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RANDOM VARIABLE DILATION EQUATION AND MULTIDIMENSIONAL PRESCALE FUNCTIONS

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ABSTRACT. A random variable Z satisfying the random variable dilation equation $MZ \overset{d}{=} Z + G$, where G is a discrete random variable independent of Z with values in a lattice $\Gamma \subset \mathbf{R}^d$ and weights $\{c_k\}_{k\in\Gamma}$ and M is an expanding and Γ-preserving matrix, if absolutely continuous with respect to Lebesgue measure, will have a density φ which will satisfy a dilation equation

$$\varphi(x) = |\det M| \sum_{k\in\Gamma} c_k \varphi(Mx - k).$$

We have obtained necessary and sufficient conditions for the existence of the density φ and a simple sufficient condition for φ's existence in terms of the weights $\{c_k\}_{k\in\Gamma}$. Wavelets in $\mathbf{R}^d$ can be generated in several ways. One is through a multiresolution analysis of $L^2(\mathbf{R}^d)$ generated by a compactly supported prescale function φ. The prescale function will satisfy a dilation equation and its lattice translates will form a Riesz basis for the closed linear span of the translates. The sufficient condition for the existence of φ allows a tractable method for designing candidates for multidimensional prescale functions, which includes the case of multidimensional splines. We also show that this sufficient condition is necessary in the case when φ is a prescale function.

1. Introduction

Multiresolution analysis on $\mathbf{R}^d$ is one possible framework for construction of wavelet bases. Let Γ be a lattice in $\mathbf{R}^d$ and let $M : \mathbf{R}^d \to \mathbf{R}^d$ be an expansive linear transformation, that is, all eigenvalues of M have modulus greater than 1, such that $M\Gamma \subseteq \Gamma$. Then $m = |\det M|$ is an integer, greater than one, equal to the order of the group $\Gamma/M\Gamma$. A *multiresolution analysis* associated to Γ and M with *prescale function* φ is an increasing sequence of subspaces of $L^2(\mathbf{R}^d)$, $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ satisfying the following four conditions:

(i) $\bigcup_j V_j$ is dense in $L^2(\mathbf{R}^d)$;

(ii) $\bigcap_j V_j = \{0\}$;

(iii) $f(\cdot) \in V_j \Leftrightarrow f(M^{-j}\cdot) \in V_0$;

(iv) $\{\varphi(\cdot - \gamma)\}_{\gamma\in\Gamma}$ is a Riesz basis for V_0.

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A *wavelet basis* associated to the multiresolution analysis is an orthonormal basis for $L^2(\mathbf{R}^d)$ of the form $\left\{m^{j/2}\psi_k\left(M^j\cdot-\gamma\right): j\in Z, \gamma\in\Gamma, 1\le k\le m\right\}$ where

$$\psi_k(x)=\sum_{\gamma\in\Gamma}a_k(\gamma)\,\varphi(Mx-\gamma)$$

and$\{a_k(\gamma)\}_{\gamma\in\Gamma}$ is square summable for $1\le k\le m$. The functions $\{\psi_k\}_{k=1}^m$ are called the *wavelet generators*. When the lattice translates of φ form an orthonormal basis of V_0 we take $\psi_1:=\varphi$.

Conditions (iii) and (iv) together imply that the set $\{\varphi(M\cdot-\gamma)\}_{\gamma\in\Gamma}$ is a Riesz basis for the subspace V_1. Since $\varphi\in V_0\subseteq V_1$, we can write

$$\varphi(x)=\sum_{\gamma\in\Gamma}a(\gamma)\,\varphi(Mx-\gamma)\,; \tag{1.1}$$

equation (1.1) is called a *dilation equation.*

One way to understand (1.1) is through a probabilistic approach. Consider a discrete random variable G with values in a subset Γ_1 of Γ and a random variable Z, independent of G, with values in $\mathbf{R}^d$, both defined on a complete probability space $(\Omega,\mathcal{F},P)$, which satisfy

$$MZ\overset{d}{=}Z+G. \tag{1.2}$$

Here, $\overset{d}{=}$ denotes equality of the corresponding laws. Assume that Z is absolutely continuous with respect to Lebesgue measure and denote its density by φ. Equation (1.2) implies that φ satisfies the dilation equation (1.1) with $a(\gamma)=|\det M|\,P(G=\gamma)$. Our approach to constructing candidates for prescale functions comes from understanding the structure of the solution of this random variable dilation equation.

In the one-dimensional case with $M=2$, Gundy and Zhang [6] proved that Z is absolutely continuous with respect to Lebesgue measure if and only if the fractional part of Z is uniform. They also gave a sufficient condition for the uniformity of the fractional part. In the higher dimensional case, we show that the statements of Gundy and Zhang hold true when a proper notion of the "fractional" part of a random variable is introduced. We have found the theory of self-affine tilings of $\mathbf{R}^d$ and use of the digit representation of the fractional part of Z to be the correct framework for the higher dimensional case. The major difficulty in generalizing the results to higher dimensions comes from the fact that M may not be merely an expansion but may include a rotation. Such an M causes a tile to have, in general, a fractal boundary. The boundary difficulties called for some new techniques of proofs beyond those used in [6].

In Section 2 we introduce notation needed to express an explicit solution Z to (1.2). Definitions of the "fractional" and "integer" parts of an $\mathbf{R}^d$-valued random variable Z are given based on concepts of self-affine tilings. We also give some basic results regarding the fractional part of Z. In Section 3 we give necessary and sufficient conditions under which the random variable Z will have a density, in terms of the fractional part of Z. In Section 4 we give a simple sufficient condition on the weights on the values of G which guarantee absolute continuity of Z. In Section 5 we give examples of density functions obtained using these results. In Section 6 we show that the sufficient condition of Section 4 is also necessary when φ is a prescale function.

2. Basic properties of a random variable dilation equation solution

In order to write an explicit solution of (1.2), some definitions are needed. Let $G_1, G_2, ...$ be an i.i.d. sequence of random variables defined on the space $(\Omega, \mathcal{F}, P)$, with $G_1 \stackrel{d}{=} G$. Recall that G is discrete with values in the lattice. Assume

$$\sum_{j=1}^{\infty} M^{-j} G_j < \infty \text{ a.s.}$$

Then the sequence $\{Z_k\}$ defined by

$$Z_k = \sum_{j=1}^{\infty} M^{-j} G_{j+k} \text{ for } k = 1, 2, \ldots \tag{2.1}$$

is a sequence of random variables. Note that the following two properties hold:

$$MZ_k = M(\sum_{j=1}^{\infty} M^{-j} G_{j+k}) = G_{k+1} + \sum_{j=2}^{\infty} M^{-j+1} G_{j+k} = G_{k+1} + Z_{k+1},$$

and

$$Z_0 \stackrel{d}{=} Z_k, \text{ and } G_k \text{ is independent of } Z_k.$$

Therefore for any k, Z_k solves the dilation equation (1.2).

The fractional part of Z will play an essential role in what follows. In order to define the fractional part of Z, we first invoke some basic facts about self-affine tilings. Let Γ_0 denote a set of coset representatives of $\Gamma/M\Gamma$, and without loss of generality, we assume $0 \in \Gamma_0$. A *self-affine tiling of* $\mathbf{R}^d$ consists of a closed set T with nonempty interior such that

$$\bigcup_{\gamma \in \Gamma} (T + \gamma) = \mathbf{R}^d \text{ and } \bigcup_{\gamma \in \Gamma_0} (T + \gamma) = MT. \tag{2.2}$$

Clearly a tiling depends on the choice of Γ_0. In dimensions $d = 2$ and 3, one can always find a Γ_0 that admits a self-affine tiling, and in higher dimensions it can be done for $m = |\det M| > d$ [10]. For the remainder of the paper, we will assume that Γ_0 admits a self-affine tiling.

The lattice translates of the interior of T are disjoint and $int\, T \neq \varnothing$ [1], so if $x \in \bigcup_{\gamma \in \Gamma} (int\, T + \gamma)$, then $x \in int\, T + \gamma_x$ where γ_x denotes the unique element of Γ giving the location of the point x. If $x \notin \bigcup_{\gamma \in \Gamma} (int\, T + \gamma)$, then we say x is a *boundary point* and note that $x \in \bigcap_{\gamma \in \Gamma_1} (T + \gamma)$, for some finite $\Gamma_1 \subseteq \Gamma$. The fact that Γ_1 is finite follows from the compactness of T.

Define $[\,\cdot\,] : \mathbf{R}^d \to \Gamma$ by

$$[x] = \begin{cases} \gamma_x & \text{if } x \in \bigcup_{\gamma \in \Gamma} (int\, T + \gamma), \\ \max_{\gamma \in \Gamma_1} \gamma & \text{if } x \text{ is a boundary point,} \end{cases}$$

where "max" is meant in the sense of the dictionary ordering of $\mathbf{R}^d$.

Proposition 1. $[\,\cdot\,]$ *is Borel-measurable.*

Proof. We only need to consider $\{x \mid [x] = \gamma\}$ for a fixed $\gamma \in \Gamma$. Since T is compact and Γ is countable,

$$[\gamma_1]^{-1} = (int\, T + \gamma_1) \cup \bigcup_{\gamma \in \Gamma} ((T + \gamma) \cap (T + \gamma_1))$$

is a Borel set. □

For any $x \in \mathbf{R}^d$ we will call $[x]$ the *integer part of* x and $(x) = x - [x]$ the fractional part of x. By Proposition 1, $[Z]$ is a random variable and therefore so is $(Z) = Z - [Z]$. Notice that (Z) takes values in the tile T.

A point $t \in \mathbf{R}^d$ is in T if and only if

$$t = \sum_{j=1}^{\infty} M^{-j} \gamma_j, \tag{2.3}$$

where for all j, $\gamma_j \in \Gamma_0$ [5]. Based on the expansion (2.3), define functions $\xi_j : \Omega \to \Gamma_0$, $j = 1, 2, ...$, by

$$(Z_0) = \sum_{j=1}^{\infty} M^{-j} \xi_j; \tag{2.4}$$

that is, $\xi_j(\omega)$ is the element of Γ_0 which appears in the jth term of the tile expansion of $(Z_0)(\omega)$. If there is more than one expansion for a tile point, simply choose one of them.

Proposition 2. *Assume that* $P((Z_0) \in \partial T) = 0$. *Then* $\{\xi_j\}_{j=1}^{\infty}$ *is a sequence of random variables and for each* k

$$(Z_k) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+k} \quad a.s.$$

Proof. From the dilation equation (1.2) and from the decomposition of Z_0 into its fractional and integer parts, we obtain

$$M[Z_0] + M(Z_0) = MZ_0 = G_1 + Z_1 = G_1 + [Z_1] + (Z_1).$$

Using (2.4) it follows that

$$M[Z_0] + \xi_1 + \sum_{j=1}^{\infty} M^{-j} \xi_{j+1} = G_1 + [Z_1] + (Z_1). \tag{2.5}$$

The definition of a lattice tiling implies $(\gamma + T) \cap (\gamma' + int\, T) = \emptyset$ if and only if $\gamma \neq \gamma'$. So, if $(Z_1) \in int\, T$, then by (2.5), we have

$$M[Z_0] + \xi_1 = G_1 + [Z_1] \text{ and } (Z_1) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+1}. \tag{2.6}$$

Since $P((Z_0) \in \partial T) = 0$ and since $Z_1 \stackrel{d}{=} Z_0$, it follows that $P((Z_1) \in int\, T) = 1$, and therefore

$$(Z_1) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+1} \ a.s.$$

By (2.6) $\xi_1 = G_1 + [Z_1] - M[Z_0]$ almost surely and so ξ_1 is a random variable.

The proof is completed by induction on k. □

Define $h : \Gamma \to \Gamma_0$ to be the map which assigns to each element of Γ its coset representative.

Proposition 3. *Suppose $P((Z_0) \in \partial T) = 0$. Then for*

$$k = 1, 2, \ldots, \qquad \xi_k = h([Z_k] + G_k) \ \text{a.s.}$$

Proof. $P((Z_0) \in \partial T) = 0$ implies that (2.6) holds. So $\xi_1 = h(G_1 + [Z_1])$ since coset representatives are unique.

Proposition 2, the fact that $Z_k = [Z_k] + (Z_k)$, and the dilation equation (1.2) together lead to

$$M[Z_k] + \xi_{k+1} + (Z_{k+1}) = G_{k+1} + [Z_{k+1}] + (Z_{k+1}) \ \text{a.s.}$$

This implies that $M[Z_k] + \xi_{k+1} = G_{k+1} + [Z_{k+1}]$ a.s. since $P((Z_{k+1}) \in \partial T) = 0$. The uniqueness of coset representatives ensures $\xi_{k+1} = h(G_{k+1} + [Z_{k+1}])$ a.s. $\square$

Define $g : (\mathbf{R}^d)^\infty \to \mathbf{R}^d$ by

$$g(x_1, x_2, \ldots) = x_1 + \left[\sum_{j=1}^{\infty} M^{-j} x_{j+1}\right].$$

The measurability of g follows from Proposition 1 and from the fact that the projection map is a measurable function.

Proposition 4. *Let $Y_k := (h \circ g)(G_k, G_{k+1}, \ldots)$. Then $Y_1, Y_2, \ldots$ is a stationary and ergodic sequence of random variables.*

Proof. The proof follows from the fact that $h \circ g$ is measurable and $\{G_k\}_{k=1}^{\infty}$ is i.i.d. $\square$

Corollary 1. *If $P((Z_0) \in \partial T) = 0$, the sequence $\xi_1, \xi_2, \ldots$ is stationary and ergodic.*

Proof. If $P((Z_0) \in \partial T) = 0$, Proposition 4 implies $\xi_k = Y_k$ a.s . $\square$

3. Necessary and sufficient conditions for absolute continuity of Z

Throughout this section let $\lambda_T := \frac{\lambda}{\lambda(T)}$ denote Lebesgue measure normalized by the measure of the tile T (if $\Gamma = \mathbf{Z}^d$, then $\lambda(T) = 1$)).

Theorem 1. *Let M, Γ, Γ_0 and random variables G, Z and ξ_k be as defined in the previous sections. Suppose G has values in a finite set Γ_1 such that $\Gamma_0 \subseteq \Gamma_1 \subset \Gamma$. Then the following are equivalent:*
1) The law of (Z) is λ_T on T;
2) The ξ_k are independent and uniformly distributed on Γ_0;
3) The law of Z is absolutely continuous with respect to λ.

Proof. (1⇒3) Since G is bounded, so is Z, and therefore $[Z]$ takes on only finitely many values. Let Γ_2 be the range of $[Z]$. One solution of equation (1.2) is $Z \overset{d}{=} \sum_{k=1}^{\infty} M^{-k} G_k$. Jessen and Wintner's theorem [8] implies that the law of Z must be either purely discrete, purely singular, or purely absolutely continuous. We will rule out the discrete and singular cases.

First, suppose Z is purely discrete. Then $P(Z=z)>0$ for some z. Now,

$$\begin{aligned} 0 &< P(Z=z) = P\left([Z]+(Z)=z\right) \\ &= \sum_{\gamma\in\Gamma_2} P\left([Z]+(Z)=z \mid [Z]=\gamma\right) P\left([Z]=\gamma\right) \end{aligned}$$

implies that there exists a $\gamma\in\Gamma_1$ such that

$$P\left((Z)=z-\gamma \mid [Z]=\gamma\right) P\left([Z]=\gamma\right)>0,$$

contradicting the assumption that (Z) is uniform.

Second, suppose Z is purely singular with respect to Lebesgue measure. Then there exists B such that $P(Z\in B)=1$ and $\lambda_T(B)=0$. So

$$P([Z]+(Z)\in B)=\sum_{\gamma\in\Gamma_2} P\left([Z]+(Z)\in B \mid [Z]=\gamma\right) P\left([Z]=\gamma\right)=1,$$

which implies that there exists a $\gamma\in\Gamma_2$ such that

$$P\left((Z)\in B-\gamma \mid [Z]=\gamma\right) P\left([Z]=\gamma\right)\geq \frac{1}{|\Gamma_2|}.$$

But under the assumption that (Z) is uniform, $P\left((Z)\in B-\gamma\right)=\lambda_T\left(B-\gamma\right)=\lambda_T(B)=0$, a contradiction.

Next, 2) $\Rightarrow$ 1). This proof will be broken into three main steps:

(i) assumption 2) implies $P\left((Z_0)\in\partial T\right)=0$;

(ii) $\nu:=\pounds\left(Z\right)$ and λ_T agree on sets of the type $M^{-k}T+M^{-k}\gamma, \gamma\in\Gamma$;

(iii) ν and λ_T agree on all closed balls.

Remark. The first step is trivial in one dimension. For example, if $M=2$, $\Gamma=\mathbf{Z}$ and $\Gamma_0=\{0,1\}$, then $T=[0,1]$ and

$$P\left(\sum_{k=1}^{\infty}2^{-k}\xi_k\in\partial T\right)=P\left(\xi_k=0 \text{ for all } k \text{ or } \xi_k=1 \text{ for all } k\right)=0.$$

i) For each $n=0,1,2,...$ let

$$W_n=\sum_{k=1}^{\infty}M^{-k}\xi_{k+n}.$$

Notice that the range of W_n is in T and since the sequence $\{\xi_k\}_{k=1}^{\infty}$ is i.i.d., $W_n\overset{d}{=}W_0$, $n=1,2,....$

Claim. $P\left(W_0\in int\, T\right)>0$.

Proof. Since $int\, T\neq\emptyset$ [10], let $B\left(x;r\right)\subset int\, T$ be an open ball centered at x with radius r. Then $x=\sum_{i=1}^{\infty}M^{-i}\gamma_i\left(x\right)$, where $\gamma_i\left(x\right)\in\Gamma_0$ for all i [5]. Choose k large enough so that

$$\sum_{i=k}^{\infty}\left\|M^{-i}\right\|\max\left\{\|\gamma\| \mid \gamma\in\Gamma_0\right\}<\frac{r}{2}.$$

Let $y=\sum_{i=1}^{k-1}M^{-i}\gamma_i\left(x\right)$. Note that $y\in B\left(x;\frac{r}{2}\right)$. Let

$$S=\left\{t\in T \mid \gamma_i\left(t\right)=\gamma_i\left(x\right) \text{ for } i=1,2,\ldots,k-1\right\}.$$

Then $S \subseteq B(x;r)$ and

$$P(W_0 \in S) = P(\xi_1 = \gamma_1(x), ..., \xi_{k-1} = \gamma_{k-1}(x)) = \frac{1}{m^{k-1}}.$$

So $P(W_0 \in S) > 0$, which together with $S \subset int\,T$ implies $P(W_0 \in int\,T) > 0$. □

One property of a tiling is that distinct tiles may only intersect on their boundaries. If we set $\Gamma_\partial = \{\gamma \in \Gamma \setminus \{0\} \mid T \cap (T+\gamma) \neq \emptyset\}$, then

$$\partial T = \bigcup_{\gamma \in \Gamma_\partial} (T \cap (T+\gamma)). \tag{3.1}$$

Claim. $\{W_n \in \partial T\} \subseteq \{W_{n+1} \in \partial T\}$ for $n = 0, 1, 2, \ldots$.

Proof. Suppose $\omega \in \{W_0 \in \partial T\}$; that is,

$$\sum_{k=1}^{\infty} M^{-k} \xi_k(\omega) \in \partial T.$$

Applying M to both sides and using properties of tiles yields

$$W_1(\omega) = \sum_{k=1}^{\infty} M^{-k} \xi_{k+1}(\omega) \in \partial MT - \xi_1(\omega). \tag{3.2}$$

Set $\gamma_1 := \xi_1(\omega)$. By the self-affine property of the tiling, $\partial MT \subseteq \bigcup_{\gamma \in \Gamma_0} (\gamma + \partial T)$. Therefore, (3.2) becomes

$$W_1(\omega) \in \bigcup_{\gamma \in \Gamma_0} ((\gamma - \gamma_1) + \partial T),$$

implying that for at least one $\gamma \in \Gamma_0$, $W_1(\omega) \in (\gamma - \gamma_1) + \partial T$. So

$$W_1(\omega) \in ((\gamma - \gamma_1) + \partial T) \cap T.$$

If $\gamma = \gamma_1$, then $W_1(\omega) \in \partial T$; if $\gamma \neq \gamma_1$, then $int\,T \cap (\gamma - \gamma_1 + int\,T) = \emptyset$, so $W_1(\omega) \in \partial T$. We have shown that $\{W_0 \in \partial T\} \subseteq \{W_1 \in \partial T\}$. By the same argument, $\{W_n \in \partial T\} \subseteq \{W_{n+1} \in \partial T\}$ for each n. □

Claim. $\mathrm{P}(W_0 \in \partial T) = 0$.

Suppose not. Set $B_k = \{W_k \in \partial T\}$ and $B = \bigcup_{k=0}^{\infty} B_k$. Notice that since the B_k are nested, $B \in \bigcap_{n=1}^{\infty} \sigma(\xi_n, \xi_{n+1}, ...)$. By the Kolmogorov 0-1 law for independent random variables $P(B) = 1$, because $\{W_0 \in \partial T\} \subset B$ and $P(W_0 \in \partial T) > 0$. Furthermore,

$$1 = P(B) = \lim_{k \to \infty} P(W_k \in \partial T) = P(W_0 \in \partial T),$$

with the last equality following from the fact that the sequence $\xi_1, \xi_2, \ldots$ is i.i.d. But this is a contradiction of the fact that $P(W_0 \in int\,T) > 0$. So $P(W_0 \in \partial T) = 0$. □

Since $W_0 = (Z_0)$ almost surely we have shown that $P((Z_0) \in \partial T) = 0$, concluding the first step.

(ii) To begin the second step of the proof, fix $\gamma \in \Gamma$ and $k \in \mathbf{N}$. Then

$$\lambda(M^{-k}T + M^{-k}\gamma) = \lambda(M^{-k}T) = \frac{\lambda(T)}{m^k}.$$

By Proposition 2 and (i) $(Z_k) = \sum_{i=1}^{\infty} M^{-i}\xi_{i+k}$ a.s. Now,

$$\begin{aligned} P\left((Z_0) \in M^{-k}T + M^{-k}\gamma\right) &= P\left(M^k (Z_0) \in T+\gamma\right) \\ &= P\left(M^k \sum_{i=1}^{\infty} M^{-i}\xi_i \in T+\gamma\right) \\ &= P\left(\sum_{j=1-k}^{0} M^{-j}\xi_{j+k} + \sum_{j=1}^{\infty} M^{-j}\xi_{j+k} \in T+\gamma\right) \\ &= P\left(L(k) + (Z_k) \in T+\gamma\right), \end{aligned}$$

where $L(k) := \sum_{j=1-k}^{0} M^{-j}\xi_{j+k}$. Notice that $L(k)$ is a function of finitely many ξ_i and has values in the lattice; therefore,

$$\begin{aligned} P\left(L(k) + (Z_k) \in T+\gamma\right) &= \sum_{\gamma'} P\left((Z_k) \in T+\gamma-\gamma',\, L(k) = \gamma'\right) \\ &= P\left((Z_k) \in T, L(k) = \gamma\right). \end{aligned}$$

The last equality follows since all the terms in the sum are zero except when $\gamma' = \gamma$ as a consequence of $P((Z_k) \in \partial T) = 0$. Furthermore,

$$\begin{aligned} P\left((Z_k) \in T, L(k) = \gamma\right) &= P\left(L(k) = \gamma\right) \\ &= P\left(M^{k-1}\xi_1 + \cdots + M\xi_{k-1} + \xi_k = \gamma\right) \\ &= P\left(M\left(M^{k-2}\xi_1 + \cdots + \xi_{k-1}\right) + \xi_k = \gamma\right). \end{aligned} \tag{3.3}$$

Since each $\gamma \in \Gamma$ has a unique representation $\gamma = \gamma_0 + M\gamma''$, (3.3) becomes

$$\begin{aligned} &P\left(\xi_k = \gamma_0\ ,\ M^{k-2}\xi_1 + \cdots + \xi_{k-1} = \gamma''\right) \\ &\quad = P\left(\xi_k = \gamma_0\ ,\ \xi_{k-1} = \gamma_1, M^{k-3}\xi_1 + \cdots + \xi_{k-2} = \gamma'''\right) \\ &\quad = P\left(\xi_k = \gamma_0\ ,\ \xi_{k-1} = \gamma_1, \ldots, \xi_1 = \gamma_{k-1}\right) \\ &\quad = \prod_{i=1}^{k} P\left(\xi_i = \gamma_{k-i}\right) = \frac{1}{m^k}. \end{aligned}$$

So $\pounds((Z))$ and $\frac{\lambda}{\lambda(T)}$ are equal on sets of the type $M^{-k}T + M^{-k}\gamma$, $\gamma \in \Gamma$ and $k \in N$.

(iii) We now show that $\pounds((Z))$ and λ_T agree on all closed balls in $\mathbf{R}^d$.

Set $\nu := \pounds((Z))$, and suppose there is a closed ball $B(x,r)$ on which the measures do not agree. Assume first that $\nu(B(x,r) \cap T) > \lambda_T(B(x,r) \cap T)$. There exists $\eta > 0$, such that $\nu(B(x,r) \cap T) > \lambda_T(B(x, r+\eta) \cap T)$. Choose k_0 such that $diam(M^{-k_0}T) < \frac{\eta}{2}$. Set

$$D = \bigcup \left\{ M^{-k_0}T + M^{-k_0}\gamma \mid \gamma \in M^{k_0} B\left(x, r+\frac{\eta}{2}\right) \right\}.$$

Claim. $B(x,r) \subseteq D \subseteq B(x, r+\eta)$.

Proof. Let $y \in B(x,r)$. Since $\mathbf{R}^d = \bigcup_{\gamma \in \Gamma} \left(M^{-k_0}T + M^{-k_0}\gamma\right)$, there is a $\gamma \in \Gamma$ such that $y \in M^{-k_0}T + M^{-k_0}\gamma$. So $y = z + M^{-k_0}\gamma$, for some $z \in M^{-k_0}T$. If $z \in M^{-k_0}T$, then $\|z\| \le diam(M^{-k_0}T)$ since $0 \in T$. Now

$$\left\|M^{-k_0}\gamma - x\right\| \le \|y - x\| + \|z\| \le r + diam(M^{-k_0}T) \le r + \frac{\eta}{2},$$

that is,

$$M^{-k_0}\gamma \in B\left(x, r+\frac{\eta}{2}\right),$$

which means $y \in D$.

Now suppose that $y \in D$. Then $y = z + M^{-k_0}\gamma$ for some $z \in M^{-k_0}T$ and $\gamma \in B\left(x, r+\frac{\eta}{2}\right)$, and

$$\|y-x\| \leq \left\|M^{-k_0}\gamma - x\right\| + \|z\| \leq r+\eta;$$

so $y \in B(x, r+\eta)$. This completes the proof of the claim. □

Thus $\lambda_T(B(x, r+\eta) \cap T) \geq \lambda_T(D \cap T)$ and $\nu(D \cap T) \geq \nu(B(x,r) \cap T)$. If we can show that $\lambda_T(D \cap T) = \nu(D \cap T)$, we will obtain a contradiction. To see this, recall that by the self-affine property of the tiling, we can write

$$T = \bigcup_{\gamma \in \Gamma_{k_0}} M^{-k_0}T + M^{-k_0}\gamma\ , \tag{3.4}$$

where $\Gamma_{k_0} = \Gamma_0 + M\ \Gamma_0 + \cdots + M^{k_0-1}\Gamma_0$. If $\gamma \in \Gamma_{k_0}$, then $M^{-k_0}T + M^{-k_0}\gamma \subset T$, so $int(M^{-k_0}T + M^{-k_0}\gamma) \subset T$. If $\gamma \notin \Gamma_{k_0}$, then $T \cap int\big(M^{-k_0}T + M^{-k_0}\gamma\big) = \emptyset$. If not, there is an x in $T \cap int\big(M^{-k_0}T + M^{-k_0}\gamma\big)$. Since $x \in T$, x is in one of the sets in the right-hand side of (3.4); that is, $x \in M^{-k_0}T + M^{-k_0}\gamma'$, where $\gamma' \in \Gamma_{k_0}$. So

$$x \in \big(M^{-k_0}T + M^{-k_0}\gamma'\big) \cap int\big(M^{-k_0}T + M^{-k_0}\gamma\big)\,,$$

which implies $M^{k_0}x \in (T+\gamma') \cap int(T+\gamma)$. This contradicts the fact that distinct translates of T are disjoint except at the boundary. So,

$$\text{either } int\big(M^{-k_0}T + M^{-k_0}\gamma\big) \subset T \text{ or } int\big(M^{-k_0}T + M^{-k_0}\gamma\big) \subset T^c.$$

Set $C = \Gamma_{k_0} \cap M^{k_0}B\left(x, r+\frac{\eta}{2}\right)$ and $C' = (\Gamma \setminus \Gamma_{k_0}) \cap M^{k_0}B\left(x, r+\frac{\eta}{2}\right)$. Then

$$D \cap T = \left(\left(\bigcup_{\gamma \in C} M^{-k_0}T + M^{-k_0}\gamma\right) \cap T\right) \cup \left(\left(\bigcup_{\gamma \in C'} M^{-k_0}T + M^{-k_0}\gamma\right) \cap T\right).$$

The second intersection consists only of boundary points of T. Since $\nu(\partial T) = 0$, then

$$\nu(D \cap T) = \nu\left(\left(\bigcup_{\gamma \in C} M^{-k_0}T + M^{-k_0}\gamma\right) \cap T\right)$$

and $\nu\left(\partial\left(M^{-k_0}T+M^{-k_0}\gamma\right)\right)=0$. The Lebesgue measure of ∂T is zero [10], so $\lambda\left(\partial M^{-k_0}T\right)=0$. Thus we have

$$\begin{aligned}
\lambda_T\left(D\cap T\right) &= \lambda_T\left(\bigcup_{\gamma\in\Gamma_{k_0}} M^{-k_0}T+M^{-k_0}\gamma\right)\\
&= \sum_{\gamma\in\Gamma_{k_0}}\lambda_T\left(M^{-k_0}T+M^{-k_0}\gamma\right)\\
&= \sum_{\gamma\in\Gamma_{k_0}}\nu\left(M^{-k_0}T+M^{-k_0}\gamma\right)\\
&= \nu\left(\bigcup_{\gamma\in\Gamma_{k_0}} M^{-k_0}T+M^{-k_0}\gamma\right)=\nu\left(D\cap T\right).
\end{aligned}$$

As mentioned above, the fact that $\lambda_T\left(D\cap T\right)=\nu\left(D\cap T\right)$ implies

$$\lambda_T\left(B\left(x,r+\eta\right)\cap T\right)\geq\nu\left(B\left(x,r\right)\cap T\right)$$

which contradicts $\nu\left(B\left(x,r\right)\cap T\right)>\lambda_T\left(B\left(x,r\right)\cap T\right)$. So we conclude that $\lambda_T\leq\nu$ on all closed balls. Repeating the proof with the roles of ν and λ_T reversed yields that ν and λ_T agree on all closed balls.

Hoffmann-Jørgensen proved that Radon probabilities which agree on all closed balls in $\mathbf{R}^d$ agree on all Borel sets. (Corollary 5 in [7]), which completes the proof that 2) $\Rightarrow$ 1).

In order to prove 3) $\Rightarrow$ 2), we need a version of the Kakutani Dichotomy for stationary ergodic sequences.

Lemma 1. *Let $\{\xi_k'\}_{k=1}^{\infty}$ be a stationary, ergodic sequence, such that each ξ_k' is uniform with values in Γ_0. Let $\{\xi_k\}_{k=1}^{\infty}$ be a stationary, ergodic sequence, such that each ξ_k has values in Γ_0, but is not uniform. Then $\mu=\pounds\left(\xi_1,\xi_2,...\right)$ and $\mu'=\pounds\left(\xi_1',\xi_2',...\right)$ are mutually singular.*

Proof. Let $\mu=\mu_a+\mu_s$, where $\mu_a<<\mu'$ and $\mu_s\perp\mu'$. Suppose $\mu_a\left(\Omega\right)>0$.

Since $\mu\neq\mu'$, there must be a cylindrical set A such that $\mu_a(A)\neq\mu'(A)$. (If not, then $\mu_a=\mu'$, which implies $\mu=\mu'$, contradicting the assumption that $\mu\neq\mu'$.) Let $f=1_A$, then we get

$$\begin{aligned}
\int_\Omega f(x_1,..,x_n)d\mu_a(x) &\neq \int_\Omega f(x_1,..,x_n)d\mu'\left(x\right),\\
E_{\mu_a}(f) &\neq E_{\mu'}(f).
\end{aligned}$$

Set $c=E_{\mu_a}(f)$ and $c'=E_{\mu'}(f)$. The fact that $\{\xi_k\}_{k=1}^{\infty}$ and $\{\xi_k'\}_{k=1}^{\infty}$ are ergodic sequences means that the shift operator is an ergodic operator for (Ω,S,μ) and (Ω,S,μ') respectively, where $\Omega=\Gamma_0^{\infty}$. Applying the Ergodic Theorem (with f) and the fact that the sequences are stationary, it follows that

1) $\quad \frac{1}{k}\sum\limits_{i=0}^{k-1}f(x_{1+i},\ldots,x_{n+i})\overset{k\to\infty}{\to}c$ a.s. μ_a,

2) $\quad \frac{1}{k}\sum\limits_{i=0}^{k-1}f(x_{1+i},\ldots,x_{n+i})\overset{k\to\infty}{\to}c'$ a.s. μ'.

So, 1) is true for all $\{x_i\}_{i=1}^{\infty}\in\Omega\setminus N$, where $\mu_a(N)=0$ and 2) is true for all $\{x_i\}_{i=1}^{\infty}\in\Omega\setminus N'$, where $\mu'(N')=0$.

Define $M := N \cup N'$. Notice that $\mu_a(M) \leq \mu_a(N) + \mu_a(N') = \mu_a(N')$. Since $\mu_a << \mu'$ and $\mu'(N') = 0$, we have $\mu_a(N') = 0$ and so $\mu_a(M) = 0$. We have assumed that $\mu_a(\Omega) > 0$; therefore, $\mu_a(M) = 0$ implies that $\mu_a(\Omega \setminus M) > 0$; that is, $\mu_a((\Omega \setminus N) \cap (\Omega \setminus N')) > 0$, which means that there is a sequence $\{x_i\}_{i=1}^{\infty} \in (\Omega \setminus N) \cap (\Omega \setminus N')$ such that

$$\frac{1}{k}\sum_{i=0}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \overset{k\to\infty}{\to} c \text{ and } \frac{1}{k}\sum_{i=1}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \overset{k\to\infty}{\to} c'.$$

This is a contradiction, since $c \neq c'$. Therefore, $\mu_a = 0$ and thus, $\mu \perp \mu'$. ∎

Now we are ready to show that 3) ⇒ 2).

First, we note that $\pounds(Z) << \lambda_T$ implies that $\pounds((Z)) << \lambda_T$. To see this, observe that for $E \in \mathcal{B}(\mathbf{R}^d)$,

$$\begin{aligned} P((Z) \in E) &= P(Z - [Z] \in E) = \sum_{\gamma\in\Gamma} P(Z \in E + \gamma, [Z] = \gamma) \\ &\leq \sum_{\gamma\in\Gamma} P(Z \in E + \gamma). \end{aligned} \tag{3.5}$$

If $\lambda_T(E) = 0$, then $\lambda_T(E + \gamma) = 0$ and so $P(Z \in E + \gamma) = 0$ for all $\gamma \in \Gamma$ by the assumption of absolute continuity of $\pounds(Z)$. Then (3.5) implies $P((Z) \in E) = 0$. So $\pounds((Z)) << \lambda_T$.

Since $\lambda(\partial T) = 0$, $\pounds((Z)) << \lambda_T$ implies that $P((Z) \in \partial T) = 0$. Therefore, if we define $s : \Gamma_0^{\infty} \to \mathbf{R}$ by

$$s(x_1, x_2, ...) := \sum_{i=1}^{\infty} M^{-i} x_i,$$

If Γ_0^{∞} is equipped with the product topology, s is continuous. By Proposition 2, for every Borel set F the following holds true:

$$\begin{aligned} \pounds((Z))(F) &= \pounds\left(\sum_{i=1}^{\infty} M^{-i}\xi_i\right)(F) = P(s(\xi_1, \xi_2, ...) \in F) \\ &= P\left((\xi_1, \xi_2, ...) \in s^{-1}(F)\right) = \mu\left(s^{-1}(F)\right), \end{aligned} \tag{3.6}$$

where $\mu = \pounds(\xi_1, \xi_2, ...)$. Corollary 1 assures that the sequence $\{\xi_k\}_{k=1}^{\infty}$ is stationary and ergodic. Let $\mu' = \pounds(\xi_1', \xi_2', ...)$, where $\{\xi_k'\}_{k=1}^{\infty}$ is an i.i.d. sequence with ξ_1' uniform on Γ_0. Suppose that $\mu \neq \mu'$. Then by Lemma 1, $\mu \perp \mu'$. So there is a set $B \subset \mathfrak{B}(\Gamma_0^{\infty})$ such that $\mu(B) = 1$ and $\mu'(B) = 0$. Set $A = s(B)$. Since Γ_0^{∞} is a Polish space and s is continuous, A being the continuous image of a Borel set, is an analytic set. As such, A is universally measurable [13]. Let C and D be Borel sets so that $C \subseteq A \subseteq D$ and $\lambda(C) = \lambda(A) = \lambda(D)$. Since the Lebesgue measure of boundary of a tile is 0, we may assume that C does not contain any points on the boundary of tiles (the union of the tiles boundaries is a Borel set). This implies that $s^{-1}(C) \subseteq B$. From the proof of 2) ⇒ 1) it follows that $\pounds(s(\xi_1', \xi_2', ...)) = \lambda_T$. Now

$$\begin{aligned} 0 &= \mu'(B) = P((\xi_1', \xi_2', \ldots) \in B) \geq P((\xi_1', \xi_2', \ldots) \in s^{-1}(C)) \\ &= P(s(\xi_1', \xi_2', \ldots) \in C) = \lambda_T(C) = \lambda_T(A). \end{aligned}$$

We also have that

$$1 = \mu(B) = P((\xi_1, \xi_2, \dots) \in B) \leq P(s(\xi_1, \xi_2, \dots) \in D) = \pounds((Z))(A)$$

where the last equality follows from (3.6). This contradicts the fact that $\pounds((Z)) << \lambda_T$. Therefore $\mu = \mu'$, i.e. ξ_i, $i = 1, 2, \dots$, are i.i.d. and ξ_1 is uniform on Γ_0. This completes the proof of 3) $\Rightarrow$ 2) and thus of Theorem 1. □

4. Conditions for independence of $\{\xi_k\}$

In Theorem 1, the existence of a density of the solution Z to (1.2) is equivalent to the fact that the stationary, ergodic sequence $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables and that ξ_1 is uniform on Γ_0. In this section we first investigate the effects of uniformity of ξ_1 on the distributions of G_1 and $[Z_1]$; the results are then summarized in Theorem 2. In Theorem 3, we give a sufficient condition on G_1 for the independence and uniformity of the sequence $\{\xi_k\}_{k=1}^{\infty}$.

By Proposition 3, $\xi_k = h(G_k + [Z_k])$ a.s., provided that $P((Z_0) \in \partial T) = 0$. In order to describe the effects of uniformity of ξ_k, it suffices to consider the relationship between G_1, $[Z_1]$ and ξ_1.

Let $p_i = P([Z_1] \cong \gamma_i)$ and $q_i = P(G_1 \cong \gamma_i)$ for $i = 0, 1, \dots, m-1$, where $\gamma \cong \gamma_i$ means that the lattice point γ is in the coset represented by γ_i. Recalling that G_1 and Z_1 are independent we have

$$\begin{aligned} P(\xi_1 = \gamma_k) &= P(h(G_1 + [Z_1]) = \gamma_k) \\ &= \sum_{\gamma_i + \gamma_j \cong \gamma_k} P(G_1 \cong \gamma_i, [Z_1] \cong \gamma_j) \\ &= \sum_{\gamma_i + \gamma_j \cong \gamma_k} P(G_1 \cong \gamma_i) P([Z_1] \cong \gamma_j) \\ &= \sum_{\gamma_i + \gamma_j \cong \gamma_k} q_i p_j. \end{aligned}$$

Due to the uniqueness of equivalence class representatives, there are exactly m terms in the right-hand sides the equation. Now if $p_i = \frac{1}{m}$ for all i or $q_i = \frac{1}{m}$ for all i then ξ_i are uniform. Assuming that ξ_1 is uniform on $\Gamma_0 = \{\gamma_0, \dots, \gamma_{m-1}\}$ we have,

$$\begin{aligned} \frac{1}{m} &= q_0 p_0 + q_1 p_1 + \cdots + q_{m-1} p_{m-1}, \\ \frac{1}{m} &= q_{m-1} p_0 + q_0 p_1 + \cdots + q_{m-2} p_{m-1}, \\ \frac{1}{m} &= q_{m-2} p_0 + q_{m-1} p_1 + \cdots + q_{m-3} p_{m-1}, \\ &\cdots \\ \frac{1}{m} &= q_1 p_0 + q_2 p_1 + \cdots + q_0 p_{m-1}, \end{aligned}$$

or, in matrix form, $QX = \frac{1}{m}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$, where $X = \begin{bmatrix}p_0 & p_1 & \cdots & p_{m-1}\end{bmatrix}^T$. Notice that the rows as well as the columns of Q sum to 1. Without loss of generality, we may assume that $q_0 \geq q_1 \geq \cdots \geq q_{m-1}$; if not, just reindex Γ_0 so that this ordering holds. It is obvious that $p_i = \frac{1}{m}, i = 0, \dots, m-1$, is a solution of the system; we will show that it is unique by showing that the eigenvalues of the matrix Q are different from zero.

Let α_k be the kth root of $z^m = 1$. Direct computation shows that the eigenvalues of Q are $\eta_k = \sum_{j=0}^{m-1} q_j \alpha_k^j$ and the associated eigenvectors are

$$v_k = \left[\begin{array}{ccccc} 1 & \alpha_k & \alpha_k^2 & \cdots & \alpha_k^{m-1} \end{array} \right]^T,$$

for $k = 0, 1, ..., m-1$.

Remark 1. If $k \in \{0, 1, ..., m-1\}$ and m are relatively prime, then

$$\left\{ e^{\frac{2\pi ijk}{m}} | j = 0, 1, ..., m-1 \right\}$$

is equal to the set of distinct roots of $z^m = 1$.

Definition 1. We say that the set $\{q_j\}_{j=0}^{m-1}$ has a *cycle of length* r if

$$\begin{aligned} q_0 = q_1 = \cdots &= q_{r-1}, \\ q_r = q_{r+1} = \cdots &= q_{2r-1}, \\ &\cdots, \\ q_{m-r} = \cdots &= q_{m-1}. \end{aligned}$$

The trivial case $r = 1$ is excluded.

So, for example, the set $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{20}, \frac{1}{20}\right\}$ has a cycle of length 2 while the set $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$ has no cycle. Note that if $\{q_j\}_{j=0}^{m-1}$ has a cycle of length r, then r divides m.

Lemma 2. *Zero is an eigenvalue of Q if and only if $\{q_j\}_{j=0}^{m-1}$ has a cycle.*

Proof. ($\Leftarrow$) The case of a cycle of length m is trivial.

Now consider a cycle of length r, where $r < m$. Denote the greatest common divisor of m and r by (m, r). We claim that $\eta_{\frac{m}{r}} = 0$ (recall that r divides m). Since$\{q_j\}_{j=0}^{m-1}$ has a cycle of length r and $\frac{2\pi ij\frac{m}{r}}{m} = \frac{2\pi ij}{r}$, we have

$$\begin{aligned} \eta_{\frac{m}{r}} &= \sum_{j=0}^{r-1} q_j e^{\frac{2\pi ij}{r}} + \sum_{j=r}^{2r-1} q_j e^{\frac{2\pi ij}{r}} + \cdots + \sum_{j=(m,r)-1}^{m-1} q_j e^{\frac{2\pi ij}{r}} \\ &= q_0 \sum_{j=0}^{r-1} e^{\frac{2\pi ij}{r}} + q_r \sum_{j=r}^{2r-1} e^{\frac{2\pi ij}{r}} + \cdots + q_{(m,r)-1} \sum_{j=(m,r)-1}^{m-1} e^{\frac{2\pi ij}{r}} \\ &= 0. \end{aligned}$$

($\Rightarrow$) The assumption that there is no cycle implies $q_0 > q_{m-1}$. First we will show that $\eta_k \neq 0$ in the case that $(k, m) = 1$. Define $l_0 := \max\{k : q_k = q_0\}$ and inductively $l_{i+1} := \max\{k : q_k = q_{l_i+1}\}, i = 0, 1, 2, \ldots, n-2$, that is, there are n different values in the set of q's. Notice that $l_{n-1} = m-1$ and $q_{l_0} = q_0$. Now,

$$\begin{aligned} &\eta_k \left(1 - e^{\frac{2\pi ik}{m}}\right) \\ &\quad = q_0 + (q_1 - q_0) e^{\frac{2\pi ik}{m}} + \cdots + (q_{m-1} - q_{m-2}) e^{\frac{2\pi ik(m-1)}{m}} - q_{m-1} e^{\frac{2\pi ikm}{m}} \\ &\quad = (q_0 - q_{m-1}) + (q_1 - q_0) e^{\frac{2\pi ik}{m}} + \cdots + (q_{m-1} - q_{m-2}) e^{\frac{2\pi ik(m-1)}{m}} \\ &\quad = \left(q_{l_0} - q_{l_{n-1}}\right) + \left(q_{l_0+1} - q_{l_0}\right) e^{\frac{2\pi ik}{m}(l_0+1)} + \left(q_{l_1+1} - q_{l_1}\right) e^{\frac{2\pi ik}{m}(l_1+1)} \\ &\qquad + \cdots + \left(q_{l_{n-2}+1} - q_{l_{n-2}}\right) e^{\frac{2\pi ik}{m}(l_{n-2}+1)}. \end{aligned}$$

If we set

$$z_0 = \left(q_{l_0} - q_{l_0+1}\right) e^{\frac{2\pi ik}{m}(l_0+1)} + \cdots + \left(q_{l_{n-2}} - q_{l_{n-2}+1}\right) e^{\frac{2\pi ik}{m}(l_{n-2}+1)},$$

we have $\eta_k \left(1 - e^{\frac{2\pi ik}{m}}\right) = q_0 - q_{m-1} - z_0$, and

$$\left|\eta_k \left(1 - e^{\frac{2\pi ik}{m}}\right)\right| \geq q_0 - q_{m-1} - |z_0| .$$

We claim $|z_0| < q_0 - q_{m-1}$. Observe

$$\begin{aligned} |z_0| &\leq q_{l_0} - q_{l_0+1} \\ &\quad + \left|\left(q_{l_1} - q_{l_1+1}\right) e^{\frac{2\pi ik}{m}(l_1+1)} + \cdots + \left(q_{l_{n-2}} - q_{l_{n-2}+1}\right) e^{\frac{2\pi ik}{m}(l_{n-2}+1)}\right| \\ &\leq q_{l_0} - q_{l_{n-4}+1} + \left|\left(q_{l_{n-3}} - q_{l_{n-3}+1}\right) + \left(q_{l_{n-2}} - q_{l_{n-2}+1}\right) e^{\frac{2\pi ik}{m}(l_{n-2}-l_{n-3})}\right| . \end{aligned}$$

Since $k > 0$, and k and m are relatively prime, $e^{\frac{2\pi ik}{m}(l_{n-2}-l_{n-3})}$ has a nonzero imaginary part and so $\left(q_{l_{n-2}} - q_{l_{n-2}+1}\right) e^{\frac{2\pi ik}{m}(l_{n-2}-l_{n-3})}$ cannot be a positive scalar multiple of $q_{l_{n-3}} - q_{l_{n-3}+1}$. Therefore

$$\begin{aligned} |z_0| &< q_{l_0} - q_{l_{n-4}+1} + q_{l_{n-3}} - q_{l_{n-3}+1} + \left|\left(q_{l_{n-2}} - q_{l_{n-2}+1}\right) e^{\frac{2\pi ik}{m}(l_{n-2}-l_{n-3})}\right| \\ &= q_0 - q_{m-1}. \end{aligned}$$

So $\left|\eta_k \left(1 - e^{\frac{2\pi ik}{m}}\right)\right| \geq q_0 - q_{m-1} - |z_0| > 0$, which completes the case $(k,m) = 1$.

Suppose now that $(k,m) > 1$. Set

$$m_1 := \frac{m}{(k,m)}, k_1 := \frac{k}{(k,m)} \text{ and } q'_j = \sum_{i \cong j} q_i,$$

where $i \cong j$ means $i = j \bmod m_1$. Notice that $q'_0 \geq q'_1 \geq \cdots \geq q'_{m_1-1}$and $(k_1, m_1) = 1$. Rewriting η_k as

$$\eta_k = \sum_{j=0}^{m-1} q_j e^{\frac{2\pi ijk}{m}} = \sum_{j=0}^{m_1-1} q'_j e^{\frac{2\pi ijk_1}{m_1}},$$

we may apply the previous case because the absence of a cycle implies $q'_0 > q'_{m_1-1}$. □

We can summarize the above in the following theorem: (Recall that $p_i = P\left([Z_1] \cong \gamma_i\right)$ and $q_i = P\left(G_1 \cong \gamma_i\right)$.)

Theorem 2. *The random variable ξ_1 is uniform on Γ_0 if and only if one of the following two statements holds:*

(i) *if $\{q_j\}_{j=1}^{m-1}$ has no cycles, then $p_i = \frac{1}{m}$ for $i = 0, \ldots, m-1$, or*

(ii) *if $\{p_j\}_{j=1}^{m-1}$ has no cycles, then $q_i = \frac{1}{m}$ for $i = 0, \ldots, m-1$.*

The next theorem gives a condition on the distribution of G_1 which will guarantee the independence of the sequence $\{\xi_k\}_{k=1}^{\infty}$.

Theorem 3. *If $P\left(G_1 \cong \gamma_i\right) = \frac{1}{m}$ for $i = 1, \ldots, m$, then $\xi_1, \xi_2, \ldots$ are independent and ξ_1 is uniform.*

Proof. Uniformity of ξ_1 follows from Theorem 2. Let $k_1 < k_2 < \cdots < k_n$. We proceed by induction on n.

Suppose $n = 2$. Then

$$\begin{aligned} P\left(\xi_{k_1} = \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2}\right) &= P\left(G_{k_1} + [Z_{k_1}] \cong \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2}\right) \\ &= \sum_{i=0}^{m-1} P\left(G_{k_1} \cong \gamma_i, [Z_{k_1}] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2}\right) \\ &= \sum_{i=0}^{m-1} P\left(G_{k_1} \cong \gamma_i\right) P\left([Z_{k_1}] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2}\right) \end{aligned}$$

where $\gamma_i + \gamma_{j(i)} \cong \gamma_{k_1}$, and the last equality is due to the fact that G_{k_1} is independent of $[Z_{k_1}]$ and of ξ_{k_2}. Notice that when γ_i runs through Γ_0, so does $\gamma_{j(i)}$, and since $P\left(G_{k_1} \cong \gamma_i\right) = \frac{1}{m}$, we obtain

$$\begin{aligned} P\left(\xi_{k_1} = \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2}\right) &= \frac{1}{m} \sum_{i=0}^{m-1} P\left([Z_{k_1}] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2}\right) \\ &= \frac{1}{m} P\left(\xi_{k_2} = \gamma_{k_2}\right) \\ &= P\left(\xi_{k_1} = \gamma_{k_1}\right) P\left(\xi_{k_2} = \gamma_{k_2}\right). \end{aligned}$$

Now assume $P\left(\xi_{k_1} = \gamma_{k_1}, ..., \xi_{k_n} = \gamma_{k_n}\right) = \prod_{i=1}^{n} P\left(\xi_{k_i} = \gamma_{k_i}\right)$. Consider

$$\begin{aligned} &P\left(\xi_{k_1} = \gamma_{k_1}, ..., \xi_{k_n} = \gamma_{k_n}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = P\left(\xi_{k_1} = \gamma_{k_1}, ..., G_{k_n} + [Z_{k_n}] \cong \gamma_{k_n}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = \sum_{i=0}^{m-1} P\left(\xi_{k_1} = \gamma_{k_1}, ..., G_{k_n} \cong \gamma_i, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right), \end{aligned}$$

where $\gamma_i + \gamma_{j(i)} \cong \gamma_{k_n}$. Now, G_{k_n} is independent of $[Z_{k_n}]$ and of $\xi_{k_{n+1}}$; by the inductive hypothesis, G_{k_n} is also independent of $\xi_{k_1}, \ldots, \xi_{k_{n-1}}$. Thus

$$\begin{aligned} &\sum_{i=0}^{m-1} P\left(\xi_{k_1} = \gamma_{k_1}, ..., G_{k_n} \cong \gamma_i, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = \sum_{i=0}^{m-1} P\left(G_{k_n} \cong \gamma_i\right) P\left(\xi_{k_1} = \gamma_{k_1}, ..., [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = \frac{1}{m} \sum_{i=0}^{m-1} P\left(\xi_{k_1} = \gamma_{k_1}, ..., [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = \frac{1}{m} P\left(\xi_{k_1} = \gamma_{k_1}, ..., \xi_{k_{n-1}} = \gamma_{k_{n-1}}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}\right) \\ &\quad = P\left(\xi_{k_n} = \gamma_{k_n}\right) \prod_{i \neq n, i=1}^{n+1} P\left(\xi_{k_i} = \gamma_{k_i}\right) \end{aligned}$$

by the inductive hypothesis. So by induction, we have shown that $\{\xi_k\}_{k=1}^{\infty}$ is an independent sequence. □

Remark. Theorem 2 is symmetric in $[Z]$ and G, but Theorem 3 is not; that is, if $P([Z_k] = \gamma) = \frac{1}{m}$ for all $\gamma \in \Gamma_0$ but $P(G_k = \gamma_0) > \frac{1}{m}$ for some $\gamma_0 \in \Gamma_0$, the sequence $\{\xi_k\}_{k=1}^{\infty}$ is not necessarily independent. This is illustrated in the following example: $M = 2$, $\Gamma = \mathbf{Z}$ (the integers) and $\Gamma_0 = \{0, 1\}$. Let G be such that $P(G = 0) = P(G = 1) = P(G = 2) = \frac{1}{3}$. So $P(G \cong 0) = P(G \text{ is even}) = \frac{2}{3}$ and $P(G \cong 1) = P(G \text{ is odd}) = \frac{1}{3}$. Then we have

$$\begin{aligned} P([Z_1] \text{ even}) &= P(0 \leq Z_1 < 1) + P(Z_1 = 2) \\ &= P(G_1 = 0) + P(G_1 = 1, G_2 = 0) + \cdots \\ &= \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{2}. \end{aligned}$$

Therefore ξ_1 is uniform on Γ_0 by Theorem 2. However, the sequence $\{\xi_k\}_{k=1}^{\infty}$ is not independent. Consider $P(\xi_1 = 0, \xi_2 = 0)$:

$$\begin{aligned} &P(\xi_1 = 0, \xi_2 = 0) \\ &\quad = P(G_1 + [Z_1] \cong 0, \xi_2 = 0) \\ &\quad = P(G_1 \cong 0, [Z_1] \cong 0, \xi_2 = 0) + P(G_1 \cong 1, [Z_1] \cong 1, \xi_2 = 0) \\ &\quad = \frac{2}{3} P([Z_1] \cong 0, \xi_2 = 0) + \frac{1}{3} P([Z_1] \cong 1, \xi_2 = 0). \end{aligned}$$

To compute the two remaining probabilities, note that

$$[Z_1] = \left[\sum_{k=1}^{\infty} 2^{-k} G_{k+1}\right] = \left[\frac{G_2 + [Z_2]}{2} + \frac{(Z_2)}{2}\right].$$

If $\xi_2 = 0$, $G_2 + [Z_2]$ is even, so in this case, $[Z_1] = \frac{G_2 + [Z_2]}{2}$, and if $[Z_1] = \frac{G_2 + [Z_2]}{2}$, then $\xi_2 = 0$. This implies that

$$\begin{aligned} &\frac{2}{3} P([Z_1] \cong 0, \xi_2 = 0) + \frac{1}{3} P([Z_1] \cong 1, \xi_2 = 0) \\ &\quad = \frac{2}{3} P\left([Z_1] \cong 0, [Z_1] = \frac{G_2 + [Z_2]}{2}\right) + \frac{1}{3} P\left([Z_1] \cong 1, [Z_1] = \frac{G_2 + [Z_2]}{2}\right) \\ &\quad = \frac{2}{3} P(G_2 + [Z_2] \cong 0 \bmod 4) + \frac{1}{3} P(G_2 + [Z_2] \cong 2 \bmod 4) \\ &\quad = \frac{2}{3} (P(G_2 = 0, [Z_2] = 0) + P(G_2 = 2, [Z_2] = 2)) \\ &\qquad + \frac{1}{3} (P(G_2 = 0, [Z_2] = 2) + P(G_2 = 2, [Z_2] = 0) + P(G_2 = 1, [Z_2] = 1)) \\ &\quad = \frac{2}{3} \left(\frac{1}{2} \cdot \frac{1}{3} + 0\right) + \frac{1}{3} \left(0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}\right) \\ &\quad = \frac{2}{9} \neq \frac{1}{4} = P(\xi_1 = 0) P(\xi_2 = 0). \end{aligned}$$

Since the sequence $\{\xi_k\}_{k=1}^{\infty}$ is not independent, by Theorem 1, $\mathcal{L}(Z)$ is not absolutely continuous with respect to Lebesgue measure for this example. Thus, the assumption that ξ_1 is uniform does not necessarily imply the independence of the sequence $\{\xi_k\}_{k=1}^{\infty}$.

If the range of G is Γ_0, then $[Z] = 0$. In this case $G = \xi_1$ and the application of Theorem 3 yields the following:

Corollary 2. *Suppose that the range of* G *is* Γ_0*. Then*

$$\varphi(x) = \sum_{\gamma\in\Gamma_0} c(\gamma)\,\varphi(Mx-\gamma).$$

has a functional solution if and only if $P(G=\gamma) = c(\gamma) = \frac{1}{m}$.

The result of Corollary 2 is known. It was first proved by Grochenig and Madych (Theorem 2 in [5]) using different methods. The solution of the dilation equation in this case is $\varphi = \frac{1}{\sqrt{\lambda(T)}} 1_T$. Scaling functions that are indicator functions over the tile are used to construct "Haar-type" wavelet bases as discussed in detail in [5].

5. Examples

In this section we give several examples of density functions obtained by assigning probabilities so that the hypotheses of Theorem 3 are satisfied.

In most cases, there is no closed form for the density function [14]; those which cannot be computed explicitly can be numerically approximated by computing the function values on the points of $\left\{M^{-k}\Gamma \mid k=0,\ldots,k_0\right\}$ for some k_0, via the dilation equation. To obtain the approximation of the graph of φ, first the values of φ at the integers are found by considering the vector of integer values as an eigenvector of eigenvalue 1 for a matrix of coefficients [14]. Then, using the scaling relation (1.1), the values of φ can be found at all points in $M^{-1}\Gamma$. Repeatedly applying (1.1) k_0 times and plotting the results gives an approximation to the graph of φ. Questions of convergence of the approximations are discussed in [3].

For each of the following examples, the eigenvalue problem for a matrix corresponding to a set containing the support of φ was solved to obtain the values at the lattice points. Then the above algorithm was applied, resulting in approximately 2000 points plotted for each graph approximation.

Example 1. Let $d=1$, $M=2$, $\Gamma=\mathbf{Z}$ and $\Gamma_0=\{0,1\}$. Suppose the range of G is $\Gamma_1=\{0,1,2,3\}$ with the following weight assignments: $c(0)=.2$, $c(1)=.4$, $c(2)=.3$, $c(3)=.1$. Then the density function φ is continuous [2] and is pictured in Figure 1 along with a four-coefficient spline function for comparison.

Example 2. Suppose $d=2$, $\Gamma=\mathbf{Z}^2$, $\Gamma_0=\{(0,0),(1,0),(0,1),(1,1)\}$ and

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Define G to have values in $\Gamma_1 = [0,2]^2 \cap \mathbf{Z}^2$ with the following probability distribution:

$$\begin{aligned} c((0,0)) &= c((2,0)) = c((0,2)) = c((2,2)) = \frac{1}{16}, \\ c((1,0)) &= c((0,1)) = c((2,1)) = c((1,2)) = \frac{1}{8}, \\ c((1,1)) &= \frac{1}{4}. \end{aligned}$$

Since G is clearly the convolution of two independent copies of a uniform random variable on the unit square, φ is continuous. The graph of the density function is pictured in Figure 2.

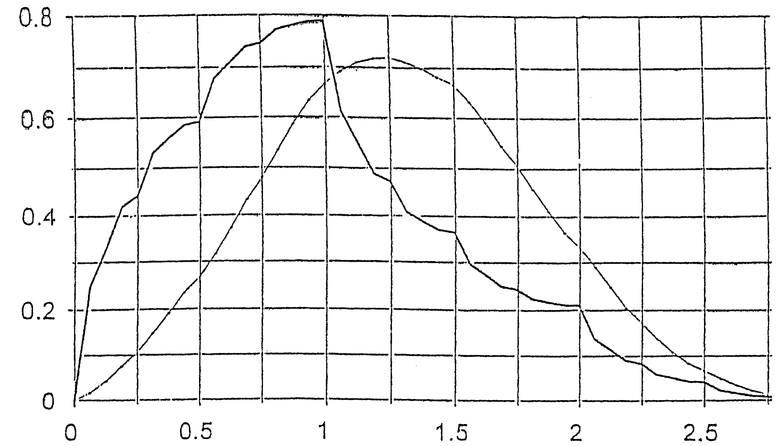

FIGURE 1.

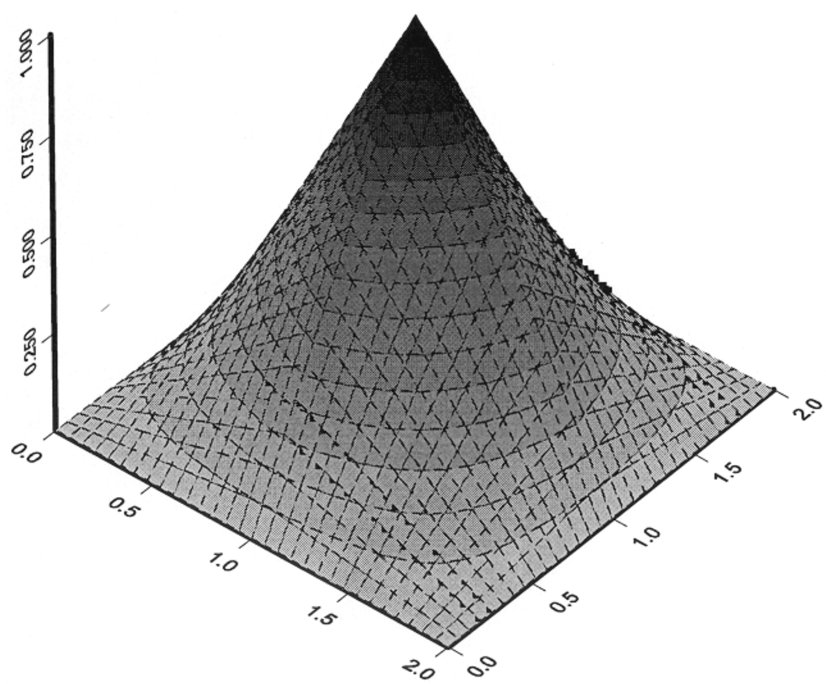

FIGURE 2.

Example 3. Let $d = 2$, $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Gamma = \mathbf{Z}^2$ and $\Gamma_0 = \{(0,0), (1,0)\}$. Define G to have values in $\Gamma_1 = \{(0,0), (1,0), (2,0)\}$ with the following distribution: $c((0,0)) = \frac{1}{4}$, $c((1,0)) = \frac{1}{2}$, $c((2,0)) = \frac{1}{4}$. The graph of the density function is pictured in Figure 3. The density is a convolution of two indicator functions of the twin dragon tile and therefore it is continuous.

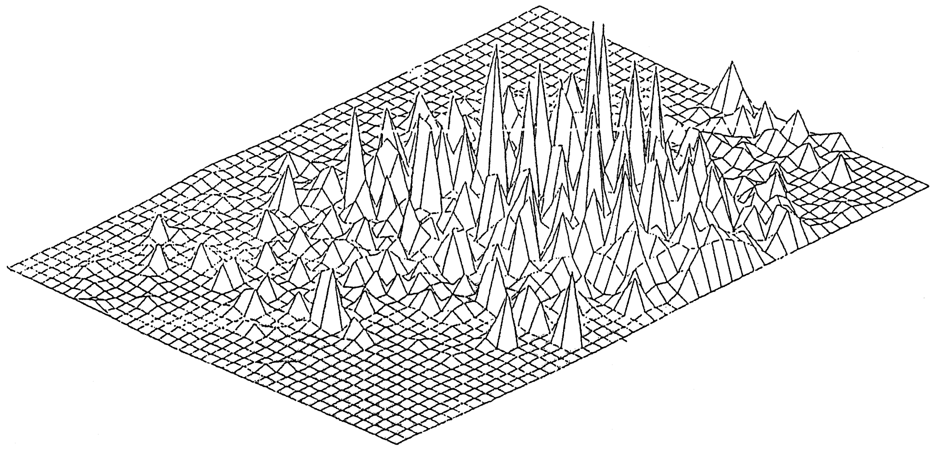

FIGURE 3.

6. A NECESSARY CONDITION FOR MULTIDIMENSIONAL PRESCALE FUNCTIONS

Suppose φ is a functional solution of the dilation equation (1.1). If the lattice translates of φ form a Riesz basis, that is, for some positive constants C_1, C_2

$$C_1\sqrt{\sum (a(\gamma))^2} \le \left\|\sum a(\gamma)\,\varphi(\cdot-\gamma)\right\|_{L^2(\mathbf{R}^d)} \le C_2\sqrt{\sum (a(\gamma))^2},$$

then φ is said to be *stable*. We show that the condition $\sum\limits_{\gamma\cong\delta} c(\gamma) = \frac{1}{m}$ for each $\delta \in \Gamma_0$, where $m = |\det M|$, which was sufficient for the existence of a functional solution to (1.1), is necessary for the stability of φ.

The Fourier transform version of the dilation equation (1.1) is

$$\widehat{\varphi}(\zeta) = \widehat{\varphi}\left(M^{*-1}\zeta\right) A\left(M^{*-1}\zeta\right), \tag{6.1}$$

where $A(\zeta) = \sum\limits_{\gamma\in\Gamma} c(\gamma)\, e^{-i\gamma\cdot\zeta}$. Stability of φ is equivalent to

$$0 < C_1 \le \sum_{k\in\Gamma} |\widehat{\varphi}(\zeta + 2\pi k)|^2 \le C_2 \text{ a.e.} \tag{6.2}$$

In the case that the coefficient sequence $c := \{c(\gamma)\}_{\gamma\in\Gamma}$ is finitely supported, the function in (6.2) is a polynomial [12] and so the inequality must hold everywhere. In the theorem below, which is known (see, for example, [9]), we will assume that the equation holds everywhere. This is not a restriction as proved in [4]. For completeness we include a short proof.

Theorem 4. *Let $\varphi \in L^2\left(\mathbf{R}^d\right)$ be a solution of the dilation equation* (1.1). *Suppose φ is stable and that equation* (6.2) *holds everywhere. Then $\sum\limits_{\gamma\cong\gamma_0} c(\gamma) = \frac{1}{m}$ for each $\gamma_0 \in \Gamma_0$.*

Proof. Without loss of generality, we assume $\Gamma = \mathbf{Z}^d$. Since φ is stable, (6.2) holds. Applying equation (6.1) we obtain

$$\begin{aligned} 0 &< C_1 \leq \sum_{k\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(\zeta+2\pi k\right)\right|^2 \\ &= \sum_{\gamma\in\Gamma_0} \left|A\left(M^{*-1}\zeta+2\pi M^{*-1}\gamma\right)\right|^2 \sum_{k'\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(M^{*-1}\zeta+2\pi\left(M^{*-1}\gamma+k'\right)\right)\right|^2. \end{aligned}$$

For $\zeta = 0$, we get

$$\begin{aligned} \sum_{k\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(2\pi k\right)\right|^2 &= \sum_{\gamma\in\Gamma_0\setminus\{0\}} \left|A\left(2\pi M^{*-1}\gamma\right)\right|^2 \sum_{k'\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(2\pi\left(M^{*-1}\gamma+k'\right)\right)\right|^2 \\ &\quad + \left|A\left(0\right)\right|^2 \sum_{k'\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(2\pi k'\right)\right|^2. \end{aligned}$$

Since by [3] $A(0) = 1$, and since $\sum_{k'\in\mathbf{Z}^d} \left|\widehat{\varphi}\left(2\pi\left(M^{*-1}\gamma+k'\right)\right)\right|^2 \geq C_1 > 0$, we have

$$\sum_{k\in\mathbf{Z}^d} c\left(k\right) e^{-i2\pi\left(M^{*-1}\gamma\right)\cdot k} = 0 \text{ for each } \gamma \in \Gamma_0 \setminus \{0\},$$

which, after letting $k = \gamma_k + Mn_k$, $\gamma_k \in \Gamma_0$, $n_k \in \mathbf{Z}^d$ and setting $\sum_{k\cong\delta} c\left(\gamma_k + Mn_k\right) = q_\delta$ leads to

$$0 = \sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}\delta} q_\delta. \tag{6.3}$$

Claim. $\sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}\delta} = 0$ for each $\gamma \in \Gamma_0 \setminus \{0\}$. □

Notice that the set $\left\{e^{-i2\pi\gamma\cdot M^{-1}\delta} \mid \delta \in \Gamma_0\right\}$ is a group on the unit circle. If

$$\sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}\delta} = r \neq 0, \tag{6.4}$$

then for every $\gamma \in \Gamma_0 \setminus \{0\}$ there is a $\delta \in \Gamma_0$ so that $e^{-i2\pi\gamma\cdot M^{-1}\delta} \neq 1$. (If not, $e^{-i2\pi\gamma\cdot M^{-1}\delta} = 1$ for all $\delta \in \Gamma_0$ implies that

$$0 = \sum_{\delta\in\Gamma_0} \sum_{k:k\cong\delta} c\left(\gamma_k + Mn_k\right) = \sum c\left(\gamma\right),$$

contradicting $\sum c(\gamma) = 1$.) Multiplying both sides of (6.4) by $e^{-i2\pi\gamma\cdot M^{-1}p}$ where $p \in \Gamma_0$ is such that $e^{-i2\pi\gamma\cdot M^{-1}p} \neq 1$, we obtain

$$\sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}(\delta+p)} = re^{-i2\pi\gamma\cdot M^{-1}p}.$$

Note that since $\delta + p = \delta' + Mk$, where $\delta' \in \Gamma_0$ and $k \in \mathbf{Z}^d$, then $\sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}(\delta+p)}$ includes all the elements of the group and nothing more, and therefore it is equal to r. So $r = re^{-i2\pi\gamma\cdot M^{-1}p}$, contradicting $e^{-i2\pi\gamma\cdot M^{-1}p} \neq 1$. □

The set of $m-1$ equations $\sum_{\delta\in\Gamma_0} e^{-i2\pi\gamma\cdot M^{-1}\delta} q_\delta = 0$ for each $\gamma \in \Gamma_0 \setminus \{0\}$, along with the constraint $\sum_{\delta\in\Gamma_0} q_\delta = 1$, comprises a system of m equations with m variables q_δ. Notice that $q_\delta = \frac{1}{m}$ for each $\delta \in \Gamma_0$ is a solution. The coefficient matrix for this system is given by

$$U = \left(e^{-i2\pi\gamma_i\cdot M^{-1}\gamma_j}\right)_{0\le i,j\le m-1}.$$

By (6.3), $UU^* = mI_m$, and so $\det U \neq 0$. Therefore $q_\delta = \frac{1}{m}$ for all $\delta \in \Gamma_0$ is the unique solution of the system, which concludes the proof of the theorem. □

If, as in the previous section, we let $c(\gamma) = P(G = \gamma)$, the above theorem says that $P(G \cong \gamma) = \frac{1}{m}$ for each $\gamma \in \Gamma_0$ is necessary in order for the density φ to be stable. However, this condition, which by Theorems 1 and 3 guarantees that φ is absolutely continuous, is not sufficient for the stability of φ. Consider the following example: $\Gamma = \mathbf{Z}$, $M = 2$ with the constants assigned as follows:

$$\begin{aligned} c(0) &= c(2) = c(3) = c(5) = \frac{1}{8}, \\ c(1) &= c(4) = \frac{1}{4}. \end{aligned}$$

Notice that the two cosets have equal weight and so by Theorems 1 and 3 the solution φ of the dilation equation will be a density function. However, it is shown in [11] that φ is not stable.

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