I$ (i.e. $g_2 = c(g_2) = 0$).

Lemma 2.20. *When $c(g_1) = I$ (with g_1 as in Lemma 2.19) we have, for all $a \in pMp$ and $x \in pM(I-p)$,*

$$T(ax) = \alpha(a)T(x).$$

Moreover, we have now $r_0 = 0$, α is a *-isomorphism of pMp onto rNr and T maps $pM(I-p)$ onto $rN(I-r)$ (i.e. $r = q$).

Proof. Now, that $g_2 = 0$, α is injective and, thus, a *-isomorphism onto $(r-r_0)N(r-r_0)$. But then we can repeat the argument of the proof of Lemma 2.19(3)(ii) (with p replacing h) to show that $r_0 v_i = 0$ for all i. hence $r_0 = 0$ and α maps onto rNr. Now fix $x \in pM(I-p)$ with $x = u_i u_i^* x$. Write $x = x_1 + x_2$ with $x_1 = u_i u_i^* x u_i^* u_i$ and $x_2 = u_i u_i^* x(I - u_i^* u_i)$. We have $T(x_1) \in T(u_i u_i^* M u_i^* u_i) = r_i N d_i$ (Lemma 2.9) and $T(x_2) = T\big(u_i u_i^* x_2 + x_2 u_i^* u_i\big) = r_i T(x_2) + T(x_2) d_i$. However, for all $j \neq i$, $x_2 u_j^* = 0$ (as $u_j^* u_j = u_i^* u_i$) and $u_j^* x_2 = 0$ (as $u_j u_j^* u_i u_i^* = 0$); hence (Lemma 2.9(3)) $T(x_2) v_j^* = 0$ and also $T(x_2) d_j = T(x_2) v_j^* v_j = 0$. But $d_j = d_i$ (as $d_j(I - d_i) = v_j^* r_i v_j = v_j^* r_0 v_j = 0$) and we conclude that

$$T(x_2) = r_i T(x_2) \subseteq rN.$$

Hence $T(pM(I-p)) \subseteq rN$. But $T(pM(I-p)) = qN(I-q)$; hence $qN(I-q) \subseteq rN$ while $r \leq q$. If $r \neq q$ then it follows from the fact that $c(I-q) = I$ that $(q-r)N(I-q) \neq 0$ but this contradicts $qN(I-q) \subseteq rN$ and we get $q = r$.

We then conclude, from Corollary 2.18, that, given i, j, k in I, $a \in u_i u_i^* M u_j u_j^*$ and $x \in u_k u_k^* M(I-p)$, we have

$$T(ax) = \alpha(a)T(x).$$

This equality then holds for finite sums of such a, x. Since T is σ-weakly continuous by [Ho, Corollary 3.22] and α is σ-weakly continuous the equality holds for all $a \in pMp$ and $x \in pM(I-p)$. ■

Corollary 2.21. *Assume $c(g_1) = I$. For all $x, y \in pM(I-p)$*

$$T(x)T(y)^* = \alpha(xy^*).$$

Proof. This is [MS, Lemma 5.10] (which generalizes the result of Lance [L2, Theorem 3.5]). ■

Proposition 2.22. *Assume $c(g_1) = I$ as above. Then there is a *-isomorphism*

$$\gamma : (I-p)M(I-p) \longrightarrow (I-q)N(I-q)$$

with

(i) *γ is surjective.*
(ii) *For $a \in (I-p)M(I-p)$ and $x \in pM(I-p)$,*

$$T(xa) = T(x)\gamma(a).$$

(iii) *For* $x, y \in pM(I-p)$,

$$T(x)^*T(y) = \gamma(x^*y).$$

Proof. Suppose $N \subseteq B(H)$ (and the unit of N is I_H) and write $H_0 = \overline{\text{span}}\{T(x)^*h : x \in pM(I-p),\ h \in H\}$. Since $T(pM(I-p)) = qN(I-q)$ and $c(q) = I$, it follows that $H_0 = (I-q)(H)$.

For $a \in (I-p)M(I-p)$ we define $\gamma(a)$ as an operator in $B(H_0)$ and assume that it is defined to be zero on $H \ominus H_0$. We define

$$\gamma(a)\left(\sum_{i=1}^{n} T(x_i)^*h_i\right) = \sum_{i=1}^{n} T(x_i a^*)^*h_i$$

where $x_i \in pM(I-p),\ h_i \in H$.

Note the following

$$\begin{aligned}\left\langle \sum T(x_i a^*)^*h_i, \sum T(x_i a^*)^*h_i \right\rangle &= \sum_{i,j} \langle T(x_j a^*)T(x_i a^*)^*h_i, h_j\rangle \\ &= \sum_{i,j} \langle \alpha(x_j a^* a x_i^*)h_i, h_j\rangle .\end{aligned}$$

Since the matrix $(x_j a^* a x_i^*) \in M_n(pMp)$ is majorized by the matrix $\|a\|^2(x_j x_i^*)$ and α is a *-isomorphism,

$$\big(\alpha(x_j a^* a x_i^*)\big) \le \|a\|^2\big(\alpha(x_j x_i^*)\big).$$

Hence

$$\begin{aligned}\left\|\sum T(x_i a^*)^*h_i\right\|^2 &\le \|a\|^2 \sum \langle \alpha(x_j x_i^*)h_i, h_j\rangle \\ &= \left\|\sum T(x_i)^*h_i\right\|^2 \|a\|^2.\end{aligned}$$

It follows that $\gamma(a)$ is well defined and can be extended to an operator in $B(H)$ with $\|\gamma(a)\| \le \|a\|$. For $x_i \in pM(I-p),\ h_i \in H,\ i = 1, 2$, we have

$$\begin{aligned}&\langle \gamma(a)T(x_i)^*h_1,\ T(x_2)^*h_2\rangle = \langle T(x_1 a^*)^*h_1,\ T(x_2)^*h_2\rangle \\ &= \langle \alpha(x_2 a x_1^*)h_1, h_2\rangle = \langle T(x_1)^*h_1, T(x_2 a)^*h_2\rangle \\ &= \langle T(x_1)^*h_1, \gamma(a^*)T(x_2)^*h_2\rangle .\end{aligned}$$

Hence $\gamma(a^*) = \gamma(a)^*$. It is easy to check that γ is multiplicative and injective. Now suppose $b \in N'$ then for every $x \in pM(I-p),\ h \in H$,

$$\gamma(a)bT(x)^*h = \gamma(a)T(x)^*bh = T(xa^*)^*bh = bT(xa^*)^*h = b\gamma(a)T(x)^*h.$$

Hence $\gamma(a) \in N$. Since $\gamma(a)$ is zero on $H \ominus H_0,\ \gamma(a) \in (I-q)N(I-q)$.

It follows from the definition that, for $a \in (I-p)M(I-p)$ and $x \in pM(I-p),\ \gamma(a)T(x)^* = T(xa^*)^*$; hence

$$T(x)\gamma(a) = T(xa).$$

This proves part (ii).

Now choose $z, t \in qN(I-q)$ and write $a = T^{-1}(z)^*T^{-1}(t) \in (I-p)M(I-p)$. Compute, for $x \in pM(I-p)$ and $h \in H$,

$$\begin{aligned}\gamma(a)T(x)^*h &= T\big(xT^{-1}(t)^*T^{-1}(z)\big)^*h = [\alpha(xT^{-1}(t)^*)z]^*h \\ &= [T(x)t^*z]^*h = z^*tT(x)^*h = z^*tT(x)^*h.\end{aligned}$$

Hence $z^*t \in \gamma\big((I-p)M(I-p)\big)$. Since products of this form generate $(I-q)N(I-q)$ as a von Neumann algebra and the image of γ is a von Neumann algebra, γ is surjective. This proves (i). We can now apply [MS, Lemma 5.10] to get (iii). ∎

Remark. Note that γ is in fact equivalent to the representation on the internal tensor product $(I-p)Mp \otimes_\alpha H$.

Corollary 2.23. *Assume $c(g_1) = I$. Then there is a *-isomorphism*

$$\Psi : M \longrightarrow N$$

such that, for $a \in M$,

(i) $\Psi(pap) = \alpha(pap) \in qNq$,
(ii) $\Psi(pa(I-p)) = T(pa(I-p)) \in qN(I-q)$,
(iii) $\Psi((I-p)a(I-p)) = \gamma((I-p)a(I-p)) \in (I-q)N(I-q)$,
(iv) $\Psi((I-p)ap) = T(pa^*(I-p))^* \in (I-q)Nq$.

Proof. The equations (i)-(iv) define Ψ and the properties of α and γ (see Lemma 2.20, Corollary 2.21 and Proposition 2.22) show that Ψ is indeed a *-isomorphism of M onto N. ∎

Proof of Theorem 2.1. It was shown in Lemma 2.4 that, to prove Theorem 2.1, it will suffice to write the algebra as a direct sum (using central projections) of algebras for which Theorem 2.1 holds. In Lemma 2.5 we saw that we can assume that the algebra M and the projections p and $I-p$ satisfy one of the conditions ((1), (2) or (3)) stated in that lemma. If condition (1) is satisfied then Theorem 2.1 follows from Lemma 2.6. So we can assume that either condition (2) or condition (3) is satisfied. Condition (3) is, in fact, condition (2) for $p, I-p$ interchanged. It will suffice, therefore, to assume condition (2). We then find, in Lemma 2.19, two central projections, $c(g_1)$ and $c(g_2)$, with $c(g_1)+c(g_2) = I$ and (again, by referring to Lemma 2.4) we can assume that either $c(g_1) = I$ or $c(g_2) = I$. For the first case the theorem is proved in Corollary 2.23. In this case we get, in fact, that $e_2 = 0$ and the map, extending T, is a *-isomorphism. The proof of the other case, when $c(g_2) = I$, is similar and is omitted. In that case the map turns out to be a *-anti-isomorphism. ∎

Remark 2.24. The map $\Phi + \Psi$ of Theorem 2.1 is an isometry of M onto N that extends T and maps $pMp + (I-p)M(I-p)$ onto $qNq + (I-q)N(I-q)$ and $pM(I-p)$ onto $qN(I-q)$. We write Λ for $\Phi + \Psi$.

Recall from the introduction that given a right self-dual Hilbert C^*-module X over a von Neumann algebra A, we can form the von Neumann linking algebra which can be written

$$\mathcal{L}_w(X) = \begin{pmatrix} \mathbb{B}(X) & X \\ \bar{X} & A \end{pmatrix}$$

where $\mathbb{B}(X)$ is the algebra of all bounded, adjointable A-linear maps on X and $\bar{X}$ is the conjugate module (which is a left Hilbert C^*-module over A). It is known that this algebra is indeed a von Neumann algebra. We assume that our C^*-modules are full and this implies that we can write X as $p\mathcal{L}_w(X)(I-p)$ for a projection p with $c(p) = c(I-p) = I$. The following corollary then follows immediately from the theorem.

Corollary 2.25. *If X and Y are right selfdual C^*-modules over (possibly different) von Neumann algebras A, B and if $\mathcal{L}_w(X)$ and $\mathcal{L}_w(Y)$ are the von Neumann linking algebras of X and Y respectively, then every linear surjective isometry to a linear surjective isometry $\Lambda : \mathcal{L}_w(X) \to \mathcal{L}_w(Y)$.*

Moreover, there is a central projection $z \in \mathcal{L}_w(X)$ such that if we write $\Psi = \Lambda | z\mathcal{L}_w(X)$ and $\Phi = \Lambda|(I-z)\mathcal{L}_w(X)$ then

(1) Ψ is a *-isomorphism onto $\Lambda(z)\mathcal{L}_w(Y)$. It defines a *-isomorphism $\Psi_{11} : z\mathbb{B}(X) \to \Lambda(z)\mathbb{B}(Y)$ and a *-isomorphism $\Psi_{22} : zA \to \Lambda(z)B$ such that, for $L \in \mathbb{B}(X)z,\ a \in Az$ and $x \in zX,\ y \in zX$,

$$\begin{aligned} T(Lxa) &= \Psi_{11}(L)T(x)\Psi_{22}(a), \\ T(x)T(y)^* &= \Psi_{11}(x \otimes y^*), \\ T(x)^*T(y) &= \Psi_{22}(\langle x, y\rangle_A) \end{aligned}$$

and

(2) Φ is a *-anti-isomorphism onto $\Lambda(I-z)\mathcal{L}_w(Y)$. It defines *-anti-isomorphisms $\Phi_{11} : (I-z)\mathbb{B}(X) \to \Lambda(I-z)B$ and $\Phi_{22} : (I-z)A \to \Lambda(I-z)\mathbb{B}(Y)$ such that, for $L \in \mathbb{B}(X)(I-z),\ x, y \in (I-z)X,\ a \in (I-z)A$,

$$\begin{aligned} T(Lxa) &= \Phi_{22}(a)T(x)\Phi_{11}(L), \\ T(x)T(y)^* &= \Phi_{22}(\langle x, y\rangle), \\ T(x)^*T(y) &= \Phi_{11}(y \otimes x^*). \qquad \blacksquare \end{aligned}$$

Corollary 2.26. *If X, Y, A, B are as in Corollary 2.24 but we assume also that A is a factor then the surjective linear isometry $T : X \to Y$ can be extended to a map*

$$\Lambda : \mathcal{L}_w(X) \to \mathcal{L}_w(Y)$$

which is either a *-isomorphism of or a *-anti-isomorphism.

Moreover, if X^t is any C^*-module that is isometric to X and such that the induced isometry from $\mathcal{L}_w(X^t)$ to $\mathcal{L}_w(X)$ is a *-anti-isomorphism, then Y is completely isometric to either X or X^t. ■

3. Isometries of Hilbert C^*-modules

Now let X and Y be right (full) Hilbert C^*-modules over the C^*-algebras A and B respectively and let $T : X \to Y$ be a surjective linear isometry. Write $\mathcal{L}(X)$ (and $\mathcal{L}(Y)$) for the linking algebra of X (and Y); i.e.

$$\mathcal{L}(X) = \begin{pmatrix} \mathbb{K}(X) & X \\ \bar{X} & A \end{pmatrix};\ \mathcal{L}(Y) = \begin{pmatrix} \mathbb{K}(Y) & Y \\ \bar{Y} & B \end{pmatrix}$$

where $\bar{X}$ and $\bar{Y}$ are the conjugate modules. $\bar{X}$ is a left Hilbert C^*-module over A and $\bar{Y}$ is over B. (See Section 1 for the definitions.) It is known that $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are C^*-algebras. Now the isometry $T : X \to Y$ induces an isometry $T^{**} : X^{**} \to Y^{**}$ (where X^{**} is the second dual of X) extending T. Write

$$M = \mathcal{L}(X)^{**},\ N = \mathcal{L}(Y)^{**}.$$

Then M (respectively N) is a von Neumann algebra. In fact it can be identified with the universal enveloping algebra of $\mathcal{L}(X)$ (respectively $\mathcal{L}(Y)$). To continue we note the following.

Lemma 3.1. *There is a projection $p \in \mathcal{L}(X)^{**}$ such that $p\mathcal{L}(X)^{**}(I-p)$ is the w^*-closure of $X \subseteq \mathcal{L}(X) \subseteq \mathcal{L}(X)^{**}$. Similarly, the w^*-closure of Y in $\mathcal{L}(Y)^{**}$ is $q\mathcal{L}(Y)^{**}(I-p)$ for some projection $q \in \mathcal{L}(Y)^{**}$. Moreover, $c(p) = c(I-p) = I$ and similarly for q.*

Proof. We can identify $\mathcal{L}(X)^{**}$ with the σ-weak closure of $\pi_u(\mathcal{L}(X))$ where (π_u, H_u) is the universal representation of $\mathcal{L}(X)$. The w^*-topology of $\mathcal{L}(X)^{**}$ is then the σ-weak topology of $\pi_u(\mathcal{L}(X))$. Write

$$\mathcal{I}_1 = \begin{pmatrix} \mathbb{K}(X) & X \\ 0 & 0 \end{pmatrix} \subseteq \mathcal{L}(X); \;\; \mathcal{I}_2 = \begin{pmatrix} 0 & 0 \\ \bar{X} & A \end{pmatrix} \subseteq \mathcal{L}(X).$$

Write M for the σ-weak closure of $\pi_u(\mathcal{L}(X))$. $\mathcal{I}_1$ and $\mathcal{I}_2$ are right ideals in $\mathcal{L}(X)$ and the σ-weak closures of $\mathcal{I}_1$ and $\mathcal{I}_2$ are of the form p_1M and p_2M respectively. In fact, $p_1 = \bigvee\{r(y) : y \in \pi_u(\mathcal{I}_1)\}$ and $p_2 = \bigvee\{r(z) : z \in \pi_u(\mathcal{I}_2)\}$ where $r(y)$ is the range projection (in M) of y. The σ-weak closure of $\pi_u(X)$ is then $p_1M \cap (p_2M)^* = p_1Mp_2$. But it is clear that $p_1 + p_2 = I$ and, thus, writing $p = p_1$ we find that the σ-weak closure of $\pi_u(X)$ is $pM(I-p)$. A similar argument works for Y. Finally note that MpM contains $\pi_u(\mathcal{L}(X)\mathcal{I}_1) = \pi_u(\mathcal{L}(X))$; hence is σ-weakly dense in M and it follows that $c(p) = I$. The argument for $I - p (= p_2)$ is similar. ■

Since it follows from Banach space theory that X^{**} is isometrically isomorphic to the w^*-closure of X in $\mathcal{L}(X)^{**}$ we conclude that T^{**} induces a surjective linear isometry

$$S : p\mathcal{L}(X)^{**}(I-p) \to q\mathcal{L}(Y)^{**}(I-q)$$

and S extends T (when we view X, Y as subspaces of $\mathcal{L}(X)^{**}$ and $\mathcal{L}(Y)^{**}$ respectively). In particular, S maps X onto Y. Applying Theorem 2.1 to S we obtain the following.

Theorem 3.2. *Let X be a right full Hilbert C^*-module over the C^*-algebra A and Y be a right full Hilbert C^*-module over B. Let $T : X \to Y$ be a surjective linear isometry. Then*

(1) *There is a surjective linear isometry*

$$\Lambda_0 : \mathcal{L}(X) \to \mathcal{L}(Y)$$

extending T and mapping $\mathbb{K}(X) \oplus A$ onto $\mathbb{K}(Y) \oplus B$.

(2) *There is a projection f in the center of $\mathcal{L}(Y)^{**}$ such that the map*

$$\Psi_0(a) = \Lambda_0(a)f, \quad a \in \mathcal{L}(X),$$

is a *-homomorphism and the map

$$\Phi_0(a) = \Lambda_0(a)(I-f), \quad a \in \mathcal{L}(X),$$

is a *-antihomomorphism.

Proof. Part (2) follows from Theorem 2.1 but in fact it is known whenever Λ_0 is an isometry of C^*-algebras (see [K] and [T, p. 188]). For part (1) we need only to notice that the isometry $\Lambda = \Psi + \Phi$, given by Theorem 2.1 and Remark 2.24 (mapping $\mathcal{L}(X)^{**}$ onto $\mathcal{L}(Y)^{**}$) maps $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$. But we know from Corollary 2.25 that Ψ maps $\mathbb{K}(X)$ onto $\mathbb{K}(Y)$ since

$$\Psi(x \otimes y^*) = S(x) \otimes S(y)^* = T(x) \otimes T(y)^* \in \mathbb{K}(Y)$$

and similarly Ψ maps A onto B, X onto Y and $\bar{X}$ onto $\bar{Y}$. The statements for Φ are similar (although here $\Phi(\mathbb{K}(X)) = B$, $\Phi(A) = \mathbb{K}(Y)$). ■

Remark 3.3. If we have X, Y and T as in Theorem 3.2 and we let Ψ_{11} define the restriction of Ψ_0 to $\mathbb{K}(X)$ and Ψ_{22} be the restriction of Ψ_0 to A, then it follows from the properties of Ψ_0 that both Ψ_{11} and Ψ_{22} are *-homomorphisms and

$$T(Kxa)f = \Psi_{11}(K)T(x)\Psi_{22}(a), \quad a \in A,\ K \in \mathbb{K}(X),\ x \in X.$$

Similarly, we get *-antihomomorphisms Φ_{11} and Φ_{22} with

$$T(Kxa)(I-f) = \Phi_{22}(a)T(x)\Phi_{11}(K), \quad a \in A,\ K \in \mathbb{K}(X),\ x \in X.$$

Corollary 3.4. *Suppose X and Y are as in Theorem 3.2 and $T : X \to Y$ is a 2-isometry (i.e. the map $I \otimes T$ that maps $M_2 \otimes X$ onto $M_2 \otimes Y$ is an isometry). Then the map Λ_0 of Theorem 3.2 is a *-isomorphism (i.e. $\Lambda_0 = \Psi_0,\ f = I$) and if we let Ψ_{11} and Ψ_{22} be the maps induced by $\Lambda_0 = \Psi_0$ on $\mathbb{K}(X)$ and on A respectively (so that $\Psi_{11} : \mathbb{K}(X) \to \mathbb{K}(Y)$ and $\Psi_{22} : A \to B$), then Ψ_{11} and Ψ_{22} are *-isomorphisms and*

(i) $T(Lxa) = \Psi_{11}(L)T(x)\Psi_{22}(a), \quad L \in \mathbb{K}(X),\ a \in A,\ x \in X,$
(ii) $T(x)^*T(y) = \Psi_{22}(\langle x, y\rangle), \quad x, y \in X,$
(iii) $T(x)T(y)^* = \Psi_{11}(x \otimes y^*), \quad x, y \in X.$

Proof. Write T_2 for the isometry $T_2 : M_2 \otimes X \to M_2 \otimes Y$. Fix

$$x = e_{12} \otimes x_0,\ y = e_{11} \otimes y_0,\ z = e_{21} \otimes z_0 \ \text{ in } \ M_2 \otimes X.$$

Since T_2 satisfies Lemma 2.2,

$$T_2(xy^*z + zy^*x) = T_2(x)T_2(y)^*T_2(z) + T_2(z)T_2(y)^*T_2(x).$$

But $xy^*z = 0$ and $T_2(x)T_2(y)^*T_2(z) = 0$. This implies that $T_2\big(e_{22} \otimes z_0y_0^*x_0\big) = T_2(zy^*x) = T_2(z)T_2(y)^*T_2(x) = e_{22} \otimes T(z_0)T(y_0)^*T(x_0)$. Hence $T(z_0y_0^*x_0) = T(z_0)T(y_0)^*T(x_0)$ for all x_0, y_0, z_0 in X. We again write X^{**} as $pM(I-p)$ ($M = \mathcal{L}(X)^{**}$) and Y^{**} as $qN(I-q)$ ($N = \mathcal{L}(Y)^{**}$). We get an isometry $S : pM(I-p) \to qN(I-q)$ extending T. Clearly this extension still satisfies

$$S(zy^*x) = S(z)S(y)^*S(x), \quad x, y, z \in pM(I-p).$$

We can now apply the results of the previous section to S. As in Lemma 2.5, M can be decomposed into a direct sum of von Neumann algebras each satisfying one of the conditions stated in Lemma 2.5. If condition (1) holds then, using the proof of Lemma 2.6 we see that the map induced on the linking algebra is a *-isomorphism. Suppose now that condition (2) holds. Let $\{u_i\}$ be as in this condition and write $v_i = S(u_i),\ r_i = S(u_i)S(u_i)^*$ and $d_i = S(u_i)^*S(u_i)$. We have, for $i \neq j$,

$$0 = S(u_iu_i^*u_j) = S(u_i)S(u_i)^*S(u_j).$$

Hence $r_ir_j = 0$. It follows form Lemma 2.7 that $d_i = d_j$ for all i, j. It then follows that the map β, defined in the discussion following Lemma 2.13, vanishes; i.e. $\theta = \alpha$. Hence the map induced on the linking algebra M is a *-isomorphism. A similar argument works if condition (3) (of Lemma 2.5) holds. Statements (i)-(iii) now follow. ∎

Remark 3.5. Corollary 3.4 shows that a (surjective) 2-isometry from one C^*-Hilbert module to another is necessarily completely isometric. I do not know to what extent it holds for larger classes of operator spaces. The argument above (combined with Corollary 2.10 of [AS]) can be used to show that unital 2-isometries of operator algebras are multiplicative and for some classes of operator algebras this would imply that they are complete isometries.

Given a Hilbert space H there is more than one way to represent it as an operator space; i.e. one can find different operator spaces that are all isometric to H but they are pairwise non-completely-isometric. One representation of H as an operator space is when you fix an orthonormal basis $\{e_i\}$ for H and consider the space of all bounded operators in $B(H)$ whose matrix with respect to this basis has non-zero entries only in the first column. This subspace of $B(H)$ is isometric to H and is called the Hilbert column space, written H^c. One can write $H^c = B(\mathbb{C}\, e_1, H)$. Replacing the word "column" by the word "row" we get the Hilbert row space H^r (or $B(H, \mathbb{C}\, e_1)$). It is known that these two operator spaces are isometric but not completely isometric. Both operator spaces have a natural Hilbert C^*-module structure. H^c is a C^*-module over the algebra $\mathbb{C}$ and H^r is a C^*-module over the algebra $K(H)$, the compact operators on H. But in addition to H^c and H^r there are many other different (i.e. not completely isometric) representations of H as an operator space (see [Pi]). The following corollary shows that none of these is a Hilbert C^*-module.

Corollary 3.6. *Let H be a fixed Hilbert space and X be a Hilbert C^*-module over a C^*-algebra A that is isometric (as a Banach space) to H. Then either A is isomorphic to $\mathbb{C}$ and X is completely isometrically isomorphic to H^c or A is *-isomorphic to $K(H)$ and X is completely isometrically isomorphic to H^r.*

Proof. Assume, for simplicity, that H is infinite dimensional. Write $T : H^c \to X$ for the linear surjective isometry of H^c onto X. From Theorem 3.2 it follows that we can extend T to an isometry

$$\Lambda_0 : \mathcal{L}(H^c) = \begin{pmatrix} K(H) & H^c \\ \overline{H^c} & \mathbb{C} \end{pmatrix} \longrightarrow \mathcal{L}(X) = \begin{pmatrix} \mathbb{K}(X) & X \\ \bar{X} & A \end{pmatrix}.$$

But $\mathcal{L}(H^c)$ is *-isomorphic to $K(H)$ and thus $\mathcal{L}(H^c)^{**}$ is a factor. Hence Λ_0 is either a *-isomorphism or a *-anti-isomorphism. In the former case Λ_0 maps $\mathbb{C}$ onto A and H^c onto X and is a complete isometry. In the latter case we consider the map τ

$$\tau : \begin{pmatrix} \mathbb{C} & H^r \\ \overline{H^r} & K(H) \end{pmatrix} \longrightarrow \begin{pmatrix} K(H) & H^c \\ \overline{H^c} & \mathbb{C} \end{pmatrix}$$

defined by

$$\tau \begin{pmatrix} \lambda & y \\ \bar{z} & K \end{pmatrix} = \begin{pmatrix} K^t & y^t \\ \overline{z^t} & \lambda \end{pmatrix}, \qquad y, z \in H^r,\ K \in K(H),\ \lambda \in \mathbb{C}$$

(where K^t is the transpose of K and y^t is the transpose of y). Then τ is a *-anti-isomorphism and $\Lambda_0 \circ \tau$ is then a *-isomorphism that maps, completely isometrically, $K(H)$ onto A and H^r onto X. ■

References

[AS] J. Arazy and B. Solel, Isometries of non-self-adjoint operator algebras, *J. Funct. Anal.* **90** (1990), 284-305. MR **91c:**47085

[B1] D. P. Blecher, A new approach to Hilbert C^*-modules, *Math. Ann.* **307** (1997), 253-290. MR **98d:**46063

[B2] D. P. Blecher, On selfdual Hilbert modules, in "Operator Algebras and their Applications", Fields Institute Comm. vol. 13, pp. 65-80, Amer. Math. Soc.,Providence, RI, 1997. MR **97k:**46070

[F] M. Frank, A multiplier approach to the Lance-Blecher theorem, *Zeit. Anal. Anwendungen* **16** (1997), 565-573. MR **98j:**46056

[Ha] M. Hamana, Triple envelopes and Silov boundaries of operator spaces, *Math. J. Toyama Univ.* **22** (1999), 77-93. MR **2001a:**46057

[H] L. A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, in "Proceedings on Infinite Demensional Holomorphy (Kentucky 1973)", Lecture Notes in Mathematics vol. 364, pp. 13-40, Springer Verlag, Berlin–Heidelberg–New York, 1974. MR **53:**11106

[Ho] G. Horn, Characterization of the predual and ideal structure of a JBW^*-triple, *Math. Scand.* **61** (1987), 117-133. MR **89a:**46140

[K] R. V. Kadison, Isometries of operator algebras, *Ann. Math.* **54** (1951), 325-338. MR **13:**256a

[KR] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, vol. II, Academic Press, New York 1986. MR **98f:**46001b

[Ka] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183** (1983), 503-529. MR **85c:**46040

[L1] E. C. Lance, Unitary operators on Hilbert C^*-modules, *Bull. London Math. Soc.* **26** (1994), 363-366. MR **95i:**46088

[L2] E. C. Lance, "Hilbert C^*-modules – A Toolkit for Operator Algebraists", London Math. Soc. Lecture Notes, Cambridge Univ. Press, Cambridge, UK, 1995. MR **96k:**46100

[MS] P. S. Muhly and B. Solel, On the Morita equivalence of tensor algebras, *Proc. London Math. Soc.* **81** (2000), 113-118. CMP 2000:12

[P] W. L. Paschke, Inner product modules over B^*-algebras, *Trans. Amer. Math. Soc.* **182** (1973), 443-468. MR **50:**8087

[Pi] G. Pisier, The operator Hilbert space OH, complex interpolation and tensor norms, *Mem. Amer. Math. Soc.* No. **585** (1996). MR **97a:**46024

[RW] I. Raeburn and D. P. Williams, Morita equivalence and continuous trace C^*-algebras, Math. Surveys and Monogr. vol. 60, Amer. Math. Soc., Providence RI, 1998. MR **2000c:**46108

[S] B. Solel, Isometries of CSL algebras, *Trans. Amer. Math. Soc.* **332** (1992), 595-606. MR **92j:**47082

[T] M. Takesaki, "Theory of operator algebras I", Springer–Verlag, 1979. MR **81e:**46038

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `mabaruch@techunix.technion.ac.il`