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SHELLABILITY IN REDUCTIVE MONOIDS

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ABSTRACT. The purpose of this paper is to extend to monoids the work of Björner, Wachs and Proctor on the shellability of the Bruhat-Chevalley order on Weyl groups. Let M be a reductive monoid with unit group G, Borel subgroup B and Weyl group W. We study the partially ordered set of $B \times B$-orbits (with respect to Zariski closure inclusion) within a $G \times G$-orbit of M. This is the same as studying a $W \times W$-orbit in the Renner monoid R. Such an orbit is the retract of a 'universal orbit', which is shown to be lexicograhically shellable in the sense of Björner and Wachs.

INTRODUCTION

The combinatorial concept of shellability of a simplicial complex provides a powerful link between algebra, topology of geometry, [3], [6], [21]. A shellable complex has the homotopy type of a wedge of r-spheres and its Stanley-Reisner ring is Cohen-Macaulay. Björner [1] and Björner and Wachs [2] have introduced the stronger concept of lexicographic shellability of a poset. It has been shown in [2], [13] that the Bruhat-Chevalley order on a Weyl group is lexicographically shellable. This in turn has connections to the geometry of Schubert varieties [8], [9]. In this paper we apply the Björner-Wachs approach to reductive monoids.

Reductive monoids are Zariski closures of reductive groups. They arise naturally connection with embeddings of some symmetric spaces [7], the behaviour at infinity of a Lie group [22] and Schur algebras [10]. They have been studied for the last 20 years by Lex Renner and the author. There is a monograph [15] on the earlier work. There is also an excellent expository paper by Solomon [20].

Our focus in this paper is on the Bruhat decomposition for reductive monoids [17], where the Weyl group W is replaced by the Renner monoid R. The Bruhat-Chevalley order on R, first studied by Renner [17], [18], remains quite mysterious. We studied this order in detail in an earlier paper [12] (with Pennell and Renner). In particular, we obtained an algebraic description of the order. The main purpose of the present paper is to study this order on the $W \times W$-orbits of R. We show that such an orbit is isomorphic to a nicely constructed poset $\mathcal{W}_{I,K}$, where I is a set of simple reflections and K is a union of some components of I. $\mathcal{W}_{I,K}$ is a retract of a universal orbit $\mathcal{W}_I = \mathcal{W}_{I,\emptyset}$ (which arise as maximal orbits of some Renner monoid). Making use of the methods of Björner and Wachs, we show that a universal orbit $\mathcal{W}_I$ and its dual are lexicographically shellable, Eulerian posets. In particular their Stanley-Reisner rings are Gorenstein. The question of whether the Bruhat-Chevalley order is shellable on R, remains open.

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1. Reductive monoids

Let k be an algebraically closed field. By a *reductive monoid* M we will mean an irreducible linear algebraic monoid M defined over k such that the unit group G is reductive. Let T be a maximal torus contained in a Borel subgroup B of G. Let $W = N_G(T)/T$ denote the Weyl group of G and let S denote the generating set of simple reflections of W. Then G has the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB. \tag{1}$$

By the theory of torus embeddings [11], the diagonal idempotents (i.e. the idempotents in $\bar{T}$) form a finite lattice that is isomorphic to the face lattice of a rational polytype $\mathcal{P}$. We have shown in [14] that there is a diagonal idempotent cross-section Λ of $G \times G$-orbits of M such that for all $e, f \in \Lambda$,

$$e \leq f \Leftrightarrow e \in MfM. \tag{2}$$

Here as usual [5], $e \leq f$ means that $ef = e = fe$. Λ is a finite lattice called the *cross-section lattice* of M. Λ may also be viewed as the quotient of the face lattice of $\mathcal{P}$ by the action of W. All maximal chains of Λ have the same length. Λ is unique up to conjugacy by an element of W. We note that in the case of the multiplicative monoid $M_n(k)$ of all $n \times n$ matrices,

$$\Lambda = \left\{ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \middle| \, 0 \leq r \leq n \right\}$$

is the usual set of idempotent representatives of matrices of different ranks. In general, determining the possible lattices Λ (in terms of face lattices of polytopes) remains a difficult open problem. However, when M is the Zariski closure of the image of an irreducible representation of a reductive group, the problem has been solved in [16].

Example 1.1. The table in Figure 1 lists the cross-section lattice Λ and the polytope $\mathcal{P}$ when M is the closure of the image of a representation of $M_4(k)$.

In [17] the Bruhat decomposition (1) is extended to M as

$$M = \bigsqcup_{\sigma \in R} B\sigma B \tag{3}$$

where $R = \overline{N_G(T)}/T$ is the *Renner monoid*. W is the unit group of R. If $\mathcal{W}(e) = WeW$, $e \in \Lambda$, then

$$R = \bigsqcup_{e \in \Lambda} \mathcal{W}(e). \tag{4}$$

By a *maximal* $W \times W$-orbit, we will mean an orbit maximal in $R \backslash W$. If R has a zero, then by a *minimal* $W \times W$-orbit we will mean an orbit minimal in $R \backslash \{0\}$. We note that for $M_n(k)$, W is the symmetric group of permutation matrices, R is the symmetric inverse semigroup of all partial permutation matrices, and a $W \times W$-orbit $\mathcal{W}(e)$ consists of partial permutation matrices of a particular rank.

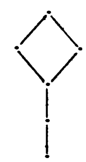
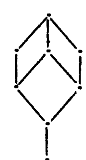
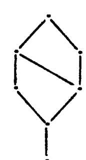
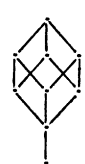
Representation	$\boldsymbol{\Lambda}$	$\boldsymbol{\mathcal{P}}$
$A \to A$		Tetrahedron
$A \to \wedge^2 A$		Octahedron
$A \to A \otimes \wedge^3 A$		Cuboctahedron
$A \to A \otimes \wedge^2 A$		Truncated tetrahedron
$A \to A \otimes \wedge^2 A \otimes \wedge^3 A$		Truncated octahedron

FIGURE 1.

2. Bruhat-Chevalley order

The Bruhat-Chevalley order on the Weyl group W, first studied in the 1950s by Chevalley [4], is defined as

$$x \leq y \quad \text{if } BxB \subseteq \overline{ByB}. \tag{5}$$

As is well known, this is equivalent to x being a subword of a reduced expression $y = s_1 \cdots s_m$, $s_1, \ldots, s_m \in S$. The length $\ell(y)$ is defined to be m. If $w_1, \ldots, w_n \in$

W, let

$$w_1 * \cdots * w_n = \begin{cases} w_1 \cdots w_n & \text{if } \ell(w_1 \cdots w_n) = l(w_1) + \cdots + \ell(w_n), \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let x, y, z, $winW$. Then*:
(i) *If $x \leq y$ and $xw = x * w$, then $x * w \leq y * u$ for some $u \leq w$.*
(ii) *If $x * w \leq x * z$, then $w \leq z$.*

Proof. (i) We proceed by induction on $\ell(w)$. If $\ell(w) = 0$, this is clear. So let $\ell(w) > 0$. Then $w = w_1 * s, s \in S$. By induction hypothesis, $xw_1 = x * w_1 \leq y * u_1$ for some $u_1 \leq w_1$. Since $xw = xw_1 * s$, we see that either $xw \leq yu_1$ or else $xw \leq yu_1 * s = y * u_1 s$.

(ii) We proceed by induction on $\ell(x)$. If $\ell(x) = 0$, this is clear. So let $\ell(x) > 0$. Then $x = s * x_1$, $s \in S$. Then either $x * w \leq x_1 * z$ or else $x_1 * w \leq x_1 * z$. In either case $x_1 * w \leq x_1 * z$. By induction hypothesis, $w \leq z$. □

For $I \subseteq S$, let W_I denote the parabolic subgroup of W generated by I and let

(6) $$D_I = \{x \in W \mid xw = x * w \quad \text{for all } w \in W_I\}.$$

Let v_0, w_0 denote the longest elements of W_I and W respectively. Then for $x \in D_I$, $w_0 x v_0 \in D_I$ and

(7) $$x \leq y \Leftrightarrow w_0 y v_0 \leq w_0 x v_0 \quad \text{for all } x, y \in D_I.$$

Moreover for all $x \in D_I$,

(8) $$\ell(w_0 x v_0) = \ell(w_0) - l(v_0) - \ell(x) = \ell(w_0 v_0) - \ell(x).$$

Lemma 2.2. *Let $x, y \in D_I$ $w, u \in W_I$ such that $xw \leq yu$. Then $w = w_1 * w_2$ with $xw_1 \leq y$ and $w_2 \leq u$.*

Proof. We proceed by induction on $\ell(u)$. If $\ell(u) = 0$, this is clear. So let $\ell(u) > 0$. Then $u = u_1 * s$, $s \in I$. If $xw \leq yu_1$, then we are done by the induction hypothesis. Otherwise, since $yu = yu_1 * s$, $xws < xw$. Since $xws = x * ws$ and $xw = x * w$, we see by Lemma 2.1 (ii) that $ws < w$. So $w = w' * x$ and $xw' \leq yu_1$. By induction hypothesis $w' = w_1 * w_2$ with $xw_1 \leq y$ and $w_2 \leq u_1$. Then $w = w_1 * w_2 * s$ and $w_2 * s \leq u$. □

For $K \subseteq I \subseteq S$, we will write $K \triangleleft I$ if K is a union of some components of I (including the possibility that $K = \emptyset$). In such a case

(9) $$W_I = W_K \times W_{I \setminus K}, \quad D_K = D_I W_{I \setminus K}.$$

Now for monoids. The order (5) on W extends naturally to R if we define

(10) $$\sigma \leq \theta \quad \text{if } B\sigma B \subseteq \overline{B\theta B}.$$

Renner [17] has shown that all the maximal chains in R have the same length.

Example 2.3. For $M = M_2(k)$, the poset $(R, \leq)$ is given in Figure 2.

In general the order $\leq$ on R is much more subtle than on W. We have studied this order in [12] (with Pennell and Renner). In particular we found an algebraic

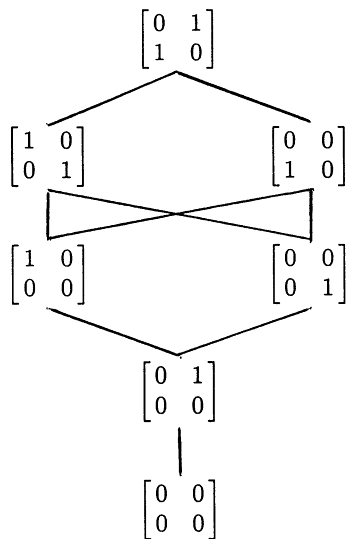


FIGURE 2.

description of this order that we now describe. For $e \in \Lambda$, we have the parabolic subgroups,

$$W(e) = \{w \in W \mid we = ew\},$$
$$W_e = \{w \in W \mid we = e = ew\}$$

of W. Then by (9),

$$W(e) = W_e \times \widetilde{W(e)}$$

from some parabolic subgroup $\widetilde{W(e)}$ of W. If $W(e) = W_I$ and $W_e = W_K$, then let

$$D(e) = D_I, \quad D_e = D_K. \tag{11}$$

Then by (9),

$$D_e \cap W(e) = \widetilde{W(e)}, \quad D_e = D(e)\widetilde{W(e)}. \tag{12}$$

We note that for $M_n(k)$, if $e = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then $W(e)$ consists of permutation matrices of the form $\left[\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right]$, W_e consists of permutation matrices of the form $\left[\begin{smallmatrix} I & 0 \\ 0 & Q \end{smallmatrix}\right]$ and $\widetilde{W(e)}$ consists of permutation matrices of the form $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$.

If $\sigma \in R$, then

$$\sigma = xey \quad \text{for unique } e \in \Lambda,\ x \in D_e,\ y \in D(e)^{-1}. \tag{13}$$

We call this the *standard form* of σ. Let $\sigma, \theta \in R$. Let $\sigma = xey$, $\sigma' = x'e'y'$ in standard form. Then by [12],

$$\sigma \leq \sigma' \Leftrightarrow e \leq e', x \leq x'w, w^{-1}y' \leq y \quad \text{for some } w \in W(e')W_e. \tag{14}$$

Fix $e \in \Lambda$. Our interest is in the poset $\mathcal{W}(e) = WeW$. Then for $\sigma = xey$, $\sigma' = x'ey'$ in standard form, (14) simplifies to

$$\sigma \leq \sigma' \Leftrightarrow x \leq x'w, w^{-1}y' \leq y \quad \text{for some } w \in W(e). \tag{15}$$

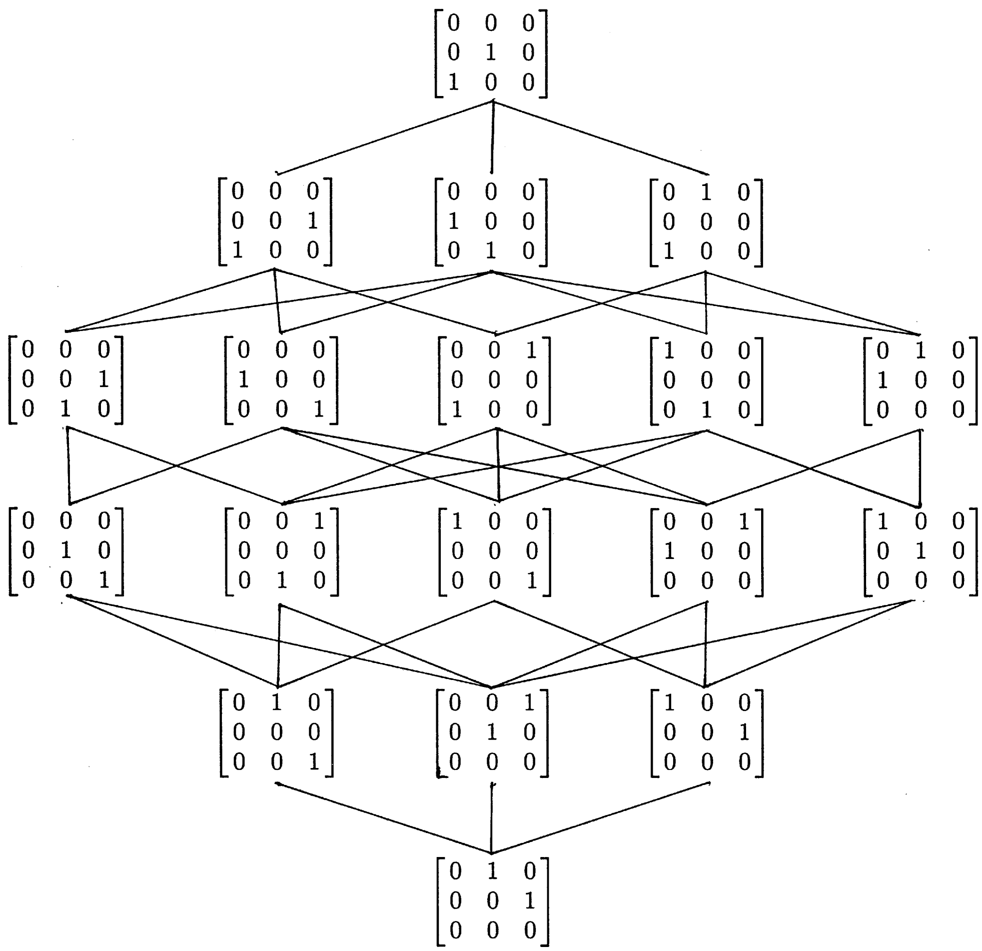


FIGURE 3.

Let $W(e) = W_I$. We call I the *type* of the $W \times W$-orbit $\mathcal{W}(e)$. If $W_e = W_K$, $K \lhd I$, then we call K the *subtype* of $\mathcal{W}(e)$. Thus the $W \times W$-orbits $\mathcal{W}(e)$ are classified first according to type, and then according to subtype. We will call $\mathcal{W}(e)$ *universal* if $W_e = 1$ and *fundamental* if $W_e = W(e)$. Let v_0, w_0 denote respectively the longest elements of W_I and W. Then $w_0 e, ev_0 w_0$ are respectively the maximum and minimum elements of $\mathcal{W}(e)$. If $\sigma = xey \in \mathcal{W}(e)$ in standard form, then any maximal chain from σ to $ev_0 w_0$ has length

$$\ell(\sigma) = \ell(x) - \ell(y) + \ell(v_0 w_0) = \ell(x) + \ell(v_0 y w_0). \tag{16}$$

This agrees with the definition of length $\ell(\sigma)$ given by Solomon [19] and Renner [18].

Example 2.4. Let $M = M_3(k)$. The poset of rank 2 elements of R is given in Figure 3 and the poset of rank 1 elements is given in Figure 4.

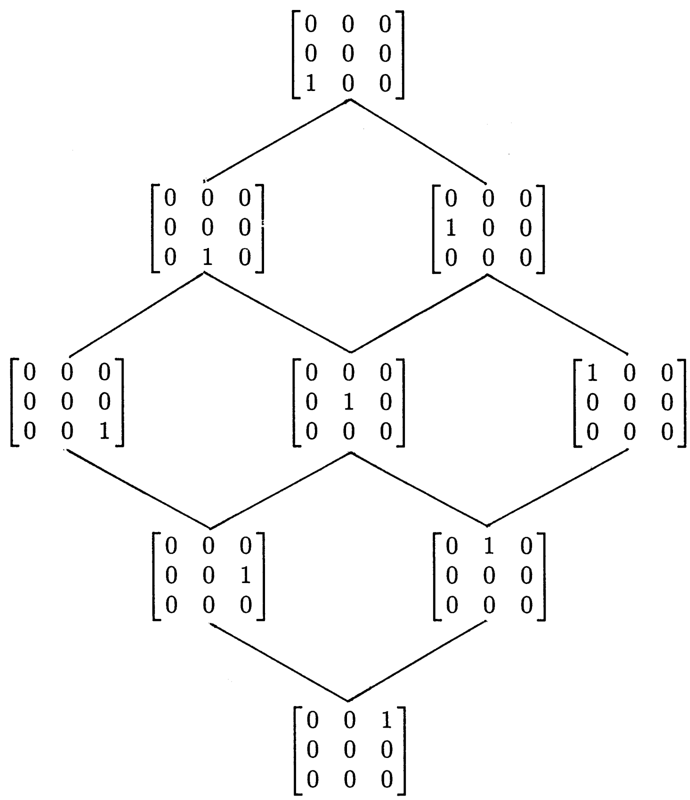


FIGURE 4.

We now proceed to obtain a more useful description of the $W \times W$-orbits $\mathcal{W}(e)$. Let $I \subseteq S$, $K \triangleleft I$. Let D_I be as in (6) and set

$$\mathcal{W}_{I,K} = D_I \times W_{I \setminus K} \times D_I^{-1}. \tag{17}$$

Let v_0 denote the longest element of W_I and let

$$\bar{w} = v_0 w v_0, \quad w \in W_I. \tag{18}$$

For $\sigma = (x, w, y)$, $\sigma' = (x', w', y') \in \mathcal{W}_{I,K}$, define

$$\sigma \leq \sigma' \quad \text{if } w = w_1 * w_2 * w_3 \quad \text{with} \quad x w_1 \leq x', w_2 \leq w', \bar{w}_3 y \leq y'. \tag{19}$$

Also define the *length*

$$\ell(\sigma) = \ell(x) + \ell(w) + \ell(y). \tag{20}$$

We call

$$\mathcal{W}_I = \mathcal{W}_{I,\emptyset}, \quad \mathcal{W}_I^{\vee} = \mathcal{W}_{I,I} \tag{21}$$

respectively the *universal* and *fundamental* orbit of type I. Clearly

$$\mathcal{W}_I^{\vee} = D_I \times D_I^{-1} \cong D_I \times D_I. \tag{22}$$

Let $\mathcal{W}_{I,K}^* = \mathcal{W}_{I,K}$ as sets. For $\sigma = (x, w, y)$, $\sigma' = (x', w', y') \in \mathcal{W}_{I,K}^*$, define

$$\sigma \leq \sigma' \quad \text{if } w = w_1 * w_2 * w_3 \quad \text{with} \quad x w_1 \leq x', w_2 \leq w', w_3 y \leq y'. \tag{23}$$

Note the subtle difference between (29) an (23).

Theorem 2.5. (i) *$\mathcal{W}^*_{I,K}$ is isomorphic to the dual of $\mathcal{W}_{I,K}$.*

(ii) *$\mathcal{W}_{I,K}$ is a retract of $\mathcal{W}_I$ and $\mathcal{W}_I^\vee$ is a retract of $\mathcal{W}_{I,K}$ with the fibres being isomorphic respectively to W_K and $W_{I\setminus K}$.*

(iii) *The orbit $\mathcal{W}(e)$ is isomorphic to $\mathcal{W}_{I,K}$ if I, K are respectively the type and subtype of $\mathcal{W}(e)$.*

(iv) *Any maximal orbit $\mathcal{W}(e)$ is universal and if R has a zero, then any minimal orbit is fundamental.*

Proof. Let u_0, v_0, w_0 denote respectively the longest elements of $W_{I\setminus K}$, W_I and W. For $w \in W_{I\setminus K}$, let $\bar{w} = v_0 w v_0 = u_0 w u_0$.

(i) For $\sigma = (x, w, y) \in \mathcal{W}_{I,K}$, let

$$\Phi(\sigma) = (w_0 x v_0, u_0 w, v_0 y w_0) \in \mathcal{W}^*_{I,K}.$$

Let $\sigma = (x, w, y)$, $\sigma' = (x', w', y') \in \mathcal{W}_{I,K}$ such that $\sigma \leq \sigma'$. Then

$$w = w_1 * w_2 * w_3 \quad \text{with } x w_1 \leq x', w_2 \leq w', \bar{w}_3 y \leq y'.$$

From $x w_1 \leq x'$, we deduce that

$$w_0 x' v_0 \cdot v_0 = w_0 x' \leq w_0 x w_1 = w_0 x v_0 \cdot v_0 w_1.$$

Since $v_0 = \bar{w}_1^{-1} * v_0 w_1$, we see by Lemma 2.1 (ii) that $w_0 x' v_0 \cdot \bar{w}_1^{-1} \leq w_0 x v_0$. Similarly from $\bar{w}_3 y \leq y'$, we deduce that $w_3^{-1} \cdot v_0 y' w_0 \leq v_0 y w_0$.

Thus

$$w_0 x' v_0 \cdot \bar{w}_1^{-1} \leq w_0 x v_0, \quad w_3^{-1} \cdot v_0 y' w_0 \leq v_0 y w_0. \tag{24}$$

Since $w_2 \leq w'$ and $w = w_1 * w_2 * w_3$, we see by Lemma 2.1 (i) that

$$w \leq w_1' * w' * w_3' \quad \text{for some } w_1' \leq w_1, w_3' \leq w_3. \tag{25}$$

Let

$$w_1'' = u_0 (w_1')^{-1} u_0, \quad w_2'' = u_0 w_1' w' w_3', \quad w_3'' = (w_3')^{-1}.$$

By computing the lengths, we see that $u_0 w' = w_1'' * w_2'' * w_3''$. By (24), (25)

$$w_0 x' v_0 \cdot w_1'' \leq w_0 x v_0, \quad w_2'' \leq u_0 w, \quad w_3'' \cdot v_0 y' w_0 \leq v_0 y w_0.$$

Hence $\Phi(\sigma') \leq \Phi(\sigma)$. Similarly $\Phi(\sigma') \leq \Phi(\sigma)$ implies that $\sigma \leq \sigma'$.

(ii) Clearly $\mathcal{W}_{I,K} \subseteq \mathcal{W}_I$. The natural map from $\mathcal{W}_I$ to $\mathcal{W}_{I\setminus K}$ in (9), yields the retracts from $\mathcal{W}_I$ to $\mathcal{W}_{I,K}$. Clearly the retraction from $\mathcal{W}_I$ to $\mathcal{W}_I^\vee$ factors through $\mathcal{W}_{I,K}$.

(iii) Define $\Phi : \mathcal{W}_{I,K} \to \mathcal{W}(e)$ as $\Phi(x, w, y) = x w e v_0 y w_0$. Let $\sigma(x, w, y)$, $\sigma' = (x', w', y') \in \mathcal{W}_{I,K}$. First suppose that $\sigma \leq \sigma'$. Then $w = w_1 * w_2 * w_3$ with $x w_1 \leq x', w_2 \leq w', \bar{w}_3 y \leq y'$. Then $x w_1 w_2 \leq x' w'$. By Lemma 2.1 (i),

$$x w = x w_1 w_2 * w_3 \leq x' w' * u \quad \text{for some } u \leq w_3. \tag{26}$$

So

$$\bar{u} y \leq \bar{w}_3 y u \leq y'.$$

So

$$v_0 \cdot v_0 y' w_0 = y' w_0 \leq \bar{u} y w_0 = v_0 u \cdot v_0 y w_0.$$

Since $v_0 = v_0 u * u^{-1}$, Lemma 2.1 (ii) implies that $u^{-1} \cdot v_0 y' w_0 \leq v_0 y w_0$. Combined with (26), we see by (15) that $\Phi(\sigma) \leq \Phi(\sigma')$.

Assume conversely that $\Phi(\sigma) \leq \Phi(\sigma')$. Then by (15), there exists $u \in W_I$ such that

$$xw \leq x'w'u, \quad u^{-1}v_0y'w_0 \leq v_0yw_0. \tag{27}$$

Since $x, x' \in D_I$, $w, w'u \in W_I$, we see by Lemma 2.2 that $w = w_1 * w_2 * w_3$ with $xw_1 \leq x'$, $w_2 \leq w'$, $w_3 \leq u$. Hence by (27)

$$w_3^{-1} * v_0y'w_0 \leq u^{-1} * v_0y'w_0 \leq v_0yw_0. \tag{28}$$

By comparing lengths we see that $v_0 = w_3^{-1}v_0 * \bar{w}_3$. Hence by (28),

$$w_3^{-1}v_0 * \bar{w}_3y = v_0y \leq w_3^{-1}v_0 * y'.$$

By Lemma 2.1 (ii), $\bar{w}_3y \leq y'$. Hence $\sigma \leq \sigma'$.

(iv) If $W(e)$ is maximal, then by [15, Chapter 10], $W_e = 1$. Hence $\mathcal{W}(e)$ is universal. If $\mathcal{W}(e)$ is minimal, then by [15] $W_e = W(e)$ and $\mathcal{W}(e)$ is fundamental This completes the proof. □

Remark 2.6. (i) the isomorphism in Theorem 2.5 (iii) preserves length as defined in (16), (20),

(ii) Let G_0 be a semisimple group with Weyl group W and let $K \lhd I \subseteq S$. Let θ be an irreducible representation of G_0 such that K is the set of simple reflections fixing the highest weight vector. Let $M = \overline{k\theta(G_0)}$. By [16], there exists $e \in \Lambda$ such that $\mathcal{W}(e)$ is of type I and subtype K. Hence by Theorem 2.5 (iii), $\mathcal{W}(e) \cong \mathcal{W}_{I,K}$. Thus each of the synthetically constructed posets $\mathcal{W}_{I,K}$ arise naturally. For this reason we call $\mathcal{W}_{I,K}$, $W \times W$-orbits.

(iii) The orbit in Figure 3 is universal while the orbit in Figure 4 is fundamental.

(iv) If M is a canonical monoid (unique minimal orbit and this orbit is of type $\emptyset$), then by [16], every $W \times W$-orbit is universal.

(v) If M is a dual canonical monoid (unique maximal orbit and this orbit is of type $\emptyset$), then every $W \times W$-orbit is fundamental.

(vi) By (7), (22), any fundamental orbit is isomorphic to its dual.

(vii) If no component of $\widetilde{\mathcal{W}(e)} = W_{I\setminus K}$ is of type A_ℓ ($\ell > 1$), D_ℓ (ℓ odd) or E_6, then $\bar{w} = w$ in (18), for all $w \in W_{I\setminus K}$. Hence (19), (23) are identical in this situation and $\mathcal{W}_{I,K} = \mathcal{W}^*_{I,K}$. Hence by Theorem 2.5, $\mathcal{W}(e)$ is isomorphic to its dual. This is what is happening in Example 2.4.

Example 2.7. Let $M = M_4(k)$. The universal $W \times W$-orbit of rank 3 partial permutations is not isomorphic to its dual. This is most easily seen using Theorem 2.5. Let $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$, $I = \{s_1, s_2\}$. We claim that $\mathcal{W}_I \ncong \mathcal{W}^*_I$. The elements of length 1 are

$$(s_3, 1, 1),\ (1, s_1, 1),\ (1, s_2, 1),\ (1, 1, s_3).$$

In $\mathcal{W}^*_I$, $(1, s_2, 1)$ is covered by 6 elements:

$$(s_2s_3, 1, 1),\ (s_3, s_2, 1),\ (1, s_1s_2, 1),\ (1, s_2s_1, 1),\ (1, 1, s_3s_2),\ (1, s_2, s_3).$$

In $\mathcal{W}_1$, $(s_3, 1, 1)$ is covered by

$$(s_2s_3, 1, 1),\ (s_3, s_1, 1),\ (s_3, s_2, 1),\ (s_3, 1, s_3)$$

$(1, s_1, 1)$ is covered by

$$(s_3, s_1, 1),\ (1, s_1s_2, 1),\ (1, s_2s_1, 1),\ (1, s_1, s_3),\ (1, 1, s_3s_2)$$

$(1, s_2, 1)$ is covered by

$$(s_2s_3, 1, 1), (s_3, s_2, 1),\ (1, s_1s_2, 1),\ (1, s_2s_1, 1),\ (1, s_2, s_3)$$

$(1, 1, s_3)$ is covered by

$$(s_3, 1, s_3),\ (1, s_2, s_3),\ (1, s_1, s_3),\ (1, 1, s_3s_2).$$

Thus in $\mathcal{W}_I$, no element of length 1 is covered by 6 elements. Hence $\mathcal{W}_I$ is not isomorphic to $\mathcal{W}_I^*$.

3. Shellability

Let P be a finite partially ordered set with a maximum element $\mathbf{1}$ and minimum element $\mathbf{0}$, and so that all maximal chains have the same length. If $a, b \in P$, write $a \to b$ if a *covers* b (i.e. $a > b$ and there is no c such that $a > c > b$). For an $a \in P$, let $\ell(a)$ denote the length of a maximal chain from a to $\mathbf{0}$. P is said to be *Eulerian* (cf. [21]) if for $a \le b$, the Möbius function $\mu(a, b) = (-1)^{\ell(a)+\ell(b)}$. Of much importance in the study of P has been the topological concept of shellability of the order complex of all chains in P. We now briefly review the stronger concept of *lexicographic shellability* introduced by Björner and Wachs [2]. The edges of P are labeled recursively starting from the top, whereby for $a \to b$ the label depends on the choice of a maximal chain from $\mathbf{1}$ to a. Fix $a > b$ and a maximal chain from $\mathbf{1}$ to a. the labeling must be such that there is a unique maximal chain from a to b with increasing labels and so that this chain is lexicograhically less than any other maximal chain from a to b.

It is shown in [2] that D_I is lexicographically shellable. It therefore follows from (22) that the fundamental orbit $\mathcal{W}_1^\vee$ is lexicographically shellable and hence that its Stanley-Reisner ring is Cohen-Macaulay. Figure 4 shows that in general $\mathcal{W}_I^\vee$ is not Eulerian and the Stanley-Reisner ring is not Gorenstein.

We proceed to show that $\mathcal{W}_I$ is an Eulerian lexicographically shellable poset. Let v_0, w_0 denote the longest elements of $\mathcal{W}_I$, $\mathcal{W}$, respectively. for $w \in \mathcal{W}_I$, let $\bar{w} = v_0wv_0 \in \mathcal{W}_I$. Let $\sigma = (x, w, y) \in \mathcal{W}_I$. Let $\sigma' \in \mathcal{W}_I, \sigma \to \sigma'$. We will say that the edge is of *type* 1 if

$$\sigma' = (x', u * w, y), \quad x \to x'u \quad \text{in } W. \tag{29}$$

We will say that the edge is of *type* 2 if

$$\sigma' = (x, w', y), \quad w \to w' \quad \text{in } W_I. \tag{30}$$

We will say that the edge is of *type* 3 if

$$\sigma' = (x, w * \bar{v}, y'), \quad y \to vy' \quad \text{in } W. \tag{31}$$

We see by (19), (20) that exactly one of these cases occurs.

Let $s_1, \ldots, s_m, s_{m+1}, \ldots, s_n \in S$. Then

$$s_1 \cdots s_m = xw_1, \quad x \in D_I, w_1 \in W_I,$$
$$s_{m+1} \cdots s_n = w_2y, \quad y \in D_T^{-1}, w_2 \in W_I.$$

Let

$$\Phi(s_1 \cdots s_m; s_{m+1} \cdots s_n) = (x, w_1\bar{w}_2, y) \in \mathcal{W}_I. \tag{32}$$

Let $\sigma(x, w, y)$, $w = w_1\bar{w}_2$. Then by (20),

$$\ell(\sigma) = \ell(x) + \ell(w) + \ell(y) \le \ell(x) + \ell(w_1) + \ell(w_2) + \ell(y) \le n. \tag{33}$$

We will say that $s_1 \cdots s_m;\ s_{m+1} \cdots s_n$ is an *expression* for σ. If $m = 0$, we write the expression as $1;\ s_1 \cdots s_n$. If $m = n$, we write the expression as $s_1 \cdots s_n;\ 1$. If $\ell(\sigma) = n$, we will say that the expression is *reduced*. By (33), this happens if and only if

$$\ell(xw_1) = m, \quad \ell(w_2 y) = n - m \quad \text{and} \quad w = w_1 * \bar{w}_2. \tag{34}$$

Lemma 3.1. *Let $s_1 \cdots s_m;\ s_{m+1} \cdots s_n$ be a reduced expression for σ and let $\sigma \to \sigma'$. Then for some $1 \le i \le n$, $s_1 \cdots \hat{s}_i \cdots s_m;\ s_{m+1} \cdots s_n$ or $s_1 \cdots s_m;\ s_{m+1} \cdots \hat{s}_i \cdots s_n$ is a reduced expression for σ'. Moreover, i is unique.*

Proof. The existence of the reduced expression for σ' follows from (32), (33), and (34). We claim that i is unique. So suppose that σ' is obtained by deleting either s_i or s_j from the expression for σ. First suppose that the $\sigma \to \sigma'$ is of type 1. Let σ' be as in (29). Then $i, j \le m, \ell(x'uw_1) = m - 1$ and

$$x'uw_1 = s_1 \cdots \hat{s}_i \cdots s_m = s_1 \cdots \hat{s}_j \cdots s_m$$

which implies $i = j$. Next assume that the $\sigma \to \sigma'$ is of type 2 and that σ' is as in (30). Now $w = w_1 * \bar{w}_2 \to w'$. So

$$w' = w_1' * \bar{w}_2, w_1 \to w_1' \quad \text{or} \quad w' = w_1 * \bar{w}_2', w_2 \to w_2'$$

with the two cases being exclusive. in the first case $i, j \le m, \ell(xw_1') = m - 1$, and

$$xw_1' = s_1 \cdots \hat{s}_i \cdots s_m = s_1 \cdots \hat{s}_j \cdots s_m$$

which implies $i = j$. In the second case, $i, j > m, \ell(w_2'y) = n - m - 1$ and

$$w_2'y = s_{m+1} \cdots \hat{s}_i \cdots s_n = s_{m+1} \cdots \hat{s}_j \cdots s_n$$

and this too implies $i = j$. Finally assume that $\sigma \to \sigma'$ is of type 3 and that σ' is in (31). Then $i, j > m$, $\ell(w_2vy') = n - m - 1$ and

$$w_2vy' = s_{m+1} \cdots \hat{s}_i \cdots s_n = s_{m+1} \cdots \hat{s}_j \cdots s_n$$

which implies $i = j$. Thus i is unique. □

The maximum element of $\mathcal{W}_I$ is $\mathbf{1} = (w_0v_0, v_0, v_0w_0)$. Fix a reduced expression

$$\mathbf{1} = \Phi(s_1 \cdots s_m; s_{m+1} \cdots s_n) \tag{35}$$

for $\mathbf{1}$. Let $\sigma \in \mathcal{W}_I$ and consider a maximal chain from $\mathbf{1}$ to σ. By Lemma 3.1, this leads uniquely to a reduced expression

$$\sigma = \Phi(s_{i_1} \cdots s_{i_p}; s_{i_{p+1}} \cdots s_{i_q}) \tag{36}$$

of σ. If $\sigma \to \sigma'$, then by Lemma 3.1, a reduced expression for σ' is obtained by deleting some s_{i_j} from the reduced expression for σ. We attach the label i_j to the edge. We proceed to show that this labeling process leads to lexicographic shelling.

Let σ', $\sigma \in \mathcal{W}_I, \sigma' < \sigma$. Fix a maximal chain from $\mathbf{1}$ to σ resulting in the reduced expression $t_1 \cdots t_p;\ t_{p+1} \cdots t_q$ for σ where $t_j = s_{i_j}$. We will need the following analogue of [2, Lemma 4.3].

Lemma 3.2. *Let $\sigma_1 \in [\sigma', \sigma]$ such that $\ell(\sigma_1) = \ell(\sigma) - 2$. Then the open interval $(\sigma_1, \sigma) = \{\sigma_2, \sigma_3\}$ such that:*

(i) *$\sigma \to \sigma_2 \to \sigma_1$ has increasing labels.*
(ii) *$\sigma \to \sigma_3 \to \sigma_1$ has decreasing labels.*
(iii) *The label for $\sigma \to \sigma_2$ is less than the label for $\sigma \to \sigma_3$.*

Proof. Let $\sigma = (x, w, y), \sigma_1 = (x', w', y')$. Then by (32),

$$w = w_1 * \bar{w}_2 \quad \text{with } xw_1 = t_1 \cdots t_p, w_2 y = t_{p+1} \cdots t_q. \tag{37}$$

Suppose first that $x = x'$ and $y = y'$. Then $w' < w$, $\ell(w) - \ell(w') = 2$. If $(w', w) = \{u_1, u_2\}$, then $(\sigma_1, \sigma) = \{(x, u_1, y),\ (x, u_2, y)\}$. For $i = 1, 2$,

$$\begin{aligned} xw &= t_1 \cdots t_p * \bar{w}_2 \to xu_i \to xw', \\ \bar{w}y &= \bar{w}_1 * t_{p+1} \cdots t_q \to \bar{u}_i y \to \bar{w}' y. \end{aligned} \tag{38}$$

Fix reduced expressions for $\bar{w}_1$ and $\bar{w}_2$. A deletion in $\bar{w}_2$ in the first sequence corresponds to deleting some t_μ, $\mu > p$, in the second sequence. A deletion in $\bar{w}_1$ in the second sequence corresponds to deleting t_μ, $\mu \le p$, in the first sequence. Thus applying [2, Lemma 4.3] to (38), we see that the lemma is valid.

Suppose next that $x = x'$ and $y \ne y'$. Then $w'y' < wy$, $\ell(\bar{w}y) - \ell(\bar{w}'y) = 2$. So

$$(\bar{w}y', \bar{w}y) = \{u_1 y_1, u_2 y_2\}, \quad u_1, u_2 \in W_I, y_1, y_2 \in D_I^{-1}.$$

Correspondingly $(\sigma_1, \sigma) = \{(x, \bar{u}_1, y_1),\ (x, \bar{u}_2, y_2)\}$. So for $i = 1, 2$,

$$\bar{w}y = \bar{w}_1 t_{p+1} \cdots t_q \to u_i y_i \to \bar{w}' y'. \tag{39}$$

Fix a reduced expression for $\bar{w}_1$. A deletion in $\bar{w}_1$ corresponds to deleting some t_μ, $\mu \le p$, in $xw_1 = t_1 \cdots t_p$. Since $y \ne y$, not both the deletions in (39) can be from $\bar{w}_1$. Again applying [2, Lemma 4.3] to (39) yields the lemma. The case when $x \ne x'$ and $y = y'$ is handled similarly.

Finally, let $x' \ne x$ and $y' \ne y$. Then

$$w' = w_1' * w * \bar{w}_2', \quad x \to x' w_1', \quad y \to w_2' y'.$$

Then by (37),

$$xw_1 = t_1 \cdots t_p \to x' w_1' * w_1, \quad w_2 y = t_{p+1} \cdots t_q \to w_2 * w_2' y'.$$

So we see that the lemma is valid with

$$\sigma_2 = (x', w_1' * w, y), \quad \sigma_3 = (x, w * \bar{w}_2', y').$$

Hence the lemma is valid in all cases. □

It follows from Lemma 3.2 and induction that the maximal chain of $[\sigma', \sigma]$ with lexicographically minimal labeling has increasing labels. Let $\sigma = (x, w, y)$ be as in (37) and suppose that there are two maximal chains

$$\begin{aligned} \sigma &= \sigma_r \to \sigma_{r-1} \to \cdots \to \sigma_1 \to \sigma', \\ \sigma &= \theta_r \to \theta_{r-1} \to \cdots \to \theta_1 \to \sigma', \end{aligned} \tag{40}$$

both with increasing labels. Suppose σ' is obtained from σ_1 by deleting t_β and that σ' is obtained from θ_1 by deleting t_α. Let $\alpha \le \beta$. Let $\sigma = (x', w', y)$ and let $w' = w_1' * \bar{w}_2'$ analogous to (37). If $\beta \le p$, then $w_2' = w_2$ and (40) yields two maximal chains from xw_1 to $x'w_1'$ with increasing labels. So by [2, Theorem 5.1], the two chains are identical. So assume $\beta > p$. Then $\sigma_1 = (x', w_1, y_1)$ and

$$\bar{w}_1 y_1 = \bar{w}' y' \cdot t_q \cdots t_{\beta+1} \cdot t_\beta \cdots t_q.$$

So if $\alpha < \beta$, then since σ' is also obtained from θ_1 by deleting t_α,

$$\bar{w}' y' = \cdots t_\beta t_{\beta+1} \cdots t_q,$$

and hence $\ell(\bar{w}_1 y_1) < \ell(\bar{w}' y')$. This implies that $\ell(\sigma_1) < \ell(\sigma)$, a contradiction. Hence $\alpha = \beta$ and $\theta_1 = \sigma_1$. By induction, the two maximal chains in (40) are identical. Hence $\mathcal{W}_I$ is lexicographically shellable. Our proof shows that there is also a unique maximal chain in $[\sigma', \sigma]$ with decreasing labels. Hence by [2, Theorem 3.4], $\mathcal{W}_I$ is Eulerian and its Stanley-Reisner ring is Gorenstein. Using Theorem 2.5 (i) and an almost identical (and slightly easier) argument as above, we see that $\mathcal{W}_I^*$ is also an Eulerian lexicographically shellable poset. Hence we have proved

Theorem 3.3. *The universal orbit $\mathcal{W}_I$ and its dual $\mathcal{W}_I^*$ are Eulerian, lexicographically shellable posets. In particular, their Stanley-Reisner rings are Gorenstein.*

Many problems remain open.

Problem 3.4. Is the Renner monoid R shellable with respect to the Bruhat-Chevalley order?

Problem 3.5. Is the cross-section lattice Λ always shellable?

Remark 3.6. Suppose $\Lambda\backslash\{0\}$ has a minimum element e. This happens when M is the Zariski closure of the image of an irreducible representation of a reductive group. Let I be the type of WeW. Then by [16, Theorem 4.16], $\Lambda\backslash\{0\}$ is isomorphic to the poset (with respect to inclusion) of all subsets of S with no components contained in I. Being closed under taking unions, this poset is easily seen to be a semimodular lattice. Hence by [1], Λ is shellable.

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