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INTERTWINING OPERATOR SUPERALGEBRAS AND VERTEX TENSOR CATEGORIES FOR SUPERCONFORMAL ALGEBRAS, II

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Abstract. We construct the intertwining operator superalgebras and vertex tensor categories for the $N = 2$ superconformal unitary minimal models and other related models.

0. Introduction

It has been known that the $N = 2$ Neveu-Schwarz superalgebra is one of the most important algebraic objects realized in superstring theory. The $N = 2$ superconformal field theories constructed from its discrete unitary representations of central charge $c < 3$ are among the so-called "minimal models." In the physics literature, there have been many conjectural connections among Calabi-Yau manifolds, Landau-Ginzburg models and these $N = 2$ unitary minimal models. In fact, the physical construction of mirror manifolds [GP] used the conjectured relations [Ge1] [Ge2] between certain particular Calabi-Yau manifolds and certain $N = 2$ superconformal field theories (Gepner models) constructed from unitary minimal models (see [Gr] for a survey). To establish these conjectures as mathematical theorems, it is necessary to construct the $N = 2$ unitary minimal models mathematically and to study their structures in detail.

In the present paper, we apply the theory of intertwining operator algebras developed by the first author in [H3], [H5] and [H6] and the tensor product theory for modules for a vertex operator algebra developed by Lepowsky and the first author in [HL1]–[HL6], [HL8] and [H1] to construct the intertwining operator algebras and vertex tensor categories for $N = 2$ superconformal unitary minimal models. The main work in this paper is to verify that the conditions to use the general theories are satisfied for these models. The main technique used is the representation theory of the $N = 2$ Neveu-Schwarz algebra, which has been studied by many physicists and mathematicians, especially by Eholzer and Gaberdiel [EG], Feigin, Semikhatov, Sirota and Tipunin [FSST] [FST] [FS], and Adamović [A1] [A2].

The present paper is organized as follows: In Section 1, we recall the notion of $N = 2$ superconformal vertex operator superalgebra. In Section 2, we recall and

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prove some basic results on representations of unitary minimal $N = 2$ superconformal vertex operator superalgebras and on representations of $N = 2$ superconformal vertex operator superalgebras in a much more general class. Section 3 is devoted to the proof of the convergence and extension properties for products of intertwining operators for unitary minimal $N = 2$ superconformal vertex operator superalgebras and for vertex operator superalgebras in the general class. Our main results on the intertwining operator superalgebra structure and vertex tensor category structure are given in Section 4.

1. $N = 2$ superconformal vertex operator superalgebras

In this section we recall the notion of $N = 2$ superconformal vertex operator algebra and basic properties of such an algebra. These algebras have been studied extensively by physicists. The following precise version of the definition is from [HZ]:

Definition 1.1. An $N = 2$ *superconformal vertex operator superalgebra* is a vertex operator superalgebra $(V, Y, \mathbf{1}, \omega)$ together with odd elements τ^+, τ^- and an even element μ satisfying the following axioms: Let

$$\begin{aligned} Y(\tau^+, x) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} G^+(r)x^{-r-3/2}, \\ Y(\tau^-, x) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} G^-(r)x^{-r-3/2}, \\ Y(\mu, x) &= \sum_{n\in\mathbb{Z}} J(n)x^{-n-1}. \end{aligned}$$

Then V is a direct sum of eigenspaces of $J(0)$ with integral eigenvalues which modulo $2\mathbb{Z}$ give the $\mathbb{Z}_2$ grading for the vertex operator superalgebra structure, and the following $N = 2$ Neveu-Schwarz relations hold: For $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$

$$\begin{aligned} [L(m), L(n)] &= (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [J(m), J(n)] &= \frac{c}{3}m\delta_{m+n,0}, \\ [L(m), J(n)] &= -nJ_{m+n}, \\ [L(m), G^\pm(r)] &= \left(\frac{m}{2} - r\right) G^\pm(m+r), \\ [J(m), G^\pm(r)] &= \pm G^\pm(m+r), \\ [G^+(r), G^-(s)] &= 2L(r+s) + (r-s)J(r+s) + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\ [G^\pm(r), G^\pm(s)] &= 0 \end{aligned}$$

where $L(m)$, $m \in \mathbb{Z}$, are the Virasoro operators on V and c is the central charge of V.

Modules and *intertwining operators* for an $N = 2$ superconformal vertex operator superalgebra are modules and intertwining operators for the underlying vertex operator superalgebra.

Note that this definition is slightly different from the one in [A1]; in [A1], V is not required to be a direct sum of eigenspaces of $J(0)$.

The $N = 2$ superconformal vertex operator superalgebra defined above is denoted by $(V, Y, \mathbf{1}, \tau^+, \tau^-, \mu)$ (without ω since

$$\omega = L(-2)\mathbf{1} = \frac{1}{2}[G^+(-\frac{3}{2}), G^-(-\frac{1}{2})]\mathbf{1} + \frac{1}{2}J(-2)\mathbf{1})$$

or simply by V. Note that a module W for a vertex operator superalgebra (in particular the algebra itself) has a $\mathbb{Z}_2$-grading called *sign* in addition to the $\mathbb{C}$-grading by weights. We shall always use W^0 and W^1 to denote the even and odd subspaces of W. If W is irreducible, there exists $h \in \mathbb{C}$ such that $W = W^0 \oplus W^1$ where $W^0 = \bigoplus_{n\in h+\mathbb{Z}} W_{(n)}$ and $W^1 = \bigoplus_{n\in h+\mathbb{Z}+1/2} W_{(n)}$. We shall always use the notation $|\cdot|$ to denote the map from the union of the even and odd subspaces of a vertex operator superalgebra or of a module for such an algebra to $\mathbb{Z}_2$ by taking the signs of elements in the union.

The notion of $N = 2$ superconformal vertex operator superalgebra can be reformulated using odd formal variables. (In the $N = 1$ case, this reformulation was given by Barron [Ba1] [Ba2].)

We need some spaces which play the role of spaces of algebraic functions and fields on certain "superspaces." For l symbols $\varphi_1, \dots, \varphi_l$, consider the exterior algebra of the vector space over $\mathbb{C}$ spanned by these symbols and denote this exterior algebra by $\mathbb{C}[\varphi_1, \dots, \varphi_l]$. For any vector space E, we also have the vector space

$$\begin{gathered}
E[\varphi_1, \dots, \varphi_l], \\
E[x_1, \dots, x_k][\varphi_1, \dots, \varphi_l], \\
E[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}][\varphi_1, \dots, \varphi_l], \\
E[[x_1, \dots, x_k]][\varphi_1, \dots, \varphi_l], \\
E[[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]][\varphi_1, \dots, \varphi_l], \\
E\{x_1, \dots, x_k\}[\varphi_1, \dots, \varphi_l]
\end{gathered}$$

and

$$E((x_1, \dots, x_k))[\varphi_1, \dots, \varphi_l].$$

If E is a $\mathbb{Z}_2$-graded vector space, then there are natural structures of modules over the ring

$$\mathbb{C}[x_1, \dots, x_k][\varphi_1, \dots, \varphi_l]$$

on these spaces. The ring

$$\mathbb{C}[x_1, \dots, x_k][\varphi_1, \dots, \varphi_l]$$

can be viewed as the space of algebraic functions on a "superspace" consisting of elements of the form $(x_1, \dots, x_k; \varphi_1, \dots, \varphi_l)$ where $x_1, \dots, x_k$ are commuting coordinates and $\varphi_1, \dots, \varphi_l$ are anticommuting coordinates. The other spaces can be viewed as spaces of suitable fields over this "superspace."

Let $(V, Y, \mathbf{1}, \tau^+, \tau^-, \mu)$ be an $N = 2$ superconformal vertex operator superalgebra. Let

$$\begin{aligned}
\tau_1 &= \frac{1}{\sqrt{2}}(\tau^+ + \tau^-), \\
\tau_2 &= \frac{1}{\sqrt{-2}}(\tau^+ - \tau^-)
\end{aligned}$$

and

$$\begin{aligned} Y(\tau_1, x) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_1(r)x^{-r-3/2}, \\ Y(\tau_2, x) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_2(r)x^{-r-3/2}. \end{aligned}$$

We define the *vertex operator map with odd variables*

$$\begin{aligned} Y : V\otimes V &\to V((x))[\varphi_1, \varphi_2] \\ u\otimes v &\mapsto Y(u, (x, \varphi_1, \varphi_2))v \end{aligned}$$

by

$$\begin{aligned} Y(u, (x, \varphi_1, \varphi_2))v &= Y(u, x)v + \varphi_1 Y(G_1(-1/2)u, x)v \\ &\quad +\varphi_2 Y(G_2(-1/2)u, x)v \\ &\quad +\varphi_1\varphi_2 Y(G_1(-1/2)G_2(-1/2)u, x)v \end{aligned}$$

for $u, v \in V$. (We use the same notation Y to denote the vertex operator map and the vertex operator map with odd variables.) In particular, we have

$$\begin{aligned} Y(\mu, (x, \varphi_1, \varphi_2)) &= Y(\mu, x) - \sqrt{-1}\varphi_1 Y(\tau_2, x) \\ &\quad -\sqrt{-1}\varphi_2 Y(\tau_1, x) - 2\sqrt{-1}\varphi_1\varphi_2 Y(\omega, x). \end{aligned}$$

Also, if we introduce

$$\begin{aligned} \varphi^+ &= \frac{-\varphi_1 + \sqrt{-1}\varphi_2}{\sqrt{2}}, \\ \varphi^- &= \frac{\varphi_1 + \sqrt{-1}\varphi_2}{\sqrt{2}}, \end{aligned}$$

then we can write

$$\begin{aligned} Y(\mu, (x, \varphi_1, \varphi_2)) &= Y(\mu, x) + \varphi^+ Y(\tau^+, x) \\ &\quad +\varphi^- Y(\tau^-, x) + 2\varphi^+\varphi^- Y(\omega, x). \end{aligned}$$

We have

Proposition 1.2. *The vertex operator map with odd variables satisfies the following properties:*

1. *The vacuum property:*

$$Y(\mathbf{1}, (x, \varphi_1, \varphi_2)) = 1$$

   where 1 *on the right-hand side is the identity map on* V.

2. *The creation property: For any* $v \in V$,

$$Y(v, (x, \varphi_1, \varphi_2))\mathbf{1} \in V[[x]][\varphi_1, \varphi_2],$$

$$\lim_{(x,\varphi_1,\varphi_2)\mapsto(0,0,0)} Y(v, (x, \varphi_1, \varphi_2))\mathbf{1} = v.$$

3. *The Jacobi identity: In*

$$(\mathrm{End}\ V)[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]][\varphi_1, \varphi_2, \psi_1, \psi_2],$$

we have

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2-\varphi_1\psi_1-\varphi_2\psi_2}{x_0}\right)Y(u,(x_1,\varphi_1,\varphi_2))Y(v,(x_2,\psi_1,\psi_2))\\
&\qquad -(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2-x_1+\varphi_1\psi_1+\varphi_2\psi_2}{-x_0}\right)\\
&\qquad\qquad \cdot Y(v,(x_2,\psi_1,\psi_2))Y(u,(x_1,\varphi_1,\varphi_2))\\
&\quad = x_2^{-1}\delta\left(\frac{x_1-x_0-\varphi_1\psi_1-\varphi_2\psi_2}{x_2}\right)\\
&\qquad\qquad \cdot Y(Y(u,(x_0,\varphi_1-\psi_1,\varphi_2-\psi_2))v,(x_2,\psi_1,\psi_2)),
\end{aligned}$$

for $u, v \in V$ which are either even or odd.

4. *The $G_i(-1/2)$-derivative property: For any $v \in V$, $i = 1, 2$,*

$$Y(G_i(-1/2)v,(x,\varphi_1,\varphi_2)) = \left(\frac{\partial}{\partial\varphi_i}+\varphi_i\frac{\partial}{\partial x}\right)Y(v,(x,\varphi_1,\varphi_2)).$$

5. *The $L(-1)$-derivative property: For any $v \in V$,*

$$Y(L(-1)v,(x,\varphi_1,\varphi_2)) = \frac{\partial}{\partial x}Y(v,(x,\varphi_1,\varphi_2)).$$

6. *The skew-symmetry: For any $u, v \in V$ which are either even or odd,*

$$\begin{aligned}
&Y(u,(x,\varphi_1,\varphi_2))v\\
&\quad = (-1)^{|u||v|}e^{xL(-1)+\varphi_1G_1(-1/2)+\varphi_2G_2(-1/2)}Y(v,(-x,-\varphi_1,-\varphi_2))u. \quad \square
\end{aligned}$$

The proof of this result is straightforward and we omit it.

We can also reformulate the data and axioms for modules and intertwining operators for an $N = 2$ superconformal vertex operator superalgebra using odd variables as in the $N = 1$ case in [Ba2] and [HM]. Here we give the details of the corresponding reformulation of the data and axioms for intertwining operators.

Let W_1, W_2 and W_3 be modules for an $N = 2$ superconformal vertex operator superalgebra V and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1W_2}$. We define the corresponding *intertwining operator map with odd variables*

$$\begin{aligned}
\mathcal{Y} : W_1\otimes W_2 &\to W_3\{x\}[\varphi_1,\varphi_2]\\
w_{(1)}\otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)},(x,\varphi_1,\varphi_2))w_{(2)}
\end{aligned}$$

by

$$\begin{aligned}
\mathcal{Y}(w_{(1)},(x,\varphi_1,\varphi_2))w_{(2)} &= \mathcal{Y}(w_{(1)},x)w_{(2)}+\varphi_1\mathcal{Y}(G_1(-1/2)w_{(1)},x)w_{(2)}\\
&\quad +\varphi_2\mathcal{Y}(G_2(-1/2)w_{(1)},x)w_{(2)}\\
&\quad +\varphi_1\varphi_2\mathcal{Y}(G_1(-1/2)G_2(-1/2)w_{(1)},x)w_{(2)}
\end{aligned}$$

for $u, v \in V$. Then we have

Proposition 1.3. *The intertwining operator map with odd variables satisfies the following properties:*

1. *The Jacobi identity: In*

$$\mathrm{Hom}(W_1\otimes W_2,W_3)\{x_0,x_1,x_2\}[\varphi_1,\varphi_2,\psi_1,\psi_2],$$

we have

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2-\varphi_1\psi_1-\varphi_2\psi_2}{x_0}\right)Y(u,(x_1,\varphi_1,\varphi_2))\mathcal{Y}(w_{(1)},(x_2,\psi_1,\psi_2))\\
&\quad -(-1)^{|u||w_{(1)}|}x_0^{-1}\delta\left(\frac{x_2-x_1+\varphi_1\psi_1+\varphi_2\psi_2}{-x_0}\right)\\
&\qquad \cdot\mathcal{Y}(w_{(1)},(x_2,\psi_1,\psi_2))Y(u,(x_1,\varphi_1,\varphi_2))\\
&\quad = x_2^{-1}\delta\left(\frac{x_1-x_0-\varphi_1\psi_1-\varphi_2\psi_2}{x_2}\right)\\
&\qquad \cdot\mathcal{Y}(Y(u,(x_0,\varphi_1-\psi_1,\varphi_2-\psi_2))w_{(1)},(x_2,\psi_1,\psi_2)),
\end{aligned}$$

for $u\in V$ and $w_{(1)}\in W_1$ which are either even or odd.

2. *The $G_i(-1/2)$-derivative property: For any $v\in V$, $i=1,2$,*

$$\mathcal{Y}(G_i(-1/2)v,(x,\varphi_1,\varphi_2))=\left(\frac{\partial}{\partial\varphi_i}+\varphi_i\frac{\partial}{\partial x}\right)\mathcal{Y}(v,(x,\varphi_1,\varphi_2)).$$

3. *The $L(-1)$-derivative property: For any $v\in V$,*

$$\mathcal{Y}(L(-1)w_{(1)},(x,\varphi_1,\varphi_2))=\frac{\partial}{\partial x}\mathcal{Y}(w_{(1)},(x,\varphi_1,\varphi_2)).$$

4. *The skew-symmetry: There is a linear isomorphism*

$$\Omega:\mathcal{V}_{W_1W_2}^{W_3}\to\mathcal{V}_{W_2W_1}^{W_3}$$

such that

$$\begin{aligned}
&\Omega(\mathcal{Y})(w_{(1)},(x,\varphi_1,\varphi_2))w_{(2)}\\
&\quad=(-1)^{|w_{(1)}||w_{(2)}|}e^{xL(-1)+\varphi_1G_1(-1/2)+\varphi_2G_2(-1/2)}\\
&\qquad\cdot\mathcal{Y}(w_{(2)},(e^{-\pi i}x,-\varphi_1,-\varphi_2))w_{(1)}
\end{aligned}$$

for $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$ which are either even or odd. □

The proof of this result is similar to the proof of Proposition 1.2 and is omitted.

2. Unitary minimal $N=2$ superconformal vertex operator superalgebras

In this section, we recall the constructions and results on unitary minimal $N=2$ superconformal vertex operator superalgebras and their representations. New results needed in later sections are also proved. We then introduce in this section a class of $N=2$ superconformal vertex operator superalgebras and generalize most of the results for unitary minimal $N=2$ superconformal vertex operator superalgebras to algebras in this class.

The $N=2$ *Neveu-Schwarz Lie superalgebra* is the vector space

$$\bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n\oplus\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^+\oplus\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^-\oplus\bigoplus_{n\in\mathbb{Z}}\mathbb{C}J_n\oplus\mathbb{C}C$$

equipped with the following $N = 2$ Neveu-Schwarz relations:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0},\\
[J_m, J_n] &= \frac{C}{3} m\delta_{m+n,0},\\
[L_m, J_n] &= -nJ_{m+n},\\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right) G_{m+r}^{\pm},\\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm},\\
[G_r^+, G_s^-] &= 2L_{r+s} + (r-s)J_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0},\\
[G_r^{\pm}, G_s^{\pm}] &= 0
\end{aligned}$$

for $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$. For simplicity, we shall simply denote the $N = 2$ Neveu-Schwarz Lie superalgebra by $\mathfrak{ns}(2)$ in this paper.

We now construct certain representations of the $N = 2$ Neveu-Schwarz Lie superalgebra. Consider the two subalgebras

$$\begin{aligned}
\mathfrak{ns}^+(2) &= \bigoplus_{n>0} \mathbb{C}L_n \oplus \bigoplus_{r>0} \mathbb{C}G_r^+ \oplus \bigoplus_{r>0} \mathbb{C}G_r^- \oplus \bigoplus_{n>0} \mathbb{C}J_n,\\
\mathfrak{ns}^-(2) &= \bigoplus_{n<0} \mathbb{C}L_n \oplus \bigoplus_{r<0} \mathbb{C}G_r^+ \oplus \bigoplus_{r<0} \mathbb{C}G_r^- \oplus \bigoplus_{n<0} \mathbb{C}J_n
\end{aligned}$$

of $\mathfrak{ns}(2)$. Let $U(\cdot)$ be the functor from the category of Lie superalgebras to the category of associative algebras obtained by taking the universal enveloping algebras of Lie superalgebras. For any representation of $\mathfrak{ns}(2)$, we shall use $L(n)$, $n \in \mathbb{Z}$, $G^{\pm}(r)$, $r \in \mathbb{Z} + \frac{1}{2}$, and $J(n)$, $n \in \mathbb{Z}$, to denote the representation images of L_n, $G_r^{\pm}$ and J_n.

For any $c, h, q \in \mathbb{C}$, the Verma module $M_{\mathfrak{ns}(2)}(c, h, q)$ for $\mathfrak{ns}(2)$ is a free $U(\mathfrak{ns}^-(2))$-module generated by $\mathbf{1}_{c,h,q}$ such that

$$\begin{aligned}
\mathfrak{ns}^+(2)\mathbf{1}_{c,h,q} &= 0,\\
L(0)\mathbf{1}_{c,h,q} &= h\mathbf{1}_{c,h,q},\\
C\mathbf{1}_{c,h,q} &= c\mathbf{1}_{c,h,q},\\
J(0)\mathbf{1}_{c,h,q} &= q\mathbf{1}_{c,h,q}.
\end{aligned}$$

There exists a unique maximal proper submodule $J_{\mathfrak{ns}(2)}(c, h, q)$ of the Verma module $M_{\mathfrak{ns}(2)}(c, h, q)$. It is easy to see that when $c \neq 0$, $\mathbf{1}_{c,0,0}$, $G^{\pm}(-3/2)\mathbf{1}_{c,0,0}$ and $L(-2)\mathbf{1}_{c,0,0}$ are not in $J_{\mathfrak{ns}(2)}(c, 0, 0)$. Let

$$L_{\mathfrak{ns}(2)}(c, h, q) = M_{\mathfrak{ns}(2)}(c, h, q)/J_{\mathfrak{ns}(2)}(c, h, q)$$

and

$$V_{\mathfrak{ns}(2)}(c, 0, 0) = M_{\mathfrak{ns}(2)}(c, 0, 0)/\langle G^+(-1/2)\mathbf{1}_{c,0,0}, G^-(-1/2)\mathbf{1}_{c,0,0}\rangle$$

where

$$\langle G^+(-1/2)\mathbf{1}_{c,0,0}, G^-(-1/2)\mathbf{1}_{c,0,0}\rangle$$

is the submodule of $M_{\mathfrak{ns}(2)}(c, 0, 0)$ generated by $G^{\pm}(-1/2)\mathbf{1}_{c,0,0}$. Then $L_{\mathfrak{ns}(2)}(c, 0, 0)$ and $V_{\mathfrak{ns}(2)}(c, 0, 0)$ have the structures of vertex operator superalgebras with the vacuum $\mathbf{1}_{c,0,0}$, the Neveu-Schwarz elements $G^{\pm}(-3/2)\mathbf{1}_{c,0,0}$ and the Virasoro element $L(-2)\mathbf{1}_{c,0,0}$ (see [A1]).

In [EG], Eholzer and Gaberdiel showed, among other things, that among vertex operator superalgebras of the form $L_{\mathfrak{ns}(2)}(c,0,0)$, the only ones having finitely many irreducible modules are the "unitary" ones $L_{\mathfrak{ns}(2)}(c_m,0,0)$ for nonnegative integers m, where $c_m = \frac{3m}{m+2}$. The following result was proved by Adamović in [A1] and [A2] using the results obtained Adamović and Milas [AM], Feigin, Semikhatov and Tipunin [FST] and Doerrzapf [D]:

Theorem 2.1. *The vertex operator superalgebra $L_{\mathfrak{ns}(2)}(c,0,0)$ has only finitely many irreducible modules (up to isomorphisms) and every module for $L_{\mathfrak{ns}(2)}(c,0,0)$ is completely reducible if and only if*

$$c = c_m = \frac{3m}{m+2}$$

where m is a nonnegative integer. A set of representatives of the equivalence classes of irreducible modules for $L_{\mathfrak{ns}(2)}(c_m,0,0)$ is

$$\{L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k})\}_{j,k\in\mathbb{N}_{\frac{1}{2}},\ 0\le j,k,j+k<m}$$

where $\mathbb{N}_{\frac{1}{2}} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$ and

$$\begin{aligned} h_m^{j,k} &= \frac{jk-\frac{1}{4}}{m+2}, \\ q_m^{j,k} &= \frac{j-k}{m+2}. \qquad \square \end{aligned}$$

For any $m \ge 0$, we call the vertex operator algebra $L_{\mathfrak{ns}(2)}(c_m,0,0)$ a *unitary minimal $N=2$ superconformal vertex operator superalgebra.*

Proposition 2.2. *Let $j_i, k_i \in \mathbb{N}_{\frac{1}{2}}$, $i=1,2,3$, satisfying $0 \le j_i, k_i, j_i+k_i < m$ and $\mathcal{Y}$ an intertwining operator of type*

$$\binom{L_{\mathfrak{ns}(2)}(c_m, h_m^{j_3,k_3}, q_m^{j_3,k_3})}{L_{\mathfrak{ns}(2)}(c_m, h_m^{j_1,k_1}, q_m^{j_1,k_1}) L_{\mathfrak{ns}(2)}(c_m, h_m^{j_2,k_2}, q_m^{j_2,k_2})}. \tag{2.1}$$

Then we have:

1. *For any*

$$w_{(1)} \in L_{\mathfrak{ns}(2)}(c_m, h_m^{j_1,k_1}, q_m^{j_1,k_1})$$

and

$$w_{(2)} \in L_{\mathfrak{ns}(2)}(c_m, h_m^{j_2,k_2}, q_m^{j_2,k_2}),$$

$$\mathcal{Y}(w_{(1)},x)w_{(2)} \in x^{h_m^{j_3,k_3}-h_m^{j_1,k_1}-h_m^{j_2,k_2}} L_{\mathfrak{ns}(2)}(c_m, h_m^{j_3,k_3}, q_m^{j_3,k_3})((x^{1/2})).$$

2. *Let $\Delta = h_m^{j_3,k_3} - h_m^{j_1,k_1} - h_m^{j_2,k_2}$ and $w_{(i)} = \mathbf{1}_{c_m, h_m^{j_i,k_i}, q_m^{j_i,k_i}}$, $i=1,2,3$, the lowest weight vectors in $L_{\mathfrak{ns}(2)}(c_m, h_m^{j_i,k_i}, q_m^{j_i,k_i})$. Then the map $\mathcal{Y}$ is uniquely determined by the maps*

$$\begin{gathered} (w_{(1)})_{-\Delta-1}, \\ (G^+(-1/2)w_{(1)})_{-\Delta-1/2}, \\ (G^-(-1/2)w_{(1)})_{-\Delta-1/2}, \\ (G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta} \end{gathered}$$

from the 1-dimensional subspace of W_2 spanned by $w_{(2)}$ to the 1-dimensional subspace of W_3 spanned by $w_{(3)}$. That is, if these maps are 0, then $\mathcal{Y}=0$.

3. *If $q_m^{j_3,k_3}$ is not equal to one of the numbers $q_m^{j_1,k_1}+q_m^{j_2,k_2}$, $q_m^{j_1,k_1}+q_m^{j_2,k_2}-1$ and $q_m^{j_1,k_1}+q_m^{j_2,k_2}+1$, then the space*

$$\mathcal{V}^{L_{\mathfrak{ns}(2)}(c_m,h_m^{j_3,k_3},q_m^{j_3,k_3})}_{L_{\mathfrak{ns}(2)}(c_m,h_m^{j_1,k_1},q_m^{j_1,k_1})L_{\mathfrak{ns}(2)}(c_m,h_m^{j_2,k_2},q_m^{j_2,k_2})}$$

of intertwining operators of type (2.1) is 0. If $q_m^{j_3,k_3}=q_m^{j_1,k_1}+q_m^{j_2,k_2}\pm 1$, it is at most 1-dimensional. If $q_m^{j_3,k_3}=q_m^{j_1,k_1}+q_m^{j_2,k_2}$, it is at most 2-dimensional.

Proof. Conclusion 1 is clear since the three modules are irreducible.

Conclusion 2 can be proved similarly to the proof of the similar statement in the $N=1$ case in [HM]. Here we give a different proof.

Suppose that

$$(w_{(1)})_{-\Delta-1}w_{(2)}, \tag{2.2}$$

$$(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)}, \tag{2.3}$$

$$(G^-(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)}, \tag{2.4}$$

$$(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} \tag{2.5}$$

are all equal to 0 but $\mathcal{Y}\neq 0$.

Using the associator formula (obtained by taking residue in x_1 in the Jacobi identity defining intertwining operators)

$$\begin{aligned}\mathcal{Y}(Y(u,x_0)w,x_2) &= Y(u,x_0+x_2)\mathcal{Y}(w,x_2)\\ &\quad -(-1)^{|u||w|}\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(w,x_2)Y(u,x_1)\end{aligned}$$

repeatedly, we see that $\mathcal{Y}=0$ if $\mathcal{Y}(w_{(1)},x)=0$. Thus $\mathcal{Y}(w_{(1)},x)\neq 0$.

Using the commutator formula (obtained by taking residue in x_0 in the Jacobi identity defining intertwining operators)

$$\begin{aligned}&Y(u,x_1)\mathcal{Y}(w,x_2)-(-1)^{|u||w|}\mathcal{Y}(w,x_2)Y(w,x_1)\\ &\quad=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)w,x_2)\end{aligned}$$

together with the $N=2$ Neveu-Schwarz algebra relations, the $L(-1)$-derivative property and the definition of the lowest weight vector $w_{(1)}$ repeatedly, we see that $\mathcal{Y}(w_{(1)},x)=0$ if (2.2)–(2.5) are all equal to 0. Thus these four vectors cannot be all 0. We have a contradiction.

We prove Conclusion 3 now. By Conclusion 2, we need only estimate the number of nonzero vectors in the set of four vectors (2.2)–(2.5).

We need the following:

Lemma 2.3. *The following equalities hold:*

$$q_m^{j_3,k_3}(w_{(1)})_{-\Delta-1}w_{(2)}=(q_m^{j_1,k_1}+q_m^{j_2,k_2})(w_{(1)})_{-\Delta-1}w_{(2)}, \tag{2.6}$$

$$\begin{aligned}&q_m^{j_3,k_3}(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)}\\ &\quad=(q_m^{j_1,k_1}+q_m^{j_2,k_2}+1)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)},\end{aligned} \tag{2.7}$$

$$\begin{aligned}&q_m^{j_3,k_3}(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)}\\ &\quad=(q_m^{j_1,k_1}+q_m^{j_2,k_2}-1)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)},\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
(2.9)\qquad & q_m^{j_3,k_3}(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} \\
&= (q_m^{j_1,k_1}+q_m^{j_2,k_2})(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} \\
&\quad +(2h_m^{j_1,k_1}+q_m^{j_1,k_1})(w_{(1)})_{-\Delta-1}w_{(2)}.
\end{aligned}
$$

Proof. A straightforward calculation gives

$$
J(0)(w_{(1)})_{-\Delta-1}w_{(2)} = (q_m^{j_1,k_1}+q_m^{j_2,k_2})(w_{(1)})_{-\Delta-1}w_{(2)},
$$

$$
\begin{aligned}
& J(0)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)} \\
&\quad = (q_m^{j_1,k_1}+q_m^{j_2,k_2}+1)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)},
\end{aligned}
$$

$$
\begin{aligned}
& J(0)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)} \\
&\quad = (q_m^{j_1,k_1}+q_m^{j_2,k_2}-1)(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_{(2)},
\end{aligned}
$$

$$
\begin{aligned}
& J(0)(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} \\
&\quad = (q_m^{j_1,k_1}+q_m^{j_2,k_2})(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} \\
&\qquad +(2h_m^{j_1,k_1}+q_m^{j_1,k_1})(w_{(1)})_{-\Delta-1}w_{(2)}.
\end{aligned}
$$

But on the other hand, note that (2.2)–(2.5) are all (zero or nonzero) constant multiples of $w_{(3)}$ and thus all have $U(1)$ charge $q_m^{j_3,k_3}$. So we have (2.6)–(2.9). □

We prove Conclusion 3 using this lemma now. If $q_m^{j_3,k_3}$ is not equal to one of the numbers $q_m^{j_1,k_1}+q_m^{j_2,k_2}$ $q_m^{j_1,k_1}+q_m^{j_2,k_2}-1$ and $q_m^{j_1,k_1}+q_m^{j_2,k_2}+1$, then from (2.6)–(2.9), we conclude that (2.2)–(2.5) are all equal to 0. Thus the space of intertwining operators is 0.

If $q_m^{j_3,k_3}=q_m^{j_1,k_1}+q_m^{j_2,k_2}+1$, then by (2.6), (2.8) and (2.9), (2.2), (2.4) and (2.5) must be 0. Thus there is at most one nonzero vector (2.3). So the dimension is at most 1.

Similarly in the case of $q_m^{j_3,k_3}=q_m^{j_1,k_1}+q_m^{j_2,k_2}-1$, we can show that $(w_{(1)})_{-\Delta-1}w_{(2)}$, (2.2), (2.3) and (2.5) must be 0 and thus the dimension is at most 1.

If $q_m^{j_3,k_3}=q_m^{j_1,k_1}+q_m^{j_2,k_2}$, then by (2.7) and (2.8), (2.3) and (2.4) must both be 0. Thus we have at most two nonzero vectors (2.2) and (2.5) and the dimension is at most 2. □

Remark 2.4. In the proofs above we do not use the particular properties, except the irreducibility, of $L_{\mathfrak{ns}(2)}(c_m,h_m^{j_i,k_i},q_m^{j_i,k_i})$, $i=1,2,3$. Thus the conclusions of Proposition 2.2 remain true if we replace c_m by an arbitrary c and $L_{\mathfrak{ns}(2)}(c_m,h_m^{j_i,k_i},q_m^{j_i,k_i})$, $i=1,2,3$, by $L_{\mathfrak{ns}(2)}(c,0,0)$-modules $L_{\mathfrak{ns}(2)}(c,h_i,q_i)$, $i=1,2,3$, if $L_{\mathfrak{ns}(2)}(c,h_i,q_i)$ are irreducible.

Definition 2.5. An irreducible module for $L_{\mathfrak{ns}(2)}(c,0,0)$ is said to be chiral (anti-chiral) if

$$
G^+(-1/2)w=0,\quad (G^-(-1/2)w=0)
$$

where w is a nonzero lowest weight vector of the module.

Note that in the case $c=c_m$ we have only finitely many chiral (anti-chiral) modules.

Corollary 2.6. *Assume that* $L_{\mathfrak{ns}(2)}(c_m, h_m^{j_1,k_1}, q_m^{j_1,k_1})$ *is chiral or anti-chiral. Then the dimension of the space*

$$\mathcal{V}^{L_{\mathfrak{ns}(2)}(c_m,h_m^{j_3,k_3})}_{L_{\mathfrak{ns}(2)}(c_m,h_m^{j_1,k_1})L_{\mathfrak{ns}(2)}(c_m,h_m^{j_2,k_2})}$$

is at most 1.

Proof. Assume that $L_{\mathfrak{ns}(2)}(c_m, h_m^{j_1,k_1}, q_m^{j_1,k_1})$ is chiral. Then

$$(G^+(-1/2)w_{(1)})_{-\Delta-1/2}w_2 = 0.$$

We claim that in this case

$$(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_{(2)} = 2(w_{(1)})_{-\Delta-1}w_{(2)}. \tag{2.10}$$

In fact, the commutator formula for $G^+(-1/2)$ and $G^-(-1/2)$ gives

$$G^+(-1/2)G^-(-1/2) = -G^-(-1/2)G^+(-1/2) + 2L(-1).$$

Thus

(2.11)

$$\begin{aligned}(G^+(-1/2)G^-(-1/2)w_{(1)})_{-\Delta}w_2 &= -(G^-(-1/2)G^+(-1/2)w_{(1)})_{-\Delta}w_2\\ &\quad +2(L(-1)w_{(1)})_{-\Delta}w_2\\ &= 2(L(-1)w_{(1)})_{-\Delta}w_2.\end{aligned}$$

From the $L(-1)$-derivative property for intertwining operators, we obtain

$$(L(-1)w_{(1)})_{-\Delta} = (w_{(1)})_{-\Delta-1}.$$

Thus the right-hand side of (2.11) becomes $2(w_{(1)})_{-\Delta-1}w_{(2)}$, proving (2.10). On the other hand, from (2.6) and (2.8), we see that the vectors $(w_{(1)})_{-\Delta-1}w_2$ and $(G^-(-1/2)w_{(1)})_{-\Delta-1/2}w_2$ cannot be nonzero at the same time. Thus in this case, the dimension of the space spanned by the four vectors (2.2)–(2.5) is at most 1. Equivalently, the corollary is proved. □

Remark 2.7. Note that the operator $J(0)$ plays an essential role in the proofs of Conclusion 3 in Proposition 2.2 and Corollary 2.6.

Remark 2.8. After the first version of the present paper was finished, we received a preprint [A3] from Adamović in which the fusion rules for $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$ are calculated explicitly and a stronger complete reducibility theorem is proved. But for the purpose of the present paper, we shall not need these stronger results.

Combining Theorem 2.1 and the second or third conclusion of Proposition 2.2, we obtain

Corollary 2.9. *The unitary minimal* $N = 2$ *superconformal vertex operator superalgebras are rational in the sense of* [HL1]*, that is, the following three conditions are satisfied:*

1. *Every module for such an algebra is completely reducible.*
2. *There are only finitely many inequivalent irreducible modules for such an algebra.*
3. *The fusion rules among any three (irreducible) modules are finite.* □

We also have

Proposition 2.10. *Any finitely-generated lower truncated generalized module* W *for* $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$ *is an ordinary module.*

Proof. The proof is the same as the corresponding result in [HM]. We repeat it here since it is simple. Suppose that W is generated by a single vector $w \in W$. Then by the Poincaré-Birkhoff-Witt theorem and the lower truncation condition, every homogeneous subspace of $U(\mathfrak{ns}(2))w$ is finite-dimensional, proving the result. $\square$

Let m_i, $i = 1, \dots, n$, be positive integers and let

$$V = L_{\mathfrak{ns}(2)}(c_{m_1}, 0, 0) \otimes \cdots \otimes L_{\mathfrak{ns}(2)}(c_{m_n}, 0, 0).$$

From the trivial generalizations of the results proved in [FHL] and [DMZ] to vertex operator superalgebras, V is a rational $N = 2$ superconformal vertex operator superalgebra, a set of representatives of equivalence classes of irreducible modules for V can be listed explicitly and the fusion rules for V can be calculated easily.

We introduce a class of $N = 2$ superconformal vertex operator superalgebras:

Definition 2.11. Let m_i, $i = 1, \dots, n$, be positive integers. An $N = 2$ superconformal vertex operator superalgebra V is said to be *in the class* $\mathcal{C}_{m_1;\dots;m_n}$ if V has a vertex operator subalgebra isomorphic to $L_{\mathfrak{ns}(2)}(c_{m_1}, 0, 0) \otimes \cdots \otimes L_{\mathfrak{ns}(2)}(c_{m_n}, 0, 0)$.

Proposition 2.12. *Let V be an $N = 2$ superconformal vertex operator superalgebra in the class $\mathcal{C}_{m_1;\dots;m_n}$. Then any finitely-generated lower truncated generalized V-module W is an ordinary module.*

Proof. The proof is similar to the proofs of Proposition 3.7 in [H2] and Proposition 2.7 in [HM]. Here we only point out the main difference. As in [H2] and [HM], we discuss only the case $n = 2$. Similar to the proofs of Proposition 3.7 in [H2] and Proposition 2.7 in [HM], using the Jacobi identity, the $N = 2$ Neveu-Schwarz relations, in particular, the formulas $[G^+(-1/2), G^-(-1/2)] = 2L(-1)$, $(G^+(-1/2))^2 = (G^-(-1/2))^2 = 0$, and Theorem 4.7.4 of [FHL], we can reduce our proof in the case of $n = 2$ to the finite-dimensionality of the space spanned by the elements of the form

(2.12)
$$\begin{aligned} &A(L(-1)^{l_1} J(-1)^{m_1} G^+(-1/2)^{k_1} G^-(-1/2)^{k_2} u^{(1)}_{(j)})_{j_1} B w^{(1)}_{(t)} \\ &\qquad \otimes C(L(-1)^{l_2} J(-1)^{m_2} G^+(-1/2)^{k_3} G^-(-1/2)^{k_4} u^{(2)}_{(j)})_{j_2} D w^{(2)}_{(t)}, \end{aligned}$$

$l_1, l_2 \in \mathbb{N}$, $m_1, m_2, k_1, k_2, k_3, k_4 = 0, 1$, $j_1, j_2 \in \mathbb{Z}$, $t = 1, \dots, c$, $j = 1, \dots, d$, where A (or C) are products of operators of the forms $L(-a)_1$ (or $L(-a)_2$), $J(-a)_1$ (or $J(-a)_2$), $a \in \mathbb{Z}_+$, and $G^\pm(-b)_1$ (or $G^\pm(-b)_2$), $b \in \mathbb{Z} + \frac{1}{2}$, B (or D) are products of operators of the forms $L(a)_1$ (or $L(a)_2$), $J(a)_1$ (or $J(a)_2$), $a \in \mathbb{Z}_+$, and $G^\pm(b)_1$ (or $G^\pm(b)_2$), $b \in \mathbb{Z} + \frac{1}{2}$, where $u^{(i)}_{(j)}$, $j = 1, \dots, d$, $i = 1, 2$, are homogeneous elements of V such that the $L_{\mathfrak{ns}(2)}(c_{m_i}, 0, 0)$-submodules generated by them are isomorphic to $L(c_{m_i}, h^{r_j,s_j}_{m_i}, q^{r_j,s_j}_{m_i})$ for some $r_j, s_j \in \mathbb{N}_{\frac{1}{2}}$ satisfying $0 \le r_j, s_j, r_j + s_j < m_i$ with the images of $u^{(i)}_{(j)}$, $j = 1, \dots, d$, $i = 1, 2$, as the lowest weight vectors and such that V is isomorphic to the direct sum of these submodules, and where $w^{(i)}_{(t)}$, $t = 1, \dots, c$, $i = 1, 2$, are homogeneous elements of some irreducible $L_{\mathfrak{ns}(2)}(c_{m_i}, 0, 0)$-modules. Because of the $L(-1)$-property, we may assume that $l_1 = l_2 = 0$. Because W is lower truncated, there are only finitely many elements of the form (2.12) of the fixed weight. Thus W is a V-module. $\square$

3. The convergence and extension property for products of intertwining operators

In this section, we study products of intertwining operators for the unitary minimal $N = 2$ superconformal vertex operator superalgebras. The main result is the following:

Theorem 3.1. *Let m be a positive integer. Then intertwining operators for the vertex operator superalgebra $L(c_m, 0, 0)$ satisfy the convergence and extension property for products of intertwining operators introduced in* [H1].

An immediate consequence is a similar result for the vertex operator algebras in the class $\mathcal{C}_{m_1;\dots;m_n}$. See Theorem 3.7.

Instead of proving Theorem 3.1 by deriving differential equations with regular singularities satisfied by the matrix elements of products of intertwining operators of lowest weight vectors, as in [H1], [HL7] and [HM], we use the so-called anti-Kazama-Suzuki mapping [FST], which reduces the problem to the study of intertwining operators for a vertex operator algebra constructed from an $\widehat{\mathfrak{sl}}_2$ integrable lowest weight representation.

First we give some auxiliary constructions and discuss some results obtained using the so-called "anti-Kazama-Suzuki mapping" (introduced in [FST]).

Fix a positive integer m. As before, we let $c_m = \frac{3m}{m+2}$ and denote the unitary minimal $N = 2$ superconformal vertex operator superalgebra by $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$.

Let L be a rank one lattice generated by α with the bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle \alpha, \alpha \rangle = -1,$$

and let $l = L \otimes_{\mathbb{Z}} \mathbb{C}$. As in [FLM], we have a vertex superalgebra

$$V_L \cong S(\hat{l}_-) \otimes \mathbb{C}[L].$$

Note that V_L is super since L is odd and V_L does not satisfy the grading-restriction conditions for the grading obtained from the usual Virasoro elements for lattice vertex algebras since the bilinear form is not positive definite.

Let

$$\tilde{\omega}_{V_L} = -\frac{\alpha(-1)^2}{2} + \frac{\alpha(-2)}{2}.$$

It can be verified easily that the component operators of the vertex operator associated to $\tilde{\omega}_{V_L}$ satisfy the Virasoro relations with central charge 4. In particular, the component operator for the -2nd power of x gives a $\mathbb{Z}/2$-grading for V_L. With this grading, V_L is a $\mathbb{Z}/2$-graded vertex superalgebra. However, it is easy to see that this grading is not truncated from below. Thus V_L with $\tilde{\omega}_{V_L}$ as its Virasoro element fails to be a vertex operator superalgebra.

We also need the following construction of the so-called "Liouville scalar model:" Let $\hat{\mathbb{R}}$ be the Lie algebra with a basis $a(n)$, $n \in \mathbb{Z}$, and d, satisfying the bracket relations

$$\begin{aligned} [a(m), a(n)] &= m\delta_{m+n,0} d, \\ [a(m), d] &= 0 \end{aligned}$$

for $m, n \in \mathbb{Z}$. Let $M(1, s)$ be the corresponding irreducible highest weight module with central charge 1 and highest weight s. It is well-known that $M(1, 0) = S(\hat{\mathbb{R}}_-)$ has a vertex operator algebra structure with the Virasoro element $\omega_{M(1,0)} = \frac{a(-1)^2}{2}$.

Consider the vertex algebra structure on $M(1,0)$ together with a different Virasoro element

$$\tilde{\omega}_{M(1,0)} = \frac{a(-1)^2}{2} + \frac{ia(-2)}{2}.$$

A straightforward calculation shows that the vertex algebra structure on $M(1,0)$ together with the Virasoro element $\tilde{\omega}_{M(1,0)}$ is a vertex operator algebra with the central charge 4 (see, for example, [L] and [FF]). We shall denote this vertex operator algebra by V_{Liou} (the vertex operator algebra associated to the Liouville scalar model). In addition, every $M(1,s)$, $s \in \mathbb{C}$, is an irreducible V_{Liou}-module and any V_{Liou}-module on which $a(0)$ acts semisimply is completely reducible. We will work only with such V_{Liou}-modules, which are enough for our purposes.

The anti-Kazama-Suzuki mapping gives us a structure of an $\widehat{\mathfrak{sl}}_2$-module on $L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k}) \otimes V_L$ for $j,k \in \mathbb{N}_{\frac{1}{2}}$, $0 < j,k,j+k < m$. Consider the vectors (as in [FST] and [A1])

$$\begin{aligned}
\mathfrak{e} &= G^+(-3/2)\mathbf{1}_{c_m,0,0} \otimes e^{-\alpha},\\
\mathfrak{f} &= \frac{m+2}{2} G^-(-3/2)\mathbf{1}_{c_m,0,0} \otimes e^{\alpha},\\
\mathfrak{h} &= -m\mathbf{1}_{c_m,0,0} \otimes \alpha(-1) + (m+2)J(-1)\mathbf{1}_{c_m,0,0} \otimes e^0
\end{aligned}$$

in $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$. Then the vertex operators $Y(\mathfrak{e},x)$, $Y(\mathfrak{f},x)$ and $Y(\mathfrak{h},x)$ for the $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$-module $L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k}) \otimes V_L$ give a representation of $\widehat{\mathfrak{sl}}_2$ of level m on $L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k}) \otimes V_L$. The main observation in [FST] is that $L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k}) \otimes V_L$ is completely reducible as an $\widehat{\mathfrak{sl}}_2$-module. In the special case $h_m^{j,k} = q_m^{j,k} = 0$, we obtain a vertex subalgebra of $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$ which is isomorphic as a vertex algebra to the underlying vertex algebra of the vertex operator algebra $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ on the integrable highest-weight $\mathfrak{sl}(2)$-module of level m and highest weight 0 (as in [A1]). The Virasoro element for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ is given by the Sugawara-Segal construction, and if we identify this vertex subalgebra with $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$, then

$$\begin{aligned}
\omega_{L_{\widehat{\mathfrak{sl}}(2)}(m,0)} &= \omega_{L_{\mathfrak{ns}(2)}(c_m,0,0)} \otimes e^0 + \frac{m+2}{4} J(-1)^2 \mathbf{1}_{c_m,0,0} \otimes e^0\\
&\quad - \frac{m+2}{2} J(-1)\mathbf{1}_{c_m,0,0} \otimes \alpha(-1) + \frac{m}{4}\mathbf{1}_{c_m,0,0} \otimes \alpha(-1)^2,
\end{aligned}$$

where $\omega_{L_{\widehat{\mathfrak{sl}}(2)}(m,0)}$ and $\omega_{L_{\mathfrak{ns}(2)}(c_m,0,0)}$ are the Virasoro elements for the vertex operator (super)algebras $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ and $L_{\mathfrak{ns}(2)}(c_m,0,0)$, respectively. We see that under the isomorphism from this vertex subalgebra of $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$ to $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$, the Virasoro element in $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ is not the image of the Virasoro element of $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$.

To get the correct Virasoro element (as in [FST]), we consider the vertex subalgebra of $L_{\mathfrak{ns}(2)}(c_m,0,0) \otimes V_L$ generated by the element

$$\rho = \sqrt{\frac{m+2}{2}}(J(-1)\mathbf{1}_{c_m,0,0} \otimes e^0 - \mathbf{1}_{c_m,0,0} \otimes \alpha(-1)).$$

It is straightforward to verify that this vertex subalgebra is actually isomorphic to V_{Liou}. Straightforward calculations also show that $Y(\rho,x)$ commutes with $\widehat{\mathfrak{sl}}_2$ gener-

ators. So $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ is isomorphic to a vertex subalgebra of $L_{\mathfrak{ns}(2)}(c_m,0,0)\otimes V_L$ as well and we shall, for convenience, identify $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ with this vertex subalgebra. It is easy to see that the Virasoro element of $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ is identified with the Virasoro element of $L_{\mathfrak{ns}(2)}(c_m,0,0)\otimes V_L$. Thus $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ is a vertex operator subalgebra.

Now we use a key result in [FST] and [FSST] (in the case of unitary modules), slightly reformulated in the language of vertex operator algebras:

Theorem 3.2. *As a generalized $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-module,*

$$L_{\mathfrak{ns}(2)}(c_m,h,q)\otimes V_L$$

decomposes as

$$\bigoplus_{k\in\{0,1,\dots,m\}}\bigoplus_{s\in I_s} L_{\widehat{\mathfrak{sl}}(2)}(m,k)\otimes M(1,s). \tag{3.1}$$

where s runs through a certain infinite index set I_s.

Proof. The main result from [FSST], applied to the case of irreducible admissible modules (cf. [AM]), yields a decomposition similar to the above decomposition but with twisted admissible modules for $\widehat{\mathfrak{sl}}(2)$ also appearing in (3.1). In the case of unitary modules (which are admissible modules), the situation is simpler because only ordinary unitary modules appear (see also Remark 4.7 in [FS]). So the theorem holds. □

Remark 3.3. The proof of the theorem above illustrates why the non-unitary minimal models for the superalgebra $\mathfrak{ns}(2)$ are harder to study than the unitary ones (see also [A1]).

We know that $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ is rational (see [FZ]). Although V_{Liou} is not rational, any module on which $a(0)$ acts semisimply is completely reducible, as we mentioned above. Any irreducible module for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ is isomorphic to $L_{\widehat{\mathfrak{sl}}(2)}(m,i)$ for some $i\in\{1,\dots,m\}$ and any irreducible module for V_{Liou} is isomorphic to $M(1,s)$ for some $s\in\mathbb{C}$. Thus by the result in [FHL] on modules for a tensor product of vertex operator algebras, any irreducible module for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ is isomorphic to $L_{\widehat{\mathfrak{sl}}(2)}(m,i)\otimes M(1,s)$ for some $i\in\{0,...,m\}$, $s\in\mathbb{C}$.

Proposition 2.7 in [DMZ] and its proof can be generalized trivially to the case where one of the vertex operator algebra is an irrational vertex operator algebra like V_{Liou}, such that in particular, any $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-module with $\mathbf{1}_{L_{\widehat{\mathfrak{sl}}(2)}(m,0)}\otimes a(0)$ ($\mathbf{1}_{L_{\widehat{\mathfrak{sl}}(2)}(m,0)}$ being the vacuum vector of $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$) acting semi-simply is completely reducible. Now suppose that M is such an $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-module. Then it follows that M is completely reducible. So it has a decomposition

$$\bigoplus_{\beta\in B} M_\beta$$

where B is an index set. Note that here the sum might be infinite (comparing with the rational case where this sum is always finite). Since any irreducible $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-module is isomorphic to $L_{\widehat{\mathfrak{sl}}(2)}(m,i)\otimes M(1,s)$ for some $i\in\{0,...,m\}$, $s\in\mathbb{C}$, M_β for any $\beta\in B$ is isomorphic to such a module.

We need the following:

Lemma 3.4. *Let $\mathcal{Y}$ be an intertwining operator of type*

$$\begin{pmatrix} L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3) \\ L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1) \quad L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2) \end{pmatrix}.$$

Then

$$\mathcal{Y}=\mathcal{Y}'\otimes\mathcal{Y}''$$

where $\mathcal{Y}'$ and $\mathcal{Y}''$ are intertwining operators of types

$$\begin{pmatrix} L_{\widehat{\mathfrak{sl}}(2)}(m,i_3) \\ L_{\widehat{\mathfrak{sl}}(2)}(m,i_1) \ L_{\widehat{\mathfrak{sl}}(2)}(m,i_2) \end{pmatrix}$$

and

$$\begin{pmatrix} M(1,s_3) \\ M(1,s_1) \ M(1,s_2) \end{pmatrix},$$

respectively. In particular, all fusion rules for irreducible modules for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$ are 0 or 1.

Proof. It is enough to show that there is a linear injective map from

$$\mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3)}_{(L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1))(L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2))},$$

the space of intertwining operators of type

$$\begin{pmatrix} L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3) \\ L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1) \quad L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2) \end{pmatrix},$$

to

$$\mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)}_{L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)}\otimes\mathcal{V}^{M(1,s_3)}_{M(1,s_1)M(1,s_2)},$$

where $\mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)}_{L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)}$ and $\mathcal{V}^{M(1,s_3)}_{M(1,s_1)M(1,s_2)}$ are the space of intertwining operators of type $\binom{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)}{L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)}$ and $\binom{M(1,s_3)}{M(1,s_1)M(1,s_2)}$, respectively. But this follows from Proposition 2.10 in [DMZ] which in turn is a consequence of a result in [FHL] on irreducible modules for a tensor product vertex operator algebra and a result in [FZ] giving an isomorphism between a space of intertwining operators and a certain vector space. Since the fusion rules for irreducible $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$-modules are 0 and 1 (see [FZ]) and the same is true for irreducible V_{Liou}-modules, the lemma is proved. □

Remark 3.5. Note that this result should be very useful in calculating the fusion algebra for $L_{\mathfrak{ns}(2)}(c_m,0,0)$ since every such intertwining operator for $L_{\mathfrak{ns}(2)}(c_m,0,0)$ factors as a tensor product of an intertwining operator for vertex operator algebra $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ and an intertwining operator for the vertex operator algebra associated to the Heisenberg algebra. In the present paper, the exact values of the fusion rules are not what we are interested in and thus we shall not calculate them here. (After the first version of the present paper was finished, we received a preprint [A3] from Adamović in which the fusion rules for $L_{\mathfrak{ns}(2)}(c_m,0,0)$ are calculated explicitly.)

Using Lemmas 3.4, we obtain

Proposition 3.6. *For fixed $i_1, i_2 \in \{1, \dots, m\}$ and $s_1, s_2 \in \mathbb{C}$, if $s_3 \neq s_1 + s_2$, the space*

$$\mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3)}_{(L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1))(L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2))}$$

is 0. *In particular, there are only finitely many pairs*

$$(i_3, s_3) \in \{1, \dots, m\} \times \mathbb{C}$$

such that the space

$$\mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3)}_{(L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1))(L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2))}$$

are not 0.

Proof. Let

$$\mathcal{Y} \in \mathcal{V}^{L_{\widehat{\mathfrak{sl}}(2)}(m,i_3)\otimes M(1,s_3)}_{(L_{\widehat{\mathfrak{sl}}(2)}(m,i_1)\otimes M(1,s_1))(L_{\widehat{\mathfrak{sl}}(2)}(m,i_2)\otimes M(1,s_2))}.$$

By Lemma 3.4,

$$\mathcal{Y} = \mathcal{Y}' \otimes \mathcal{Y}''$$

where $\mathcal{Y}'$ and $\mathcal{Y}''$ are intertwining operators of types

$$\begin{pmatrix} L_{\widehat{\mathfrak{sl}}(2)}(m, i_3) \\ L_{\widehat{\mathfrak{sl}}(2)}(m, i_1) \ L_{\widehat{\mathfrak{sl}}(2)}(m, i_2) \end{pmatrix}$$

and

$$\begin{pmatrix} M(1, s_3) \\ M(1, s_1) \ M(1, s_2) \end{pmatrix},$$

respectively. It is clear that if $s_3 \neq s_1 + s_2$, $\mathcal{Y}'' = 0$, proving the result. □

Proof of Theorem 3.1. Let $W_1, \dots, W_5$ be irreducible $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$-modules and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ intertwining operators of types $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively. Consider the (formal) matrix coefficients

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle, \tag{3.2}$$

where $w_{(l)} \in W_l$, $l = 1, 2, 3$ and $w'_{(4)} \in W'_4$. We shall identify W_l, $l = 1, 2, 3$, with $W_l \otimes e^0$ in $W_l \otimes V_L$ and W'_4 with $W'_4 \otimes e^0$ in $W'_4 \otimes V_L$. In particular, we use the same notations $w_{(l)}$, $l = 1, 2, 3$, to denote $w_{(l)} \otimes e^0$, and $w'_{(4)}$ to denote $w'_{(4)} \otimes e^0$.

We extend intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ uniquely to intertwining operators (denoted by the same notations $\mathcal{Y}_1$ and $\mathcal{Y}_2$) of types

$$\begin{pmatrix} W_4 \otimes V_L \\ W_1 \otimes V_L \quad W_5 \otimes V_L \end{pmatrix},$$

and

$$\begin{pmatrix} W_5 \otimes V_L \\ W_2 \otimes V_L \quad W_3 \otimes V_L \end{pmatrix},$$

respectively. By Theorem 3.2, $W_l \otimes V_L$, $l = 1, 2, 3$, and $W'_4 \otimes V_L$ are generalized modules for $L_{\widehat{\mathfrak{sl}}(2)}(m, 0) \otimes V_{\mathrm{Liou}}$ and are completely reducible. So $w_{(l)} = \sum_{k=1}^{p_l} w^{(k)}_{(l)}$, $l = 1, 2, 3$, and $w'_{(4)} = \sum_{k=1}^{p_4} w^{(k)}_{(4)}$ where $w^{(k)}_{(l)}$, $k = 1, \dots, p_i$, $l = 1, 2, 3, 4$, are

elements of direct summands $M_l^{(k)}$ (irreducible $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-modules) in W_i for $i=1,2,3$ or W_4' for $i=4$. Thus (3.2) is equal to

$$\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}\sum_{k_3=1}^{p_3}\sum_{k_4=1}^{p_4}\langle w_{(4)}^{(k_4)},\mathcal{Y}_{k_1W_5}^{k_4}(w_{(1)}^{(k_1)},x_1)\mathcal{Y}_{k_2k_3}^{W_5}(w_{(2)}^{(k_2)},x_2)w_{(3)}^{(k_3)}\rangle, \tag{3.3}$$

where $\mathcal{Y}_{k_1W_5}^{k_4}$, $\mathcal{Y}_{k_2k_3}^{W_5}$ are intertwining operators of types $\binom{M_4^{(k_4)}}{M_1^{(k_1)}W_5}$, $\binom{W_5}{M_2^{(k_2)}M_3^{(k_3)}}$, respectively.

By Theorem 3.2, W_5 is a completely reducible generalized module for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$. By Proposition 3.6, (3.3) is equal to

$$\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}\sum_{k_3=1}^{p_3}\sum_{k_4=1}^{p_4}\sum_{k_5=1}^{p_5}\langle w_{(4)}^{(k_4)},\mathcal{Y}_{k_1k_5}^{k_4}(w_{(1)}^{(k_1)},x_1)\mathcal{Y}_{k_2k_3}^{k_5}(w_{(2)}^{(k_2)},x_2)w_{(3)}^{(k_3)}\rangle, \tag{3.4}$$

where $\mathcal{Y}_{k_1k_5}^{k_4}$, $\mathcal{Y}_{k_2k_3}^{k_5}$ are intertwining operators of types $\binom{M_4^{(k_4)}}{M_1^{(k_1)}M_5^{(k_5)}}$, $\binom{M_5^{(k_5)}}{M_2^{(k_2)}M_3^{(k_3)}}$, respectively, and $M_5^{(k_5)}$, $k_5=1,\dots,p_5$, are irreducible $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-submodules of W_5 as $L_{\widehat{\mathfrak{sl}}(2)}(m,0)\otimes V_{\mathrm{Liou}}$-modules.

By Proposition 3.4,

$$\mathcal{Y}_{k_1k_5}^{k_4}=(\mathcal{Y}_{k_1k_5}^{k_4})'\otimes(\mathcal{Y}_{k_1k_5}^{k_4})''$$

and

$$\mathcal{Y}_{k_2k_3}^{k_5}=(\mathcal{Y}_{k_2k_3}^{k_5})'\otimes(\mathcal{Y}_{k_2k_3}^{k_5})'',$$

where $(\mathcal{Y}_{k_1k_5}^{k_4})'$ and $(\mathcal{Y}_{k_2k_3}^{k_5})'$ are intertwining operators for the vertex operator algebras $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ and $(\mathcal{Y}_{k_1k_5}^{k_4})''$ and $(\mathcal{Y}_{k_2k_3}^{k_5})''$ are intertwining operators for the vertex operator algebras V_{Liou}. Thus (3.4) is equal to a finite sum of series of the form

$$\langle\tilde{w}_{(4)},\widetilde{\mathcal{Y}}_1(\tilde{w}_{(1)},x_1)\widetilde{\mathcal{Y}}_2(\tilde{w}_{(2)},x_2)\tilde{w}_{(3)}\rangle\langle\tilde{w}_{(4)},\widetilde{\widetilde{\mathcal{Y}}}_1(\tilde{\tilde{w}}_{(1)},x_1)\widetilde{\widetilde{\mathcal{Y}}}_2(\tilde{\tilde{w}}_{(2)},x_2)\tilde{\tilde{w}}_{(3)}\rangle, \tag{3.5}$$

where $\widetilde{\mathcal{Y}}_1$ and $\widetilde{\mathcal{Y}}_2$ are intertwining operators among irreducible modules for $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ and $\widetilde{\widetilde{\mathcal{Y}}}_1$ and $\widetilde{\widetilde{\mathcal{Y}}}_2$ are intertwining operators among irreducible modules for V_{Liou}.

In [HL7], it was proved that intertwining operators for the vertex operator algebra $L_{\widehat{\mathfrak{sl}}(2)}(m,0)$ satisfy the convergence and extension property for products using the Knizhnik-Zamolodchikov equations. The convergence and extension property for products of intertwining operators for the vertex operator algebra V_{Liou} can be proved trivially by a straightforward calculation. Using the convergence and extension properties for products of intertwining operators for these vertex operator algebras and using the fact proved above that (3.2) is a finite sum of series of the form (3.5), we conclude that (3.2) is convergent when we substitute $e^{n_i\log z_i}$ for x_i^n with $z_1,z_2\in\mathbb{C}$ satisfying $|z_1|>|z_2|>0$ and it can be analytically extended to an analytic function in the region $|z_2|>|z_1-z_2|>0$ of the form

$$\sum_{i=1}^{j}z_2^{r_i}(z_1-z_2)^{s_i}f_i\left(\frac{z_1-z_2}{z_2}\right). \tag{3.6}$$

We still need to prove the following: There exists N (which does not depend on $w_{(1)}$ and $w_{(2)}$) such that

$$\mathrm{wt}(w_{(1)}) + \mathrm{wt}(w_{(2)}) + s_i > N, \tag{3.7}$$

for $i = 1, ..., j$. The existence of N follows (as in the cases in [H2] and [HM], respectively) from an induction argument for the $N = 2$ superconformal algebra. Since any $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$-module is completely reducible and since there are only finitely many irreducible $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$-modules, we need only prove the existence in the case where W_1 and W_2 are irreducible. When $w_{(1)}$ and $w_{(2)}$ are lowest weight vectors, (3.2) is absolutely convergent in the region $|z_1| > |z_2| > 0$ and can be analytically extended to an analytic function in in the region $|z_2| > |z_1 - z_2| > 0$ of the form (3.6). We choose an N such that for these lowest weight vectors, (3.7) holds. For general $w_{(1)}$ and $w_{(2)}$, we use induction instead of the proof above to show that (3.2) converges absolutely in the region $|z_1| > |z_2| > 0$ and can be analytically extended to an analytic function in the region $|z_2| > |z_1 - z_2| > 0$ of the form (3.6). In addition, the induction also shows that (3.7) holds for the N we choose. □

An immediate consequence of Theorem 3.1 is the following:

Theorem 3.7. *Let m_i, $i = 1, \dots, n$, be positive integers and V an $N = 2$ superconformal vertex operator superalgebra in the class $\mathcal{C}_{m_1;\dots,m_n}$. Then intertwining operators for V satisfy the convergence and extension property for products of intertwining operators introduced in* [H1]. □

We omit the proof since it is the same as the corresponding result in [H2] and [HL7].

4. Intertwining operator superalgebras and vertex tensor categories for $N = 2$ unitary minimal models

Let m_i, $i = 1, \dots, n$, be n nonnegative integers and V a vertex operator superalgebra in the class $\mathcal{C}_{m_1;\dots;m_n}$. Using Corollary 2.9, Proposition 2.12 and Theorem 3.7 above, and Theorems 3.1 and 3.2 in [H2], which are proved in [H2] using results in [HL1]–[HL6] and [H1], we obtain the following:

Theorem 4.1 (associativity for intertwining operators)**.**

1. *For any V-modules W_0, W_1, W_2, W_3 and W_4, any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, and any choice of* $\log z_1$ *and* $\log z_2$*,*

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1^n = e^{n \log z_1},\ x_2^n = e^{n \log z_2},\ n \in \mathbb{C}}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. For any modules W_0, W_1, W_2, W_3, and W_5 and any intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, and any choice of $\log z_2$ *and* $\log(z_1 - z_2)$*,*

$$\langle w'_{(0)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0^n = e^{n \log(z_1 - z_2)},\ x_2^n = e^{n \log z_2},\ n \in \mathbb{C}}$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$ for $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$.

2. *For any V-modules W_0, W_1, W_2, W_3 and W_4, any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0}{W_1W_4}$ and $\binom{W_4}{W_2W_3}$, respectively, there exist a module W_5 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_5}{W_1W_2}$ and $\binom{W_0}{W_5W_3}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,*

$$\begin{aligned}&\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle|_{x_1^n=e^{n\log z_1}, x_2^n=e^{n\log z_2}, n\in\mathbb{R}}\\ &\quad = \langle w'_{(0)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle|_{x_0^n=e^{n\log(z_1-z_2)}, x_2^n=e^{n\log z_2}, n\in\mathbb{R}},\end{aligned} \tag{4.1}$$

where $\log z_1 = |z_1| + i\arg z_1$, $\log z_2 = |z_2| + i\arg z_2$ and $\log(z_1 - z_2) = |z_1 - z_2| + i\arg(z_1 - z_2)$ are the values of the logarithms of z_1, z_2 and $z_1 - z_2$ such that $0 \le \arg z_1, \arg z_2, \arg(z_1 - z_2) \le 2\pi$.

3. *For any modules W_0, W_1, W_2, W_3, and W_5, any intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_5}{W_1W_2}$ and $\binom{W_0}{W_5W_3}$, respectively, there exist a module W_4 and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0}{W_1W_4}$ and $\binom{W_4}{W_2W_3}$, respectively, such that for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the equality (4.1) holds.* □

Theorem 4.2 (commutativity for intertwining operators)**.** *For any V-modules W_0, W_1, W_2, W_3 and W_4 and any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_0}{W_1W_4}$ and $\binom{W_4}{W_2W_3}$, respectively, there exist a module W_5 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_0}{W_2W_5}$ and $\binom{W_5}{W_1W_3}$, respectively, such that for any homogeneous $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, the multivalued analytic function*

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$

of z_1 and z_2 in the region $|z_1| > |z_2| > 0$ and the multivalued analytic function

$$(-1)^{|w_{(1)}||w_{(2)}|}\langle w'_{(0)}, \mathcal{Y}_3(w_{(2)}, x_2)\mathcal{Y}_4(w_{(1)}, x_1)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$ are analytic extensions of each other. □

The notions of intertwining operator algebra in [H3] (see also [H5] and [H6]) and $N = 1$ superconformal intertwining operator superalgebra in [HM] can be generalized easily to the following notion:

Definition 4.3. An $N = 2$ *superconformal intertwining operator superalgebra* is an intertwining operator superalgebra W together with three elements τ^+, τ^- and μ such that $(W^e, Y, \mathbf{1}, \tau^+, \tau^-, \mu)$ is an $N = 2$ superconformal vertex operator algebra.

Then we have

Theorem 4.4. *Assume in addition that V is rational. Let $\mathcal{A} = \{a_i\}_{i=1}^m$ be the set of all equivalence classes of irreducible V-modules. Let $W^{a_1}, \dots, W^{a_m}$ be representatives of $a_1, \dots, a_m$, respectively. Let*

$$W = \coprod_{i=1}^m W^{a_i},$$

and let $\mathcal{V}_{a_1a_2}^{a_3}$, for $a_1, a_2, a_3 \in \mathcal{A}$, be the space of intertwining operators of type $\binom{W^{a_3}}{W^{a_1}W^{a_2}}$. Then $(W, \mathcal{A}, \{\mathcal{V}_{a_1a_2}^{a_3}\}, \mathbf{1}, \tau^+, \tau^-, \mu)$ (where $\mathbf{1}$, τ^+, τ^- and μ are the

distinguished elements of V) is an $N = 2$ superconformal intertwining operator superalgebra. □

In particular, we have

Theorem 4.5. *For any nonnegative integer m, the direct sum*

$$\coprod_{j,k\in\mathbb{N}_{\frac{1}{2}},\ 0\le j,k,j+k<m} L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k})$$

together with the finite set

$$\{j,k\in\mathbb{N}_{\frac{1}{2}} \mid\ 0\le j,k,j+k<m\},$$

the spaces of intertwining operators of type

$$\begin{pmatrix} & L_{\mathfrak{ns}(2)}(c_m, h_m^{j_3,k_3}, q_m^{j_3,k_3}) & \\ L_{\mathfrak{ns}(2)}(c_m, h_m^{j_1,k_1}, q_m^{j_1,k_1}) & & L_{\mathfrak{ns}(2)}(c_m, h_m^{j_2,k_2}, q_m^{j_2,k_2}) \end{pmatrix}$$

for $j_i, k_i \in \mathbb{N}_{\frac{1}{2}}$, $0 \le j_i, k_i, j_i + k_i < m$, $i = 1, 2, 3$, and the vacuum and the Neveu-Schwarz elements of $L_{\mathfrak{ns}(2)}(c_m, 0, 0)$ is an $N = 2$ superconformal intertwining operator superalgebra. □

Recall the sphere partial operad $K = \{K(j)\}_{j\in\mathbb{N}}$, the vertex partial operads $\tilde{K}^c = \{\tilde{K}^c(j)\}_{j\in\mathbb{N}}$ of central charge $c \in \mathbb{C}$ constructed in [H4] and the definition of vertex tensor category in [HL3] and [HL8]. For any $c \in \mathbb{C}$ and $j \in \mathbb{N}$, $\tilde{K}^c(j)$ is a trivial holomorphic line bundle over $K(j)$ and we have a canonical holomorphic section ψ_j. Given a vertex tensor category, we have, among other things, a tensor product bifunctor $\boxtimes_{\tilde{Q}}$ for each $\tilde{Q} \in \tilde{K}^c(2)$. In particular, $\psi_2(P(z)) \in \tilde{K}^c(2)$ and thus there is a tensor product bifunctor $\boxtimes_{\psi_2(P(z))}$.

Note that $\boxtimes_{P(z)}$ constructed in [HL6] can be generalized without any difficulty to categories of modules for vertex operator superalgebras. By Proposition 2.12 and Theorem 3.7, and Theorem 3.2 and Corollary 3.3 in [H2], we obtain

Theorem 4.6. *Let c be the central charge of V. Then the category of V-modules has a natural structure of vertex tensor category of central charge c such that for each $z \in \mathbb{C}^\times$, the tensor product bifunctor $\boxtimes_{\psi_2(P(z))}$ associated with $\psi_2(P(z)) \in \tilde{K}^c(2)$ is equal to the generalization to the category of V-modules of $\boxtimes_{P(z)}$ constructed in* [HL6]. □

Combining Theorem 4.6 with Theorem 4.4 in [HL3] (see [HL8] for the proof), we obtain

Corollary 4.7. *The category of V-modules has a natural structure of braided tensor category such that the tensor product bifunctor is $\boxtimes_{P(1)}$. In particular, the category of $L_{\mathfrak{ns}(2)}(c_{m_1}, 0, 0) \otimes \cdots \otimes L_{\mathfrak{ns}(2)}(c_{m_n}, 0, 0)$-modules has a natural structure of braided tensor category.* □

In particular, the special case $V = L_{\mathfrak{ns}(2)}(c_m, 0, 0)$ gives

Theorem 4.8. *For any nonnegative integer m, the category of modules for the $N = 2$ Neveu-Schwarz Lie superalgebra isomorphic to finite direct sums of*

$$L_{\mathfrak{ns}(2)}(c_m, h_m^{j,k}, q_m^{j,k}), \qquad j,k\in\mathbb{N}_{\frac{1}{2}}, 0\le j,k,j+k<m,$$

has a natural structure of braided tensor category such that the tensor product bifunctor is $\boxtimes_{P(1)}$. □

References

[A1] D. Adamović, Representations of the $N = 2$ vertex operator superalgebra, *Internat. Math. Res. Notices* **1999** (1999), 62-79. MR **99m:**17032

[A2] D. Adamović, Rationality of unitary $N = 2$ vertex operator superalgebras, `math.QA/9909055`, to appear.

[A3] D. Adamović, Vertex algebra approach to fusion rules for $N = 2$ superconformal minimal models, to appear.

[AM] D. Adamović and A. Milas, Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$, *Math. Res. Lett.* **2** (1995), 563–575. MR **96m:**17047

[Ba1] K. Barron, The supergeometric interpretation of vertex operator superalgebras, Ph.D. thesis, Rutgers University, 1996.

[Ba2] K. Barron, $N = 1$ Neveu-Schwarz vertex operator superalgebras over Grassmann algebras and with odd formal variables, in: *Representations and quantizations* (*Shanghai, 1998*), China High. Educ. Press, Beijing, 2000, pp. 9–35. CMP 2001:06

[D] M. Doerrzapf, The embedding structure of unitary $N = 2$ minimal models, *Nucl. Phys.* **B529** (1998), 639–655.

[DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, in: *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, Proc. Sympos. Pure Math., Vol. 56, Part 2, Amer. Math. Soc., Providence, RI, 1994, 295–316. MR **95c:**17043

[EG] W. Eholzer and M. R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, *Comm. Math. Phys.* **186** (1997), 61–85. MR **99e:**81075

[FF] B. Feigin and E. Frenkel, Integrals of motion and quantum groups, in: *Integrable systems and quantum groups (Montecatini Terme, 1993),* Lecture Notes in Math., Vol. 1620, Springer, Berlin, 1996, 349–418. MR **97d:**58093

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993. MR **94a:**17007

[FLM] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Pure and Appl. Math., **134**, Academic Press, New York, 1988. MR **90h:**17026

[FS] B. L. Feigin and A. M. Semikhatov, Free-field resolutions of the unitary $N = 2$ super-Virasoro representations, `hep-th/9810059`, to appear.

[FSST] B. L. Feigin, A. M. Semikhatov, V. A. Sirota and I. Y. Tipunin, Resolutions and characters of irreducible representations of the $N = 2$ superconformal algebra, *Nucl. Phys.* **B536** (1999), 617–656. MR **2000a:**81077

[FST] B. L. Feigin, A. M. Semikhatov and I. Y. Tipunin, Equivalence between chain categories of representations of affine $\mathfrak{sl}(2)$ and $N = 2$ superconformal algebras, *J. Math. Phys.* **39** (1998), 3865–3905. MR **2000i:**81047

[FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168. MR **93g:**17045

[Ge1] D. Gepner, Exactly solvable string compactifications on manifolds of rm SU(N) holonomy, *Phys. Lett.* **B199** (1987), 380–388. MR **89h:**83035

[Ge2] D. Gepner, Space-time supersymmetry in compactified string theory and superconformal models, *Nucl. Phys.* **B296** (1988), 757–778. MR **89b:**81189

[Gr] B. R. Greene, Aspects of quantum geometry, in: *Mirror Symmetry III, Proceedings of the Conference on Complex Geometry and Mirror Symmetry, Montréal, 1995,* Stud. Adv. Math., Vol. 10, Amer. Math. Soc., Internat. Press and Centre de Recherches Math., 1–67. MR **2000e:**81256

[GP] B. R. Greene and M. R. Plesser, Duality in Calabi-Yau moduli space, *Nucl. Phys.* **B338** (1990), 15–37. MR **91h:**32018

[H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.*, **100** (1995), 173-216. MR **98a:**17050

[H2] Y.-Z. Huang, Virasoro vertex operator algebras, (nonmeromorphic) operator product expansion and the tensor product theory, *J. Alg.* **182** (1996), 201–234. MR **97h:**17029

[H3] Y.-Z. Huang, Intertwining operator algebras, genus-zero modular functors and genus-zero conformal field theories, in: *Operads: Proceedings of Renaissance Conferences*, ed. J.-L. Loday, J. Stasheff, and A. A. Voronov, Contemporary Math., **202**, Amer. Math. Soc., Providence, 1997, 335–355. MR **98a:**17051

[H4] Y.-Z. Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997. MR **98i:**17037

[H5] Y.-Z. Huang, Genus-zero modular functors and intertwining operator algebras, *Internat. J. Math.* **9** (1998), 845–863. MR **99i:**17031

[H6] Y.-Z. Huang, Generalized rationality and a generalized Jacobi identity for intertwining operator algebras, *Selecta Math.* (*N.S.*) **6** (2000), 225–267.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354. MR **94k:**17045

[HL2] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, in: *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups, Proc. Joint Summer Research Conference, Mount Holyoke, 1992*, ed. P. Sally, M. Flato, J. Lepowsky, N. Reshetikhin and G. Zuckerman, Contemporary Math., Vol. 175, Amer. Math. Soc., Providence, 1994, 131-148. MR **95m:**17022

[HL3] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebra and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, and V. Kac, Birkhäuser, Boston, 1994, 349–383. MR **96e:**17061

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica, New Series* **1** (1995), 699-756. MR **98a:**17047

[HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica, New Series* **1** (1995), 757–786. MR **98a:**17047

[HL6] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995), 141-171. MR **98a:**17049

[HL7] Y.-Z. Huang and J. Lepowsky, Intertwining operator algebras and vertex tensor categories for affine Lie algebras, *Duke Math. J.* **99** (1999), 113–134. CMP 99:16

[HL8] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, V, to appear.

[HM] Y.-Z. Huang and A. Milas, Intertwining operator superalgebras and vertex tensor categories for superconformal algebras, I, `math.QA/9909039`, IHES preprint IHES/M/99/69, to appear;

[HZ] Y.-Z. Huang and W. Zhao, Semi-infinite forms and topological vertex operator algebras, *Commun. Contemp. Math.* **2** (2000), 191–241. CMP 2000:13

[L] B. Lian, On the classification of simple vertex operator algebras, *Comm. Math. Phys.* **163** (1994), 307–357. MR **95i:**17033

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