TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 1, Pages 1–45
S 0002-9947(01)02790-8
Article electronically published on May 17, 2001

RATIONAL S^1-EQUIVARIANT HOMOTOPY THEORY

LAURA SCULL

Abstract. We give an algebraicization of rational S^1-equivariant homotopy theory. There is an algebraic category of "$\mathbb{T}$-systems" which is equivalent to the homotopy category of rational S^1-simply connected S^1-spaces. There is also a theory of "minimal models" for $\mathbb{T}$-systems, analogous to Sullivan's minimal algebras. Each S^1-space has an associated minimal $\mathbb{T}$-system which encodes all of its rational homotopy information, including its rational equivariant cohomology and Postnikov decomposition.

1. Introduction

The theory of rational homotopy is one of the most elegant and best understood areas of topology. Sullivan's theory of minimal models [DGMS, S] provides an encoding of the entire rational homotopy structure of a space, the sort of complete algebraic description that topologists always attempt and rarely achieve. Sullivan's theory uses a piecewise linear version of differential forms to model the Postnikov decomposition of a space, algebraically encoding all of its rational homotopy groups and rational k-invariants, and thus capturing all of the rational structure of the space. Algebraically, this theory is based on commutative differential graded algebras (DGAs). A particularly nice class of DGAs is identified, the "minimal" ones, and it is shown that an arbitrary DGA has a minimal approximation, its "minimal model". The PL de Rham forms $\mathcal{E}(X)$ of a space X is the DGA used to connect geometry and algebra, and the minimal model of a space X is defined to be the minimal model of its de Rham DGA. These minimal models have allowed many concrete calculations, and have led to general theories on the classification of manifolds and to the idea of "formal" spaces which are completely determined rationally by their cohomology.

An equivariant analogue was developed by Triantafillou for actions of finite groups [T1] and has proved extremely useful in studying rational equivariant homotopy. This theory has led to new results about the structure of equivariant H-spaces, equivariant formality, and the classification of G-manifolds up to finite ambiguity ([FT, RT1, RT2, T2]). To account for the added structure introduced by the G-action on the space, we consider not only the space itself but also the fixed point subspaces X^H for subgroups $H \leq G$, with the inclusions and relations induced by the group action. These relations are encoded using the homotopy orbit category $h\mathcal{O}_G$, and the algebraic category used is "systems of DGAs", which are functors

Received by the editors April 26, 2000 and, in revised form, September 27, 2000.
2000 *Mathematics Subject Classification.* Primary 55P91, 55P62; Secondary 55R35, 55S45.
Key words and phrases. Equivariant homotopy, minimal model, rationalization.

from $h\mathcal{O}_G$ to DGAs. These functors introduce considerable algebraic complications, and even deciding what the correct equivariant analogue of "minimal" is for a functor into DGAs is not trivial. In fact, the original definition used is incorrect. The theory contains an error concerning the properties of these "minimal" systems. To correct this, it is necessary to redefine equivariant minimality. This paper gives a new definition for the simply connected case. The new minimal functors have weaker algebraic properties, but retain the most important features. In particular, the PL de Rham functor, which has as values the DGAs $\mathcal{E}(X^H)$ associated to each fixed point space X^H, has a minimal model, the equivariant minimal model for the G-space X; and the correspondence between G-spaces and their equivariant minimal models gives a bijection between rational homotopy types of simply connected G-spaces and isomorphism classes of minimal systems of DGAs. This paper considers the algebraic theory of these functors in some detail, discussing the needed modifications.

The main part of this paper involves extending the theory of minimal models to consider actions by the circle group $\mathbb{T}$. In addition to being the natural next step towards more general Lie groups, $\mathbb{T}$-equivariant theory is also of considerable interest in itself. Circle actions occur naturally in a number of contexts; in particular free loop spaces, which come equipped with a circle action, have been much studied. $\mathbb{T}$-equivariant theory is also used in studying cyclic cohomology, leading to results in algebraic K-theory. Thus a theory of $\mathbb{T}$-equivariant minimal models could be applied in a variety of areas.

Circle actions are considerably harder than those of finite groups, since the circle has its own topology which must be taken into account. The recent work of Greenlees [G] studies the stable case, analyzing rational $\mathbb{T}$-spectra, and producing an algebraic structure which completely describes stable rational homotopy theory. When considering the unstable world of $\mathbb{T}$-spaces, the techniques used by Greenlees no longer apply. Instead, we extend the models of the finite group actions to circles. There are fundamental problems in doing this as well, however. In the case of finite groups, the underlying strategy is to apply the standard non-equivariant constructions and results to the category of functors from $h\mathcal{O}_G$, using the diagrams of spaces given by the fixed point sets. Although the algebra is more difficult, the underlying geometric ideas are the same.

For the actions of the circle group, this is no longer sufficient. It is necessary to find an algebraic way of describing the action of the circle. A finite group can act on a DGA simply by permuting generators around, but for a connected group like the circle a new approach is needed. Instead, we make use of the classical Borel construction $X \times_{\mathbb{T}} E\mathbb{T}$, and the natural fibration $X \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$. Algebraically, this induces a $\mathbb{Q}[c]$-module structure on its cohomology, and also on its PL de Rham algebra; this module structure carries information about the fibration, and thus about the original action of $\mathbb{T}$ on the space X. This provides a way of encoding the $\mathbb{T}$-action algebraically, and the $\mathbb{T}$-equivariant theory uses functors into DGAs which have a $\mathbb{Q}[c]$-module structure. Another complication which arises is that the internal structure of the circle group matters when studying $\mathbb{T}$-equivariant cohomology theories and Postnikov decompositions, and we must look at the orbit spaces $X^H/\mathbb{T}$, and not just the fixed point spaces themselves. To do this while still retaining the structure associated to the Borel construction, we define a substitute for the orbit space $X//\mathbb{T}$. This is the orbit space of a "semifree approximation of X", defined as a quotient of the Borel space $X//\mathbb{T} = X \times_{\mathbb{T}} E\mathbb{T}/[x,e] \sim [x,e']$

for $x \in X^{\mathbb{T}}$, where we collapse the fibre over the fixed points. This is rationally equivalent to the orbit space $X/\mathbb{T}$.

To develop models for rational $\mathbb{T}$-equivariant homotopy theory, we consider the diagram of Borel spaces of the fixed point sets $X^H \times_{\mathbb{T}} E\mathbb{T}$ together with their projections down to $X//\mathbb{T}$. This is the motivation for the algebraic category used, and a $\mathbb{T}$-system is defined to be a functor into $\mathbb{Q}[c]$-DGAs which has properties that mimic the structure of the system of de Rham DGAs obtained by applying the functor of PL de Rham forms to this entire structure. This complicated category is the basic tool for describing the topology of $\mathbb{T}$-spaces. This paper develops the necessary algebra of minimal models for $\mathbb{T}$-systems, closely analogous to the (revised) theory of minimal functors for finite group actions. The connection with geometry comes through the $\mathbb{T}$-system given by a version of the de Rham algebra on $X^H \times_{\mathbb{T}} E\mathbb{T}$, and we define its minimal model $\mathfrak{M}_X$ to be the minimal model of X. This correspondence is shown to induce a bijection between rational homotopy types of simply connected $\mathbb{T}$-spaces having finitely many orbit types on the one hand, and isomorphism classes of minimal $\mathbb{T}$-systems on the other. In addition, $\mathfrak{M}_X$ encodes the rational Postnikov decomposition of X, including all of its rational homotopy groups and k-invariants, and various equivariant rational cohomologies of X can be computed from $\mathfrak{M}_X$.

The paper begins with a brief sketch of background material on equivariant homotopy theory in Section 2, and then gives an overview of the major results in Sections 3 - 6. The algebraic theory of the relevant functor categories is developed in Sections 7 - 13, and Sections 14 - 19 contain the topological theory. Section 20 gives an application of the theory to some classical results, and Section 21 is a discussion of the mistake in the original theory.

2. Equivariant homotopy theory

Equivariant homotopy theory studies the category of G-spaces, that is, spaces X with an action of a compact Lie group G, with G-equivariant maps between them. We will be considering the specific cases when G is either a finite group or the circle group $\mathbb{T}$, but the following is a sketch of the general theory for any compact Lie group. Many of the ideas and definitions in equivariant homotopy theory are motivated by the following equivariant version of the Whitehead theorem ([B2]).

Theorem 2.1 (Bredon). *An equivariant map $f : X \to Y$ between two G-complexes is an equivariant homotopy equivalence if and only if $f^H : X^H \to Y^H$ is a weak equivalence for each closed subgroup $H \subseteq G$.*

Because of this result, when studying G-spaces it is natural to consider not only the structure of the space itself but also that of each of its fixed point sets, and to define algebraic invariants accordingly. To organize this information, we use the orbit category $\mathcal{O}_G$ of G, whose objects are the canonical orbits G/H with equivariant maps between them, and also the homotopy orbit category $h\mathcal{O}_G$, which has the same objects with homotopy classes of maps between them. (The precise structure of $h\mathcal{O}_G$ and its relationship to the orbit category $\mathcal{O}_G$ are discussed at the beginning of Section 8.) Associated to any G-space X there is a contravariant fixed point functor from $\mathcal{O}_G$ to spaces. This functor has the value X^H at G/H, with morphisms induced by the G-action on the space X. This also gives a functor from $h\mathcal{O}_G$ to the homotopy category of spaces. Any of the usual abelian group valued homotopy invariants may be composed with this fixed point functor. Thus the algebraic structures suited to equivariant homotopy theory are contravariant functors from $h\mathcal{O}_G$ to abelian groups, called *coefficient systems*. Some important examples are $\underline{\pi}_*(X)(G/H) = \pi_*(X^H)$, $\underline{H}_*(X)(G/H) = H_*(X^H)$, and $\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H)$, where X^n is the (non-equivariant) n-skeleton of X^H.

The category of coefficient systems is abelian, since the set $\mathrm{Hom}(\underline{A}, \underline{B})$ of natural transformations between coefficient systems is an abelian group; kernels and cokernels are defined entrywise, and this category is suitable for homological algebra. This leads to the definition of ordinary equivariant cohomology theories, developed by Bredon ([B1]). The boundary map $d : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ is a map of coefficient systems, with $d^2 = 0$. For any coefficient system $\underline{A}$, $\mathrm{Hom}(\underline{C}_*(X), \underline{A})$ is a cochain complex with coboundary operator induced by the boundary operator of $\underline{C}_*$. We define the equivariant cohomology of X by

$$H^n_G(X; \underline{A}) = H^n(\mathrm{Hom}(\underline{C}_*(X), \underline{A})).$$

Standard homological algebra yeilds a universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(\underline{J}_t(X), \underline{A}) \Rightarrow H^{s+t}_G(X; \underline{A}),$$

where $\underline{H}_*(\underline{C}_*) \cong \underline{J}_*(X)$. This is constructed by taking an injective resolution $\underline{I}^*$ of the coefficient system $\underline{A}$, and considering the double complex $\mathrm{Hom}(\underline{C}_*(X), \underline{I}^*)$. Grading one way gives the interpretation of E^2 as Ext; grading the other, the spectral sequence collapses because $\underline{C}_*(X)$ is a projective object in the category of coefficient systems, and this identifies the E^∞ term ([B1, M, W]).

Note that the system $\underline{C}_*$ used above is the G-cellular chain complex, and not simply the ordinary non-equivariant chain complex of the spaces involved. The theory of Lie groups allows us to identify its homology:

$$\underline{J}_*(X)(G/H) = H_*(X^H/W_0H),$$

where W_0H is the identity component of $WH = NH/H$. For discrete groups W_0H is always trivial, so when dealing with finite group actions we can use the ordinary chain complex. For more general Lie groups, however, this presents an added complexity for understanding and using these cohomology groups.

An important property of these cohomology theories is that they can be represented. Let $\underline{A}$ be a coefficient system. An *Eilenberg-Mac Lane G-complex* of type $(\underline{A}, n)$ is a G-complex K with the property that $\underline{\pi}_n(K) = \underline{A}$ and $\underline{\pi}_i(K) = 0$ for $i \neq n$. Note that this implies that K^H is an Eilenberg-Mac Lane space of dimension n for every $H \subseteq G$. Such G-complexes exist and satisfy the relation

$$[X, K]_G \cong \widetilde{H}^n_G(X; \underline{A});$$

see [E]. This allows us to define equivariant principal fibrations as pullbacks of maps into $K(\underline{\pi}, n)$'s, and develop equivariant obstruction theory and G-Postnikov decompositions.

A G-Postnikov decomposition for a based simple G-space X consists of a sequence of G-spaces X_n, together with based G-maps

$$\alpha_n : X \to X_n \quad \text{and} \quad p_{n+1} : X_{n+1} \to X_n$$

for $n \geq 0$ such that X_0 is a point, α_n induces an isomorphism $\underline{\pi}_i(X) \to \underline{\pi}_i(X_n)$ for $i \leq n$, $\pi_i(X_n) = 0$ for $i > n$, $p_{n+1}\alpha_{n+1} = \alpha_n$, and p_{n+1} is a principal G-fibration. The fibre of p_n is an Eilenberg-Mac Lane G-complex $K(\underline{\pi}_n(X), n)$, and this fibration is characterized by a cohomology class $k^{n+1} \in H^{n+1}_G(X_{n-1}; \underline{\pi}_n(X))$, which we call the $(n+1)$*st equivariant k-invariant of* X. G-Postnikov decompositions exist for any G-simple space, and encode all of its homotopy information.

A G-space X is said to be *rational* if the homotopy groups $\pi_i(X^H)$ are $\mathbb{Q}$-vector spaces for each $H \subseteq G$. For each G-simple G-space X there exists a rational G-space X_0 and a rationalization G-map $f : X \to X_0$ such that f induces rationalization on homotopy and homology groups; and f is universal among maps into rational spaces. The existence of a G-rationalization of X can be shown by inductively constructing G-rationalizations of the G-spaces X_n in the G-Postnikov decomposition, by localizing the Eilenberg-Mac Lane spaces and the k-invariants. We consider the G-rational homotopy type of a G-space X to be the G-homotopy type of its rationalization X_0. Further discussion of equivariant homotopy theory may be found in [T1, B2, M, tD]

Throughout this paper, all G-spaces are assumed to be G-CW complexes; note that this ensures that all orbit spaces and related constructions like Borel spaces are also CW complexes. In addition, all G-spaces are assumed to have finitely many orbit types. We also assume that all G-spaces are G-simply connected in the sense that the fixed point spaces X^H are all connected and simply connected (and also non-empty). Lastly, we assume that the rational cohomology of each X^H is of finite type. We will refer to spaces satisfying all of these conditions as $\mathbb{Q}$-good.

3. Algebra of systems of DGAs

Non-equivariant minimal models use differential graded $\mathbb{Q}$-algebras, or DGAs. In addition, all DGAs are based, meaning they come with an injection from $\mathbb{Q}$ taken to be in degree 0. We now define the equivariant analogue of such a structure for the actions of finite groups.

Definition 3.2. A *system of DGAs* $\mathfrak{A}$ is a covariant functor from the orbit category of G to the category of based DGAs which is injective when regarded as a dual rational coefficient system by neglect of structure.

The restriction to injective systems is important. Not only is it necessary for understanding maps between systems of DGAs, but this condition reflects the fundamental geometric fact that $\underline{C}_*(X)$ is *projective* for any space. Restricting to injective systems will, however, make many constructions considerably more complicated; much of the technical work is a thorough analysis of exactly what it means to be injective.

A homotopy of systems of DGAs is defined as in the non-equivariant case. Let $\mathbb{Q}(t, dt)$ be the free DGA generated by t in degree 0 and dt in degree 1, with $d(t) = dt$.

Definition 3.3. Two maps $f, g : \mathfrak{A} \to \mathfrak{B}$ are homotopic if there is a morphism $H : \mathfrak{A} \to \mathfrak{B} \otimes \mathbb{Q}(t, dt)$ such that $p_0 H = f$ and $p_1 H = g$, where $p_0 p_1 : \mathfrak{B} \otimes \mathbb{Q}(t, dt) \to \mathfrak{B}$ are defined by $p_0(t) = 0, p_0(dt) = 0$ and $p_1(t) = 1, p_1(dt) = 0$.

We now wish to establish the equivariant analogue of minimality. Our definition is based on the idea of an "elementary extension" (defined in Section 11), which builds systems of DGAs out of simple pieces generated by systems of vector spaces.

Definition 3.4. A system of DGAs $\mathfrak{M}$ is *minimal* if $\mathfrak{M} = \bigcup_n \mathfrak{M}(n)$, where $\mathfrak{M}(0) = \mathfrak{M}(1) = \mathbb{Q}$ and $\mathfrak{M}(n) = \mathfrak{M}(n-1)(\underline{V}_n)$ is an elementary extension for some system of vector spaces $\underline{V}_n$ of degree n.

Non-equivariantly, there are two ways of defining minimal DGAs. One is to use the intrinsic algebraic condition that the differential is decomposible; this is the original approach taken by Sullivan ([S]). The other is to define minimal to be a union of a sequence of elementary extensions of DGAs ([H]). It is not hard to show that these two definitions are equivalent. Equivariantly, however, these two approaches are not the same. Triantafillou's original work claims erroneously that the two approaches are still equivalent, and defines a minimal system using an analogue of the algebraic condition of decomposible differential. This is not true; Section 21 discusses this and gives counterexamples. Because the theory used to model the Postnikov tower of a space is based around extensions, in order to obtain the connection to geometry, it is necessary to redefine minimal and abandon the original definition, and redevelop the algebraic properties using the new definition. Here is a summary of the algebraic properties retained by the new minimal systems.

Proposition 3.5. *Homotopy is an equivalence relation between maps $\mathfrak{M} \to \mathfrak{A}$ whenever $\mathfrak{M}$ is minimal.*

Proposition 3.6. *If $\rho : \mathfrak{A} \to \mathfrak{B}$ is a quasi-isomorphim and $f : \mathfrak{M} \to \mathfrak{B}$ is any map from a minimal system $\mathfrak{M}$, then there is a lift $g : \mathfrak{M} \to \mathfrak{A}$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & \mathfrak{A} \arrow[d, "\rho"] \\ \mathfrak{M} \arrow[ur, dashed, "g"] \arrow[r, "f"] & \mathfrak{B} \end{tikzcd}$$

Furthermore, this lift is unique up to homotopy.

Corollary 3.7. *If $\rho : \mathfrak{A} \to \mathfrak{B}$ is a quasi-isomorphism, then for any minimal system $\mathfrak{M}$, $\rho^* : [\mathfrak{M}, \mathfrak{A}] \cong [\mathfrak{M}, \mathfrak{B}]$.*

Theorem 3.8 (Uniqueness of Minimal Systems)**.** *If $f : \mathfrak{M} \to \mathfrak{N}$ is a quasi-isomorphism between minimal systems, then $f \simeq g$, where g is an isomorphism.*

The earlier definition of minimal satisfied a stronger uniqueness theorem, which stated that any quasi-isomorphism of minimal systems is actually an isomorphism; this is not true for the new definition.

Corollary 3.9. *If $\mathfrak{M}$ and $\mathfrak{M}'$ are two minimal systems and $\rho : \mathfrak{M} \to \mathfrak{A}$ and $\rho' : \mathfrak{M}' \to \mathfrak{A}$ are quasi-isomorphisms, then there is an isomorphism $f : \mathfrak{M} \cong \mathfrak{M}'$ such that $\rho' f \simeq \rho$.*

This uniqueness of minimal systems allows us to make the following definition.

Definition 3.10. If $\mathfrak{M}$ is minimal and $\rho : \mathfrak{M} \to \mathfrak{A}$ is a quasi-isomorphism, we say that $\mathfrak{M}$ is the *minimal model* of the system $\mathfrak{A}$.

Observe that any minimal system is cohomologically 1-connected, that is, it satisfies $\underline{H}^0(\mathfrak{M}) = \mathbb{Q}$ and $\underline{H}^1(\mathfrak{M}) = 0$. This is the only requirement needed for a system to have a minimal model.

Theorem 3.11 (Existence of Minimal Models)**.** *If $\mathfrak{A}$ is a system of DGAs which is cohomologically 1-connected, then there exists a minimal model of $\mathfrak{A}$, that is, a minimal system of DGAs $\mathfrak{M}$ and a quasi-isomorphism $\rho : \mathfrak{M} \to \mathfrak{A}$*

This is the theorem which contains the error with the original definition of minimal.

4. Main results for finite group actions

Here is a brief summary of the results of [T1], developing the theory of equivariant minimal models for G-spaces with G finite. All of these results can be obtained with the modified algebraic theory.

The basic tool for modeling equivariant spaces is an equivariant version of the PL de Rham algebra of differential forms. Non-equivariantly, we pass from geometry to algebra using the DGA of PL differential forms of a simplicial complex X. On any n-simplex σ^n, a PL form of degree p is a polynomial form $\sum_I f_I(t_0, ..., t_n) dt_{i_1} \wedge ... \wedge dt_{i_p}$, where f_I is a polynomial with coefficients in $\mathbb{Q}$. A global PL form on X is a collection of polynomial forms, one for each simplex of X, which coincide on common faces. The PL forms of a simplicial complex X form a DGA over $\mathbb{Q}$ which is denoted by $\mathcal{E}(X)$. Triantafillou's theory uses the DGA $\mathcal{E}(X)$ as the tool for understanding G-simplicial complexes. However, equivariant triangulation theorems are harder and less powerful than non-equivariant results, and so the distinction between G-spaces and G-simplicial complexes is more important. To avoid this problem we modify the theory slightly, developing it in the more general context of Alexander-Spanier cohomology.

If X is a topological space and $\mathcal{U}$ is an open covering of X, then the Vietoris nerve $\overline{\mathcal{U}}$ of $\mathcal{U}$ is the simplicial complex with vertices the points of X and simplices all finite subsets $\{x_0, \dots, x_n\}$ of X such that there is a set $U \in \mathcal{U}$ containing x_i for $0 \le i \le n$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then inclusion gives a simplicial map $\overline{\mathcal{V}} \to \overline{\mathcal{U}}$. Now we can apply the functor of de Rham PL forms, and obtain $\{\mathcal{E}(\overline{\mathcal{U}})\}$, a direct system of DGAs.

Definition 4.12. For a topological space X, the de Rham-Alexander-Spanier algebra of X, denoted $\mathcal{A}(X)$, is defined to be

$$\mathcal{A}(X) = \mathrm{colim}_{\mathcal{U}} \mathcal{E}(\overline{\mathcal{U}}),$$

where $\mathcal{U}$ ranges over all open coverings of X.

Equivariantly, we define a system of DGAs $\underline{\mathcal{E}}(X)$ by $\underline{\mathcal{E}}(X)(G/H) = \mathcal{A}(X^H)$, and observe that the geometry of X ensures that this is injective (see [T1]). The main geometric results are as follows.

Theorem 4.13. *Let $\mathfrak{M}_X$ be the minimal model of $\underline{\mathcal{E}}(X)$. Then the correspondence $X \to \mathfrak{M}_X$ induces a bijection between rational homotopy types of $\mathbb{Q}$-good spaces and isomorphism classes of minimal systems of DGAs. Furthermore, $\mathfrak{M}_X$ computes the equivariant cohomology and also encodes the G-Postnikov decomposition.*

This will be made more precise later. We say that $\mathfrak{G}$ is *geometric* for X if there is a quasi-isomorphism $\mathfrak{G} \to \underline{\mathcal{E}}(X)$; so $\mathfrak{M}_X$ is the minimal system of DGAs geometric for X, unique up to isomorphism. Note that although X determines the isomorphism class of the minimal model, there is no canonical choice of model or of quasi-isomorphism $\mathfrak{M}_X \to \underline{\mathcal{E}}(X)$. Moreover, the association is not functorial, but it does capture homotopy classes of maps.

Theorem 4.14. *If X is a $\mathbb{Q}$-good G-space and Y is a rational $\mathbb{Q}$-good G-space, then there is a bijection $[X, Y]_G \cong [\mathfrak{M}_Y, \mathfrak{M}_X]$.*

Because systems of DGAs are modelled on differential forms, they are covariant functors rather than the usual contravariant coefficient systems like $\underline{\pi}_n(X)$, and the homotopy classes of maps are modelled contravariantly. In order to prove these results we show how the algebraic structure of the minimal model closely corresponds to the geometric structure of the equivariant Postnikov decomposition. A minimal system of DGAs is made from a sequence of elementary extensions (defined in Section 11), while a Postnikov tower is a sequence of principal fibrations; we develop an equivalence between the algebraic extensions and the geometric fibrations. The first step is to show how to model the basic pieces which compose the tower, the Eilenberg-Mac Lane spaces.

For a coefficient system $\underline{V}$, let $\underline{V}^*$ denote the dual covariant functor. If $\underline{V}^* \to I_0 \to I_1 \to \dots$ is an injective resolution, we can form the free injective system of DGAs generated by $\underline{V}^*$ in degree n by defining $\mathfrak{V} = \bigotimes_i \mathbb{Q}(I_i)$, where $\mathbb{Q}(I_i)$ is the free graded commutative algebra generated at G/H by the vector space $\underline{I}_i(G/H)$ in degree $n+i$ with differential coming from the maps in the resolution.

Theorem 4.15. *If $\mathfrak{V}$ is the free injective system of DGAs generated by $\underline{V}^*$ in degree n, then $\mathfrak{V}$ is geometric for $K(\underline{V}, n)$. Futhermore, if $\mathfrak{G}$ is geometric for X, then homotopy classes of maps $[X, K]$ are in bijective correspondence with homotopy classes of maps of systems of DGAs $[\mathfrak{V}, \mathfrak{G}]$.*

Now that we can model the k-invariants $X \to K$ which determine the fibrations in the Postnikov tower, we prove that principal fibrations determine the algebraic elementary extensions.

Theorem 4.16. *Suppose $\mathfrak{G}$ is a system of DGAs which is geometric for X. Then isomorphism classes of principal fibrations $K(\underline{V}, n) \to Y \to X$ are in bijective correspondence with isomorphism classes of elementary extensions $\mathfrak{G}(\underline{V}^*_n)$ of degree n; the fibration induced by $X \xrightarrow{k} K(\underline{V}, n{+}1)$ corresponds to the elementary extension induced by the corresponding map $\mathbb{Q}(\underline{V}^*) \to \mathfrak{G}$, and $\mathfrak{G}(\underline{V}^*_n)$ is geometric for Y.*

Passing to limits, we get a correspondence between a tower of principal fibrations and a sequence of elementary extensions, giving an exact correspondence between Postnikov decompositions and minimal systems of DGAs.

Theorem 4.17. *Suppose X is the limit of a countable sequence of principal fibrations $\cdots \to X_2 \to X_1 \to X_0$, $\mathfrak{M}_0$ is geometric for X_0 and $\mathfrak{M}_n = \mathfrak{M}_{n-1}(\underline{V}_n)$ is an elementary extension geometric for X_n. Then the colimit $\mathfrak{M} = \bigcup \mathfrak{M}_n$ is geometric for X.*

5. Algebra of $\mathbb{T}$-systems

We now wish to develop a suitable algebraic category for modelling $\mathbb{T}$-spaces. This presents a more difficult problem, since the circle has its own topology which must be accounted for. In the case of finite groups, we apply the standard non-equivariant constructions and results to the category of functors from $h\mathcal{O}_G$, using the diagrams of spaces given by the fixed point sets. Although the algebra is more difficult, the underlying geometric ideas are the same. For the actions of the circle group, this is no longer sufficient. We still need to consider all the fixed point spaces, so the algebraic context is still functors from the homotopy orbit category; the need for all such functors to be injective, and the consequent algebraic difficulties, occur as before. In addition, it is necessary to find an algebraic way of describing the action of the circle. A finite group can act on a DGA simply by permuting generators around, but for a connected group like the circle a new approach is needed.

In order to handle this problem, we use the Borel construction $X \times_{\mathbb{T}} E\mathbb{T}$ rather than the space itself. There is a natural fibration $X \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$. Algebraically, this induces a $\mathbb{Q}[c]$-module structure on its cohomology, and also on its PL de Rham algebra, where $\mathbb{Q}[c] = H^*(B\mathbb{T})$ is the free polynomial algebra on one generator c of degree 2. This module structure carries information about the fibration, and thus about the action of $\mathbb{T}$ on the space X; this is the way we algebraically encode the $\mathbb{T}$-action.

In using the Borel construction, we have replaced the space X with a free approximation $X \times E\mathbb{T}$. Finite isotropy spaces are rationally equivalent to their free approximations, so only the fixed points $X^{\mathbb{T}} \subset X$ have been seriously affected. To retain the true structure of X, we keep track of this fixed point set and the trivial fibration $X^{\mathbb{T}} \times B\mathbb{T} \to X^{\mathbb{T}}$. Algebraically, this corresponds to a sub-DGA which generates the cohomology as a $\mathbb{Q}[c]$-module. These considerations motivate the following definition.

Definition 5.18. A $\mathbb{T}$-*system* consists of

1. A covariant functor $\mathfrak{A}$ from the homotopy orbit category of $\mathbb{T}$ to the category of DGAs under $\mathbb{Q}[c]$ such that $\mathfrak{A}$ is of finite orbit type and injective when regarded as a dual rational coefficient system by neglect of structure.
2. A distinguished sub-DGA $\mathfrak{A}_{\mathbb{T}}$ of $\mathfrak{A}(\mathbb{T}/\mathbb{T})$ such that the induced map $\mathfrak{A}_{\mathbb{T}} \otimes \mathbb{Q}[c] \to \mathfrak{A}(\mathbb{T}/\mathbb{T})$ induces an isomorphism on cohomology.

A morphism between $\mathbb{T}$-systems $\mathfrak{A}$ and $\mathfrak{B}$ is a natural transformation such that $\mathfrak{A}_{\mathbb{T}}$ lands in $\mathfrak{B}_{\mathbb{T}}$.

The category is necessarily complicated, representing the rich structure of $\mathbb{T}$-spaces; the example which motivates this definition is given at the beginning of Section 6. Nonetheless, it is possible to develop a theory of the algebra of $\mathbb{T}$-systems which is very close to that of systems of DGAs. Once again the key idea is that of an elementary extension generated by a system of vector spaces, defined in Section 4 .

Definition 5.19. A $\mathbb{T}$-system $\mathfrak{M}$ is *minimal* if $\mathfrak{M} = \bigcup_n \mathfrak{M}(n)$, where $\mathfrak{M}(0) = \mathfrak{M}(1) = \mathbb{Q}[c]$ and $\mathfrak{M}(n) = \mathfrak{M}(n-1)(\underline{V}_n)$ is an elementary extension for a system of vector spaces $\underline{V}_n$ of degree n.

We get the following analogous properties of such minimal objects.

Proposition 5.20. *Homotopy is an equivalence relation between maps $\mathfrak{M} \to \mathfrak{A}$ whenever $\mathfrak{M}$ is minimal.*

Proposition 5.21. *If $\rho : \mathfrak{A} \to \mathfrak{B}$ is a quasi-isomorphim and $f : \mathfrak{M} \to \mathfrak{B}$ is any map from a minimal system $\mathfrak{M}$, then there is a lift $g : \mathfrak{M} \to \mathfrak{A}$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd} & \mathfrak{A} \arrow[d, "\rho"] \\ \mathfrak{M} \arrow[r, "f"'] \arrow[ur, dashed, "g"] & \mathfrak{B} \end{tikzcd}$$

Furthermore, this lift is unique up to homotopy.

Corollary 5.22. *If $\rho : \mathfrak{A} \to \mathfrak{B}$ is a quasi-isomorphism, then for any minimal system $\mathfrak{M}$, $\rho^* : [\mathfrak{M}, \mathfrak{A}] \cong [\mathfrak{M}, \mathfrak{B}]$.*

Theorem 5.23 (Uniqueness of Minimal $\mathbb{T}$-systems)**.** *If $f : \mathfrak{M} \to \mathfrak{N}$ is a quasi-isomorphism between minimal $\mathbb{T}$-systems, then $f \simeq g$, where g is an isomorphism.*

Corollary 5.24. *If $\mathfrak{M}$ and $\mathfrak{M}'$ are two minimal systems and $\rho : \mathfrak{M} \to \mathfrak{A}$ and $\rho' : \mathfrak{M}' \to \mathfrak{A}$ are quasi-isomorphisms, then there is an isomorphism $f : \mathfrak{M} \cong \mathfrak{M}'$ such that $\rho' f \simeq \rho$.*

Again, we use this to make the following definition.

Definition 5.25. If $\mathfrak{M}$ is minimal and $\rho : \mathfrak{M} \to \mathfrak{A}$ is a quasi-isomorphism, we say that $\mathfrak{M}$ is the *minimal model* of the system $\mathfrak{A}$.

Theorem 5.26 (Existence of Minimal Models)**.** *If $\mathfrak{A}$ is a $\mathbb{T}$-system, then there exists a minimal model of $\mathfrak{A}$, that is, a minimal $\mathbb{T}$-system $\mathfrak{M}$ and a quasi-isomorphism $\rho : \mathfrak{M} \to \mathfrak{A}$.*

These results are proved in in Section 13.

6. Main results for $\mathbb{T}$-spaces

In studying $\mathbb{T}$-spaces, the main tool will be the following $\mathbb{T}$-system.

Definition 6.27. Let X be a $\mathbb{Q}$-good $\mathbb{T}$-space, and consider the Borel construction $X \times_{\mathbb{T}} E\mathbb{T}$. Let $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ be the $\mathbb{T}$-system defined by

$$\underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}/H) = \mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T}),$$

with special sub-DGA

$$\mathcal{E}_{\mathbb{T}} = \mathcal{A}(X^{\mathbb{T}}) \subset \mathcal{A}(X^{\mathbb{T}} \times B\mathbb{T}) = \underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}/\mathbb{T}),$$

where the inclusion $\mathcal{A}(X^{\mathbb{T}}) \subset \mathcal{A}(X^{\mathbb{T}} \times B\mathbb{T})$ is induced by the projection $p_1 : X^{\mathbb{T}} \times B\mathbb{T} \to X^{\mathbb{T}}$.

To see that this is a $\mathbb{T}$-system, note that the projection $p : X^H \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$ induces a DGA map $\mathbb{Q}[c] \to \mathcal{A}(B\mathbb{T}) \to \mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T})$. The fact that the sub-DGA $\mathcal{E}_{\mathbb{T}}$ generates the cohomology as a $\mathbb{Q}[c]$-module comes from the Künneth formula. Furthermore, any $\mathbb{T}$-equivariant map from X to Y will induce a map from $X \times_{\mathbb{T}} E\mathbb{T}$ to $Y \times_{\mathbb{T}} E\mathbb{T}$ which has the form $f \times id$ on $X^{\mathbb{T}} \times B\mathbb{T} \to Y^{\mathbb{T}} \times B\mathbb{T}$, and so will induce a map $f^* : \mathcal{E}_{\mathbb{T}}(Y) \to \mathcal{E}_{\mathbb{T}}(X)$ which is a morphism of $\mathbb{T}$-systems. In order to show that this is a $\mathbb{T}$-system, all that is required is to show that it is injective as a functor to $\mathbb{Q}$-vector spaces; this will be proved in Section 8, where we analyze the structure of such injective systems.

The $\mathbb{T}$-system $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ provides the tool for modelling the structure of a $\mathbb{T}$-space.

Theorem 6.28. *Let X be a $\mathbb{Q}$-good $\mathbb{T}$-space, and let $\mathfrak{M}_X$ be the minimal model of $\underline{\mathcal{E}}_{\mathbb{T}}(X)$. Then the correspondence $X \to \mathfrak{M}_X$ induces a bijection between rational homotopy types of $\mathbb{Q}$-good spaces and isomorphism classes of minimal $\mathbb{T}$-systems. Furthermore, $\mathfrak{M}_X$ computes the equivariant cohomology and also encodes the $\mathbb{T}$-Postnikov decomposition.*

For a $\mathbb{T}$-space X, we say $\mathfrak{G}$ is geometric for X if there is a quasi-isomorphism of $\mathbb{T}$-systems $\mathfrak{G} \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$; again, $\mathfrak{M}_X$ is a minimal system which is geometric for X. As before, this association is not functorial, but we do have the following correspondence.

Theorem 6.29. *If X is a $\mathbb{Q}$-good $\mathbb{T}$-space and Y is a rational $\mathbb{Q}$-good $\mathbb{T}$-space, then there is a bijection $[X, Y]_{\mathbb{T}} \cong [\mathfrak{M}_Y, \mathfrak{M}_X]$.*

As before, we develop a relationship between the algebraic structure of a minimal $\mathbb{T}$-system and the geometric structure of a Postnikov decomposition. We begin with the Eilenberg-Mac Lane spaces which are used to build the Postnikov tower. It is possible to produce a "free injective" $\mathbb{T}$-system generated by a system of vector spaces; this construction is defined in Section 17.

Theorem 6.30. *If $\mathfrak{V}$ is the free injective $\mathbb{T}$-system generated by $\underline{V}^*$ in degree n, then $\mathfrak{V}$ is geometric for $K(\underline{V}, n)$. Futhermore, if $\mathfrak{G}$ is geometric for X, then homotopy classes of maps $[X, K]$ are in bijective correspondence with homotopy classes of maps of $\mathbb{T}$-systems $[\mathfrak{V}, \mathfrak{G}]$.*

Once we have maps into Eilenberg-Mac Lane spaces, and thus k-invariants, we show that principal fibrations determine elementary extensions.

Theorem 6.31. *Suppose $\mathfrak{G}$ is a $\mathbb{T}$-system which is geometric for X. Then isomorphism classes of principal fibrations $K(\underline{V}, n) \to Y \to X$ are in bijective correspondence with isomorphism classes of elementary extensions $\mathfrak{G}(\underline{V}_n^*)$ of degree n; the fibration induced by $X \xrightarrow{k} K(\underline{V}, n+1)$ corresponds to the elementary extension induced by the corresponding map $\mathbb{Q}(\underline{V}^*) \to \mathfrak{G}$, and $\mathfrak{G}(\underline{V}_n^*)$ is geometric for Y.*

Passing to limits completes the model.

Theorem 6.32. *Suppose X is the limit of a countable sequence of principal fibrations $\cdots \to X_2 \to X_1 \to X_0$, $\mathfrak{M}_0$ is geometric for X_0 and $\mathfrak{M}_n = \mathfrak{M}_{n-1}(\underline{V}_n)$ is an elementary extension geometric for X_n. Then the colimit $\mathfrak{M} = \bigcup_n \mathfrak{M}_n$ is geometric for X.*

7. Injective systems for finite groups

In defining systems of DGAs and $\mathbb{T}$-systems, we restrict to functors which are injective as systems of vector spaces. Algebraically, this requirement provides the main source of complications when moving from non-equivariant to equivariant spaces. We now examine more closely what this condition entails. For finite groups, a thorough analysis of the projective objects in the dual category of contravariant functors was carried out by Triantafillou in [T1], and her analysis yields the following results about injective dual systems.

First, recall that the orbit category $\mathcal{O}_G$ consists of canonical orbits G/H with equivariant maps between them. Any map between orbits $G/H \to G/K$ is given by $\hat{a} : gH \to gaK$ for $a \in G$ such that $a^{-1}Ha \subseteq K$; two such maps are equal if and only if $ab^{-1} \in K$. In particular, the equivariant self-maps $G/H \to G/H$ are identified with NH/H. So if A is a dual rational coefficient system, its value at G/H, $A(G/H)$, is a $\mathbb{Q}(NH/H)$-module.

On the other hand, if V is any $\mathbb{Q}(NH/H)$-module, we define the injective dual coefficient system it generates, denoted $\underline{V}_H$, by

$$\underline{V}_H(G/K) = \mathrm{Hom}_{\mathbb{Q}(N(H)/H)}(\mathbb{Q}(G/H)^K, V), \tag{1}$$

with structure maps induced by the maps of fixed points $(G/H)^K \to (G/H)^{K'}$. Under the action of NH/H, $(G/H)^K$ splits into free NH/H orbits, one for each conjugate of H which contains K. So as a set, $(G/H)^K$ is the disjoint union of copies of NH/H, indexed by certain conjugates of H; and the maps of fixed point sets $\hat{a} : G/H^K \to G/H^{K'}$ act on these copies by taking one indexed by gHg^{-1} to one indexed by $agHg^{-1}a^{-1}$. This implies that

$$\begin{aligned}\underline{V}_H(G/K) &= \mathrm{Hom}_{\mathbb{Q}(NH/H)}(\mathbb{Q}(G/H)^K, V) \\ &= \mathrm{Hom}_{\mathbb{Q}(NH/H)}\left(\bigoplus_{(\text{conj of } H)\supset K} \mathbb{Q}(NH/H), V\right) \\ &= \bigoplus_{(\text{conj of } H)\supset K} V^*,\end{aligned}$$

where $V^* = \mathrm{Hom}_{\mathbb{Q}(NH/H)}(\mathbb{Q}(NH/H), V)$ and the structure maps permute the copies by conjugation. It turns out that these are the basic building blocks for all injective systems.

Proposition 7.33 (Triantafillou)**.** *A dual coefficient system A is injective if and only if it is of the form $A = \bigoplus_H \underline{V}_H$ for some collection of $\mathbb{Q}(N(H)/H)$-modules V_H.*

Proposition 7.34 (Triantafillou)**.** *Any coefficient system $\underline{A}$ can be embedded in an injective system.*

Proof. We produce an embedding $\underline{A} \to \underline{I}$ to an injective. Define

$$V_H = \bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K}),$$

where $\hat{e}_{H,K} : G/H \to G/K$ is the projection and $\underline{A}(\hat{e}_{H,K})$ is the induced structure map on the functor $\underline{A}$. We understand this to mean that V_G is all of $\underline{A}(G/G)$. Let $\underline{I} = \bigoplus_H \underline{V}_H$. Then there is an an injection $\underline{A} \hookrightarrow \underline{I}$ extending the natural inclusions of $\bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K})$. This is exactly dual to the construction of the projective cover given by Triantafillou in [T1]. □

The above construction has several features worth pointing out. First, the injection $\underline{A} \hookrightarrow \underline{I}$ induces an isomorphism

$$\bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K}) \cong \bigcap_{K\supset H} \ker\, \underline{I}(\hat{e}_{H,K}).$$

To see this, observe that $\mathbb{Q}(G/L)^K = 0$ unless $L \supset gKg^{-1}$ for some g. Thus the only factors contributing to $\underline{I}(G/K)$ come from $\underline{V}_L$ for some L with K subconjugate to L, and $\bigcap_{K\supset H} \ker\, I(\hat{e}_{H,K}) = V_H$. But the injection $\underline{A} \hookrightarrow \underline{I}$ was constructed to extend the isomorphism $\bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K}) \cong V_H$.

In particular, the inclusion is always an isomorphism at G/G. This also implies that if, for some H, $\underline{A}(G/K) = 0$ for all $K \supset H$, then the inclusion is an isomorphism $\underline{A}(G/K) \to \underline{I}(G/K)$ for all $K \supseteq H$, including H itself. This follows from the fact that if $\underline{A}(G/K)$ is 0 for all $K \supset H$, then $\underline{A}(G/K)$ is also 0 for all $K \supset gHg^{-1}$, since the structure maps give an isomorphism between the values of the functor at conjugate subgroups. So V_K is 0 for all $K \supset gHg^{-1}$, and any summand which could contribute to $\underline{I}(G/K)$ vanishes for $K \supset H$. Therefore $\underline{A}(G/K) = \underline{I}(G/K) = 0$ for $K \supset H$. Moreover, $\underline{A}(G/H) = \bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K})$, since all structure maps land in zero vector spaces; similarly $\underline{I}(G/H) = \bigcap_{K\supset H} \ker\, \underline{I}(\hat{e}_{H,K})$. We have already noted that there is an isomorphism $\bigcap_{K\supset H} \ker\, \underline{A}(\hat{e}_{H,K}) \cong \bigcap_{K\supset H} \ker\, \underline{I}(\hat{e}_{H,K})$; this is the claimed isomorphim $\underline{A}(G/H) \cong \underline{I}(G/H)$.

Corollary 7.35. *Any dual coefficient system has an injective resolution which is of finite length.*

Proof. We can form an injective resolution $\underline{V} \hookrightarrow \underline{V}_0 \xrightarrow{v_0} \underline{V}_1 \xrightarrow{v_1} \cdots$ by defining $\underline{V}_0$ to be the the injective defined in Proposition 7.34 and then successively embedding coker v_i in $\underline{V}_{i+1}$ using the same method. To show this is of finite length, observe that if $\underline{V}_i(G/K) = 0$ for all $K \supset H$, then coker $v_{i-1}(G/K) = 0$ for $K \supset H$ as well, and so coker $v_{i-1}(G/K) \hookrightarrow \underline{V}_i(G/K)$ is an isomorphism at all $K \supseteq H$, including H itself. Thus coker $v_i(G/K) = 0$ for all $K \supseteq H$, and the injective $\underline{V}_{i+1}$ vanishes here also. G contains only finite length chains of subgroups, so eventually $\underline{V}_n$ vanishes for high enough n. □

Proposition 7.36 (Golasinski)**.** *If A and B are injective, then so is $A \otimes B$.*

There has been some controversy surrounding the status of this proposition. Triantafillou's original work [T1] assumed but did not prove it, as did Fine's thesis [F] for the more general disconnected case. It was proved for the connected case treated here by Golasinski [Go].

The result is not true in the disconnected case as it stands.

8. Injective systems for the circle

We now consider the situation for the circle group, and develop an analogous analysis of injective systems of vector spaces.

First, we need to take a closer look at the structure of the homotopy orbit category $h\mathcal{O}_{\mathbb{T}}$. Recall that this consists of canonical orbits $\mathbb{T}/H$ with homotopy classes of equivariant maps between them. Any equivariant map between orbits $\mathbb{T}/H \to \mathbb{T}/K$ is given by $\hat{a} : gH \to gaK$ for some a for which $a^{-1}Ha \subseteq K$; since $\mathbb{T}$ is abelian this is equivalent to $H \subseteq K$. Two such maps $\hat{a}$ and $\hat{b}$ are the same if and only if $aK = bK$, i.e. $ab^{-1} \in K$. So the orbit category has morphisms

$$\mathrm{Hom}\,(\mathbb{T}/H, \mathbb{T}/K) = \begin{cases} \mathbb{T}/K & \text{if } H \subseteq K, \\ \emptyset & \text{otherwise.} \end{cases}$$

All the maps from $\mathbb{T}/H$ to $\mathbb{T}/K$ are homotopic, since $\mathbb{T}$ is connected; so the homotopy orbit category has exactly one morphism from $\mathbb{T}/H$ to $\mathbb{T}/K$ if $H \subseteq K$, and no other morphisms.

Now we consider the category of dual coefficient systems. We assume that all vector spaces are finitely generated and that all functors from $h\mathcal{O}_{\mathbb{T}}$ have only finitely many orbit types; this means that the functor A takes only finitely many different values, and that for all but finitely many subgroups $H \subset \mathbb{T}$ we have $A(\mathbb{T}/H) = A(\mathbb{T}/\mathbb{T})$, and the structure map $A(\mathbb{T}/H) \to A(\mathbb{T}/\mathbb{T})$ is the identity. We want to identify which functors are injective.

Proposition 8.37. *If A is a dual rational coefficient system for which the map $A(\mathbb{T}/H) \to \lim_{K \supset H} A(\mathbb{T}/K)$ induced by the structure maps of A is* surjective *for all H, then A is injective.*

Proof. Suppose we have morphisms of dual coefficient systems:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\alpha"'] & N \arrow[dl, dashed, "\beta"] \\ A & \end{tikzcd}$$

We will produce β by induction over subgroups, beginning at $\mathbb{T}/\mathbb{T}$. The finitely many orbit types condition ensures that there are only a finite number of subgroups we need to consider in the induction, and we can define β on the rest by using $\beta(\mathbb{T}/\mathbb{T})$.

At $\mathbb{T}/\mathbb{T}$, we have a diagram of rational vector spaces, and there is no problem producing the map. Now suppose that we have maps $\beta(\mathbb{T}/K) : N(\mathbb{T}/K) \to A(\mathbb{T}/K)$ for all $K \supset H$ which fill in the diagram and commute with the structure maps of the functors. By the univeral property these induce a map $\tilde{\beta} : N(\mathbb{T}/H) \to \lim_{K\supset H} A(\mathbb{T}/K)$. By assumption, as vector spaces

$$A(\mathbb{T}/H) \cong \lim_{K\supset H} A(\mathbb{T}/K) \oplus V,$$

where $V = \ker\{A(\mathbb{T}/H) \to \lim_{K\supset H} A(\mathbb{T}/K)\}$; so composing with projection gives a map $p\alpha : M(\mathbb{T}/H) \to V$, and we can produce a vector space map $\phi : N(\mathbb{T}/H) \to V$ extending $p\alpha$. Define

$$\beta(\mathbb{T}/H) = \tilde{\beta} \oplus \phi : N(\mathbb{T}/H) \to A(\mathbb{T}/H);$$

for this to be part of a natural transformation, it must commute with structure maps. But a structure map $A(\mathbb{T}/H) \to A(\mathbb{T}/K)$ factors through $\lim_{K\supset H} A(\mathbb{T}/K)$, so the choice of the splitting of $A(\mathbb{T}/H)$ and of the map $\tilde{\beta}$ is unimportant, and by examining the diagram

$$\begin{tikzcd}
N(\mathbb{T}/H) \arrow[rr] \arrow[d, "\beta"'] \arrow[dr, "\tilde{\beta}"] & & N(\mathbb{T}/K) \arrow[d, "\beta"] \\
A(\mathbb{T}/H) \arrow[r] & \lim_{K\supset H} A(\mathbb{T}/K) \arrow[r] & A(\mathbb{T}/K)
\end{tikzcd}$$

we see that β commutes with all structure maps between values of the functor for which it is defined. Inducting through all subgroups, we produce a natural transformation as required. $\square$

The vector space $V = \ker\{A(\mathbb{T}/H) \to \lim_{K\supset H} A(\mathbb{T}/K)\}$ is actually equal to $\bigcap_{K\supset H} \ker A(\hat{e}_{H,K})$, where $\hat{e}_{H,K} : \mathbb{T}/H \to \mathbb{T}/K$ is the projection map and $A(\hat{e}_{H,K})$ is the induced structure map of the functor A; this more concrete description will be useful for the analysis of injective systems given below.

Before continuing our analysis of the structure of injective systems, we show the following.

Corollary 8.38. *For any space X, the $\mathbb{T}$-system $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ is injective.*

Proof. Recall that $\underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}/H)$ is defined to be the de Rham-Alexander-Spanier algebra of $X^H \times_{\mathbb{T}} E\mathbb{T}$, that is, $\operatorname{colim}_{\mathcal{U}} \mathcal{E}(\overline{\mathcal{U}})$, where $\mathcal{U}$ ranges over all open coverings of $X^H \times_{\mathbb{T}} E\mathbb{T}$. For any $H \in \mathbb{T}$, the inclusions $X^K \subseteq \bigcup X^{K_i}$ for $K_i \subset H$ induce a map $\mathcal{A}(\bigcup X^{K_i} \times_{\mathbb{T}} E\mathbb{T}) \to \lim_{K\supset H} \underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}/K)$ by the universal property. This map is surjective because any open cover $\mathcal{U}$ of $(\bigcup X^{K_i} \times_{\mathbb{T}} E\mathbb{T})$ restricts to an open cover of each $X^{K_i} \times_{\mathbb{T}} E\mathbb{T}$. It is injective because any element in $\lim_{K\supset H} \underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}/K)$ is represented by open covers $\mathcal{U}_i$ of $X^{K_i} \times_{\mathbb{T}} E\mathbb{T}$ for $K_i \supset H$, and there are only a finite number of these so they have a common refinement which is an open cover of $\bigcup X^{K_i} \times_{\mathbb{T}} E\mathbb{T}$. Therefore this map is an isomorphism.

Any cover $\mathcal{U}_K$ of $\bigcup X^{K_i} \times_{\mathbb{T}} E\mathbb{T}$ can be extended to an open cover $\mathcal{U}_H$ of $X^H \times_{\mathbb{T}} E\mathbb{T}$, and so $\mathcal{E}(\overline{\mathcal{U}}_H) \to \mathcal{E}(\overline{\mathcal{U}}_K)$ is onto; taking a filtered colimit over all covers, we see that

$\mathcal{E}(\mathbb{T}/H) \to \mathcal{A}(\bigcup X^{K_i} \times_{\mathbb{T}} E\mathbb{T})$ is onto. But by the previous observation, this is $\lim_{K \supset H} \mathcal{E}(\mathbb{T}/H)$, and so $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ is injective. $\square$

Now we return to considering the structure of $\mathbb{T}$-systems. We construct the injective systems $\underline{V}_H$ generated by a vector space V at a given subgroup $H \subseteq \mathbb{T}$ by defining

$$\underline{V}_H(\mathbb{T}/K) = \begin{cases} V & \text{if } K \subseteq H, \\ 0 & \text{otherwise,} \end{cases}$$

with structure maps equal to either the identity or 0, as appropriate. These are precisely analogous to the injective systems $\underline{V}_H$ defined for finite groups, but here they are significantly simpler because the category under consideration has fewer internal structure maps. Once again, these are the basic building blocks for all injective systems.

Proposition 8.39. *A dual coefficient system A is injective if and only if it is of the form $A = \bigoplus_H \underline{V}_H$ for some collection of vector spaces V_H.*

Proof. First, suppose A is of the form described. Then $A(\mathbb{T}/H) = \bigoplus_{K \supseteq H} V_K$ and $\lim_{K \supset H} A(\mathbb{T}/K) = \bigoplus_{K \supset H} V_K$, so the map

$$A(\mathbb{T}/H) \to \lim_{K \supset H} A(\mathbb{T}/K)$$

is surjective and the system is injective by Proposition 8.37.

Conversely, suppose that A is injective. For each $H \subseteq \mathbb{T}$, we define the vector space

$$V_H = \bigcap_{K \supset H} \ker A(\hat{e}_{H,K}),$$

where we interpret this as meaning $V_{\mathbb{T}} = A(\mathbb{T}/\mathbb{T})$. The fact that A has only finitely many orbit types implies that $V_H = 0$ for all but a finite number of subgroups $H \subseteq \mathbb{T}$. We produce a morphism $f : A \to \bigoplus_H \underline{V}_H$ by induction on subgroups. We let $f(\mathbb{T}/\mathbb{T}) = id$ on $A(\mathbb{T}/\mathbb{T}) \cong \bigoplus \underline{V}_H(\mathbb{T}/\mathbb{T}) = V_{\mathbb{T}}$. Suppose $f(\mathbb{T}/K)$ is defined for $K \supset H$. These induce a map

$$\tilde{f} : A(\mathbb{T}/H) \to \lim_{K \supset H} \bigoplus \underline{V}_H(\mathbb{T}/K) = \bigoplus_{K \supset H} V_K.$$

Let $h : A(\mathbb{T}/H) \to \bigcap_{K \supset H} \ker A(\hat{e}_{H,K}) = V_H$ be the map induced by the structure maps, and define

$$f(\mathbb{T}/H) = \tilde{f} \oplus h : A(\mathbb{T}/H) \to \bigoplus_{K \supseteq H} V_K = \bigoplus_K \underline{V}_K(\mathbb{T}/H).$$

Then f is part of a natural transformation, making the appropriate diagrams commute.

We now show that f is an injection. At $\mathbb{T}/\mathbb{T}$ it is an isomorphism. Assume that f is injective for $K \supset H$ and suppose $a_H \in \ker f(\mathbb{T}/H)$. Since f is a natural transformation, $A(\hat{e}_{H,K}) : a_H \to a_K \in \ker f(\mathbb{T}/K)$ for any $K \supset H$, so, by the inductive assumption, $a_K = 0$. Therefore $a_H \in \bigcap_{K \supset H} \ker A(\hat{e}_{H,K})$, and f is injective on this intersection of kernels by construction; so $a_H = 0$. Thus f is injective on $A(\mathbb{T}/H)$, and, by induction, on all of A.

The fact that A is an injective system means that there is a splitting of f, $s : \bigoplus \underline{V}_H \to A$, such that $sf = id$. We show that s is also an injection. It is an isomorphism at $\mathbb{T}/\mathbb{T}$; assume s is injective for all $K \supset H$, and suppose $v \in \ker s(\mathbb{T}/H)$. As above, the fact that s is a natural transformation means that $v \in \bigcap_{K \supset H} \ker \bigoplus \underline{V}_H(\hat{e}_{H,K})$. Futhermore, s restricts to a map

$$\bigcap_{K \supset H} \ker \bigoplus \underline{V}_H(\hat{e}_{H,K}) = V_H \to \bigcap_{K \supset H} \ker A(\hat{e}_{H,K}).$$

Now f was defined so that it took $\bigcap_{K \supset H} \ker A(\hat{e}_{H,K})$ isomorphically onto V_H, so there is an $a \in \bigcap_{K \supset H} \ker A(\hat{e}_{H,K}) \subset A(\mathbb{T}/H)$ for which $f(a) = v$. Then $a = sf(a) = s(v) = 0$, so $f(a) = v = 0$ also.

Now we have $sf = id$, where both f and s are injections; therefore f is actually an isomorphism and $A \cong \bigoplus \underline{V}_H$. □

This explicit form of any injective dual rational coefficient system leads to several very useful observations.

Corollary 8.40. *If A is injective and $a_H \in A(\mathbb{T}/H)$, then for any $K \subset H$, there is always a lift $a_K \in A(\mathbb{T}/K)$ which maps to a_H under the structure map $A(\hat{e}_{H,K})$.*

Corollary 8.41. *Any coefficient system can be embedded in an injective system.*

Proof. We produce an injective by putting

$$V_H = \bigcap_{K \supset H} \ker A(\hat{e}_{H,K})$$

and defining $f : A \to \bigoplus \underline{V}_H$ as above; the proof that f is an injection does not use any assumptions about A. □

It is clear from the construction that the injective embedding has the same basic properties as were discussed in the case of finite groups. In particular, the inclusion $A \hookrightarrow I$ restricts to an isomorphism

$$\bigcap_{K \supset H} \ker A(\hat{e}_{H,K}) \cong \bigcap_{K \supset H} \ker I(\hat{e}_{H,K})$$

and is an isomorphism at $\mathbb{T}/\mathbb{T}$; furthermore, if $A(\mathbb{T}/K) = 0$ for all $K \supset H$, the inclusion $A(\mathbb{T}/K) \to I(\mathbb{T}/K)$ is an isomorphism for all $K \supseteq H$, including H. As before, these properties imply the following fact.

Corollary 8.42. *Any coefficient system has an injective resolution which is of finite length.*

Corollary 8.43. *If A and B are injective, then so is $A \otimes B$.*

Proof. Write $A = \bigoplus \underline{A}_H$ and $B = \bigoplus \underline{B}_H$ as in Proposition 8.39; then

$$A \otimes B = \bigoplus (\underline{A}_{H_1} \otimes \underline{B}_{H_2}).$$

Now

$$(\underline{A}_{H_1} \otimes \underline{B}_{H_2})(\mathbb{T}/K) = \begin{cases} A_{H_1} \otimes B_{H_2} & \text{if } K \subseteq H_1 \cap H_2, \\ 0 & \text{otherwise;} \end{cases}$$

so this is equal to $\underline{(A_{H_1} \otimes B_{H_2})}_{H_1 \cap H_2}$ and $A \otimes B$ is an injective system. □

9. Cohomology of functors

We now want to discuss cohomology in the various categories we have introduced. First, observe that a $\mathbb{T}$-system $\mathfrak{A}$ is also a system of DGAs by neglect of structure. In addition, we want to incorporate the extra structure provided by the sub-DGA $\mathfrak{A}_{\mathbb{T}}$. To do this, we define another structure associated to $\mathfrak{A}$.

Definition 9.44. Given a $\mathbb{T}$-system $\mathfrak{A}$, define the functor $\widetilde{\mathfrak{A}}$ by

$$\widetilde{\mathfrak{A}}(\mathbb{T}/H) = \{a \in \mathfrak{A}(\mathbb{T}/H) | a \to a_{\mathbb{T}} \in \mathfrak{A}_{\mathbb{T}}\},$$

the set of all elements which are taken to $\mathfrak{A}_{\mathbb{T}}$ by structure maps $\mathfrak{A}(\hat{e}_{H,\mathbb{T}})$.

Since $\mathfrak{A}_{\mathbb{T}}$ is a sub-DGA, so is $\widetilde{\mathfrak{A}}(\mathbb{T}/H)$; in fact, it is a system of DGAs, as shown by the following.

Proposition 9.45. *$\widetilde{\mathfrak{A}}$ is injective as a dual rational coefficient system.*

Proof. Applying Proposition 8.37, we check that $\widetilde{\mathfrak{A}}(\mathbb{T}/H) \to \lim_{K\supset H} \widetilde{\mathfrak{A}}(\mathbb{T}/K)$ is onto for any $H \subset \mathbb{T}$. If $x = (x_{K_1}, x_{K_2}, \dots) \in \lim_{K\supset H} \widetilde{\mathfrak{A}}(\mathbb{T}/K)$, then $x_K \to x_{\mathbb{T}} \in \mathfrak{A}_{\mathbb{T}}$ for each K. Since $\mathfrak{A}$ is injective, $\mathfrak{A}(\mathbb{T}/H) \to \lim_{K\supset H} \mathfrak{A}(\mathbb{T}/K)$ is onto, so there is an element $x_H \in \mathfrak{A}(\mathbb{T}/H)$ which maps to x; and $x_H \to x_{\mathbb{T}} \in \mathfrak{A}_{\mathbb{T}}$, since the structure map factors through x_K for $K \supset H$. This means that x_H is actually an element of $\widetilde{\mathfrak{A}}(\mathbb{T}/H)$. □

A morphism of $\mathbb{T}$-systems from $\mathfrak{A}$ to $\mathfrak{B}$ will induce a natural transformation from $\widetilde{\mathfrak{A}}$ to $\widetilde{\mathfrak{B}}$. We can thus consider a $\mathbb{T}$-system to be a system of $\mathbb{Q}[c]$-DGAs along with a distinguished sub-functor which is an ordinary system of DGAs.

Now we turn to the problem of the cohomology of systems of DGAs. There are two ways to define the cohomology of a functor. First we can take cohomology entrywise, and define the cohomology functor $\underline{H}^*(\mathfrak{A})(G/H) = H^*(\mathfrak{A}(G/H))$. All DGAs come with a basing map from a ground ring $R = \mathbb{Q}$ or $\mathbb{Q}[c]$, and all structure maps are based. So there is a copy of the constant functor $\underline{R}$ inside $\mathfrak{A}$ and also $\underline{H}^*(\mathfrak{A})$. This allows us to define the reduced cohomology functor as the graded vector space quotient

$$\widetilde{\underline{H}}^*(\mathfrak{A}) = \underline{H}^*(\mathfrak{A})/\underline{R}.$$

Note that since $\underline{R}$ is injective, there is a splitting of the inclusion, and since all structure maps respect the base map, we can make this splitting natural. Thus the reduced cohomology is the kernel of the splitting map, and $\underline{H}^*(\mathfrak{A}) \cong \widetilde{\underline{H}}^*(\mathfrak{A}) \oplus \underline{R}$ as systems of vector spaces.

If $\mathfrak{A}$ is a system of DGAs, the reduced cohomology will be taken with respect to $\mathbb{Q}$; if it is a $\mathbb{T}$-system, with respect to $\mathbb{Q}[c]$. Note that $\widetilde{\mathfrak{A}}$ does not have a $\mathbb{Q}[c]$-module structure on the entries, so when we take its reduced cohomology we mean as a system of DGAs. Since $\mathfrak{A}_{\mathbb{T}}$ generates the cohomology of $\mathfrak{A}(\mathbb{T}/\mathbb{T})$ as a $\mathbb{Q}[c]$-algebra,

$$\widetilde{H}^*(\mathfrak{A})(\mathbb{T}/\mathbb{T}) = (H^*(\mathfrak{A}_{\mathbb{T}}) \otimes \mathbb{Q}[c])/(\mathbb{Q} \otimes \mathbb{Q}[c]) = \widetilde{H}^*(\mathfrak{A}_{\mathbb{T}}) \otimes \mathbb{Q}[c].$$

The other way to approach cohomology is using homological algebra. We define the cohomology with respect to a dual coefficent system $\underline{V}$ by $H^*(\mathfrak{A}; \underline{V}) = H^*(\mathrm{Hom}(\underline{V}, \mathfrak{A}))$, where Hom means morphisms of dual coefficient systems. We relate the two definitions using a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(\underline{V}, \underline{H}^t(\mathfrak{A})) \Rightarrow H^{s+t}(\mathfrak{A}; \underline{V}). \tag{2}$$

This is constructed by taking a projective resolution $\underline{V}^*$ of $\underline{V}$ and considering the double complex $\mathrm{Hom}(\underline{V}^*, \mathfrak{A})$. Grading one way, we identify the E_2-term as given above. Grading the other way, the spectral sequence collapses and we can calculate $E_\infty = H^*(\mathfrak{A}; \underline{V})$. Note that $\mathfrak{A}$ must be injective as a dual coefficient system in order for this collapse to occur.

We will use this spectral sequence extensively; the first important implication is that a map $\mathfrak{A} \to \mathfrak{B}$ which induces an isomorphism on the cohomology functor $\underline{H}^*$ also induces an isomorphism of cohomology with respect to any dual coefficient system. We refer to such a map as a quasi-isomorphism.

Now we want to define relative cohomology by producing a cofibre object in the appropriate category. First note that we can factor any map $f : \mathfrak{A} \to \mathfrak{B}$ as $\mathfrak{A} \xrightarrow{\alpha} \mathfrak{A}' \xrightarrow{\beta} \mathfrak{B}$, where α is a quasi-isomorphism and β is surjective. To do this, we consider $\mathfrak{B}(G/H)$ as a graded $\mathbb{Q}(N(H)/H)$-module by neglect of structure (this is just a vector space for a $\mathbb{T}$-system), and we let $\underline{B}_H$ be the injective system it generates, defined in Sections 1 and 8.

Then $\Sigma\underline{B}_H$ is a copy of $\underline{B}_H$ shifted in degree by $+1$. Let $\mathbb{Q}(\underline{B}_H, \Sigma\underline{B}_H)$ denote the free acyclic system of DGAs generated by the vector space $\underline{B}_H \oplus \Sigma\underline{B}_H$, with differential $d(x) = \Sigma x$ for $x \in \underline{B}_H(G/K)$. Now define $\mathfrak{A}' = \mathfrak{A} \otimes (\bigotimes_H \mathbb{Q}(\underline{B}_H, \Sigma\underline{B}_H))$, and let α be the inclusion and β the map defined by f on $\mathfrak{A}$ and by $\beta(\underline{B}_H) = \underline{B}_H$, $\beta(\Sigma\underline{B}_H) = d\underline{B}_H$. As a system of vector spaces, $\mathfrak{A}'$ is injective by Proposition 7.36, since it is the tensor product of injective systems; thus it is a system of DGAs. Moreover, if $\mathfrak{A}$ is a $\mathbb{T}$-system then $\mathfrak{A}'$ is also a $\mathbb{T}$-system with $\mathfrak{A}'_{\mathbb{T}} = \mathfrak{A}_{\mathbb{T}} \otimes \mathbb{Q}(\tilde{B}, \Sigma\tilde{B})$ for $\tilde{B} = \mathfrak{B}_{\mathbb{T}}$; since we have tensored with acyclic algebras, $\mathfrak{A}'_{\mathbb{T}}$ generates the cohomology of $\mathfrak{A}'(\mathbb{T}/\mathbb{T})$. With this construction, the induced map of sub-DGAs $\tilde{\beta} : \widetilde{\mathfrak{A}}' \to \widetilde{\mathfrak{B}}$ is also onto.

Define $\mathfrak{R} = \ker \beta \oplus \underline{R} \subset \mathfrak{A}'$, so that $\mathfrak{R} \to \mathfrak{A}' \to \mathfrak{B}$ induces a long exact sequence on the cohomology functor $\widetilde{\underline{H}}^*$. The problem is that $\mathfrak{R}$ may not be injective; but we can produce an injective system $\mathfrak{I}$ and an inclusion $\mathfrak{R} \hookrightarrow \mathfrak{I}$ which is a quasi-isomorphism (following [FT]). To do this we define the enlargement at $H \subset G$ by

$$\mathfrak{R} \subset \mathfrak{R} \otimes_R (\mathbb{Q}(\underline{K}_H, \Sigma\underline{K}_H) \otimes R),$$

where $\underline{K}_H = \bigcap_{K \supset H} \ker \mathfrak{R}(\hat{e}_{H,K})$. Then we produce the injective system $\mathfrak{I}$ by successively taking enlargements at each subgroup. For a $\mathbb{T}$-system, this process doesn't change anything at $\mathbb{T}/\mathbb{T}$, so we can define $\mathfrak{I}_{\mathbb{T}} = \mathfrak{R}_{\mathbb{T}}$. We have simply added acyclic pieces, so we obtain a quasi-isomorphism $\mathfrak{R} \hookrightarrow \mathfrak{I}$, an extension of the inclusion to $\mathfrak{I} \to \mathfrak{A}'$ and a long exact sequence

$$\cdots \to \widetilde{\underline{H}}^n(\mathfrak{I}) \to \widetilde{\underline{H}}^n(\mathfrak{A}') \to \widetilde{\underline{H}}^n(\mathfrak{B}) \to \widetilde{\underline{H}}^{n+1}(\mathfrak{I}) \to \cdots,$$

and similarly for cohomology with coefficients ([FT]). Therefore we define the relative cohomology using this cofibre object, $\underline{H}^n(\mathfrak{A}, \mathfrak{B}) = \underline{H}^n(\mathfrak{I})$.

If $\mathfrak{A}$ is a $\mathbb{T}$-system, observe that by construction the sub-DGA $\widetilde{\mathfrak{I}}$ of $\mathfrak{I}$ has the form

$$\widetilde{\mathfrak{I}}(\mathbb{T}/H) = (\widetilde{\mathfrak{R}}(\mathbb{T}/H) \otimes \mathbb{Q}) \oplus (\mathfrak{R}(\mathbb{T}/H) \otimes \mathbb{Q}(V_H, \Sigma V_H)/\mathbb{Q})$$

for some vector space V_H. Therefore $\widetilde{\mathfrak{I}} \hookrightarrow \widetilde{\mathfrak{R}}$ is also a quasi-isomorphism, and we get a long exact sequence

$$\cdots \to \underline{\widetilde{H}}^n(\widetilde{\mathfrak{I}}) \to \underline{\widetilde{H}}^n(\widetilde{\mathfrak{A}}') \to \underline{\widetilde{H}}^n(\widetilde{\mathfrak{B}}) \to \underline{\widetilde{H}}^{n+1}(\widetilde{\mathfrak{I}}) \to \cdots,$$

and also for cohomology with coefficients. Thus we have produced a relative $\mathbb{T}$-system $\mathfrak{I}$ for which, in addition, $\widetilde{\mathfrak{I}}$ is a relative object for $\tilde{f} : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$.

10. Structure of $\mathbb{T}$-systems

We now study the relationship between $\mathfrak{A}$ and its sub-functor $\widetilde{\mathfrak{A}}$.

Theorem 10.46. *Suppose $f : \mathfrak{A} \to \mathfrak{B}$ is a map of $\mathbb{T}$-systems which is a quasi-isomorphism. Then the induced map $\tilde{f} : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$ is also a quasi-isomorphism.*

First we need a small observation.

Lemma 10.47. *If $\alpha = \mathfrak{A}(\hat{e}_{H,\mathbb{T}}) : \mathfrak{A}(\mathbb{T}/H) \to \mathfrak{A}(\mathbb{T}/\mathbb{T})$ is the structure map of the functor, the induced map $\bar{\alpha} : \mathfrak{A}(\mathbb{T}/H)/\widetilde{\mathfrak{A}}(\mathbb{T}/H) \to \mathfrak{A}(\mathbb{T}/\mathbb{T})/\mathfrak{A}_{\mathbb{T}}$ is an isomorphism.*

Proof. $\mathfrak{A}$ is injective as a dual coefficient system, so it has the form described in Proposition 8.39. In particular, the structure map $\alpha = \mathfrak{A}(\hat{e}_{H,\mathbb{T}}) : \mathfrak{A}(\mathbb{T}/H) \to \mathfrak{A}(\mathbb{T}/\mathbb{T})$ is surjective, so $\bar{\alpha}$ is surjective as well. The fact that $\widetilde{\mathfrak{A}}$ is defined to be exactly the inverse image of $\mathfrak{A}_{\mathbb{T}}$ under the structure maps implies that $\widetilde{\mathfrak{A}}$ is exactly $\ker \bar{\alpha}$, and so $\bar{\alpha}$ is also an injection. □

Proof of Theorem 10.46. $\tilde{f}^*$ is a natural transformation of the functor $\underline{H}^*$, and we need to show it is an isomorphism at each $\mathbb{T}/H$. Consider the following commutative diagram, where the indicated isomorphism comes from Lemma 10.47.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathfrak{A}}(\mathbb{T}/H) \arrow[r] \arrow[d, "\tilde{f}"] & \mathfrak{A}(\mathbb{T}/H) \arrow[r] \arrow[d, "f"] & \mathfrak{A}(\mathbb{T}/H)/\widetilde{\mathfrak{A}}(\mathbb{T}/H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\mathfrak{B}}(\mathbb{T}/H) \arrow[r] \arrow[d] & \mathfrak{B}(\mathbb{T}/H) \arrow[r] \arrow[d] & \mathfrak{B}(\mathbb{T}/H)/\widetilde{\mathfrak{B}}(\mathbb{T}/H) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathfrak{B}_{\mathbb{T}} \arrow[r] & \mathfrak{B}(\mathbb{T}/\mathbb{T}) \arrow[r] & \mathfrak{B}(\mathbb{T}/\mathbb{T})/\mathfrak{B}_{\mathbb{T}} \arrow[r] & 0
\end{tikzcd}$$

By definition, the maps f and $\tilde{f}$ are natural transformations, so they commute with structure maps and the vertical composite maps are the same as the vertical composites in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathfrak{A}}(\mathbb{T}/H) \arrow[r] \arrow[d] & \mathfrak{A}(\mathbb{T}/H) \arrow[r] \arrow[d] & \mathfrak{A}(\mathbb{T}/H)/\widetilde{\mathfrak{A}}(\mathbb{T}/H) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathfrak{A}_{\mathbb{T}} \arrow[r] \arrow[d, "\tilde{f}"] & \mathfrak{A}(\mathbb{T}/\mathbb{T}) \arrow[r] \arrow[d, "f"] & \mathfrak{A}(\mathbb{T}/\mathbb{T})/\mathfrak{A}_{\mathbb{T}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{B}_{\mathbb{T}} \arrow[r] & \mathfrak{B}(\mathbb{T}/\mathbb{T}) \arrow[r] & \mathfrak{B}(\mathbb{T}/\mathbb{T})/\mathfrak{B}_{\mathbb{T}} \arrow[r] & 0
\end{tikzcd}$$

We are given that $f(\mathbb{T}/\mathbb{T})$ induces an isomorphism on cohomology; since $\mathfrak{A}_{\mathbb{T}}$ generates the cohomology of $\mathfrak{A}(\mathbb{T}/\mathbb{T})$ for any $\mathbb{T}$-system, $\tilde{f}^* : H^*(\mathfrak{A}_{\mathbb{T}}) \to H^*(\mathfrak{B}_{\mathbb{T}})$ must also be an isomorphism. Applying the 5-lemma to the long exact sequences of the cohomology of the bottom two rows, we see that the map $\mathfrak{A}(\mathbb{T}/\mathbb{T})/\mathfrak{A}_{\mathbb{T}} \to \mathfrak{B}(\mathbb{T}/\mathbb{T})/\mathfrak{B}_{\mathbb{T}}$ is also a quasi-isomorphism; so the right vertical composite is also, and this is equal to the right vertical composite in the first diagram. Thus the map $\mathfrak{A}(\mathbb{T}/H)/\widetilde{\mathfrak{A}}(\mathbb{T}/H) \to \mathfrak{B}(\mathbb{T}/H)/\widetilde{\mathfrak{B}}(\mathbb{T}/H)$ is a quasi-isomorphism, and the long exact sequences associated to the first two rows of the first diagram imply that $\tilde{f}^*(\mathbb{T}/H) : H^*(\mathfrak{A}(\mathbb{T}/H)) \to H^*(\mathfrak{B}(\mathbb{T}/H))$ is an isomorphism for any $H \subset \mathbb{T}$. □

We remind the reader that the reduced cohomology of a $\mathbb{T}$-system $\mathfrak{A}$ is defined with respect to the ground ring $\mathbb{Q}[c]$, while the reduced cohomology of $\widetilde{\mathfrak{A}}$ is defined with respect to $\mathbb{Q}$, since it is only a system of DGAs.

Theorem 10.48. *If $\mathfrak{A}$ is a $\mathbb{T}$-system such that $\underline{\widetilde{H}}^i(\mathfrak{A}) = 0$ for $i \leq n$, then $\underline{\widetilde{H}}^i(\widetilde{\mathfrak{A}}) = 0$ for $i \leq n$ and the map $\underline{\widetilde{H}}^{n+1}(\widetilde{\mathfrak{A}}) \to \underline{\widetilde{H}}^{n+1}(\mathfrak{A})$ induced by the inclusion is an isomorphism.*

Proof. First we show that the map $i^* : \underline{\widetilde{H}}^i(\widetilde{\mathfrak{A}}) \to \underline{\widetilde{H}}^i(\mathfrak{A})$ is an injection for $i \leq n+1$. Suppose $[a] \in H^i(\widetilde{\mathfrak{A}}(\mathbb{T}/H))$ is in the kernel. Then by naturality, the image $[a_{\mathbb{T}}] \in \widetilde{H}^i(\widetilde{\mathfrak{A}}(\mathbb{T}/\mathbb{T}))$ of $[a]$ under the structure map is also in the kernel. But at $\mathbb{T}/\mathbb{T}$, the induced map

$$\widetilde{H}^i(\widetilde{\mathfrak{A}}(\mathbb{T}/\mathbb{T})) = \widetilde{H}^i(\mathfrak{A}_{\mathbb{T}}) \hookrightarrow \widetilde{H}^i(\mathfrak{A}(\mathbb{T}/\mathbb{T})) = (\widetilde{H}^*(\mathfrak{A}_{\mathbb{T}}) \otimes \mathbb{Q}[c])^i$$

is an injection, so $[a_{\mathbb{T}}] = 0$ and there is a $z_{\mathbb{T}} \in \mathfrak{A}_{\mathbb{T}}$ such that $dz_{\mathbb{T}} = a_{\mathbb{T}}$. Because $\widetilde{\mathfrak{A}}$ is injective, we may lift $z_{\mathbb{T}}$ to $z \in \widetilde{\mathfrak{A}}(\mathbb{T}/H)$. Then $[a] = [a - dz]$, and replacing a with $a - dz$, we may assume that $a_{\mathbb{T}} = 0$.

Now $[a] \in \ker i^*$ means that a is a coboundary in $\mathfrak{A}(\mathbb{T}/H)$. Let $b \in \mathfrak{A}(\mathbb{T}/H)$ be such that $db = a$; then $db_{\mathbb{T}} = a_{\mathbb{T}} = 0$ so $b_{\mathbb{T}}$ represents a cohomology class $[b_{\mathbb{T}}] \in H^{i-1}(\mathfrak{A}(\mathbb{T}/\mathbb{T}))$. But if $i \leq n+1$, then $\widetilde{H}^{i-1}(\mathfrak{A}(\mathbb{T}/\mathbb{T}))$ is zero and $[b_{\mathbb{T}}] = r[c^{i-1}]$ for some $r \in \mathbb{Q}$. Let $w_{\mathbb{T}} \in \mathfrak{A}(\mathbb{T}/\mathbb{T})$ be such that $dw_{\mathbb{T}} = b_{\mathbb{T}} - rc^{i-1}$, and lift $w_{\mathbb{T}}$ to $w \in \mathfrak{A}(\mathbb{T}/H)$. Then $d(b - dw - rc^{i-1}) = db - 0 - 0 = a$ and $(b - dw - rc^{i-1}) \to (b_{\mathbb{T}} - dw_{\mathbb{T}} - rc^{i-1}) = 0$ under the structure map, so $(b - dw - rc^{i-1}) \in \widetilde{\mathfrak{A}}(\mathbb{T}/H)$. Therefore $[a] = 0 \in H^*(\widetilde{\mathfrak{A}}(\mathbb{T}/H))$, and i^* is an injection as claimed.

Next, we must show that $\underline{\widetilde{H}}^{n+1}(\widetilde{\mathfrak{A}}) \to \underline{\widetilde{H}}^{n+1}(\mathfrak{A})$ is a surjection. Let $[a] \in \widetilde{H}^{n+1}(\mathfrak{A}(\mathbb{T}/H))$ and let $[a_{\mathbb{T}}]$ be the image of $[a]$ under the structure map; then

$$[a_{\mathbb{T}}] \in \widetilde{H}^{n+1}(\mathfrak{A}(\mathbb{T}/\mathbb{T})) = (\widetilde{H}^*(\mathfrak{A}_{\mathbb{T}}) \otimes \mathbb{Q}[c])^{n+1},$$

so $[a_{\mathbb{T}}] = \sum_j [a_j] \otimes c^j$. Let $z_{\mathbb{T}} \in \mathfrak{A}(\mathbb{T}/\mathbb{T})$ be such that $a_{\mathbb{T}} - dz_{\mathbb{T}} = \sum_j a_j \otimes c^j$, where $a_j \in \mathfrak{A}_{\mathbb{T}}$. Lifting $z_{\mathbb{T}}$ to $z \in \widetilde{\mathfrak{A}}(\mathbb{T}/H)$ and replacing a with $a - dz$, we may assume that $a_{\mathbb{T}} = \sum_j a_j \otimes c^j$. Each $[a_j] \in \widetilde{H}^{n+1-2j}(\mathfrak{A}_{\mathbb{T}})$; if $j > 0$, these groups are all zero and there exist $w_j \in \mathfrak{A}_{\mathbb{T}}$ such that $dw_j = a_j$. Lifting $\sum_{j>0} w_j \otimes c^j$ to $w \in \widetilde{\mathfrak{A}}(\mathbb{T}/H)$, we find that $[a] = [a - dw]$, where $a - dw \to a_0$ under the structure maps. But $a_0 \in \mathfrak{A}_{\mathbb{T}}$, and this means that $a - dw \in \widetilde{\mathfrak{A}}(\mathbb{T}/H)$. Thus $[a] \in \operatorname{im} \widetilde{H}^{n+1}(\widetilde{\mathfrak{A}}(\mathbb{T}/H))$. □

Corollary 10.49. *If $\underline{\widetilde{H}}^i(\mathfrak{A}) = 0$ for all i, then $\underline{\widetilde{H}}^i(\widetilde{\mathfrak{A}}) = 0$ for all i also.*

We refer to a $\mathbb{T}$-system with the above property as *acyclic*.

Corollary 10.50. *If $f : \mathfrak{A} \to \mathfrak{B}$ induces $f^* : \underline{H}^i(\mathfrak{A}) \to \underline{H}^i(\mathfrak{B})$ such that f^* is an isomorphism for $i \leq n-1$, then $\tilde{f}^* : \underline{H}^i(\widetilde{\mathfrak{A}}) \to \underline{H}^i(\widetilde{\mathfrak{B}})$ is also an isomorphism for $i \leq n-1$. Furthermore,*

$$\ker\{\underline{\widetilde{H}}^n(\mathfrak{A}) \to \underline{\widetilde{H}}^n(\mathfrak{B})\} \cong \ker\{\underline{\widetilde{H}}^n(\widetilde{\mathfrak{A}}) \to \underline{\widetilde{H}}^n(\widetilde{\mathfrak{B}})\};$$

and if f^* is an injection for $i = n$, then $\tilde{f}^*$ is also, and

$$\operatorname{coker}\{\underline{\widetilde{H}}^n(\mathfrak{A}) \to \underline{\widetilde{H}}^n(\mathfrak{B})\} \cong \operatorname{coker}\{\underline{\widetilde{H}}^n(\widetilde{\mathfrak{A}}) \to \underline{\widetilde{H}}^n(\widetilde{\mathfrak{B}})\}.$$

Proof. Produce the relative $\mathbb{T}$-system of $f : \mathfrak{A} \to \mathfrak{B}$ and examine the long exact sequences of cohomology that result; Theorem 10.48 applied to the relative $\mathbb{T}$-system gives the desired isomorphisms. □

11. Elementary extensions

Here we give the promised definition of an elementary extension which was used to define minimal models, and discuss its properties.

Given a system of DGAs $\mathfrak{A}$, a system of vector spaces $\underline{V}$ of degree n, and a map $\alpha : \underline{V} \to \underline{Z}^{n+1}(\mathfrak{A})$, we construct the *elementary extension* $\mathfrak{A}(\underline{V})$ of $\mathfrak{A}$ with respect to α as follows. Let $\underline{V} \hookrightarrow \underline{V}_0 \xrightarrow{w_0} \underline{V}_1 \xrightarrow{w_1} \cdots$ be the injective resolution of $\underline{V}$ constructed by taking $\underline{V}_i$ to be the injective embedding of coker w_{i-1} constructed in Proposition 7.34; note that it is of finite length. We construct a commutative diagram

$$\begin{tikzcd}
\underline{V} \arrow[r, hook] \arrow[d, "\alpha"] & \underline{V}_0 \arrow[r, "w_0"] \arrow[d, "\alpha_0"] & \underline{V}_1 \arrow[r, "w_1"] \arrow[d, "\alpha_1"] & \underline{V}_2 \arrow[r] \arrow[d, "\alpha_2"] & \cdots \\
\underline{Z}^{n+1}(\mathfrak{A}) \arrow[r, hook] & \mathfrak{A}^{n+1} \arrow[r, "d"] & \mathfrak{A}^{n+2} \arrow[r, "d"] & \mathfrak{A}^{n+3} \arrow[r] & \cdots
\end{tikzcd}$$

We produce the maps α_i inductively by observing that $d\alpha_i w_{i-1} = dd\alpha_{i-1} = 0$, so $d\alpha_i \mid_{\text{im } w_{i-1}} = 0$ and by the injectivity of $\mathfrak{A}$ we can fill in

$$\begin{tikzcd}
\underline{V}_i/\text{im } w_{i-1} \arrow[r, hook] \arrow[d, "d\alpha_i"] & \underline{V}_{i+1} \arrow[dl, dashed, "\alpha_{i+1}"] \\
\mathfrak{A}^{n+i+1} &
\end{tikzcd}$$

Define $\mathfrak{A}(\underline{V}) = \mathfrak{A} \otimes (\bigotimes_i \mathbb{Q}(\underline{V}_i))$, where $\mathbb{Q}(\underline{V}_i)$ is the free graded commutative algebra generated at G/H by the vector space $\underline{V}_i(G/H)$ in degree $n+i$; the differential is defined on $\mathfrak{A}$ by the original differential of $\mathfrak{A}$, and on generators of $\underline{V}_i$ by $d = (-1)^i \alpha_i + w_i$. Since all the $\underline{V}_i$ are injective by construction, as a vector space the system is the tensor product of injectives, and therefore is injective by Proposition 7.36. Thus $\mathfrak{A}(\underline{V})$ is a new system of DGAs.

For $\mathbb{T}$-systems, we modify this construction only slightly. The key observation is that if $\mathbb{Q}(\underline{V}) \otimes \mathbb{Q}[c]$ is the free $\mathbb{T}$-system generated by the system of graded vector spaces $\underline{V}$ with distinguished sub-DGA $\mathbb{Q}(V(\mathbb{T}/\mathbb{T}))$, then

$$\mathrm{Hom}_{\mathbb{T}\text{-sys}}(\mathbb{Q}(\underline{V}) \otimes \mathbb{Q}[c], \mathfrak{A}) = \mathrm{Hom}(\underline{V}, \widetilde{\mathfrak{A}}).$$

Therefore it is $\widetilde{\mathfrak{A}}$ which is relevant. To obtain an elementary extension of $\mathbb{T}$-systems, we start with a map $\alpha : \underline{V} \to \underline{Z}^{n+1}(\widetilde{\mathfrak{A}})$, and extend so that all α_i also land in $\widetilde{\mathfrak{A}} \subset \mathfrak{A}$.

Since $\widetilde{\mathfrak{A}}$ is injective, there is no difficulty doing this. Define the distinguished sub-DGA by $\mathfrak{A}(V)_{\mathbb{T}} = \mathfrak{A}_{\mathbb{T}} \otimes \mathbb{Q}(\underline{V}_i(\mathbb{T}/\mathbb{T}))$; this is closed under the differential since $\alpha_i(\mathbb{T}/\mathbb{T})$ lands in $\widetilde{\mathfrak{A}}(\mathbb{T}/\mathbb{T}) = \mathfrak{A}_{\mathbb{T}}$, and it generates the cohomology over $\mathbb{Q}[c]$. Thus $\mathfrak{A}(\underline{V})$ is a $\mathbb{T}$-system.

Note that the differential of both kinds of elementary extensions is defined by the map α on $\underline{V}$ itself, and the cohomology of $\mathfrak{A}(\underline{V})$ is equal to that of $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$ under this differential. We can also consider the relative cohomology of the inclusion $\mathfrak{A} \hookrightarrow \mathfrak{A}(\underline{V})$ and observe the following.

Lemma 11.51. *There is an isomorphism of dual coefficient systems*

$$\underline{H}^n(\mathfrak{A}(\underline{V}), \mathfrak{A}) \cong \underline{V}.$$

Proof. If $[(a,b)] \in \underline{H}^n(\mathfrak{A}(\underline{V}), \mathfrak{A})$, this cohomology class is represented by a pair $(a,b) \in \mathfrak{A}^{n+1} \times \mathfrak{A}(\underline{V})^n$ such that $a = db$, since $d(a,b) = (da, a - db) = 0$. Now $b \in \mathfrak{A}(\underline{V})^n = \mathfrak{A}^n \oplus \underline{V}_0$ must have the form $b = b' + x$ for some $b' \in \mathfrak{A}$ and $x \in \underline{V}_0$. Moreover, since $d(x) = \alpha_0(x) + w_0(x)$ with $w_0(x) \in \underline{V}_1$, if $db = a \in \mathfrak{A}$ then we must have $w_0(x) = 0$ and therefore $x \in \underline{V}$.

Now we want to show that the map $\underline{H}^n(\mathfrak{A}(\underline{V}), \mathfrak{A}) \to \underline{V}$ given by the $\underline{V}$-component of b, $[(a, b' + x)] \to x$, is well-defined. If we pick a different representative $(\tilde{a}, \tilde{b})$ for $[(a,b)]$, then there is an element $(f,g) \in \mathfrak{A}^n \times \mathfrak{A}(\underline{V})^{n-1}$ for which

$$d(f,g) = (df, f - dg) = (a - \tilde{a}, b - \tilde{b}) = (a,b) - (\tilde{a}, \tilde{b}).$$

But $\mathfrak{A}(\underline{V})^{n-1} = \mathfrak{A}^{n-1}$ since the elementary extension does not change the system in degrees less that n, so in fact $g \in \mathfrak{A}$. Thus $b - \tilde{b} = f - dg \in \mathfrak{A}$, and the $\underline{V}$-components of b and $\tilde{b}$ must be equal. The map $[(a, b' + x)] \to x$ is onto since any element $x \in \underline{V}$ gives a relative cohomology class $[(dx, x)] \in \underline{H}^n(\mathfrak{A}(\underline{V}), \mathfrak{A})$; it is injective since if $[(a,b)]$ is a cohomology class with $x = 0$, then $b \in \mathfrak{A}$ and $(b, 0) \in \mathfrak{A}^n \times \mathfrak{A}(\underline{V})^{n-1}$ satisfies $d(b,0) = (db, b) = (a,b)$. This gives the claimed isomorphism. □

Suppose $f : \mathfrak{A} \to \mathfrak{B}$ is a map of systems of DGAs and $\mathfrak{A}(\underline{V})$ is an elementary extension with respect to some α. If $f' : \underline{V} \to \mathfrak{B}$ satisfies $f\alpha = df'$, we can extend f to $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$ using f' on $\underline{V}$; the condition on f' ensures that this map respects the differential. We then extend the map to $\underline{V}_0$ by the injectivity of $\mathfrak{B}$, and define it on the rest of the resolution inductively. Given f' on $\underline{V}_i$, we must define a map $f' : \underline{V}_{i+1} \to \mathfrak{B}$ such that $f'd = df'$. To ensure this is satisfied, we consider the differential coming from $\underline{V}_i$ and landing in $\underline{V}_{i+1}$; on $\underline{V}_i$ we have $d = (-1)^i \alpha_i + w_i$, and we need a map f' such that

$$f'((-1)^i \alpha_i + w_i) = (-1)^i f' \alpha_i + f' w_i = df',$$

or equivalently, $(-1)^i f' \alpha_i - df' = f w_i$. Observe that since

$$\begin{aligned} ((-1)^i f' \alpha_i - df) w_{i-1} &= (-1)^i f \alpha_i w_{i-1} - df w_{i-1} \\ &= (-1)^i f d \alpha_{i-1} - f d w_{i-1} \\ &= f d((-1)^i \alpha_{i-1} - w_{i-1}) \\ &= f d(-d) = 0, \end{aligned}$$

the map $(-1)^i f'\alpha - df$ vanishes on im $(w_{i-1}) \subset \underline{V}_i$, and we have

$$\begin{tikzcd} \underline{V}_i/\mathrm{im}\ w_{i-1} \arrow[r, hook] \arrow[d, "(-1)^i f\alpha_i - df"'] & \underline{V}_{i+1} \arrow[dl, dashed, "f'"] \\ \mathfrak{B}^{n+i+1} & \end{tikzcd}$$

and can define f' as indicated by the injectivity of $\mathfrak{B}$. Continuing in this fashion, we extend f' to all generators and thus to a DGA map on all of $\mathfrak{A}(\underline{V})$.

If $\mathfrak{A}$ if a $\mathbb{T}$-system and f' lands in $\widetilde{\mathfrak{A}}$, we carry out exactly the same process, taking all maps of $\underline{V}_i$ to land in $\widetilde{\mathfrak{A}}$. Then the induced map takes generators $\underline{V}_i(\mathbb{T}/\mathbb{T})$ of $\mathfrak{A}(\underline{V})_{\mathbb{T}}$ to $\mathfrak{B}_{\mathbb{T}}$, and that $f : \mathfrak{A}(\underline{V}) \to \mathfrak{B}$ is a map of $\mathbb{T}$-systems.

It is important to keep in mind that although the resolution $\underline{V}_i$ is necessary to ensure that the elementary extension remains injective, it is not otherwise significant. We have observed, for example, that the cohomology of the extension $\mathfrak{A}(\underline{V})$ is just that of $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$. Moreover, we have seen that any map defined on $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$ can be extended over the entire elementary extension. The choice of maps made in creating the elementary extension is unimportant; all that matters is the structure on $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$.

Proposition 11.52. *If $[\alpha] = [\alpha'] \in H^{n+1}(\mathfrak{A}; \underline{V})$, then the elementary extensions of systems of DGAs defined by the maps α and α' are isomorphic. For $\mathbb{T}$-systems, the same thing holds if $[\alpha] = [\alpha'] \in H^{n+1}(\widetilde{\mathfrak{A}}; \underline{V})$.*

To prove this, we show that a map of elementary extensions which is an isomorphism on $\mathfrak{A} \otimes \mathbb{Q}(\underline{V})$ is actually an isomorphism on the whole extension.

Lemma 11.53. *Suppose $f : \mathfrak{A}_\alpha(\underline{V}) \to \mathfrak{A}_{\alpha'}(\underline{V}')$ is a map between two degree n elementary extensions of $\mathfrak{A}$ with the following properties*:

1. *f restricts to an isomorphism of $\mathfrak{A}$.*
2. *On $\underline{V}$, $f(x) = g(x) + a(x)$, where $g : \underline{V} \to \underline{V}'$ is an isomorphism and $a(x) \in \mathfrak{A}$.*

Then f is an isomorphism.

Proof. First observe that $\mathfrak{A}_\alpha(\underline{V})$ and $\mathfrak{A}_{\alpha'}(\underline{V}')$ are isomorphic as graded vector spaces, since the injective resolution used in defining the elementary extension is determined by $\underline{V} \cong \underline{V}'$. Recall that if $\underline{V}$ has degree n, then $\underline{V}_i$ has degree $n+i$ and $\underline{V}'_i \hookrightarrow \mathfrak{A}_{\alpha'}(\underline{V}')^{i+n}$; by injectivity, there is a splitting

$$\mathfrak{A}_{\alpha'}(\underline{V}')^{i+n} = \underline{V}'_i \oplus (\mathfrak{A} \otimes_{j<i} \mathbb{Q}(\underline{V}_j)')^{i+n}.$$

Define $g_i : \underline{V}_i \to \underline{V}'_i$ to be $f\mid_{\underline{V}_i}$ composed with projection onto $\underline{V}'_i$. Observe that if $x \in V_i(G/H)$ then

$$df(x) = d(g_i(x) + \zeta) = w_i g_i(x) \pm \alpha g_i(x) + d\zeta,$$

where ζ and thus also $\pm\alpha g_i(x) + d\zeta$ lie in $\mathfrak{A} \otimes_{j<i} \mathbb{Q}(\underline{V}'_j)$. On the other hand,

$$fd(x) = f(w_i(x) \pm \alpha(x)) = g_{i+1} w_i(x) + \zeta' \pm f\alpha(x).$$

Since $fd = df$, equating the terms lying in $\underline{V}'_{i+1}$ shows that $g_{i+1} w_i = w_i g_i$. In particular, g_i takes $\mathrm{im}(w_i)$ to $\mathrm{im}(w_i)$.

Now we show that g_i is an isomorphism on the entire resolution, by induction on i. Recall that $g_0 = g$ is an isomorphism on $\underline{V} \subset \underline{V}_0$ by assumption. $\underline{V}_0$ is the injective envelope of $\underline{V}$, so $\underline{V}(G/G) = \underline{V}_0(G/G)$ and $g_0(G/G)$ is an isomorphism. Assume that $g_0(G/K)$ is an isomorphism for all $K \supset H$. Let $x_H \in \ker g_0(G/H)$; since g_0 is a natural transformation, the structure maps take x_H to $x_K \in \ker g_0(G/K)$ for any $K \supset H$. By the inductive assumption, $g_0(G/K)$ is injective, so $x_K = 0$ and $x_H \in \bigcap_{K \supset H} \ker \underline{V}_0(\hat{e}_{H,K})$. But on this intersection of kernels, $\underline{V} \hookrightarrow \underline{V}_0$ is an isomorphism by the construction of the injective envelope. So $x_H \in \underline{V}$, where $g_0 = g$ is an isomorphism. Therefore $x_H = 0$, and $g_0(G/H)$ is an injection and therefore an isomorphism. Inducting through subgroups, we see that g_0 is an isomorphism on $\underline{V}_0(G/H)$ for all $H \subset G$.

Now suppose that g_j is an isomorphism for all $j < i$, and suppose $x \in \ker(g_i) : \underline{V}_i \to \underline{V}'_i$. If $x \in \operatorname{im}(w_{i-1})$, then

$$0 = g_i(x) = g_i(w_{i-1}(y)) = w_{i-1}g_{i-1}(y);$$

so $g_{i-1}(y) \in \ker(w_{i-1}) = \operatorname{im}(w_{i-2})$. But on $\underline{V}_{i-1}$, g_{i-1} is an isomorphism which takes $\operatorname{im}(w_{i-2})$ to itself; this implies that $y \in \operatorname{im}(w_{i-2})$ and

$$x = w_{i-1}(y) = w_{i-1}w_{i-2}(z) = 0.$$

So g_i is injective on $\operatorname{im}(w_{i-1})$. Then $\underline{V}_i$ is the injective envelope of this image, and the same argument used for g_0 implies that g_i is injective on all of $\underline{V}_i$. Thus g_i is an isomorphism.

We have shown that the map f has the form $g_i + a$ on $\underline{V}_i$, where g is an isomorphism from $\underline{V}_i$ to $\underline{V}'_i$ and a lands in $\mathfrak{A} \otimes_{j<i} \mathbb{Q}(\underline{V}'_j)$. Since the $\underline{V}'_i$ are free generators in $\mathfrak{A}(\underline{V}'')$, f must be an injection on all of $\mathfrak{A}_\alpha(\underline{V})$. So f is an isomorphism. $\square$

Proof of Proposition 11.52. If $[\alpha] = [\alpha']$, there is a map $\beta : \underline{V} \to \mathfrak{A}^n$ such that $d\beta = \alpha - \alpha'$. Then for $x \in \underline{V}$, $\alpha(x) = \alpha'(x) + d\beta(x) = d(x + \beta(x))$. So if $i : \mathfrak{A} \to \mathfrak{A}$ is the identity map, α satisfies the relationship $i\alpha = d(i + \beta)$. As discussed, this means that we can define a map $f : \mathfrak{A}_\alpha(\underline{V}) \to \mathfrak{A}_{\alpha'}(\underline{V})$ by the identity on $\mathfrak{A}$ and $id + \beta$ on $\underline{V}$, and extend f over the entire elementary extension. This map has the properties required by Lemma 11.53, since β lands in $\mathfrak{A}$. So f is an isomorphism between the elementary extensions. $\square$

Corollary 11.54. *If $\mathfrak{V} = \bigotimes_i \mathbb{Q}(\underline{V}_i)$ is the free system of DGAs or the $\mathbb{T}$-system generated by the injective resolution of $\underline{V}$, and $\alpha, \alpha' : \mathfrak{V} \to \mathfrak{A}$ are homotopic, then there is an isomorphism $\mathfrak{A}_\alpha(\underline{V}) \cong \mathfrak{A}_{\alpha'}(\underline{V})$ between the elementary extensions induced by restricting α, α' to $\underline{V}$.*

Proof. The homotopy $H : \mathfrak{V} \to \mathfrak{A}(t, dt)$ gives a cohomology class

$$[H] \in H^n(\mathfrak{A}(t, dt), \underline{V})$$

such that $[\alpha] = p_0^*[H]$ and $[\alpha'] = p_1^*[H]$ in $H^n(\mathfrak{A}, \underline{V})$. But p_0 and p_1 are quasi-isomorphisms, so $[\alpha] = [\alpha']$, and, by Proposition 11.52, we can produce an isomorphism between the elementary extensions defined by α and α'. $\square$

There is an alternate way of defining elementary extensions. Let $\underline{V}$ be a system of vector spaces and $\mathfrak{V} = \bigotimes_i (\underline{V}_i) \otimes R$ the free injective system of DGAs or the $\mathbb{T}$-system it generates, where $R = \mathbb{Q}$ or $\mathbb{Q}[c]$ as appropriate and $\underline{V} \hookrightarrow \underline{V}_0 \xrightarrow{v_0} \underline{V}_1 \xrightarrow{v_1} \cdots$ is the injective resolution of $\underline{V}$. Associated to $\mathfrak{V}$ we construct an acyclic system of DGAs $\mathfrak{W}$. Define

$$\mathfrak{W} = \bigotimes_i \mathbb{Q}(\Sigma^{-1}\underline{V}_i) \bigotimes_i \mathbb{Q}(\underline{V}_i) \otimes R.$$

Let $d = v_i$ on generators of $\mathfrak{V}$, and $d = v_i + (-1)^i \sigma$ on generators of $\Sigma^{-1}\mathfrak{V}$, where $\sigma(\Sigma^{-1}x) = x$. For a $\mathbb{T}$-system, we define

$$\mathfrak{W}_{\mathbb{T}} = \bigotimes_i \mathbb{Q}(\Sigma^{-1}\underline{V}_i(\mathbb{T}/\mathbb{T})) \bigotimes_i \mathbb{Q}(\underline{V}_i(\mathbb{T}/\mathbb{T})) \otimes \mathbb{Q}.$$

If $\alpha : \mathfrak{V} \to \mathfrak{B}$ is a map of systems of DGAs, then it restricts to a map $\underline{V} \otimes \mathbb{Q} \subset \mathfrak{V} \to \mathfrak{B}$. Since $d = 0$ on $\underline{V}$, $\alpha|_{\underline{V}}$ lands in $Z^{n+1}(\mathfrak{B})$. Now consider the system $\mathfrak{B} \otimes_{\mathfrak{V}} \mathfrak{W}$. As a vector space, this is just $\mathfrak{B} \otimes (\bigotimes_i \mathbb{Q}(\Sigma^{-1}\underline{V}_i))$. The differential is given by $d_{\mathfrak{B}}$ on $\mathfrak{B}$ and by the composite $\Sigma^{-1}\underline{V} \xrightarrow{\sigma} \underline{V} \xrightarrow{\alpha} \mathfrak{B}$ on $\Sigma^{-1}\underline{V}$. Thus $\mathfrak{B} \otimes_{\mathfrak{V}} \mathfrak{W}$ is exactly an elementary extension $\mathfrak{B}(\Sigma^{-1}\underline{V})$ with respect to the vector space map $\alpha\,|_{\underline{V}}$; the algebraic pushout $\mathfrak{B} \otimes_{\mathfrak{V}} \mathfrak{W}$ gives an elementary extension.

12. Obstruction theory

The obstruction theory which will be used to set up the theory of minimal models is based on elementary extensions and is not changed. Here's a very brief sketch, which is taken from Fine's unpublished thesis ([F]).

Lemma 12.55 (Fine)**.** *Suppose we have maps of systems of DGAs*

$$\begin{tikzcd} \mathfrak{N} \arrow[r, "g"] \arrow[d, hook] & \mathfrak{A} \arrow[d, "\phi"] \\ \mathfrak{N}(\underline{V}) \arrow[r, "f"] & \mathfrak{B} \end{tikzcd}$$

with $f|_{\mathfrak{N}} \simeq \phi g$. Then there is an extension $\hat{g} : \mathfrak{N}(\underline{V}) \to \mathfrak{A}$ of g and a homotopy $f \simeq \phi\hat{g}$ if and only if a certain obstruction class $[\mathcal{O}] \in H^{n+1}(\mathfrak{B}, \mathfrak{A}; \underline{V})$ vanishes.

Sketch of proof. Define a map $\int_0^1 : \mathfrak{A}(t, dt) \to \mathfrak{A}$ by $\int_0^1 a \otimes t^i = 0$ and $\int_0^1 a \otimes t^i dt = (-1)^{|a|} a \otimes \frac{t^{i+1}}{i+1}$. An explicit calculation shows that if $H : \mathfrak{A} \to \mathfrak{B}(t, dt)$ is a homotopy from f to g, then $\int_0^1 H = g - f$. We define

$$\mathcal{O} = (gd, f + \int_0^1 Hd) : \underline{V} \to \mathfrak{A}^{n+1} \oplus \mathfrak{B}^n$$

and compute that

$$\begin{aligned} d\mathcal{O} &= (dgd, \phi gd - fd - d\int_0^1 Hd) \\ &= (gd^2, \phi gd - fd - d\int_0^1 Hd - \int_0^1 dHd) = (0, 0), \end{aligned}$$

so $\mathcal{O}$ defines a cohomology class $[\mathcal{O}] \in H^{n+1}(\mathfrak{B}, \mathfrak{A}; \underline{V})$. If $[\mathcal{O}] = 0$, then $\mathcal{O} = d\mathcal{O}'$ for $\mathcal{O}' : \underline{V} \to \mathfrak{A}^n \oplus \mathfrak{B}^{n-1}$. Let $(b, c) = \mathcal{O}'$, so that $d(b, c) = \mathcal{O}$. We define $\tilde{g} = b$ and $\tilde{H} = f + \int_0^t hd + d(c \otimes t)$, and check explicitly that these maps commute with the differential and that $\tilde{H}$ is a homotopy $f \simeq \phi\tilde{g}$. Then with a little care we can use injectivity arguments to extend $\tilde{g}$ to a map $\hat{g} : \mathfrak{N}(\underline{V}) \to \mathfrak{A}$, and extend $\tilde{H}$ to a homotopy $f \simeq \phi\hat{g}$.

Conversely, if we have an extension $\tilde{g} : \mathfrak{N}(\underline{V}) \to \mathfrak{A}$ of g and a homotopy $\tilde{H} : f \simeq \phi\tilde{g}$, we define $\mathcal{O}' : \underline{V} \to \mathfrak{A}^n \oplus \mathfrak{B}^{n-1}$ by $\mathcal{O}' = (\tilde{g}, \int_0^1 \tilde{H})$. Then

$$\begin{aligned} d\mathcal{O}' &= (gd, \phi\tilde{g} - d\int_0^1 \tilde{H}) = (gd, \phi\tilde{g} + \int_0^1 \tilde{H}d + f - \phi\tilde{g}) \\ &= (gd, f + \int_0^1 \tilde{H}d) = \mathcal{O}. \end{aligned}$$

□

If $\mathfrak{A}$ is a $\mathbb{T}$-system, the obstruction class lies in $H^{n+1}(\mathfrak{B}, \mathfrak{A}; \underline{V})$; the proof is exactly the same. There is also a relative version of this result. This theorem requires the extra assumption of injective kernels, which is not mentioned in [F], but which is satisfied every time this lemma is applied.

Lemma 12.56. *Suppose we have a commutative diagram*

$$\begin{tikzcd} \mathfrak{N} \arrow[r, "\phi"] \arrow[d, hook] & \mathfrak{A} \arrow[d, "f"] \arrow[dr, "\mu"] & \\ & & \mathfrak{C} \\ \mathfrak{N}(\underline{V}) \arrow[r, "\psi"] & \mathfrak{B} \arrow[ur, "\nu"'] & \end{tikzcd}$$

where we assume the following:

(i) μ *is onto;*
(ii) $\mu\phi = \nu\psi\,|_{\mathfrak{N}}$*;*
(iii) $\ker(\mu)$ *and* $\ker(\nu)$ *are injective;*
(iv) *there exists a homotopy* H *from* $f \circ \phi$ *to* $\psi|_{\mathfrak{N}}$ *such that* $(\nu \otimes 1) \circ H$ *is constant; and*
(v) *there exists* $\theta : \underline{V} \to \mathfrak{A}$ *such that* $\nu \circ \psi|_{\underline{V}} = \mu \circ \theta$.

Then the obstruction class $[\mathcal{O}] \in H^{n+1}(\mathfrak{A}, \mathfrak{B}; \underline{V})$ *vanishes if and only if there are an extension* $\tilde{H} : \mathfrak{N}(\underline{V}) \to \mathfrak{B}(t, dt)$ *of* H *and an extension* $\tilde{\phi} : \mathfrak{N}(\underline{V}) \to \mathfrak{A}$ *such that* $\mu \circ \tilde{\phi} = \nu \circ \phi$ *and* $(\nu \otimes 1) \circ \tilde{H}$ *is constant.*

13. Proof of algebraic results

Now we turn to the proof of the main algebraic results stated in Sections 3 and 5. The proof that homotopy is an equivalence relation on minimal systems, Propositions 3.5 and 5.20, is a straightforward application of the obstruction theory developed in the previous section and closely follows the non-equivariant proof of [DGMS]. Next we consider the lifting theorems.

Proof of Propositions 3.6 and 5.21. $\mathfrak{M}$ is minimal, so $\mathfrak{M} = \bigcup_n \mathfrak{M}(n)$. We construct g inductively starting with $g = id$ on $\mathfrak{M}(0) = \underline{R}$; recall that all maps are based. Assume that we have defined g on $\mathfrak{M}(n-1)$ as required, and that $\mathfrak{M}(n) = \mathfrak{M}(n-1)(\underline{V})$; by Lemma 12.55, the obstruction to extending g to $\mathfrak{M}(n)$ lies in $H^{n+1}(\mathfrak{B}, \mathfrak{A}; \underline{V})$ or $H^{n+1}(\widetilde{\mathfrak{B}}, \widetilde{\mathfrak{A}}; \underline{V})$, which vanishes because ρ is a quasi-isomorphism.

To show uniqueness, we use a similar obstruction theory argument based on the relative version of Lemma 12.55 (see [F]). □

Now we wish to show the uniqueness of a minimal object. We cannot use the non-equivariant arguments here because a map between minimal systems may not be a "cocellular" map which takes $\mathfrak{M}(n)$ to $\mathfrak{N}(n)$. Non-equivariantly, the degrees of the generators of the extensions involved force every map to be cocellular. In our situation, however, we have attached injective resolutions with generators of higher degree, and these elements may not land where they belong. But the map may be adjusted on the injective resolution without changing anything essential, and every map does have a cocellular approximation. We show that any cocellular quasi-isomorphism of minimal systems is an isomorphism.

First we need the existence of cocellular approximations.

Lemma 13.57. *A map $f : \mathfrak{M} \to \mathfrak{N}$ between minimal systems is homotopic to a map g which takes $\mathfrak{M}(n)$ to $\mathfrak{N}(n)$ for all n.*

Proof. We build g inductively, starting with $g = f = id$ on $\underline{R}$. Assume we have defined $g : \mathfrak{M}(n-1) \to \mathfrak{N}(n-1)$ with $g \simeq f|_{\mathfrak{M}(n-1)}$, and consider the following diagram:

$$\begin{tikzcd} \mathfrak{M}(n-1) \arrow[r, "g"] \arrow[d, hook] & \mathfrak{N}(n-1) \arrow[r, hook] & \mathfrak{N}(n) \arrow[d, hook] \\ \mathfrak{M}(n) \arrow[rr, "f"] & & \mathfrak{N} \end{tikzcd}$$

By Lemma 12.55, there are an extension of g to $\mathfrak{M}(n)$ and an extension of the homotopy $g \simeq f$ if and only if the obstruction class $[\mathcal{O}] \in H^{n+1}(\mathfrak{N}, \mathfrak{N}(n); \underline{V}_n)$ vanishes. But $\mathfrak{N}(n) \hookrightarrow \mathfrak{N}$ is an n-isomorphism, so the relative cohomology groups $H^{n+1}(\mathfrak{N}, \mathfrak{N}(n); \underline{V}_n)$ and $H^{n+1}(\widetilde{\mathfrak{N}}, \widetilde{\mathfrak{N}}(n); \underline{V}_n)$ vanish and we can produce g as required. $\square$

Proof of Theorems 3.8 and 5.23. By Lemma 13.57, there is a map $g : \mathfrak{M} \to \mathfrak{N}$ which is homotopic to f, and which takes $\mathfrak{M}(n)$ to $\mathfrak{N}(n)$ for all n. We will show that g is an isomorphism.

Assume inductively that $g : \mathfrak{M}(n-1) \cong \mathfrak{N}(n-1)$. Now g is a quasi-isomorphism and $i : \mathfrak{M}(n) \hookrightarrow \mathfrak{M}$ is an n-isomorphism; the commutativity of

$$\begin{tikzcd} \underline{H}^i(\mathfrak{M}(n)) \arrow[r, hook, "i^*"] \arrow[d, "g^*"] & \underline{H}^i(\mathfrak{M}) \arrow[d, "g^*"] \\ \underline{H}^i(\mathfrak{N}(n)) \arrow[r, hook, "i^*"] & \underline{H}^i(\mathfrak{N}) \end{tikzcd}$$

implies that the restriction $g|_{\mathfrak{M}(n)} : \mathfrak{M}(n) \to \mathfrak{N}(n)$ is also an n-isomorphism. Then the 5-lemma applied to the long exact sequences of $\mathfrak{M}(n) \hookrightarrow \mathfrak{M}$ and $\mathfrak{N}(n) \hookrightarrow \mathfrak{N}$ implies that

$$g^* : \underline{H}^n(\mathfrak{M}(n), \mathfrak{M}(n-1)) \to \underline{H}^n(\mathfrak{N}(n), \mathfrak{N}(n-1))$$

is an isomorphism. This means that if $\mathfrak{M}(n) = \mathfrak{M}(n-1)(\underline{V})$ and $\mathfrak{N}(n) = \mathfrak{N}(n-1)(\underline{V}')$, then by Lemma 11.51, $\underline{V} \cong \underline{V}'$ and thus $\underline{V}_i \cong \underline{V}'_i$ for all i. Therefore $\mathfrak{M}(n)$ and $\mathfrak{N}(n)$ are isomorphic as graded vector spaces. Furthermore, $g^*[(dx, x)] = [(dx, x)]$ for $x \in \underline{V}$, so $[(gdx, gx)] = [(dx, x)]$. Examining the form of the isomorphism given by Lemma 11.51, we see that this implies that $g(x) = b + x$ for some $b \in \mathfrak{N}(n-1)$. So by Lemma 11.53, g is an isomorphism $\mathfrak{M}(n) \cong \mathfrak{N}(n)$. $\square$

Proof of Corollary 3.9. By Proposition 3.6, we can produce a map $g : \mathfrak{M} \to \mathfrak{M}'$ such that $\rho' g \simeq \rho$; then g must also be a quasi-isomorphism, so it is homotopic to an isomorphism f by Theorem 3.8. Then $\rho' f \simeq \rho' g \simeq \rho$. $\square$

We now turn to the proof of the other main algebraic result, the existence of minimal models. We begin by proving it for systems of DGAs.

Proof of Theorem 3.11. Assume inductively that we have produced $\rho : \mathfrak{M}(n-1) = \mathfrak{N} \to \mathfrak{A}$, where ρ is an $(n-1)$-isomorphism and $\mathfrak{N} = \bigcup_{i<n} \mathfrak{N}(i)$. Produce a relative system of DGAs $\mathfrak{R}$ for $\mathfrak{N} \to \mathfrak{A}$, so that $\mathfrak{N} \xrightarrow{\alpha} \mathfrak{N}' \xrightarrow{\beta} \mathfrak{A}$ is a factorization of ρ where α is a quasi-isomorphism, and $\mathfrak{R} \xrightarrow{\phi} \mathfrak{N}' \xrightarrow{\beta} \mathfrak{A}$ induces long exact sequences on cohomology. Then $\beta : \mathfrak{N}' \to \mathfrak{A}$ is an $(n-1)$-isomorphism since ρ is, and so the long exact sequence of cohomology shows that $\widetilde{\underline{H}}^i(\mathfrak{R}) = 0$ for $i \le n$. Let $\underline{V} = \widetilde{\underline{H}}^{n+1}(\mathfrak{R})$.

Recall that there is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(\underline{V}, \underline{H}^t(\mathfrak{R})) \Rightarrow H^{s+t}(\mathfrak{R}; \underline{V}).$$

The identity map gives an element $id \in \mathrm{Hom}(\underline{V}, \widetilde{\underline{H}}^{n+1}(\mathfrak{R})) = E_2^{0,n+1}$; since $\widetilde{\underline{H}}^t(\mathfrak{R}) = 0$ for $t \le n$, $E_2^{s,t} = \mathrm{Ext}^s(\underline{V}, \underline{H}^t(\mathfrak{R})) = 0$ for $s + t = n + 1$ and $s > 0$. Thus $E_2^{0,n+1} = E_\infty^{0,n+1}$, and id represents a class $[id] \in H^{n+1}(\mathfrak{R}; \underline{V})$. Consider $\phi^*[id] = [\phi] \in H^{n+1}(\mathfrak{N}', \underline{V})$. Since $\alpha : \mathfrak{N} \to \mathfrak{N}'$ is a quasi-isomorphism, $\alpha^* : H^{n+1}(\mathfrak{N}; \underline{V}) \cong H^{n+1}(\mathfrak{N}'; \underline{V})$ and there is a unique class $[\gamma] \in H^{n+1}(\mathfrak{N}; \underline{V})$ for which $\alpha^*[\gamma] = [\phi]$. With this definition,

$$\rho^*[\gamma] = \beta^*\alpha^*[\gamma] = \beta^*[\phi] = \beta^*\phi^*[id] = 0 \in H^{n+1}(\mathfrak{A}, \underline{V}),$$

so $\rho\gamma$ is a coboundary and there is a map $\rho' : \underline{V} \to \mathfrak{A}$ such that $d\rho' = \rho\gamma$.

Define $\mathfrak{N}(\underline{V})$ to be the elementary extension with respect to the map γ. On $\underline{V}$ we have $d = \gamma$, so $d\rho' = \rho d$ and we can extend ρ to a map $\mathfrak{N}(\underline{V}) \to \mathfrak{A}$. This extension fits into the map of long exact sequences:

$$\begin{tikzcd}
\widetilde{\underline{H}}^n(\mathfrak{N}) \arrow[r] \arrow[d, "\alpha^*"] & \widetilde{\underline{H}}^n(\mathfrak{N}(\underline{V})) \arrow[r] \arrow[d, "\rho^*"] & \widetilde{\underline{H}}^n(\mathfrak{N}(\underline{V}), \mathfrak{N}) \arrow[r] \arrow[d] & \widetilde{\underline{H}}^{n+1}(\mathfrak{N}) \arrow[r] \arrow[d, "\alpha^*"] & {} \\
\widetilde{\underline{H}}^n(\mathfrak{N}') \arrow[r, "\beta^*"] & \widetilde{\underline{H}}^n(\mathfrak{A}) \arrow[r] & \widetilde{\underline{H}}^n(\mathfrak{A}, \mathfrak{N}') \arrow[r] & \widetilde{\underline{H}}^{n+1}(\mathfrak{N}') \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \widetilde{\underline{H}}^{n+1}(\mathfrak{N}(\underline{V})) \arrow[r] \arrow[d, "\rho^*"] & \widetilde{\underline{H}}^{n+2}(\mathfrak{N}(\underline{V}), \mathfrak{N}) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \widetilde{\underline{H}}^{n+1}(\mathfrak{A}) \arrow[r] & \widetilde{\underline{H}}^{n+2}(\mathfrak{A}, \mathfrak{N}') \arrow[r] & \cdots
\end{tikzcd}$$

The map $\alpha : \mathfrak{N} \to \mathfrak{N}'$ is a quasi-isomorphism and, by construction,

$$\underline{H}^n(\mathfrak{N}(\underline{V}), \mathfrak{N}) \cong \underline{V} \cong \underline{H}^{n+1}(\mathfrak{R}) \cong \underline{H}^n(\mathfrak{A}, \mathfrak{N}').$$

Furthermore, $\mathfrak{N}$ has no elements of degree 1, so the only elements added to $\mathfrak{N}$ in degree $n+1$ are those in the $\underline{V}_1$ term of the injective resolution of $\underline{V}$, which don't affect the cohomology. So $\underline{H}^{n+1}(\mathfrak{N}) = \underline{H}^{n+1}(\mathfrak{N}(\underline{V}))$ and $\widetilde{\underline{H}}^{n+2}(\mathfrak{N}(\underline{V}), \mathfrak{N}) = 0$. By the 5-lemma, $\widetilde{\underline{H}}^n(\mathfrak{N}(\underline{V})) \to \widetilde{\underline{H}}^n(\mathfrak{A})$ is an isomorphism, and $\widetilde{\underline{H}}^{n+1}(\mathfrak{N}(\underline{V})) \to$

$\widetilde{\underline{H}}^{n+1}(\mathfrak{A})$ is an injection; thus we have the next step $\mathfrak{M}(n) = \mathfrak{N}(\underline{V})$ in the inductive construction of the minimal model. $\square$

The proof of existence of minimal models for $\mathbb{T}$-systems is very similar. Once again, the main modifications needed are to consider maps landing in $\widetilde{\mathfrak{A}}$ rather than $\mathfrak{A}$; but the results outlined in Section 10 allow us to transfer cohomological properties between these where needed.

Proof of Theorem 5.26. Again, we assume inductively that we have produced $\rho : \mathfrak{M}(n-1) = \mathfrak{N} \to \mathfrak{A}$, where ρ is an $(n-1)$-isomorphism and $\mathfrak{N} = \bigcup_{i<n} \mathfrak{N}(i)$, and produce the relative $\mathbb{T}$-system $\mathfrak{R}$ and the maps $\mathfrak{R} \xrightarrow{\phi} \mathfrak{N}' \xrightarrow{\beta} \mathfrak{A}$, where $\mathfrak{N} \xrightarrow{\alpha} \mathfrak{N}'$ is a quasi-isomorphism and $\alpha\beta = \rho$. Then $\widetilde{\underline{H}}^i(\mathfrak{R}) = 0$ for $i \leq n$, and $\underline{V} = \widetilde{\underline{H}}^{n+1}(\mathfrak{R})$; by Corollary 10.50, this is equal to $\widetilde{\underline{H}}^{n+1}(\widetilde{\mathfrak{R}})$. The same corollary implies that $\widetilde{\underline{H}}^i(\widetilde{\mathfrak{A}}) = 0$ for $i \leq n$, so $id : \underline{V} \to \widetilde{\underline{H}}^{n+1}(\widetilde{\mathfrak{R}})$ defines a cohomology class $[id] \in H^{n+1}(\widetilde{\mathfrak{R}}; \underline{V})$ and thus a class $\tilde{\phi}^*[id] = [\tilde{\phi}] \in H^{n+1}(\widetilde{\mathfrak{N}}', \underline{V})$. Now $\alpha : \mathfrak{N} \to \mathfrak{N}'$ is a quasi-isomorphism, so $\tilde{\alpha} : \widetilde{\mathfrak{N}} \to \widetilde{\mathfrak{N}}'$ is also by Theorem 10.46; thus there is a unique class $[\tilde{\gamma}] \in H^{n+1}(\widetilde{\mathfrak{N}}; \underline{V})$ for which $\tilde{\alpha}^*[\tilde{\gamma}] = [\tilde{\phi}]$. Then

$$\tilde{\rho}^*[\tilde{\gamma}] = \tilde{\beta}^*\tilde{\alpha}^*[\tilde{\gamma}] = \tilde{\beta}^*[\tilde{\phi}] = \tilde{\beta}^*\tilde{\phi}^*[id] \in H^{n+1}(\mathfrak{A}, \underline{V}).$$

Recall that we produced $\mathfrak{R}$ in such a way that $\widetilde{\mathfrak{R}}$ is a relative system of DGAs for $\tilde{\rho} : \widetilde{\mathfrak{N}} \to \widetilde{\mathfrak{A}}$, and $\widetilde{\mathfrak{R}} \xrightarrow{\tilde{\phi}} \widetilde{\mathfrak{N}}' \xrightarrow{\tilde{\beta}} \widetilde{\mathfrak{A}}$ also induces an exact sequence in cohomology. This means that $\tilde{\rho}^*[\tilde{\gamma}] = 0 \in H^{n+1}(\mathfrak{A}, \underline{V})$, since $\tilde{\phi}^*\tilde{\beta}^* = 0$; and there is a $\tilde{\rho}' : \underline{V} \to \widetilde{\mathfrak{A}}$ for which $d\tilde{\rho}' = \tilde{\rho}\tilde{\gamma}$. Define $\mathfrak{N}(\underline{V})$ to be the elementary extension with respect to the map $\tilde{\gamma}$, and extend the map ρ to all of $\mathfrak{N}(\underline{V})$ using $\tilde{\rho}'$. By comparing long exact sequences of cohomology, we see that ρ is an n-isomorphism, and so $\mathfrak{M}(n) = \mathfrak{N}(\underline{V})$ is the next stage in the construction of the minimal model of $\mathfrak{A}$. $\square$

14. Finite isotropy orbit spaces

Now that we have the algebraic theory in place, we want to relate it to geometry. For the actions of finite groups, this is a matter of carefully applying the non-equivariant theory using the functors of DGAs. Recall that the equivariant cohomology defined in defined in Section 2 is calculated with a spectral sequence $E_2^{s,t} = \mathrm{Ext}^s(\underline{J}_t(X), \underline{A}) \Rightarrow H_G^{s+t}(X; \underline{A})$, where $\underline{J}_*(X)(G/H) = H_*(X^H/W_0H)$. For a discrete group G, this is just $\underline{J}_*(X)(G/H) = H_*(X^H)$. The basic structures used to pass from spaces to DGA's are $\underline{\mathcal{E}}(X)$, and integration of forms provides a natural transformation

$$\mathcal{A}(X^H) \to \operatorname*{colim}_{\overline{\mathcal{U}}_{X^H}} S^* \mid \overline{\mathcal{U}}_{X^H} \mid,$$

where S^* denotes the rational simplicial cochain complex. The usual de Rham theorem then gives an isomorphim with the Alexander-Spanier cohomology $\overline{H}$, and our assumptions ensure that all fixed point sets are CW complexes and therefore "agreement spaces" in the sense of [AH]. So there is a natural isomorphism of functors

$$\overline{\underline{H}}^*(X; \mathbb{Q}) \cong \underline{H}^*(X; \mathbb{Q})),$$

which in turn induces a natural isomorphism between the E_2 terms of the spectral sequences computing $H_G^*(X; \underline{A})$ and $H^*(\underline{\mathcal{E}}(X), \underline{A}^*)$. The important point is that the system of DGAs $\underline{\mathcal{E}}(X)$ computes the equivariant cohomology of X, and therefore can be used for understanding the equivariant Postnikov decomposition of the space.

When we turn to the study of $\mathbb{T}$-spaces, things are not so straightforward. One major complication in dealing with $\mathbb{T}$-spaces is that equivariant cohomology groups are computed using information about orbit spaces X^H/W_0H, and not just the fixed point sets X^H. The circle is connected, so the spectral sequence for cohomology now relies on $\underline{J}_*(X)(G/H) = H_*(X^H/\mathbb{T})$. Our first task is to get some way of understanding these orbit spaces.

Working rationally, we can generally neglect the effect of finite isotropy; an orbit $\mathbb{T}/H$ is essentially the same as a free orbit $\mathbb{T}/e$, and the equivariant projection $\mathbb{T}/H \to \mathbb{T}/e$ is a rational equivalence. Thus we can reduce to considering only semifree $\mathbb{T}$-spaces. The first step is to show that any finite isotropy space "may as well be" a free space. We are working in the category of based $\mathbb{T}$-spaces, so rather than looking at the usual free approximation $X \times E\mathbb{T}$, we consider instead the based version $X \wedge E\mathbb{T}_+$ (which has a fixed base point and is otherwise free) and its orbit space, the based Borel construction $X \wedge_{\mathbb{T}} E\mathbb{T}_+$.

Proposition 14.58. *Suppose X is a based finite isotropy $\mathbb{T}$-space with finitely many orbit types. Then there is a rational equivalence*

$$X \wedge_{\mathbb{T}} E\mathbb{T}_+ \simeq X/\mathbb{T}.$$

Proof. For any finite $H \subset \mathbb{T}$, X may be regarded as an H-space and the projection $X \to X/H$ induces an isomorphism $H_*(X;\mathbb{Q})^H \to H_*(X/H;\mathbb{Q})$ (see [B2]). But the action of the group H on $H_*(X;\mathbb{Q})$ is trivial, since it comes from an action of the connected group $\mathbb{T}$; so the projection induces an isomorphism on rational homology, and thus is a rational equivalence.

Choose a finite $L \cong \mathbb{Z}/l\mathbb{Z} \subset \mathbb{T}$ containing all isotropy groups of X. Then $X/\mathbb{T} = (X/L)/(\mathbb{T}/L)$, where X/L is a (based) free $\mathbb{T}/L$-space; and so

$$X/L \wedge_{\mathbb{T}/L} E(\mathbb{T}/L)_+ \simeq X/\mathbb{T}.$$

If $\mathbb{T}$ acts on $\mathbb{C}^\infty$ by $\zeta(z_1, z_2, \dots) = (\zeta z_1, \zeta z_2, ...)$, then the unit sphere $S(\infty)$ is a model for $E\mathbb{T}$. Similarly, if we consider the action of $\mathbb{T}/L$ on $\mathbb{C}^\infty$ by $\zeta(z_1, z_2, \dots) = (\zeta^l z_1, \zeta^l z_2, \dots)$ the unit sphere is a model for $E(\mathbb{T}/L)$, and the map $(z_1, z_2, \dots) \to (z_1^l, z_2^l, , \dots)$ gives a $\mathbb{T}$-equivariant map $E\mathbb{T} \to E(\mathbb{T}/L)$. We use this map to define a map $\theta : X \wedge_L E\mathbb{T}_+ \to X/L \wedge E(\mathbb{T}/L)_+$ by $[x, (z_1, z_2, \dots)] \to [x, (z_1^l, z_2^l, \dots)]$. Note that this map is well-defined and $\mathbb{T}T/L$-equivariant. With this definition, the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dr, "\mathbb{Q}"] & \\
X \wedge E\mathbb{T}_+ \arrow[ur, "\simeq"] \arrow[d, "\mathbb{Q}"'] & & X/L \\
X \wedge_L E\mathbb{T}_+ \arrow[rr, "\theta"'] & & X/L \wedge E(\mathbb{T}/L)_+ \arrow[u, "\simeq"']
\end{tikzcd}$$

The maps marked by $\mathbb{Q}$ are rational equivalences, and those marked by $\simeq$ are non-equivariant homotopy equivalences. So θ is a rational equivalence between two (based) free $\mathbb{T}/L$-spaces, and thus is a $\mathbb{T}/L$-equivariant rational equivalence; by taking $\mathbb{T}/L$ orbits, it gives a rational equivalence

$$X \wedge_{\mathbb{T}} E\mathbb{T}_+ = (X \wedge_L E\mathbb{T}_+)/(\mathbb{T}/L) \to X/L \wedge_{\mathbb{T}/L} E(\mathbb{T}/L)_+$$

Combining this with the equivalence from above, we see that the projection map $X \wedge_{\mathbb{T}} E\mathbb{T}_+ \to X/\mathbb{T}$ is a rational equivalence. $\square$

Now we apply the same idea to a general $\mathbb{T}$-space X. The following construction takes the free approximation of the space, $X \times_{\mathbb{T}} E\mathbb{T}$, but then collapses the fibres over the fixed point set $X^{\mathbb{T}}$. The result is a semifree approximation of the space X.

Definition 14.59. $X//\mathbb{T}$ is the pushout of the diagram

$$\begin{tikzcd} X^{\mathbb{T}} \times B\mathbb{T} \arrow[r, hook] \arrow[d] & X \times_{\mathbb{T}} E\mathbb{T} \arrow[d] \\ X^{\mathbb{T}} \arrow[r] & X//\mathbb{T} \end{tikzcd}$$

Note that if X is a based finite isotropy space which has only the base point fixed, then this is exactly the based Borel construction $X \wedge_{\mathbb{T}} E\mathbb{T}_+$ from above. We now show that $X//\mathbb{T}$ has the same rational relationship to X.

Proposition 14.60. *If X is a $\mathbb{T}$-space, then $X/\mathbb{T}$ is rationally equivalent to $X//\mathbb{T}$.*

Proof. Projecting $E\mathbb{T} \to *$ induces a map $\alpha : X//\mathbb{T} \to X/\mathbb{T}$ for any X. Now consider the diagram

$$\begin{tikzcd} X^{\mathbb{T}} \arrow[r, hook] \arrow[d, equal] & X//\mathbb{T} \arrow[r] \arrow[d, "\alpha"] & (X//\mathbb{T})/X^{\mathbb{T}} \arrow[d, "\beta"] \\ X^{\mathbb{T}} \arrow[r, hook] & X/\mathbb{T} \arrow[r] & (X/\mathbb{T})/X^{\mathbb{T}} \end{tikzcd}$$

Since $X/X^{\mathbb{T}}$ is a based finite isotropy space, the previous proposition gives a rational equivalence between $(X/X^{\mathbb{T}}) \wedge_{\mathbb{T}} E\mathbb{T}_+$ and $(X/X^{\mathbb{T}})/\mathbb{T}$. But

$$(X/X^{\mathbb{T}})/\mathbb{T} = (X/\mathbb{T})/X^{\mathbb{T}},$$

and

$$\begin{aligned}(X/X^{\mathbb{T}}) \wedge_{\mathbb{T}} E\mathbb{T}_+ &= (X/X^{\mathbb{T}}) \times_{\mathbb{T}} E\mathbb{T}/ * \times B\mathbb{T} \\ &= X \times_{\mathbb{T}} E\mathbb{T}/X^{\mathbb{T}} \times B\mathbb{T} = (X//\mathbb{T})/X^{\mathbb{T}}.\end{aligned}$$

So β is a rational equivalence; the long exact sequence of $H_*(-, \mathbb{Q})$ implies that α also induces an isomorphism in $\mathbb{Q}$-homology, and so $\alpha : X//\mathbb{T} \to X/\mathbb{T}$ is a rational equivalence. $\square$

This implies that $X^H/\mathbb{T}$ is rationally equivalent to $X^H//\mathbb{T}$ for any $H \subset \mathbb{T}$. Furthermore, the construction is natural with respect to the inclusion maps. Thus the system of spaces $X^H/(\mathbb{T}/H)$, which is used for computing equivariant cohomology, is rationally equivalent to the system of subspaces $X^H//\mathbb{T}$ of $X//\mathbb{T}$. Because of its close connection with the Borel space, $X//\mathbb{T}$ is a much more tractable space; we will use it as a convenient substitute for the orbit space $X/\mathbb{T}$.

15. De Rham $\mathbb{T}$-systems

The topological constructions in the previous section are mirrored in the structure of a $\mathbb{T}$-system. Recall that to each $\mathbb{T}$-space X we associated a $\mathbb{T}$-system $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ with values $\mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T})$, defined in Section 6. Just as the $\mathbb{T}$-system $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ reflects information about the space $X \times_{\mathbb{T}} E\mathbb{T}$, the space $X//\mathbb{T}$ is described by the sub-system of DGAs $\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$; the category of $\mathbb{T}$-systems is designed to contain information about both spaces and their relationship.

Proposition 15.61. *Consider the functor $\underline{\mathcal{A}}(X//\mathbb{T})$ that takes the value $\mathcal{A}(X^H//\mathbb{T})$ at $\mathbb{T}/H$. Then, as functors into DGAs, $\underline{\mathcal{A}}(X//\mathbb{T}) \cong \underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$.*

Proof. The projection $p : X \times_{\mathbb{T}} E\mathbb{T} \to X//\mathbb{T}$ gives a map $\underline{\mathcal{A}}(X//\mathbb{T}) \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$. At $\mathbb{T}/\mathbb{T}$, the map $\mathcal{A}(X^{\mathbb{T}}) \to \mathcal{A}(X^{\mathbb{T}} \times B\mathbb{T})$ is induced by the projection. Since $\mathcal{A}$ is functorial, the splitting of the projection shows that the map is in fact an injection. Away from $X^{\mathbb{T}}$ the map is the identity, so we get an injection $\underline{\mathcal{A}}(X//\mathbb{T}) \hookrightarrow \underline{\mathcal{E}}_{\mathbb{T}}(X)$ at all H. We show that the image is exactly $\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$. Observe that

$$(X^H) \times_{\mathbb{T}} E\mathbb{T}/(X^{\mathbb{T}} \times B\mathbb{T}) = [(X^H)//\mathbb{T}]/X^{\mathbb{T}},$$

and consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}((X^H//\mathbb{T})/X^{\mathbb{T}}) \arrow[r, hook] \arrow[d, equal] & \mathcal{A}(X^H//\mathbb{T}) \arrow[r, two heads, "res"] \arrow[d, hook, "p^*"'] & \mathcal{A}(X^{\mathbb{T}}) \arrow[d, hook, "p^*"'] \\
\mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T}/X^{\mathbb{T}} \times B\mathbb{T}) \arrow[r, hook] & \mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T}) \arrow[r, two heads, "res"] & \mathcal{A}(X^{\mathbb{T}} \times B\mathbb{T})
\end{tikzcd}$$

The surjections are induced by restricting to a subspace. Since $\mathcal{A}(X)$ is defined as a filtered colimit of DGAs and filtered colimits are exact, and inclusion of the simplicial complexes $\overline{\mathcal{U}}_X$ of a subspace is injective, restriction of PL differential forms gives a surjection of $\mathcal{E}(\overline{\mathcal{U}}_X)$. Passing to the colimit gives a surjection of $\mathcal{A}(X)$.

As a matter of pure algebra, this commutativity implies that $\omega \in \operatorname{im}(p^*) \subset \mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T})$ if and only if $res(\omega)$ lands in $\operatorname{im}(p^*) \subset \mathcal{A}(X^{\mathbb{T}} \times B\mathbb{T})$. But by definition, an element is in $\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$ exactly when it restricts to an element in $\mathcal{E}_{\mathbb{T}} = \mathcal{A}(X^{\mathbb{T}})$. So $\underline{\mathcal{A}}(X//\mathbb{T}) = \underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$. ☐

The point of relating the structure of $\mathbb{T}$-systems to this orbit space substitute is that we can now show that $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ computes the $\mathbb{T}$-equivariant cohomology of X. This will be pivotal in using $\mathbb{T}$-systems to model the structure of $\mathbb{T}$-spaces.

Theorem 15.62 (de Rham Theorem)**.** *If X is a $\mathbb{Q}$-good $\mathbb{T}$-space, then*

$$\underline{H}^*(X/\mathbb{T};\mathbb{Q}) \cong \underline{H}^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X))$$

and

$$H^*_{\mathbb{T}}(X, \underline{N}) \cong H^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X), \underline{N}^*)$$

for any coefficient system $\underline{N}$ and its dual covariant system of vector spaces $\underline{N}^*$.

Proof. Define $\underline{\mathcal{A}}(X/\mathbb{T}) = \mathcal{A}(X^H/(\mathbb{T}/H))$ and $\underline{J}^*(X)(G/H) = \underline{H}^*(X^H/(\mathbb{T}/H))$. Standard integration of forms provides a natural transformation

$$\mathcal{A}(X^H/\mathbb{T}) \to \operatorname*{colim}_{\overline{\mathcal{U}}_{X^H/\mathbb{T}}} S^* \mid \overline{\mathcal{U}}_{X^H/\mathbb{T}} \mid,$$

where S^* denotes the rational simplicial cochain complex. The usual de Rham theorem then gives an isomorphism with the Alexander-Spanier cohomology $\overline{H}$,

$$\underline{H}^*(\underline{\mathcal{A}}(X/\mathbb{T})) \cong \overline{\underline{H}}^*((X/\mathbb{T};\mathbb{Q})).$$

As with finite group actions, we get a natural isomorphism $\underline{H}^*(\underline{\mathcal{A}}(X/\mathbb{T})) \cong \underline{J}^*(X)$, and consequently

$$\mathrm{Ext}^s(N^*, \underline{H}^t(\underline{\mathcal{A}}(X/\mathbb{T})) \cong \mathrm{Ext}^s(N^*, \underline{J}^t(X))$$

for all s, t.

The equivariant cohomology of a space $H_G(X;\underline{N})$ is computed by a spectral sequence with $E_2 = \mathrm{Ext}^s(\underline{J}_t(X;\mathbb{Q}), \underline{N})$, where $\underline{J}_t(X;\mathbb{Q})(\mathbb{T}/H) = H_t(X^H/(\mathbb{T}/H), \mathbb{Q})$, and duality provides an identification

$$\mathrm{Ext}^s(\underline{N}^*, \underline{J}^t(X)) \cong \mathrm{Ext}^s(\underline{J}_t(X;\mathbb{Q}), \underline{N}).$$

Therefore we have an isomorphism between the E_2-terms of the spectral sequences computing cohomology of the system $H^*((\underline{\mathcal{A}}(X/\mathbb{T}); \underline{N}^*)$ and the Bredon cohomology $H^*_{\mathbb{T}}(X;\underline{N})$. So integration of forms induces an isomorphism of these cohomologies.

Next, the projection $X//\mathbb{T} \to X/\mathbb{T}$ is a rational equivalence, so $\underline{\mathcal{A}}(X/\mathbb{T}) \to \underline{\mathcal{A}}(X//\mathbb{T}) = \underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)$ induces an isomorphism

$$p^* : \overline{\underline{H}}^*(\underline{\mathcal{E}}(X/\mathbb{T})) \cong \overline{\underline{H}}^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)) \cong \underline{H}^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X)).$$

Once again this induces an isomorphism of E_2-terms of spectral sequences computing the cohomology of these systems, and

$$H^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X), \underline{N}^*) \cong H^*((\underline{\mathcal{A}}(X/\mathbb{T}); \underline{N}^*) \cong H^*_{\mathbb{T}}(X, \underline{N}).$$

□

16. Pullbacks and Eilenberg-Moore spectral sequences

We need to understand equivariant principal fibrations. We will consider these as pullbacks over maps to $K(\underline{\pi}, n)$s. We now give some general results on equivariant pullbacks.

Definition 16.63. A commutative square of DGAs

$$\begin{array}{ccc} A & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C \end{array}$$

is an *EM square* if the induced map $\mathrm{Tor}^*_A(A', A'') \to H^*C$ is an isomorphism (see [BG], p. 13).

EM squares satisfy the following algebraic property.

Lemma 16.64. *If a map between EM squares*

$$\begin{array}{ccc} A & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C \end{array} \quad \longrightarrow \quad \begin{array}{ccc} B & \longrightarrow & B'' \\ \downarrow & & \downarrow \\ B' & \longrightarrow & D \end{array}$$

induces isomorphisms $H^*(A) \cong H^*(B)$, $H^*(A') \cong H^*(B')$ *and* $H^*(A'') \cong H^*(B'')$, *then it also induces an isomorphism* $H^*(C) \cong H^*(D)$.

Proof. The given maps induce an isomorphism of the E_2-terms of the algebraic Eilenberg-Moore spectral sequence

$$\mathrm{Tor}^{*,*}_{H^*A}(H^*A', H^*A'') \cong \mathrm{Tor}^{*,*}_{H^*B}(H^*B', H^*B'')$$

and thus an isomorphism of E_∞-terms

$$\mathrm{Tor}^*_A(A, A'') \cong \mathrm{Tor}^*_B(B', B'').$$

The fact that the squares are EM means that this is an isomorphism

$$H^*(C) \cong H^*(D).$$

□

When considering functors into DGAs, either systems of DGAs or $\mathbb{T}$-systems, we say that a commutative square is EM if it is EM at each G/H.

Corollary 16.65. *If a map of EM squares of systems of DGAs or $\mathbb{T}$-systems*

$$\begin{tikzcd} \mathfrak{A} \arrow[r] \arrow[d] & \mathfrak{A}'' \arrow[d] & & \mathfrak{B} \arrow[r] \arrow[d] & \mathfrak{B}'' \arrow[d] \\ \mathfrak{A}' \arrow[r] & F & \longrightarrow & \mathfrak{B}' \arrow[r] & \mathfrak{G} \end{tikzcd}$$

induces isomorphisms $\underline{H}^*(\mathfrak{A}) \cong \underline{H}^*(\mathfrak{B})$, $\underline{H}^*(\mathfrak{A}') \cong \underline{H}^*(\mathfrak{B}')$ and $\underline{H}^*(\mathfrak{A}'') \cong \underline{H}^*(\mathfrak{B}'')$, then it also induces an isomorphism $\underline{H}^*(\mathfrak{F}) \cong \underline{H}^*(\mathfrak{G})$.

This can immediately be applied to G-equivariant pullbacks for finite groups.

Lemma 16.66. *If Y is the pullback of the G-equivariant maps $X \to B \leftarrow E$, then*

$$\begin{tikzcd} \underline{\mathcal{E}}(B) \arrow[r] \arrow[d] & \underline{\mathcal{E}}(X) \arrow[d] \\ \underline{\mathcal{E}}(E) \arrow[r] & \underline{\mathcal{E}}(Y) \end{tikzcd}$$

is an EM square.

Proof. For any $H \subseteq T$, Y^H is the pullback of $X^H \to B^H \leftarrow E^H$, so we get a pullback after applying the Borel construction by Lemma 16.67. Lemma 16.70 will show that the Alexander-Spanier-de Rham functor takes pullbacks to EM squares, and so the diagram

$$\begin{tikzcd} \mathcal{A}(B^H \times_{\mathbb{T}} E\mathbb{T}) \arrow[r] \arrow[d] & \mathcal{A}(X^H \times_{\mathbb{T}} E\mathbb{T}) \arrow[d] \\ \mathcal{A}(E^H \times_{\mathbb{T}} E\mathbb{T}) \arrow[r] & \mathcal{A}(Y^H \times_{\mathbb{T}} E\mathbb{T}) \end{tikzcd}$$

is an EM square for each H. But this is exactly the above diagram of systems of DGAs evaluated at G/H. □

In order to prove the analogous theorem for $\mathbb{T}$-equivariant pullback fibrations and the $\mathbb{T}$-systems $\underline{\mathcal{E}}_{\mathbb{T}}(X)$, we first need to consider the effect of pullbacks on the Borel spaces $X \times_{\mathbb{T}} E\mathbb{T}$. Ordinarily limits and colimits do not commute. But in the following special case, we prove that they do.

Lemma 16.67. *If L is a free $\mathbb{T}$-space, then applying the functor $(- \times_{\mathbb{T}} L)$ to an equivariant pullback of $\mathbb{T}$-spaces gives a non-equivariant pullback of spaces.*

Proof. Let Y be the pullback of the $\mathbb{T}$-equivariant maps $X \to B \leftarrow E$, and let Z be the pullback of $X \times_{\mathbb{T}} L \to B \times_{\mathbb{T}} L \leftarrow E \times_{\mathbb{T}} L$. By the universal property, we have a continuous map $Y \times_{\mathbb{T}} L$ to the pullback making the following diagram commute:

$$\begin{tikzcd}
Y \times_{\mathbb{T}} L \arrow[dr] \arrow[drr] \arrow[ddr] & & & . \\
& Z \arrow[r] \arrow[d] & E \times_{\mathbb{T}} L \arrow[d] & \\
& X \times_{\mathbb{T}} L \arrow[r] & B \times_{\mathbb{T}} L &
\end{tikzcd}$$

Since L is a free $\mathbb{T}$-space, as a set it is just the disjoint union of free orbits $\coprod \mathbb{T}$. So $W \times_{\mathbb{T}} L = (W \times \coprod \mathbb{T})/\mathbb{T} \cong \coprod (W \times \mathbb{T})/\mathbb{T} \cong \coprod W$ for any $\mathbb{T}$-space W; this identification is natural in W. In particular, as sets the diagram above reduces to

$$\begin{tikzcd}
\coprod Y \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& Z \arrow[r] \arrow[d] & \coprod E \arrow[d] \\
& \coprod X \arrow[r] & \coprod B
\end{tikzcd}$$

Since Y is the pullback of $X \to B \leftarrow E$, the map is a bijection, and so $Z \cong Y \times_{\mathbb{T}} L$. ☐

Corollary 16.68. *Applying the Borel construction to a $\mathbb{T}$-equivariant pullback diagram gives a pullback diagram of spaces.*

Corollary 16.69. *If Y is the pullback of the T-equivariant maps $X \to B \leftarrow E$, then*

$$\begin{tikzcd}
\underline{\mathcal{E}}_{\mathbb{T}}(B) \arrow[r] \arrow[d] & \underline{\mathcal{E}}_{\mathbb{T}}(X) \arrow[d] \\
\underline{\mathcal{E}}_{\mathbb{T}}(E) \arrow[r] & \underline{\mathcal{E}}_{\mathbb{T}}(Y)
\end{tikzcd}$$

is an EM square.

Lemma 16.70. *A pullback of spaces*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X'' \arrow[d] \\
X' \arrow[r] & X
\end{tikzcd}$$

induces a diagram of DGAs

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[r] \arrow[d] & \mathcal{A}(X'') \arrow[d] \\
\mathcal{A}(X') \arrow[r] & \mathcal{A}(Y)
\end{tikzcd}$$

which is an EM square.

Proof. We have defined $\mathcal{A}(X) = \operatorname{colim}\mathcal{E}(\overline{\mathcal{U}})$. By definition, $\operatorname{Tor}_{\mathcal{A}(X)}(B, C)$ is the first derived functor of the tensor product $B \otimes_{\mathcal{A}(X)} C$, which is equal to $\operatorname{colim}_{\mathcal{U}} B \otimes_{\mathcal{E}(\overline{\mathcal{U}})} C$. Observe that if $B' \to B \to B''$ is a short exact sequence, then the long exact sequences

$$\cdots \to \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B', C) \to \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B, C) \to \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B'', C) \to B' \otimes_{\mathcal{E}(\overline{\mathcal{U}})} C \to \cdots$$

induce a long exact sequence

$$\cdots \to \operatorname{colim} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B', C) \to \operatorname{colim} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B, C)$$

$$\to \operatorname{colim} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B'', C) \to \operatorname{colim}(B' \otimes_{\mathcal{E}(\overline{\mathcal{U}})} C) \to \cdots$$

since filtered limits are exact. Therefore $\operatorname{colim} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}})}(B, C)$ is also the derived functor of the tensor product $B \otimes_{\mathcal{A}(X)} C$ and so is equal to $\operatorname{Tor}_{\mathcal{A}(X)}(B, C)$. So

$$\begin{aligned}
&\operatorname{Tor}_{\mathcal{A}(X)}(\mathcal{A}(X'), \mathcal{A}(X''))\\
&= \operatorname{colim}_{\mathcal{U}_X} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}}_X)}(\mathcal{A}(X'), \mathcal{A}(X''))\\
&= \operatorname{colim}_{\mathcal{U}_X} \operatorname{colim}_{\mathcal{U}_{X'}, \mathcal{U}_{X''}} \operatorname{Tor}_{\mathcal{E}(\overline{\mathcal{U}}_X)}(\mathcal{E}(\overline{\mathcal{U}}_{X'}), \mathcal{E}(\overline{\mathcal{U}}_{X''})\\
&= \operatorname{colim}_{\mathcal{U}_X} \operatorname{colim}_{\mathcal{U}_{X'}, \mathcal{U}_{X''}} H^*\mathcal{E}(\text{pullback of } \mid \overline{\mathcal{U}}_{X'} \mid, \mid \overline{\mathcal{U}}_{X''} \mid \text{ over } \mid \overline{\mathcal{U}}_X \mid)\\
&= H^*(\operatorname{colim}_{\mathcal{U}_X, \mathcal{U}_{X'}, \mathcal{U}_{X''}} \mathcal{E}(\text{pullback of } \mid \overline{\mathcal{U}}_{X'} \mid, \mid \overline{\mathcal{U}}_{X''} \mid \text{ over } \mid \overline{\mathcal{U}}_X \mid)
\end{aligned}$$

since the functor Tor commutes with limits, and the de Rham functor takes pullbacks to EM squares by a theorem in [BG]. Geometric realization commutes with finite limits, and therefore with pullbacks, so this is equal to

$$H^*(\operatorname{colim}_{\mathcal{U}_X, \mathcal{U}_{X'}, \mathcal{U}_{X''}} \mathcal{E}(|\text{pullback of simplicial sets } \overline{\mathcal{U}}_{X'}, \overline{\mathcal{U}}_{X''} \text{ over } \overline{\mathcal{U}}_X|)).$$

Moreover, the pullback simplicial set is simply the simplicial set associated to the pullback cover of Y given by $\mathcal{U}_{X'}$, $\mathcal{U}_{X''}$ over $\mathcal{U}_X$, and the topology on the space Y is such that any open set is the union of pullback open sets. Therefore the pullbacks of simplicial sets $\overline{\mathcal{U}}_{X'}, \overline{\mathcal{U}}_{X''}$ over $\overline{\mathcal{U}}_X$ are cofinal in $\overline{\mathcal{U}}_Y$, and

$$\begin{aligned}
&\operatorname*{colim}_{\mathcal{U}_X, \mathcal{U}_{X'}, \mathcal{U}_{X''}} \mathcal{E}(|\text{pullback of simplicial sets } \overline{\mathcal{U}}_{X'}, \overline{\mathcal{U}}_{X''} \text{ over } \overline{\mathcal{U}}_X|)\\
&\qquad = \operatorname*{colim}_{\mathcal{U}_Y} \mathcal{E}(\overline{\mathcal{U}}_Y) = \mathcal{A}(Y),
\end{aligned}$$

and so $\operatorname{Tor}_{\mathcal{A}(X)}(\mathcal{A}(X'), \mathcal{A}(X'')) = H^*\mathcal{A}(Y)$. □

17. Equivariant Eilenberg-Mac Lane spaces

We now examine equivariant Eilenberg-Mac Lane spaces and prove Theorems 4.15 and 6.30.

Proof of Theorem 4.15. If G is a finite group and $K = K(\underline{\pi}, n)$ is an Eilenberg-Mac Lane G-space, then each fixed point set $K^H = K(\underline{\pi}(G/H), n)$ is an ordinary non-equivariant Eilenberg-Mac Lane space, and so $H^*(K^H, \mathbb{Q}) = \mathbb{Q}(\underline{\pi}(G/H))$, the free DGA generated by the vector space $\underline{\pi}(G/H)$ in degree n. To produce a minimal model for $\underline{A}(K)$, we start with an elementary extension generated by $\underline{H}^n = \underline{\pi}^*$, constructed by taking an injective resolution given by $\underline{\pi}^* \to \underline{V}_0 \to \underline{V}_1 \to \dots$, and defining

$$\mathfrak{V} = \bigotimes_i \mathbb{Q}(\underline{V}_i)$$

with differential $d = v_i$ induced by the resolution. Then $\mathfrak{V} \to \underline{A}(K)$ is a quasi-isomorphism, and no further elementary extensions are needed; $\mathfrak{V}$ is a minimal model for $\underline{A}(K)$, thus also geometric for K.

Now suppose $\mathfrak{G}$ is geometric for a G-space X. Then homotopy classes of maps $[X, K]$ correspond to equivariant cohomology classes $H^*(X, \underline{\pi}) \cong H^*(\underline{A}(X), \underline{\pi}^*) \cong H^*(\mathfrak{G}, \underline{\pi}^*)$. By the obstruction theory of Lemma 12.55. this corresponds to homotopy classes of systems of DGAs $[\mathfrak{V}, \mathfrak{G}]$. □

Now we prove that the analogous $\mathbb{T}$-systems model $\mathbb{T}$-Eilenberg-Mac Lane spaces, drawing on the results about $\mathbb{T}$-spaces developed in Sections 14 and 15.

Proof of Theorem 6.30. If $K = K(\underline{\pi}, n)$ is a $\mathbb{T}$-equivariant Eilenberg-Mac Lane space, then each fixed point set $K^H = K(\underline{\pi}(\mathbb{T}/H), n)$ is again an ordinary non-equivariant Eilenberg-Mac Lane space, and $H^*(K^H, \mathbb{Q}) = \mathbb{Q}(\underline{\pi}(\mathbb{T}/H))$. The standard fibration $K^H \to K^H \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$ induces a spectral sequence

$$E_2 = H^*(B\mathbb{T}; \mathbb{Q}) \otimes H^*(K^H; \mathbb{Q}) \Rightarrow H^*(K^H \times_{\mathbb{T}} E\mathbb{T}).$$

The $\mathbb{T}$-fixed basepoint of K induces a section of this fibration, implying that $p^* : H^*(B\mathbb{T}) \to H^*(K^H \times_{\mathbb{T}} E\mathbb{T})$ is an injection. The form of the E_2-term shows that generators of $H^*(K^H)$ can only hit elements of $H^*(B\mathbb{T})$; but by the injectivity of p^*, these elements must live to E_∞. So all differentials are zero on generators of $H^*(K^H)$, and the multiplicative structure of the spectral sequence ensures that it collapses at E_2 and that

$$H^*(K^H \times_{\mathbb{T}} E\mathbb{T}) \cong H^*(K^H) \otimes H^*(B\mathbb{T}) \cong \mathbb{Q}(\underline{\pi}(\mathbb{T}/H)) \otimes \mathbb{Q}[c].$$

By definition of an equivariant $K(\underline{\pi}, n)$, if $H' \subset H$ the inclusion maps $K^H \to K^{H'}$ induce the maps $\underline{\pi}(\mathbb{T}/H) \to \underline{\pi}(\mathbb{T}/H')$ on π_n. Applying the dual of the Hurewicz isomorphism to generators of H^n, we see that the maps $i^* : H^*(K^{H'}) \to H^*(K^H)$ are just the maps $\mathbb{Q}(\underline{\pi}(\mathbb{T}/H')) \to \mathbb{Q}(\underline{\pi}(\mathbb{T}/H))$ induced on generators by the structure maps of the dual coefficient system $\underline{\pi}^*$. Furthermore, since the inclusions induce maps of the fibrations over $B\mathbb{T}$ and the spectral sequence is natural, the structure maps for the functor $\underline{H}^*(K^H \times_{\mathbb{T}} E\mathbb{T})$ are just $i^* \otimes id : H^*(K^{H'}) \otimes \mathbb{Q}[c] \to H^*(K^H) \otimes \mathbb{Q}[c]$. So, as a functor,

$$\underline{H}^*(K \times_{\mathbb{T}} E\mathbb{T}) = \mathbb{Q}(\underline{\pi}^*) \otimes \mathbb{Q}[c].$$

This implies that $\underline{H}^*(\underline{\mathcal{E}}_{\mathbb{T}}(K)) = \mathbb{Q}(\underline{\pi}^*) \otimes \mathbb{Q}[c]$. To produce a minimal model for $\underline{\mathcal{E}}_{\mathbb{T}}(K)$, we start with an injective resolution of $\underline{H}^n = \underline{\pi}^*$ given by

$$\underline{\pi}^* \to \underline{V}_0 \to \underline{V}_1 \to \dots,$$

and let

$$\mathfrak{V} = \bigotimes_i \mathbb{Q}(\underline{V}_i) \otimes \mathbb{Q}[c]$$

be the $\mathbb{T}$-system with differential $d = v_i$ induced by the resolution, with sub-DGA $\mathfrak{V}_{\mathbb{T}} = \bigotimes_i \mathbb{Q}(\underline{V}_i(\mathbb{T}/\mathbb{T})) \otimes \mathbb{Q}$. Then $\mathfrak{V} \to \underline{\mathcal{E}}_{\mathbb{T}}(K)$ is a quasi-isomorphism and no further elementary extensions are needed; $\mathfrak{V}$ is the minimal model for $\underline{\mathcal{E}}_{\mathbb{T}}(K)$, and geometric for K.

Now suppose $\mathfrak{G}$ is geometric for a $\mathbb{T}$-space X. Then homotopy classes of maps $[X, K]$ correspond to equivariant cohomology classes $H^*_{\mathbb{T}}(X, \underline{\pi})$, which is isomorphic to $H^*(\underline{\tilde{\mathcal{E}}}_{\mathbb{T}}(X), \underline{\pi}^*)$ by Theorem 15.62. The quasi-isomorphism $\mathfrak{G} \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$ ensures that this group is isomorphic to $H^*(\widetilde{\mathfrak{G}}, \underline{\pi}^*)$. By the obstruction theory of Lemma 12.55 this corresponds to homotopy classes of systems of DGAs $[\mathfrak{V}, \mathfrak{G}]$. $\square$

In both cases, the systems $\mathfrak{V}$ generated by a single elementary extension of a dual coefficient system $\underline{V}$ model the Eilenberg-Mac Lane spaces. We now make some observations about the properties possessed by these systems. The following discussion applies with minor modifications to systems of DGAs, but is only given for $\mathbb{T}$-systems. Suppose $\mathfrak{V} = \bigotimes_i \mathbb{Q}(\underline{V}_i) \otimes \mathbb{Q}[c]$. Then associated to $\mathfrak{V}$ we have an acyclic $\mathbb{T}$-system $\mathfrak{W} = \bigotimes_i \mathbb{Q}(\Sigma^{-1}\underline{V}_i) \bigotimes_i \mathbb{Q}(\underline{V}_i) \otimes \mathbb{Q}[c]$, introduced in Section 11.

Lemma 17.71. *Let $\mathfrak{V} \xrightarrow{\alpha} \mathfrak{B}$ be a map of $\mathbb{T}$-systems, and let $\mathfrak{W}$ be the acyclic $\mathbb{T}$-system described above. Suppose that $f : \mathfrak{B} \to \mathfrak{C}$ is a morphism to an acyclic $\mathbb{T}$-system $\mathfrak{C}$. Then there is a morphism of $\mathbb{T}$-systems ρ making the following diagram commute*:

$$\begin{tikzcd} \mathfrak{V} \arrow[r, "\alpha"] \arrow[d, hook] & \mathfrak{B} \arrow[d, "f"] \\ \mathfrak{W} \arrow[r, "\rho"] & \mathfrak{C} \end{tikzcd}$$

Proof. Define ρ as the composite $f\alpha$ on $\mathfrak{V}$; we need to extend it over $\mathfrak{W}$. As discussed in Section 11, $\alpha|_{\underline{\pi}^*}$ lands in $\widetilde{\mathfrak{C}}$, and α represents a cohomology class $[\alpha] \in H^n(\widetilde{\mathfrak{C}}, \underline{\pi})$. But this group is zero since $\mathfrak{C}$ is acyclic. Let $\beta : \underline{\pi} \to \widetilde{\mathfrak{C}}$ be a map such that $d\beta = \alpha$. Define $\rho : \Sigma^{-1}\underline{\pi} \to \widetilde{\mathfrak{C}}$ by $\rho(\Sigma^{-1}x) = \beta(x)$; this map commutes with d since $d = \sigma$ on $\Sigma^{-1}\underline{\pi}$. We can extend the map over the other generators of $\mathfrak{W}$ by the standard injectivity arguments, and over all of $\mathfrak{W}$ by freeness. Note that since α takes all generators to $\widetilde{\mathfrak{C}}$, the resulting map will take the sub-DGA $\mathfrak{W}_{\mathbb{T}}$ to $\mathfrak{C}_{\mathbb{T}}$ and be a morphism of $\mathbb{T}$-systems. $\square$

Lemma 17.72. *Let $\mathfrak{V}$ be the $\mathbb{T}$-system generated by a single elementary extension of $\underline{V}$, and $\mathfrak{W}$ the associated acyclic system. Then for any map of $\mathbb{T}$-systems $\mathfrak{V} \to \mathfrak{B}$,*

$$\begin{tikzcd} \mathfrak{V} \arrow[r] \arrow[d, hook] & \mathfrak{B} \arrow[d] \\ \mathfrak{W} \arrow[r] & \mathfrak{B} \otimes_{\mathfrak{V}} \mathfrak{W} \end{tikzcd}$$

is an EM square.

Proof. By construction, $\mathfrak{W}$ is a free $\mathfrak{V}$-module, so $\mathfrak{W}(\mathbb{T}/H) \to \mathbb{Q}[c]$ is a projective $\mathfrak{V}(\mathbb{T}/H)$-resolution of $\mathbb{Q}[c]$ for any H. Using this resolution to calculate Tor, we find that

$$\mathrm{Tor}^*_{\mathfrak{V}(\mathbb{T}/H)}(\mathfrak{W}(\mathbb{T}/H), \mathfrak{B}(\mathbb{T}/H)) = H^*(\mathfrak{B}(\mathbb{T}/H) \otimes_{\mathfrak{V}(\mathbb{T}/H)} \mathfrak{W}(\mathbb{T}/H)).$$

$\square$

18. Elementary extensions and principal fibrations

Now we consider the relationship between elementary extensions and equivariant principal fibrations, and prove Theorems 4.16 and 6.31. First we recall some basic facts about equivariant principal fibrations.

A based equivariant map $p : E \to X$ is a *G-fibration* if it has the equivariant homotopy lifting property. Let $e_0 \in E^G$ be a base point and let $x_0 = p(e_0)$; then the G-space $F = p^{-1}(x_0)$ is called the fibre of p. Suppose Y is a G-complex, meaning it is composed of G-cells $G/H \times D^n$, and $Y^{(n)}$ is the nth G-skeleton of Y. Suppose further that $f : Y \to X$ is a G-map and $\tilde{f} : Y^{(n)} \to E$ is an equivariant lifting of $f \mid Y^{(n)}$, so that $p\tilde{f} = f \mid Y^{(n)}$. Then there is a cohomology class

$$[c_{(f,\tilde{f})}] \in H_G^{n+1}(Y; \underline{\pi}_n(F))$$

that vanishes if and only if the lifting $\tilde{f} \mid Y^{(n-1)}$ can be extended equivariantly to a lifting $Y^{(n+1)} \to E$. If H is a G-homotopy between two such liftings $\tilde{f} \mid Y^{(n)}$ and $\tilde{f}' \mid Y^{(n)}$, the obstruction to extending $H \mid Y^{(n-1)}$ to a G-homotopy $Y^{(n+1)} \times I \to E$ lies in $H_G^n(Y; \underline{\pi}_n(F))$.

The obstruction theory sketched above leads to the following classification. Let $p : E \to X$ be a *principal* G-fibration that has fibre an Eilenberg-Mac Lane G-space $K(\underline{A}, n)$. Suppose $f : Y \to X$ is a G-map, and choose a fixed equivariant lifting $\tilde{f} : Y \to E$ of f. Then there is a bijection between the set of equivariant homotopy classes of equivariant liftings of f and elements of $H_G^n(Y; \underline{A})$; a class $[\tilde{f}']$ corresponds to the first obstruction to the existence of a G-homotopy between $\tilde{f}$ and $\tilde{f}'$.

This implies that G-equivalence classes of principal G-fibrations over X are classified by

$$H_G^{n+1}(X; \underline{A}) \cong [X, K(\underline{A}, n+1)]_G$$

as follows. Given a cohomology class in $H_G^{n+1}(X; \underline{A})$, we consider the corresponding G-map $f : X \to K(\underline{A}, n+1)$, and construct a G-fibration over X by considering the pullback by f of the path fibration

$$\Omega K(\underline{A}, n+1) \to PK(\underline{A}, n+1) \to K(\underline{A}, n+1).$$

This gives the claimed classification.

We now give a proof of the correspondence between principal fibrations and elementary extensions. The proof of the result for finite groups, Theorem 4.16, is very similar and will not be given. Note, however, that the proof follows the proof given below and is substantially different from the original proof given in [T1]; the original proof relied on erroneous facts about the structure of minimal models, related to the confusion over the correct equivariant definition discussed in Section 3.

Proof of Theorem 6.31. Suppose $K(\underline{V}, n) \to Y \to X$ is a $\mathbb{T}$-principal fibration. Then Y is the pullback of the path-space fibration $E \to K(\underline{V}, n+1)$. Let $\rho : \mathfrak{V} \to \underline{\mathcal{E}}_{\mathbb{T}}(K)$ be the minimal model, where $K = K(\underline{V}, n+1)$; then $\mathfrak{V}$ is constructed from a single elementary extension generated by $\underline{V}^*$, considered as a degree $n+1$ vector space. Let $\mathfrak{W}$ denote the associated acyclic $\mathbb{T}$-system generated by $(\Sigma^{-1}\underline{V}, \underline{V})$. E^H is contractible for any $H \subset \mathbb{T}$, so $E^H \times_{\mathbb{T}} E\mathbb{T} \cong B\mathbb{T}$ and $\underline{\mathcal{E}}_{\mathbb{T}}(E)$ is an acyclic $\mathbb{T}$-system. Thus by Lemma 17.71, there is a map α from $\mathfrak{W}$ to $\underline{\mathcal{E}}_{\mathbb{T}}(E)$ making the left part of the following diagram commute:

$$\begin{tikzcd}
\underline{\mathcal{E}}_{\mathbb{T}}(K) \arrow[rrr, "k^*"] \arrow[ddd] & & & \underline{\mathcal{E}}_{\mathbb{T}}(X) \arrow[ddd] \\
& \mathfrak{V} \arrow[ul, "\rho", "\cong_{H^*}"'] \arrow[r, "k^*\rho"] \arrow[d] & \underline{\mathcal{E}}_{\mathbb{T}}(X) \arrow[ur, equal] \arrow[d] & \\
& \mathfrak{W} \arrow[r] \arrow[dl, "\alpha"', "\cong_{H^*}"] & \underline{\mathcal{E}}_{\mathbb{T}}(X) \otimes_{\mathfrak{V}} \mathfrak{W} \arrow[dr, "\gamma"] & \\
\underline{\mathcal{E}}_{\mathbb{T}}(E) \arrow[rrr] & & & \underline{\mathcal{E}}_{\mathbb{T}}(Y)
\end{tikzcd}$$

The inner square is EM by Lemma 17.72, and the outer by Lemma 16.69; the maps labeled with $\cong_{H^*}$ are quasi-isomorphisms. Note that α is a morphism between acyclic $\mathbb{T}$-systems and must automatically be a quasi-isomorphism. By Lemma 16.64, the induced map $\gamma : \underline{\mathcal{E}}_{\mathbb{T}}(X) \otimes_{\mathfrak{V}} \mathfrak{W} \to \underline{\mathcal{E}}_{\mathbb{T}}(Y)$ is also a quasi-isomorphism.

We are given a quasi-isomorphism $f : \mathfrak{G} \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$. Since $\mathfrak{V}$ is minimal, there are a lift of $k^*\rho$ to $\alpha : \mathfrak{V} \to \mathfrak{G}$ and a homotopy $H : k^*\rho \cong f\alpha$ by Proposition 3.6. Then in the diagram

$$\begin{tikzcd}
\mathfrak{V} \arrow[rrr, "f\alpha"] \arrow[ddd] \arrow[dr, equal] & & & \underline{\mathcal{E}}_{\mathbb{T}}(X) \arrow[ddd] \\
& \mathfrak{V} \arrow[r, "\alpha"] \arrow[d] & \mathfrak{G} \arrow[ur, "f", "\cong_{H^*}"'] \arrow[d] & \\
& \mathfrak{W} \arrow[r] \arrow[dl, equal] & \mathfrak{G} \otimes_{\mathfrak{V}} \mathfrak{W} \arrow[dr, "\mu"] & \\
\mathfrak{W} \arrow[rrr] & & & \underline{\mathcal{E}}_{\mathbb{T}}(X) \otimes_{\mathfrak{V}} \mathfrak{W}
\end{tikzcd}$$

both inner and outer squares are EM by Lemma 16.69, so again Lemma 16.64 implies that the map $\mu : \mathfrak{G} \otimes_{\mathfrak{V}} \mathfrak{W} \to \underline{\mathcal{E}}_{\mathbb{T}}(X) \otimes_{\mathfrak{V}} \mathfrak{W}$ is a quasi-isomorphism.

Recall that $\underline{\mathcal{E}}_{\mathbb{T}}(X) \otimes_{\mathfrak{V}} \mathfrak{W}$ is an elementary extension induced by the map from $\mathfrak{V}$ to $\underline{\mathcal{E}}_{\mathbb{T}}(X)$. By construction, $f\alpha \simeq k^*\rho$, so Corollary 11.54 gives an isomorphism $\underline{\mathcal{E}}_{\mathbb{T}}(X)_{f\alpha}(\underline{V}^*) \to \underline{\mathcal{E}}_{\mathbb{T}}(X)_{k^*\rho}(\underline{V}^*)$ between the elementary extensions they define. Then composition gives quasi-isomorphisms

$$\mathfrak{G}(\underline{V}^*) = \mathfrak{G} \otimes_{\alpha} \mathfrak{W} \to \underline{\mathcal{E}}_{\mathbb{T}}(X)_{f\alpha}(\underline{V}^*) \to \underline{\mathcal{E}}_{\mathbb{T}}(X)_{k^*\rho}(\underline{V}^*).$$

To see uniqueness, note that the homotopy class of α must be a lift of $k^*\rho$. If α' is another such lift then $f\alpha \simeq f\alpha' \simeq k^*\rho$, and so $f^*[\alpha] = f^*[\alpha'] \in H^{n+1}(\underline{\mathcal{E}}_{\mathbb{T}}(X), \underline{V}^*)$. But f is a quasi-isomorphism and thus an isomorphism on cohomology, so $[\alpha] = [\alpha'] \in H^{n+1}(\mathfrak{G}, \underline{\pi}^*)$. By Proposition 11.52, the elementary extensions determined by α and α' are isomorphic.

Conversely, suppose $\mathfrak{G}(\underline{V})$ is an elementary extension of $\mathfrak{G}$. The map $d : \underline{V} \to \underline{Z}^{n+1}(\widetilde{\mathfrak{G}})$ inducing the elementary extension represents an element

$$[d] \in H^{n+1}(\widetilde{\mathfrak{G}}, \underline{V}) \cong H^{n+1}(\tilde{\underline{\mathcal{E}}}_{\mathbb{T}}(X), \underline{V}) \cong H^{n+1}_{\mathbb{T}}(X, \underline{V}^*).$$

This cohomology class determines a map $k : X \to K(\underline{V}^*, n+1)$; let Y be the pullback of the path-space fibration along this map. Then by Theorem 6.31, there is an elementary extension $d' : \underline{V} \to \widetilde{\mathfrak{G}}$ such that $\mathfrak{G}_{d'}(\underline{V})$ is geometric for Y. The way it was constructed forces $[d']$ to map to $[k] \in H^{n+1}(X, \underline{V}^*)$. So $[d] = [d'] \in H^{n+1}(\widetilde{\mathfrak{G}}, \underline{V})$, and Proposition 11.52 implies that the elementary extensions determined by d and d' are isomorphic. Since the k-invariant is determined, this fibration is unique up to homotopy. $\square$

Proof of 6.32. This result follows from applying Theorem 6.31 and observing that cohomolgy comutes with filtered limits. $\square$

19. Proofs of the main results

Again, the proofs for finite G and $\mathbb{T}$-spaces are very similar, and we only give the arguments for $\mathbb{T}$-spaces.

Proof of Theorem 6.28. Produce a minimal model $\mathfrak{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$; then $\widetilde{\mathfrak{M}}_X \to \tilde{\underline{\mathcal{E}}}_{\mathbb{T}}(X)$ is a quasi-isomorphism and thus an isomorphism of cohomology with respect to any dual coefficient system N. So

$$H^*(\widetilde{\mathfrak{M}}_X; N) \cong H^*(\tilde{\underline{\mathcal{E}}}_{\mathbb{T}}(X); N) \cong H^*_{\mathbb{T}}(X; N^*),$$

where the second isomorphism is Theorem 15.62, the $\mathbb{T}$-equivariant de Rham theorem. Moreover, the correspondence between principal $\mathbb{T}$-fibrations and elementary extensions of Theorems 6.31 and 6.32, proved in the previous section, imply that

(i) $\mathfrak{M}_X(n) = \mathfrak{M}_X(n-1)(W_n)$, where $W_n^* = \underline{\pi}_n(X) \otimes \mathbb{Q}$; and
(ii) $\mathfrak{M}_X$ models the equivariant Postnikov tower of X in the sense that
- $\mathfrak{M}_X(n)$ is the minimal model of X_n, and
- the map $W_n \to Z^{n+1}(\widetilde{\mathfrak{M}}_X(n-1))$ inducing the elementary extension determines the rational k-invariant

$$k \in H^*_{\mathbb{T}}(X_{n-1}; \underline{\pi}_n(X)) \otimes \mathbb{Q}.$$

$\square$

Proof of Theorem 6.29. First, suppose $f : X \to Y$ is a $\mathbb{T}$-equivariant map; composition yields a map $\mathfrak{M}_Y \xrightarrow{\gamma_Y} \underline{\mathcal{E}}_{\mathbb{T}}(Y) \xrightarrow{f^*} \underline{\mathcal{E}}_{\mathbb{T}}(X)$. Since $\gamma_X : \mathfrak{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$ is a quasi-isomorphism and $\mathfrak{M}_Y$ is minimal, by Proposition 3.6 there is a lift, unique up to homotopy, making the following diagram commute up to homotopy:

$$\begin{tikzcd} \mathfrak{M}_Y \arrow[r, "\alpha"] \arrow[d, "\gamma_Y"] & \mathfrak{M}_X \arrow[d, "\gamma_X"] \\ \underline{\mathcal{E}}_{\mathbb{T}}(Y) \arrow[r, "f^*"] & \underline{\mathcal{E}}_{\mathbb{T}}(X) \end{tikzcd}$$

Conversely, suppose $\alpha : \mathfrak{M}_Y \to \mathfrak{M}_X$ is a morphism of $\mathbb{T}$-systems. Lemma 13.57 allows us to assume that α restricted to $\mathfrak{M}_Y(n)$ lands in $\mathfrak{M}_X(n)$ for all n without changing the homotopy class of the map. We will build a map between the Postnikov towers of X and Y. Assume inductively that there is a $\mathbb{T}$-map $f_{n-1} : X_{n-1} \to Y_{n-1}$

such that the diagram

$$\begin{array}{ccc} \mathfrak{M}_Y(n-1) & \xrightarrow{\alpha} & \mathfrak{M}_X(n-1) \\ \Big\downarrow{\scriptstyle \gamma_Y} & & \Big\downarrow{\scriptstyle \gamma_X} \\ \underline{\mathcal{E}}_{\mathbb{T}}(Y_{n-1}) & \xrightarrow{f_{n-1}^*} & \underline{\mathcal{E}}_{\mathbb{T}}(X_{n-1}) \end{array}$$

commutes up to homotopy, and consider the principal fibrations

$$\begin{array}{ccccc} X_n & \overset{g}{\dashrightarrow} & Y_n & \longrightarrow & E \\ {\scriptstyle p_X^*}\Big\downarrow & & {\scriptstyle p_Y^*}\Big\downarrow & & \Big\downarrow \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} & \xrightarrow{k} & K(\underline{\pi}_n(Y), n+1) \end{array}$$

The obstruction to producing g is the cohomology class

$$p_X^* f_{n-1}^*[k] \in H_{\mathbb{T}}^{n+1}(X_n; \underline{\pi}_n(Y)).$$

Now the inclusion $i_Y : \mathfrak{M}_Y(n-1) \to \mathfrak{M}_Y(n)$ allows us to insider the class $i_Y[\mu] \in H^{n+1}(\mathfrak{M}_Y(n), \underline{\pi})$; since the model map γ_Y respects the Postnikov decomposition, $i_Y[\mu]$ corresponds to the class $p_Y^*[k] \in H_{\mathbb{T}}^{n+1}(Y_n; \underline{\pi}_n(Y))$. But $p_Y^*[k]$ is the obstruction to the existance of a lift $Y_n \to E$. Since this lift clearly exists, $p_Y^*[k] = 0$ and so $i_Y[\mu] = 0 \in H^{n+1}(\mathfrak{M}_Y(n); \underline{\pi})$.

Then $p_X^* f_{n-1}^*[k] = p_X^* \gamma_Y[\mu]$, since the top diagram commutes up to homotopy; and since α is assumed to be cocellular and respect the inclusions $\mathfrak{M}(n-1) \to \mathfrak{M}(n)$, we find that $p_X^* \gamma_Y[\mu] = p_X^* \gamma_X \alpha[\mu] = p_X^* \gamma_X \alpha i_Y[\mu] = 0$. So the lift g exists.

Now lift g^* to a map $\beta : \mathfrak{M}_Y(n) \to \mathfrak{M}_X(n)$ by minimality. Then, on $\mathfrak{M}_Y(n-1)$, β is homotopic to α by uniqueness of this lift. The obstruction to extending this homotopy to $\mathfrak{M}_Y(n)$ is $[\beta - \alpha] \in H^{n+1}(\mathfrak{M}_X(n), \underline{\pi}_n(Y)) \cong H_{\mathbb{T}}^{n+1}(X_n, \underline{\pi}_n(Y))$; as discussed, this latter group corresponds to lifts of f_{n-1} to $X_n \to Y_n$. So by taking the lift $\tilde{g}$ corresponding to the class $[\beta - \alpha]$ and defining $f_n = g - \tilde{g}$, we obtain a lift f_n for which the obstruction vanishes, and we can extend to a homotopy between f_n^* and α. Note that this condition ensures that the lift f_n is specified uniquely up to homotopy.

This process builds a map between the Postnikov towers which induces a $\mathbb{T}$-map $f : X \to Y$ such that f^* lifts to the homotopy class of α. This gives the desired bijection of homotopy classes of maps. □

20. A proof of the localization theorem

In this section we recast the classical localization theorem in the context of $\mathbb{T}$-minimal models.

Theorem 20.73. *Let X be a finite dimensional $\mathbb{T}$-CW complex, and assume X is $\mathbb{Q}$-good. Let S be the multiplicative set generated by $c \in \mathbb{Q}[c]$. Then the inclusion $X^{\mathbb{T}} \subset X$ induces an isomorphism*

$$S^{-1}H^*(X \times_{\mathbb{T}} E\mathbb{T}) \cong S^{-1}H^*(X^{\mathbb{T}} \times_{\mathbb{T}} E\mathbb{T}) = H^*(X^{\mathbb{T}}) \otimes \mathbb{Q}[c, c^{-1}].$$

Proof. Observe that if $\mathfrak{M}_X$ is the minimal model of the $\mathbb{T}$-space X, then by Theorem 6.28 the system $\underline{H}^*(\mathfrak{M}_X) = \underline{H}^*(X \times_{\mathbb{T}} E\mathbb{T})$, and the structure map

$$H^*(\mathfrak{M}) = H^*(\mathfrak{M}_X(\mathbb{T}/e)) \to H^*(\mathfrak{M}_X(\mathbb{T}/\mathbb{T})) = H^*(\mathfrak{M}_{\mathbb{T}} \otimes \mathbb{Q}[c])$$

is induced by the inclusion $X^{\mathbb{T}} \times B\mathbb{T} \to X \times_{\mathbb{T}} E\mathbb{T}$. Since localization is exact, it suffices to show that the kernel and cokernel of this map are c-torsion.

If X is a finite dimensional $\mathbb{T}$-CW complex, then its equivariant cohomology $H^N_{\mathbb{T}}(X) = 0$ for large N. Since $\widetilde{\mathfrak{M}}$ computes this cohomology (again by Theorem 6.28), we see that $\underline{H}^N(\widetilde{\mathfrak{M}}) = 0$ for large N also. Now suppose $[a] \in H^i(\mathfrak{M})$ is in the kernel of the displayed map; then $[a] \to [a_{\mathbb{T}}] = 0 \in H^*(\mathfrak{M}_{\mathbb{T}}) \otimes \mathbb{Q}[c]$ under the structure map. This means that there is an element $b_{\mathbb{T}} \in \mathfrak{M}(\mathbb{T}/\mathbb{T})$ such that $db_{\mathbb{T}} = a_{\mathbb{T}}$. Lifting $b_{\mathbb{T}}$ to an element $b \in \mathfrak{M}$ (recall that $\mathfrak{M}_X$ is injective, and apply Corollary 8.40), and replacing a with $a - db$, we may assume that $a_{\mathbb{T}} = 0$. This ensures also that for any m, the image under the structure map of that ac^m is $a_{\mathbb{T}}c^m = 0$. So $ac^m \in \widetilde{\mathfrak{M}}$ and thus represents a cohomology class $[ac^m] \in H^{i+2m}(\widetilde{\mathfrak{M}})$. For m large, the group $H^{i+2m}(\widetilde{\mathfrak{M}}) = 0$, and thus $[a]c^m = 0$ and $[a]$ is c-torsion.

Now consider an element $[a_{\mathbb{T}}] \in H^*(\mathfrak{M}_{\mathbb{T}}) \otimes \mathbb{Q}[c]$ in the cokernel. If we lift $a_{\mathbb{T}}$ to $a \in \mathfrak{M}$ (again using Corollary 8.40), then naturality ensures that the image under the structure map $da_{\mathbb{T}} = 0$, and so $da \in \widetilde{\mathfrak{M}}$ represents a class $[da] \in H^*(\mathfrak{M})$ which is in the kernel. By the argument of the previous paragraph, we know that $[da]c^m = 0 \in H^*(\widetilde{\mathfrak{M}})$ for m large. Let $b \in \widetilde{\mathfrak{M}}$ be such that $db = (da)c^m$; then $(ac^m - b)$ represents a cohomology class $[ac^m - b] \in H^*(\mathfrak{M})$, and $[ac^m - b] \to [a_{\mathbb{T}}]c^m - [b_{\mathbb{T}}]$ under the structure map. So $[a_{\mathbb{T}}]c^m$ and $[b_{\mathbb{T}}]$ represent the same element of the cokernel, and we have shown that for large m, $[a_{\mathbb{T}}]c^m$ is equivalent in the cokernel to a class $[b_{\mathbb{T}}]$ represented by an element in $\widetilde{\mathfrak{M}}(\mathbb{T}/\mathbb{T}) = \mathfrak{M}_{\mathbb{T}}$. Now observe that X is a finite dimensional $\mathbb{T}$-CW complex, and so $H^m(\mathfrak{M}_{\mathbb{T}}) = 0$ for large m. Therefore the cokernel class $[b_{\mathbb{T}}] = [a_{\mathbb{T}}]c^m = 0$. Thus the cokernel is also c-torsion. □

21. Equivariant minimality

When we defined equivariant minimality for functors in Sections 3 and 5, we used a definition based on elementary extensions. The minimal structure algebraically models the Postnikov decomposition of a space, where each elementary extension corresponds to one level of the rational Postnikov tower. Non-equivariantly, there is an alternate approach to defining minimality. The original definition given in [S, DGMS] stated that a DGA $\mathcal{A}$ is minimal if it is free and the differential on $\mathcal{A}$ is decomposible. It is not hard to show that this is equivalent to the condition that the DGA be composed of a sequence of elementary extensions.

The original approach to defining equivariant minimality for finite group actions was a variation of this decomposibility condition. In [T1], a system of DGAs $\mathfrak{M}$ is defined to be minimal if it satisfies the following properties:

1. $\mathfrak{M}(G/H)$ is free for all $H \subseteq G$,
2. $\mathfrak{M}(G/G)$ is a minimal DGA, and
3. the differential d is decomposible when restricted to
$$\bigcap_{K \supset H} \ker \mathfrak{M}(\hat{e}_{H,K}) \subset \mathfrak{M}(G/H),$$
where $\hat{e}_{H,K} : G/H \to G/K$ is the projection.

This definition is designed to have the same algebraic properties as minimal systems of DGAs, as these properties can generally be shown by induction over the subgroups of G, with the base case given by the minimality of $\mathfrak{M}(G/G)$ and the induction step using the fact that d is decomposible on the part of $\mathfrak{M}(G/H)$ which does not come from induced maps within the diagram. However, the algebraic complexity involved in keeping all functors injective at every stage means that it is not possible to have this condition and also decompose the structure as a sequence of elementary extensions. The following example demonstrates this.

Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ for two different primes p, q. Then G has exactly four subgroups, G, $H_p = \mathbb{Z}/q\mathbb{Z}$, $H_q = \mathbb{Z}/p\mathbb{Z}$ and $\{e\}$. Now let $\mathfrak{A}$ be the following system of DGAs, where each entry is a free DGA on the indicated generators, all entries are fixed under any internal structure maps, and the maps shown are the obvious inclusions:

$$\begin{tikzcd}
 & \mathfrak{A}(G/G) = \mathbb{Q}(x) \arrow[dl] \arrow[dr] & \\
\mathfrak{A}(G/H_p) = \mathbb{Q}(x, y_p) \arrow[dr] & & \mathfrak{A}(G/H_q) = \mathbb{Q}(x, y_q) \arrow[dl] \\
 & \mathfrak{A}(G/e) = \mathbb{Q}(x, y_p, y_q) &
\end{tikzcd}$$

In $\mathfrak{A}$, x has degree 3, and y_p, y_q have degree 4. The differential is zero on $\mathfrak{A}(G/G)$, takes x to y_n on $\mathfrak{A}(G/H_n)$, and takes x to $y_p + y_q$ on $\mathfrak{A}(G/e)$.

Observe that the differential is decomposible on all intersections of kernels of structure maps, and so this system satisfies the original definition of minimal. However, it is not composed of elementary extensions. The procedure for creating elementary extensions to approximate $\mathfrak{A}$ begins with an injective resolution of $\underline{H}^3$, the third degree cohomology, and this injective resolution includes an additional generator of degree 5 in $\mathfrak{A}(G/e)$. Therefore even at this initial stage in the construction it is clear that the elementary extension-composed approximation is not isomorphic to the original $\mathfrak{A}$, and that the two minimality conditions are incompatible in this case.

References

[AH] C. Allday and S. Halperin, *Sullivan-deRham Theory for Rational Alexander-Spanier Cohomology*, Houston Journal of Math. vol 10 pp 15 - 33 (1984). MR **85d:**55014

[BG] A.K. Bousfield and V.K.A.M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc., vol 8 no 179 (1976). MR **54:**13906

[B1] G. Bredon, *Equivariant Cohomology Theories*, Lecture Notes in Math., vol 34 (1967). MR **35:**4914

[B2] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press (1972). MR **54:**1265

[DGMS] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. vol 29 pp 245-274 (1975). MR **52:**3584

[tD] T. tom Dieck, *Transformation Groups*, de Gruyter Studies in Mathematics 8 (1987). MR **89c:**57048

[E] A. Elmendorf, *Systems of fixed point sets*, Trans. AMS. vol 277 pp 275-284 (1983). MR **84f:**57029

[F] B.L. Fine, *Disconnected equivariant rational homotopy theory and formality of compact G-Kähler manifolds*, thesis (1992).

[FT] B.L. Fine and G. Triantafillou, *On the equivariant formality of Kähler manifolds with finite group action*, Canad. J. Math. vol. 45 pp 1200-1210 (1993). MR **94j:**55010

[Go] M. Golasinski, *Componentwise injective models of functors to DGAs*, Colloq. Math. vol 73 pp 83-92 (1997). MR **98b:**55012

[G] J.P.C. Greenlees, *Rational S^1-equivariant stable homotopy theory*, Mem. Amer. Math. Soc. vol 138. no 661 (1998)

[H] S. Halperin, *Lectures on Minimal Models*, Publ. IRMA Lille vol 3 no 4 (1981); also Mém. Soc. Math. France (N.S.) no 9/10 (1983). MR **83j:**55008; MR **85i:**55009

[HMR] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North Holland (1975). MR **57:**17635

[M] J.P. May, *Equivariant Homotopy and Cohomology Theory*, CBMS Lectures vol 91 (1997). MR **97k:**55016

[RT1] M. Rothenberg and G. Triantafillou, *An algebraic model for G-simple homotopy types*, Math. Ann. vol 269 pp 301-331 (1984).

[RT2] M. Rothenberg and G. Triantafillou, *On the classification of G-manifolds up to finite ambiguity*, Comm. Pure App. Math, vol 44 pp 761-788 (1991). MR **92f:**57049

[S] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Etudes Sci. Publ. Math. vol 47 pp 269-332 (1977). MR **58:**31119

[T1] G. Triantafillou, *Equivariant minimal models*, Trans. Amer. Math. Soc. vol 274 pp 509-532 (1982). MR **84g:**55017

[T2] G. Triantafillou, *Rationalization of Hopf G-spaces*, Math. Z. vol 182 pp 485-500 (1983). MR **84h:**55008

[W] S. Willson, *Equivariant homology theories on G-complexes*, Trans. Amer. Math. Soc. vol 212 pp 155-171 (1975). MR **51:**14028

Department of Mathematics, The University of Michigan, East Hall, 525 East University Avenue, Ann Arbor, Michigan 48109-1109

E-mail address: `laurass@math.lsa.umich.edu`

Current address: Department of Mathematics, University of British Columbia, 1984 Mathematics Rd., Vancouver, BC V6T 1Z2, Canada