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SYMMETRIC APPROXIMATION OF FRAMES AND BASES IN HILBERT SPACES

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Abstract. We introduce the symmetric approximation of frames by normalized tight frames extending the concept of the symmetric orthogonalization of bases by orthonormal bases in Hilbert spaces. We prove existence and uniqueness results for the symmetric approximation of frames by normalized tight frames. Even in the case of the symmetric orthogonalization of bases, our techniques and results are new. A crucial role is played by whether or not a certain operator related to the initial frame or basis is Hilbert-Schmidt.

Given a Hilbert space H and a linearly independent set of vectors $\{f\}_1^n$ in H the Gram-Schmidt process is traditionally used as a means of creating an orthonormal set of vectors $\{\mu_i\}_1^n$ from $\{f_i\}_1^n$ in such a way that $\text{span}\{f_i\}_1^k = \text{span}\{\mu_i\}_1^k$ for all $1 \leq k \leq n$. This process is inherently order-dependent in that a reordering of $\{f_i\}_1^n$ generally results in an entirely new orthonormal set $\{\mu_i\}_1^n$. For some problems, it is desirable to treat the vectors $\{f_1, f_2, ..., f_n\}$ simultaneously in a method that is order-independent instead of successively as in the Gram-Schmidt process. Especially in numerical calculations the Gram-Schmidt process tends to be biased towards vectors which appear early in the given ordering. Thus it is often desirable in such applications to orthonormalize the set $\{f_i\}_1^n$ in such a way that the sum $\sum_{j=1}^n \|\mu_j - f_j\|^2$ is minimal. The resulting set $\{\mu_i\}_1^n$ is called the *symmetric* or *Löwdin orthogonalization of* $\{f_i\}_1^n$.

The original work on this subject was done by Per-Olov Löwdin, a quantum chemist, in the late 1940's ([20]). In later publications by Jerome A. Goldstein together with his coauthors J. G. Aiken, J. A. Erdos and M. Levy ([1, 2, 14]) this process was extensively studied. Also, symmetric orthogonalization has been linked to the construction of optimal algorithms for finding matrix inverse square roots of positive invertible matrices and for computing the principal square roots of invertible normal matrices in computer science ([21, 23, 24, 19]).

In his Ph.D. thesis the third author developed an operator-theoretic approach to the work of the above authors and investigated the existence and uniqueness of symmetric orthogonalization of countably infinite linearly independent sets $\{f_i\}_1^\infty$ in Hilbert spaces H. Introducing the operator $F : l_2 \to H$, $F(e_i) = f_i$ for any $i \in \mathbb{N}$ and for the standard orthonormal basis $\{e_i\}_1^\infty$ of l_2, he showed the existence of symmetric orthogonalizations of $\{f_i\}_1^\infty$ provided $(I - |F|)$ is a Hilbert-Schmidt operator

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on l_2 and the dimension of the kernel of F is less than or equal to the dimension of the orthogonal complement of the range of F. Uniqueness can be guaranteed if and only if the dimension of the kernel of F is zero, [25, Th. 4]. An alternative characterization of a symmetric orthogonalization $\{\nu_i\}_1^\infty$ of the set $\{f_i\}_1^\infty$ is the finiteness of the sum $\sum_{j=1}^{\infty} \|\nu_j - f_j\|^2$ and its equality to the square of the Hilbert-Schmidt norm of the operator $(I - |F|)$, [25, Th. 2, 3 and Cor. 5]. However, the identity of the Hilbert (sub-)spaces generated by $\{\nu_i\}_1^\infty$ and $\{f_i\}_1^\infty$, respectively, only takes place if the symmetric orthogonalization is unique, [25, Cor. 6]. These unpublished results will be presented in section three of the present paper. Also, the third author introduced the notion of a weak symmetric orthogonalization, and he showed its possible non-uniqueness in all cases by example.

The concept of frames, i.e. of sets of generators $\{f_i\}_{i\in\mathbb{N}}$ of Hilbert spaces with the property that the inequality $C \cdot \|x\|^2 \leq \sum_{j\in\mathbb{N}} |\langle x, f_j\rangle|^2 \leq D \cdot \|x\|^2$ is fulfilled for any $x \in H$ and two constants $C,\ D > 0$, generalizes the notion of a basis for Hilbert spaces. Frames play an important role in wavelet theory and its applications to signal processing, image and data compression or analysis, and others ([26]). Note that in infinite-dimensional spaces the concepts of frames and bases of Hilbert spaces do not coincide any more since some bases lack the frame property, see Example 3.1. In the setting of frames, the operator F introduced above is the adjoint of what is generally called the frame operator.

The goal of the present paper is to use this operator-theoretic approach to extend the above results to frames. We study the existence and uniqueness of symmetric approximations of frames (respectively, symmetric orthogonalizations of bases of Hilbert spaces) $\{f_i\}_{i\in\mathbb{N}}$ of Hilbert subspaces $K \subseteq H$. That means, we look for the existence and uniqueness of normalized tight frames (resp., orthonormal bases) $\{\nu_i\}_{i\in\mathbb{N}}$ of Hilbert subspaces $L \subseteq H$ such that the sum $\sum_{j\in\mathbb{N}} \|\nu_j - f_j\|^2$ is finite and admits the minimum of all finite sums $\sum_{j\in\mathbb{N}} \|\mu_j - f_j\|^2$ that might appear for any other normalized tight frame (resp., orthonormal basis) $\{\mu_i\}_{i\in\mathbb{N}}$ of any other Hilbert subspace of H. We apply the approach of the third author to symmetric approximations of frames in Hilbert spaces by normalized tight frames. We can rely on fundamental work done by David R. Larson and his collaborators Xingde Dai, Deguang Han, E. J. Ionascu and C. M. Pearcy ([11, 15, 18]), by A. Aldroubi [3], P. G. Casazza [4, 5], O. Christensen [8, 9, 10] and J. R. Holub [16, 17]. As a result we obtain that a symmetric approximation exists and is always unique if and only if the operator $(P - |F|)$ is Hilbert-Schmidt, where $(I - P)$ denotes the projection onto the kernel of $|F|$. For detailed explanations we refer the reader to section 2.

Section 1 is devoted to an alternative proof of the existence and uniqueness of the symmetric approximation of finite frames in subspaces of Hilbert spaces that remodels the proof given by the third author for the case of finite bases. Some obvious changes appear because for frames the kernel of the operator F is, generally, non-trivial.

1. Symmetric approximation of a finite frame in a subspace

Let H be a Hilbert space and $\{f_i\}_{i\in\mathbb{N}} \subset H$ be a *frame* in a separable Hilbert subspace $K \subseteq H$, i.e. there are two constants $C, D > 0$ such that the inequality

$$C \cdot \|x\|^2 \leq \sum_{j\in\mathbb{N}} |\langle x, f_j\rangle|^2 \leq D \cdot \|x\|^2$$

holds for every $x \in K \subseteq H$ and every finite or countable index set. Without loss of generality we consider only finite and countable frames in this paper. In case of uncountable frames there are no principal changes. If $C = D$ then the frame is said to be *tight*, if $C = D = 1$ then it is said to be *normalized tight.* If for two frames $\{f_i\}_{i\in\mathbb{N}}$ and $\{g_i\}_{i\in\mathbb{N}}$ of two Hilbert subspaces K and L of H, respectively, there exists an invertible bounded linear operator $T : K \to L$ such that $T(f_i) = g_i$ for every index i, then these two frames are said to be *weakly similar.* If $K = L$ then the frames are called *similar.* A *Riesz basis* of a Hilbert space K is a basis of K that is a frame at the same time. Riesz bases are precisely the images of orthonormal bases under invertible linear operators, [15, Prop. 1.5]. A frame is said to be a *near-Riesz basis* if the deletion of finitely many elements from the frame leads to a set that is a Riesz basis of the Hilbert space generated by the frame. P. G. Casazza and O. Christensen constructed in [7, Prop. 2.4] a tight frame of a separable Hilbert space that is not a near-Riesz basis. Other examples have been found by K. Seip [22]. However, any frame of a finite-dimensional Hilbert space is near-Riesz. For the details of the separable case we refer to the end of section 2.

Definition 1.1. A normalized tight frame $\{\nu_i\}_1^n$ in a finite-dimensional Hilbert subspace $L \subseteq H$ is said to be a *symmetric approximation of a finite frame* $\{f_i\}_1^n$ in a Hilbert subspace $K \subseteq H$ if the frames $\{f_i\}_1^n$ and $\{\nu_i\}_1^n$ are weakly similar and the inequality

$$\sum_{j=1}^{n} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{n} \|\nu_j - f_j\|^2$$

is valid for all normalized tight frames $\{\mu_i\}_1^n$ in Hilbert subspaces of H that are weakly similar to $\{f_i\}_1^n$.

To see why we require the frames to be weakly similar in the above definition, one only needs to consider the example of the frame $\{1/4\, e_1, e_2\}$ for $\mathbb{C}^2$. The closest frame to it, in the above sense, over the set of all possible frames for subspaces of $\mathbb{C}^2$ is $\{0, e_2\}$, which is not a frame for the same space. As we will prove shortly, the added hypothesis that the frames are weakly similar, forces the nearest frame to span the same subspace.

If the frame $\{f_i\}_1^n$ is a system of linearly independent vectors, then the set $\{\nu_i\}_1^n$ will become a symmetric orthogonalization of $\{f_i\}_1^n$, i.e. an orthonormal system, as will be shown below. For finite frames $\{f_i\}_{i=1}^n$ of a Hilbert subspace $K \subseteq H$ we have $\dim(K) \leq n$ since the reconstruction formula shows the property of a frame to be a set of generators of K. Let $\{e_i\}_1^n$ be the standard orthonormal basis of $\mathbb{C}^n$ and define an operator $F : \mathbb{C}^n \to H$ by the formula

$$F\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \sum_{j=1}^{n} \alpha_j f_j \,.$$

The operator F has a natural polar decomposition $F = W|F|$, where W is a partial isometry from $\mathbb{C}^n$ into H with initial space $\ker(F)^\perp \subseteq \mathbb{C}^n$ and range $K \subseteq H$, cf. [15, Prop. 1.10]. In particular, F possesses a closed range and the set $\{W(e_i)\}_1^n$ is a normalized tight frame in $K \subseteq H$, cf. [15, Cor. 1.2(i)]. So the frames $\{f_i\}_1^n$ and $\{W(e_i)\}_1^n$ span the same Hilbert subspace $K \subseteq H$. We want to show that the normalized tight frame $\{W(e_i)\}_1^n$ in K is a symmetric approximation of our initial frame. To proceed we need the following fact characterizing the Hilbert-Schmidt norm of operators on H.

Lemma 1.2. *Let T be a bounded linear operator on a Hilbert space H and let $\{h_i\}_{i\in\mathbb{J}_1}$ and $\{k_i\}_{i\in\mathbb{J}_2}$ be two normalized tight frames of H. If one of the sums in the following equality is finite, then the identity $\sum_{j\in\mathbb{J}_1}\|T(h_j)\|^2=\sum_{j\in\mathbb{J}_2}\|T(k_j)\|^2$ holds. If there exists at least one normalized tight frame of H for which the sum is finite, then T is a Hilbert-Schmidt operator and the square root of the sum equals its Hilbert-Schmidt norm $\|T\|_{c_2}$.*

Proof. Let $\{v_i\}_{i\in\mathbb{I}}$ be an orthonormal basis of the norm-closure of the range of T in H. Since $T(h_j)\in \operatorname{ran}(T)$ we have

$$\begin{aligned}\sum_{j\in\mathbb{J}_1}\|T(h_j)\|^2 &= \sum_{j\in\mathbb{J}_1}\sum_{i\in\mathbb{I}}|\langle T(h_j),v_i\rangle|^2=\sum_{j\in\mathbb{J}_1}\sum_{i\in\mathbb{I}}|\langle h_j,T^*(v_i)\rangle|^2\\ &= \sum_{j\in\mathbb{J}_1}\sum_{i\in\mathbb{I}}|\langle T^*(v_i),h_j\rangle|^2=\sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{J}_1}|\langle T^*(v_i),h_j\rangle|^2\\ &= \sum_{i\in\mathbb{I}}\|T^*(v_i)\|^2\ .\end{aligned}$$

The sums are all at most countable (or they are equal to infinity) since a Hilbert-Schmidt operator is compact and, hence, its domain and codomain are at most separable Hilbert subspaces. Since the same calculations are true for the other normalized tight frame $\{k_i\}_{i\in\mathbb{J}_2}$ we obtain the desired equality. □

Note that the number of elements in the normalized tight frames, and hence the dimension of H, can be finite or infinite. Moreover, if the frames are finite then they can contain different numbers of elements. Along the way we have obtained an interesting new formula for the calculation of the Hilbert-Schmidt norm of a Hilbert-Schmidt operator. The proof of the next theorem is a modification of the analogous proof for the symmetric orthogonalization of finite sets of linearly independent elements $\{f_i\}_1^n\subset H$ by the third author, [25, Th. 1]. (An alternative proof for this special case can be found in [2].)

Theorem 1.3. *Let $\{\mu_i\}_1^n$ be a normalized tight frame in a Hilbert subspace $L\subseteq H$ and let $\{f_i\}_1^n$ be a frame in a Hilbert subspace $K\subseteq H$ such that both these frames are weakly similar. Using the notations introduced after Definition 1.1 the inequality*

$$\sum_{j=1}^n\|\mu_j-f_j\|^2\geq\sum_{j=1}^n\|W(e_j)-f_j\|^2=\|(I-|F|)\|_{c_2}^2-N$$

holds for $N=\dim(\ker(|F|))$. Equality appears if and only if $\mu_j=W(e_j)$ for every $j=1,2,...,n$. Consequently, the symmetric approximation of a frame $\{f_i\}_1^n$ in a finite-dimensional Hilbert space $K\subseteq H$ is a normalized tight frame spanning the same Hilbert subspace $L\equiv K$ of H and being similar to $\{f_i\}_1^n$.

Proof. Let $\{\mu_i\}_1^n$ be a normalized tight frame for some Hilbert subspace $L\subseteq H$. Define $G:\mathbb{C}^n\to H$ by $G(e_i)=\mu_i$ for $1\leq i\leq n$. Note, that G is a partial isometry by [15, Prop. 1.1]. Since $|F|$ is compact it possesses an orthonormal basis of eigenvectors $\{h_i\}_1^n\subset\mathbb{C}^n$ with eigenvalues $\{\lambda_i\}_1^n$. By Lemma 1.2 we can proceed with the following calculations:

$$\begin{aligned}\sum_{j=1}^{n}\|\mu_j - f_j\|^2 &= \sum_{j=1}^{n}\|G(e_j) - F(e_j)\|^2 = \sum_{j=1}^{n}\|(G - W|F|)(e_j)\|^2 \\ &= \sum_{j=1}^{n}\|(G - W|F|)(h_j)\|^2 = \sum_{j=1}^{n}\|G(h_j) - \lambda_j W(h_j)\|^2 \,.\end{aligned}$$

We point out that the ranges of the (isometric) frame transforms G^* and F^* of the frames $\{\mu_j\}_1^n$ and $\{f_j\}_1^n$, respectively, coincide in $\mathbb{C}^n$ since the frames were supposed to be weakly similar, cf. [15, Cor. 2.8]. Therefore, the kernels of G and F also coincide. For eigenvalues $\lambda_i \neq 0$ of $|F|$ we have the following lower estimate:

$$\begin{aligned}\|G(h_i) - \lambda_i W(h_i)\|^2 &= \|G(h_i)\|^2 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2\|W(h_i)\|^2 \\ &= \|G^*G(h_i)\|^2 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2\|W(h_i)\|^2 \\ &= 1 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \\ &\geq 1 - 2\lambda_i + \lambda_i^2 = (1-\lambda_i)^2\end{aligned}$$

for every $i = 1, 2, ..., n$ since $h_i \in \ker(G)^\perp = \ker(W)^\perp$ and

$$\begin{aligned}\mathrm{Re}\langle G(h_i), W(h_i)\rangle &\leq |\langle G(h_i), W(h_i)\rangle| \leq \|G(h_i)\|\|W(h_i)\| \\ &= \|G^*G(h_i)\|\|W(h_i)\| = 1 \,.\end{aligned}$$

For eigenvalues $\lambda_i = 0$ the term simply equals zero. As a consequence we get the inequality

$$\sum_{j=1}^{n}\|\mu_j - f_j\|^2 \geq \sum_{j=1}^{n}(1-\lambda_j)^2 - N \tag{1}$$

valid for $N = \dim(\ker(|F|))$ and for any normalized tight frame $\{\mu_i\}_1^n$ in a Hilbert subspace $L \subseteq H$ that is weakly similar to the frame $\{f_i\}_1^n$ in K. To express the lower estimate in two other ways we transform it further using Lemma 1.2:

$$\begin{aligned}\sum_{j=1}^{n}(1-\lambda_j)^2 &= \sum_{j=1}^{n}(1-\lambda_j)^2\|h_j\|^2 = \sum_{j=1}^{n}\|(1-\lambda_j)h_j\|^2 \\ &= \sum_{j=1}^{n}\|(I - |F|)(h_j)\|^2 = \|(I - |F|)\|_{c_2}^2 \\ &= \sum_{j=1}^{n}\|(I - |F|)(h_j)\|^2 = \sum_{j=1}^{n}\|W(I - |F|)(h_j)\|^2 + N \\ &= \sum_{j=1}^{n}\|W(I - |F|)(e_j)\|^2 + N = \sum_{j=1}^{n}\|W(e_j) - f_j\|^2 + N \,.\end{aligned}$$

Summing up we get the estimate

$$\sum_{j=1}^{n}\|\mu_j - f_j\|^2 \geq \sum_{j=1}^{n}\|W(e_j) - f_j\|^2 = \|(I - |F|)\|_{c_2}^2 - N$$

that is valid for every normalized tight frame $\{\mu_i\}_1^n$ in Hilbert subspaces L of H which is weakly similar to the frame $\{f_i\}_1^n$.

Finally, we show uniqueness. Suppose, $\{\mu_i\}_1^n$ is a normalized tight frame in a Hilbert subspace L of H that realizes the equality

$$\sum_{j=1}^{n} \|\mu_j - f_j\|^2 = \sum_{j=1}^{n} \|W(e_j) - f_j\|^2$$

and is weakly similar to the frame $\{f_i\}_1^n$. Then $\mathrm{Re}\langle G(h_i), W(h_i)\rangle = 1$ for each $i \in \{1, 2, ..., n\}$ with $\lambda_i \neq 0$. (If $\lambda_i = 0$ then $G(h_i) = W(h_i) = 0$.) Since the inequality

$$1 = \mathrm{Re}\langle G(h_i), W(h_i)\rangle \leq |\langle G(h_i), W(h_i)\rangle| \leq \|G(h_i)\| \|W(h_i)\| = 1$$

holds for every eigenvector h_i with non-zero eigenvalue λ_i we conclude $G(h_i) = \alpha_i W(e_i)$ for certain complex numbers $\alpha_i \neq 0$ and $i \in \{1, 2, ..., n\}$ by the Cauchy-Schwarz inequality. Replacing $G(h_i)$ we obtain

$$1 = \langle G(h_i), W(h_i)\rangle = \alpha_i \langle W(h_i), W(h_i)\rangle = \alpha_i$$

for every $i \in \{1, 2, ..., n\}$. Therefore, $G(h_i) = W(h_i)$ for $i \in \{1, 2, ..., n\}$ forcing $G = W$ since $\{h_i\}_1^n$ has been selected as an orthonormal basis of eigenvectors of $|F|$ in H. This gives the desired result $\mu_i = W(e_i)$ for every $i \in \{1, 2, ..., n\}$, and the Hilbert subspaces L and K coincide. □

Let us remark that in case $\{f_i\}_1^n$ is a linearly independent set of elements in H then $\{W(e_i)\}_1^n$ has to be linearly independent, too, since the linear span of both these sets coincides in H and is a linear subspace of dimension n. In this case $|F|$ has no kernel.

2. Symmetric approximation of frames in separable Hilbert spaces

Let H be a separable Hilbert space and $\{f_i\}_{i\in\mathbb{N}}$ be a frame in a Hilbert subspace $K \subseteq H$.

Definition 2.1. A normalized tight frame $\{\nu_i\}_{i\in\mathbb{N}}$ in a Hilbert subspace $L \subseteq H$ is said to be a *minimal approximation of* $\{f_i\}_{i\in\mathbb{N}}$ if it is weakly similar to $\{f_i\}_{i\in\mathbb{N}}$, and the inequality

$$\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{\infty} \|\nu_j - f_j\|^2$$

is valid for all normalized tight frames $\{\mu_i\}_{i\in\mathbb{N}}$ in Hilbert subspaces of H that are weakly similar to $\{f_i\}_{i\in\mathbb{N}}$. When the sum at the right side of this inequality is finite, we say that $\{f_i\}_{i\in\mathbb{N}}$ is *quadratically close to a normalized tight frame.*

Again we do not assume that both frames $\{f_i\}_{i\in\mathbb{N}}$ and $\{\nu_i\}_{i\in\mathbb{N}}$ span the same Hilbert subspace of H. Theorem 2.3 will show that they do. We note that every set $\{g_i\}_{i\in\mathbb{N}}$ that is quadratically close to a frame $\{f_i\}_{i\in\mathbb{N}}$ of some Hilbert subspace of H (i.e. for which $\sum_j \|f_j - g_j\|^2 < \infty$) has to be a frame of the Hilbert subspace spanned by it, too, cf. [7, Th. 3]. However, it is not automatically weakly similar to the original frame since the position of possibly existing zero elements in these sequences matters. So we cannot sharpen our definition on the general level.

Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard orthonormal basis of the separable Hilbert space l_2. Given a frame $\{f_i\}_{i\in\mathbb{N}}$ of a Hilbert subspace $K \subseteq H$ we define the operator

$F : l_2 \to H$ by the formula

$$F\left(\sum_{j=1}^{\infty} \alpha_j e_j\right) = \sum_{j=1}^{\infty} \alpha_j f_j \,.$$

The operator F is the adjoint of the usual frame operator and so is bounded and has a natural polar decomposition $F = W|F|$, where W is a partial isometry from l_2 into H with initial space $\ker(F)^\perp \subseteq l_2$ and range $\mathrm{ran}(F)^- \subseteq H$, cf. [15, Prop. 1.10]. The fact that W is a coisometry follows by noting that the frame operator, F^* is one-to-one and $F^* = W^*|F^*|$ is its polar decomposition. In particular, F has closed range and the set $\{W(e_i)\}_{i\in\mathbb{N}}$ is a normalized tight frame of a separable Hilbert subspace of H, cf. [15, Cor. 1.2(i)]. If we simply try to repeat the steps of our considerations in the previous section we run into difficulties.

Example 2.2. Let $H = l_2$ with the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Set $F(e_1) = 0$ and $F(e_i) = \alpha_{i-1}e_{i-1}$ for a sequence of complex numbers $\{\alpha_i\}_{i\in\mathbb{N}}$ and any $i \geq 2$. The set $\{f_i\}_{i\in\mathbb{N}} = \{F(e_i)\}_{i\in\mathbb{N}}$ becomes a frame for the Hilbert space H if and only if both

$$\inf_{i\in\mathbb{N}} |\alpha_i|^2 > 0 \,, \quad \sup_{i\in\mathbb{N}} |\alpha_i|^2 < +\infty \,.$$

This guarantees the boundedness and surjectivity of F. Then $|F|(e_1) = 0$ and $|F|(e_i) = |\alpha_{i-1}|e_i$ for every $i \geq 2$, and consequently $W(e_1) = 0$ and $W(e_i) = \alpha_{i-1}/|\alpha_{i-1}|\, e_{i-1}$ for any $i \geq 2$. By Lemma 1.2 the operator $(I - |F|)$ is Hilbert-Schmidt if and only if the series

$$\sum_{i=1}^{\infty} \|(I-|F|)(e_i)\|^2 = 1 + \sum_{i=2}^{\infty} \|(1-|\alpha_{i-1}|)e_i\|^2 = 1 + \sum_{i=2}^{\infty}(1-|\alpha_{i-1}|)^2$$

converges. As can be easily seen the choice $\alpha_i = \mathrm{const} \neq 1$ leads to a infinite Hilbert-Schmidt norm and to a situation in which the operator $(I - |F|)$ is definitely not Hilbert-Schmidt. Beside this, even if a choice of the coefficients $\{\alpha_i\}_{i\in\mathbb{N}}$ would give rise to a Hilbert-Schmidt operator $(I - |F|)$, the operator F has still a non-trivial but finite-dimensional kernel by construction. Note also, that the frame $\{f_i\}_{i\in\mathbb{N}}$ is not a Riesz basis of H since it contains the zero vector as its first element.

For other examples the kernel of F can be infinite-dimensional. To see this take an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of the Hilbert space l_2 and consider the normalized tight frame $\{f_i\}_{i\in\mathbb{N}}$ for a subspace of l_2 defined by $f_{2i} = e_{2i}$, $f_{2i+1} = 0$. This frame is its own symmetric approximation. However, the kernel of the operator F is infinite-dimensional since the frame contains countably many zero elements.

Theorem 2.3. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame of a Hilbert subspace K of the Hilbert space H. Denote by P the projection of l_2 onto the (norm-closed) range of the operator F^*F. Then the operator $(P - |F|)$ is Hilbert-Schmidt if and only if the sum $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2$ is finite for at least one normalized tight frame $\{\mu_i\}_{i\in\mathbb{N}}$ of a Hilbert subspace L of H that is weakly similar to $\{f_i\}_{i\in\mathbb{N}}$. In this situation the estimate*

$$\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{\infty} \|W(e_j) - f_j\|^2 = \|(P - |F|)\|_{c_2}^2$$

is valid for every normalized tight frame $\{\mu_i\}_{i\in\mathbb{N}}$ of any Hilbert subspace of H that is weakly similar to $\{f_i\}_{i\in\mathbb{N}}$. (The left sum can be infinite for some choices of $\{\mu_i\}_{i\in\mathbb{N}}$.)

Equality appears if and only if $\mu_i = W(e_i)$ for any $i \in \mathbb{N}$, where W is the partial isometry of the polar decomposition $F = W|F|$ and $\{e_i\}_{i\in\mathbb{N}}$ is the orthonormal basis of l_2 used to define the operator F. Consequently, the symmetric approximation of a frame $\{f_i\}_{i\in\mathbb{N}}$ in a Hilbert space $K \subseteq H$ is a normalized tight frame spanning the same Hilbert subspace $L \equiv K$ of H and being similar to $\{f_i\}_{i\in\mathbb{N}}$.

Proof. Suppose $(P - |F|)$ is Hilbert-Schmidt. Then it is compact and possesses an orthonormal set $\{h_i\}_{i\in\mathbb{N}}$ of eigenvectors with corresponding non-zero eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$. This set forms a basis of the Hilbert subspace $(I - P)(H)$. Complete this basis of $(I - P)(H)$ to a basis of H by adding a basis $\{h'_i\}_{i\in\mathbb{N}}$ of $P(H)$. Note, that $(P - |F|)(h'_i) = 0$ for any index i.

Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a normalized tight frame of a Hilbert subspace $L \subseteq H$ that is weakly similar to the frame $\{f_i\}_{i\in\mathbb{N}}$. Define $G : l_2 \to H$ by $G(e_i) = \mu_i$ for $i \in \mathbb{N}$ and an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of l_2. Then G is a partial isometry with the same kernel as W and $|F|$, cf. [15, Cor. 2.8]. By Lemma 1.2 we have the following equality:

$$
\begin{aligned}
\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 &= \sum_{j=1}^{\infty} \|G(e_j) - F(e_j)\|^2 = \sum_{j=1}^{\infty} \|(G - W|F|)(e_j)\|^2 \\
&= \sum_{j=1}^{\infty} \|(G - W|F|)(h_j)\|^2 + \sum_{k\in\mathbb{N}} \|(G - W|F|)(h'_k)\|^2 \\
&= \sum_{j=1}^{\infty} \|G(h_j) - \lambda_j W(h_j)\|^2 + 0 \ . \qquad (2)
\end{aligned}
$$

Since $\lambda_i \neq 0$ for any index i we have the following lower estimates:

$$
\begin{aligned}
\|G(h_i) - \lambda_i W(h_i)\|^2 &= \|G(h_i)\|^2 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \|W(h_i)\|^2 \\
&= \|G^*G(h_i)\|^2 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \|W(h_i)\|^2 \\
&= 1 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \\
&\geq 1 - 2\lambda_i + \lambda_i^2 = (1 - \lambda_i)^2 \qquad (3)
\end{aligned}
$$

since

$$
\begin{aligned}
\mathrm{Re}\langle G(h_i), W(h_i)\rangle &\leq |\langle G(h_i), W(h_i)\rangle| \leq \|G(h_i)\|\|W(h_i)\| \\
&= \|G^*G(h_i)\|\|W(h_i)\| = 1 \ .
\end{aligned}
$$

Therefore, we get the estimate

$$
\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{\infty} (1 - \lambda_j)^2 = \sum_{j=1}^{\infty} \|(P - |F|)(h_j)\|^2 = \|(P - |F|)\|_{c_2}^2
$$

which is valid for all normalized tight frames $\{\mu_i\}_{i\in\mathbb{N}}$ being weakly similar to $\{f_i\}_{i\in\mathbb{N}}$. Obviously, the special choice $\mu_i = W(e_i)$ for $i \in \mathbb{N}$ gives the equality

$$
\sum_{j=1}^{\infty} \|W(e_j) - f_j\|^2 = \|(P - |F|)\|_{c_2}^2
$$

by equality (2). Uniqueness can be shown in the same way as in Theorem 1.3.

Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a normalized tight frame in a Hilbert subspace L of H that is weakly similar to the frame $\{f_i\}_{i\in\mathbb{N}}$ and for which the sum $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2$ is finite. Consider the operator $T : l_2 \to H$ defined by $T(e_i) = \mu_i - f_i$ for the fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of l_2. Then

$$\left\| T\left(\sum_j \alpha_j e_j\right)\right\| \le \sum_j |\alpha_j| \cdot \|\mu_j - f_j\| \le \|\{\alpha_j\}_j\| \sqrt{\sum_j \|\mu_j - f_j\|^2} < \infty$$

for any sequence $\{\alpha_i\}_i \in l_2$ and, hence, T can be approximated by finite rank operators in norm on l_2. So $T = G - F$ is compact for $G(e_i) = \mu_i$, $(i \in \mathbb{N})$. Counting $F^*F = I - T^*G - G^*T + T^*T$ we see that F^*F equals the identity operator minus a compact one and, hence, must be diagonalizable on l_2. So $|F|$ is diagonalizable, too.

Let $\{h_i\}_{i\in\mathbb{N}}$ be an orthonormal system of eigenvectors with non-zero eigenvalues of $|F|$, let $\{h'_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of the kernel of $|F|$. Repeating the calculations (2) and (3) we arrive at

$$\infty > \sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \ge \|(P - |F|)\|_{c_2}^2$$

and the operator $(P - |F|)$ is Hilbert-Schmidt, where P denotes the projection of l_2 onto the kernel of $|F|$. □

Definition 2.4. Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame of a Hilbert subspace K of the Hilbert space H, let F be defined as above and let $F = W|F|$ be its polar decomposition. We call $\{W(f_i)\}_{i\in\mathbb{N}}$ *the symmetric approximation* of $\{f_i\}_{i\in\mathbb{N}}$.

Thus, the above theorem can be summarized as saying that if any normalized tight frame is quadratically close to our given frame, then the symmetric approximation is the unique normalized tight frame that is a minimal approximation of our given frame.

Remark 2.5. Since $W(P - |F|)W^* = (I - |F^*|)$, it is easily seen that $(P - |F|)$ is Hilbert-Schmidt if and only if $(I - |F^*|)$ is Hilbert-Schmidt and that the two Hilbert-Schmidt norms agree. Moreover, $(P - |F|)$ is Hilbert-Schmidt if and only if $(P - |F|)(P + |F|) = (P - F^*F)$ is Hilbert-Schmidt, which can also be seen to be equivalent to $(I - FF^*)$ being Hilbert-Schmidt.

Corollary 2.6. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame of a Hilbert subspace K of the Hilbert space H. Denote by P the projection of l_2 onto the (norm-closed) range of the operator F^*F. If the operator $(P-|F|)$ is not Hilbert-Schmidt, then the sum $\sum_{j=1}^{\infty}\|\mu_j - f_j\|^2$ is infinite for any normalized tight frame $\{\mu_i\}_{i\in\mathbb{N}}$ of a Hilbert subspace L of H that is weakly similar to $\{f_i\}_{i\in\mathbb{N}}$. In other words, there does not exist any symmetric approximation of the frame $\{f_i\}_{i\in\mathbb{N}}$.*

Corollary 2.7. *If a frame $\{f_i\}_{i\in\mathbb{N}}$ of an infinite-dimensional Hilbert subspace K of a Hilbert space H is a Riesz basis and it has a symmetric orthogonalization in H by an orthonormal basis $\{\nu_i\}_{i\in\mathbb{N}}$ of a Hilbert subspace L of H, then the operator $(I-|F|)$ is Hilbert-Schmidt and the estimate $\sum_{j=1}^{\infty}\|\nu_j - f_j\|^2 < \infty$ holds. Moreover, the more general estimate $\sum_{j=1}^{\infty}\|\mu_j - f_j\|^2 \ge \|(I - |F|)\|_{c_2}^2$ is valid for any infinite orthonormal basis $\{\mu_i\}_{i\in\mathbb{N}}$ of any separable Hilbert subspace of H and, in particular, $|F|$ is injective.*

If the operator $(I-|F|)$ is Hilbert-Schmidt, then the Riesz basis $\{f_i\}_{i\in\mathbb{N}}$ admits a unique symmetric approximation by the orthonormal basis $\{W(e_i)\}_{i\in\mathbb{N}}$ of the same Hilbert subspace of $K \subseteq H$.

Conversely, if in this case the operator $(I - |F|)$ is not Hilbert-Schmidt, then the sum $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2$ is infinite for any orthonormal basis $\{\mu_i\}_{i\in\mathbb{N}}$ of any separable Hilbert subspace of H.

A reference to Definition 2.1 and to Theorem 2.3 makes the corollaries obvious.

Recall, that a frame $\{f_i\}_{i\in\mathbb{N}}$ is said to be a near-Riesz basis if there is a finite set σ for which the set $\{f_i\}_{i\in\mathbb{N}\setminus\sigma}$ is a Riesz basis of the (separable) Hilbert space $K \subseteq H$ generated by the initial frame. Near-Riesz bases have a number of alternative equivalent characterizations:

(i) the kernel of the derived operator $F : l_2 \to H$ is finite-dimensional and has dimension card(σ), ([16, Th. 2.4, 3.1]),

(ii) the sequence $\{f_i\}_{i\in\mathbb{N}}$ is Besselian, i.e. whenever $\sum_{j\in\mathbb{N}} c_j f_j$ converges in K then $\{c_i\}_{i\in\mathbb{N}} \in l_2$, ([16, Th. 2.5]),

(iii) $\sum_{j\in\mathbb{N}} c_j f_j$ converges in K if and only if the sequence of coefficients $\{c_i\}_{i\in\mathbb{N}}$ belongs to l_2, ([16, Th. 2.5]),

(iv) the sequence $\{f_i\}_{i\in\mathbb{N}}$ is unconditional, i.e. whenever $\sum_{j\in\mathbb{N}} c_j f_j$ converges for a sequence $\{c_i\}_{i\in\mathbb{N}}$ of numbers then it converges unconditionally, ([16, Th. 3.2] and [6, Th. 3.1]).

In case the frame $\{f_i\}_{i\in\mathbb{N}}$ is a near-Riesz basis we can formulate another corollary the formulation of which already predicts the result for general bases of infinite-dimensional Hilbert subspaces.

Corollary 2.8. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a near-Riesz basis of a Hilbert subspace of H. Then this frame possesses a symmetric approximation $\{\nu_i\}_{i\in\mathbb{N}}$ if and only if the operator $(I-|F|)$ is Hilbert-Schmidt. In this situation the frame $\{\nu_i\}_{i\in\mathbb{N}}$ is also a near-Riesz basis and the adjoint operators F and G of the corresponding frame transforms of both these frames have kernels of the same dimension (i.e. both these frames have the same excess). In fact, their kernels coincide.*

Proof. By [16] the operator F has a finite-dimensional kernel if and only if the frame $\{f_i\}_{i\in\mathbb{N}}$ is a near-Riesz basis. However, in case of a finite-dimensional kernel of F the operator $(I-|F|)$ is Hilbert-Schmidt if and only if $(P-|F|)$ is. So Theorem 2.3 implies the main equivalence of the corollary. The operators F and $W = G$ have the same kernel since W^* is an isometry and ran$F$ = ran$W$ as shown above. Hence, the normalized tight frame $\{W(e_i)\}_{i\in\mathbb{N}}$ has to be a near-Riesz basis, too. □

Corollary 2.9. *If for a frame $\{f_i\}_{i\in\mathbb{N}}$ of a Hilbert space K the operator $(I - |F|)$ is Hilbert-Schmidt, then the frame is a near-Riesz basis and admits a symmetric approximation.*

3. Symmetric orthogonalization of arbitrary linearly independent sets of elements

The considerations below are part of the Ph.D. thesis of the third author. For a different treatment of the infinite situation we refer to [1].

Let $\{f_i\}_{i\in\mathbb{N}}$ be an infinite set of linearly independent elements of a (separable) Hilbert space H and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of l_2. Consider the linear operator F defined by $F(e_i) = f_i$ for $i \in \mathbb{N}$. Since the set $\{f_i\}_{i\in\mathbb{N}}$ may lack the frame property for the Hilbert subspace $K \subset H$ generated by it the operator can in general be unbounded, possess a non-trivial kernel, or have non-closed range.

Note that $\{f_i\}_{i\in\mathbb{N}}$ is a frame for K exactly when F is bounded and has closed range (which is necessarily K).

Example 3.1. Let $f_i = e_1 + e_i$ for $i \geq 2$ and $f_1 = e_1$ for a fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of the Hilbert space l_2. Then F maps the element $x_n = \sum_{j=1}^{n} j^{-1} e_j$ into $F(x_n) = \sum_{j=1}^{n} j^{-1} e_1 + \sum_{j=2}^{n} j^{-1} e_j$. For $n \to \infty$ the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges in norm, whereas the sequence of images $\{F(x_n)\}_{n\in\mathbb{N}}$ diverges to infinity on the multiples of the first basis vector e_1. Therefore, the operator is not everywhere defined on l_2 and unbounded.

Example 3.2. To give an example of an operator F with non-trivial kernel consider the linearly independent set $\{f_i = e_i - i/(i-1)e_{i-1}, f_1 = e_1\}_{i\in\mathbb{N}}$ constructed from an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of l_2. The derived operator $F : e_i \to f_i$ is bounded since

$$\left\| F\left(\sum_{j=1}^{\infty} \alpha_j e_j\right)\right\| \leq \left\|\sum_{j=1}^{\infty} \alpha_j e_j\right\| + \left\|\sum_{j=2}^{\infty} \frac{j}{j-1} \alpha_j e_j\right\| \leq 3 \cdot \left\|\sum_{j=1}^{\infty} \alpha_j e_j\right\| .$$

Consider the element $x = \sum_{j=1}^{\infty} j^{-1} e_j$ and its image $F(x)$. An easy calculation shows that

$$F\left(\sum_{j=1}^{n} \frac{1}{j} e_j\right) = \sum_{j=1}^{n} \frac{1}{j} f_j = \frac{1}{n} e_n$$

forcing $F(x) = 0$ for $n \to \infty$. As an immediate consequence we obtain that the sequence $\{f_i\}_{i\in\mathbb{N}}$ is not a frame in the Hilbert space spanned by it. Note, that the operator $(I - |F|^2)$ is not Hilbert-Schmidt because Lemma 1.2 applied to the basis $\{e_i\}_{i\in\mathbb{N}}$ gives an infinite Hilbert-Schmidt norm. Since $(I - |F|^2) = (I - |F|)(I + |F|)$ and the Hilbert-Schmidt operators form an ideal the operator $(I - |F|)$ also cannot be Hilbert-Schmidt.

To give a reasonable definition of symmetric orthogonalization(s) of infinite sets of linearly independent vectors $\{f_i\}_{i\in\mathbb{N}} \subset K \subseteq H$ we have to suppose that the derived operator F is at least bounded. This condition does not depend on the choice of the orthonormal basis in l_2, but only on the set $\{f_i\}_{i\in\mathbb{N}} \subset H$.

Definition 3.3. An orthonormal basis $\{\nu_i\}_{i\in\mathbb{N}}$ for a Hilbert subspace $L \subseteq H$ is said to be a *symmetric orthogonalization of* $\{f_i\}_{i\in\mathbb{N}}$ if the inequality

$$\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{\infty} \|\nu_j - f_j\|^2$$

is valid for all orthonormal sets $\{\mu_i\}_{i\in\mathbb{N}}$ in Hilbert subspaces of H and the sum at the right side of this inequality is finite.

Proposition 3.4. *Let $\{f_i\}_{i\in\mathbb{N}}$ be an infinite set of linearly independent elements of a Hilbert space H and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of l_2. Consider the linear operator F defined by $F(e_i) = f_i$ for $i \in \mathbb{N}$.*

If $(I - |F|)$ is Hilbert-Schmidt, then $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \|I - |F|\|_{c_2}^2$ for all orthonormal subsets $\{\mu_i\}_{i\in\mathbb{N}}$ of H.

If $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 < \infty$ for some orthonormal set $\{\mu_i\}_{i\in\mathbb{N}}$, then the operator $(I - |F|)$ is Hilbert-Schmidt and the estimate $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \|I - |F|\|_{c_2}^2$ is valid.

Proof. Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a countable orthonormal subset of H and $\{e_i\}_{i\in\mathbb{N}}$ an orthonormal basis of l_2. Define the operator $G : l_2 \to H$ by $G(e_i) = \mu_i$ for $i \in \mathbb{N}$. Then G is an isometry. If $(I - |F|)$ is supposed to be Hilbert-Schmidt, then there exists an orthonormal basis of eigenvectors of the operator $|F|$ in l_2 denoted by $\{h_i\}_{i\in\mathbb{N}}$ with corresponding eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$. Moreover, since $(I - |F|)$ is supposed to be Hilbert-Schmidt the operator $(|F| - I)$ is compact and, hence, the operator $|F| = I + (|F| - I)$ is Fredholm. Therefore, the kernel of $|F|$ has to be finite-dimensional. Without loss of generality, let $\{h_1, ..., h_N\}$ be the eigenvectors corresponding to eigenvalues zero of $|F|$, where $N = \dim(\ker(|F|)) \geq 0$. By Lemma 1.2 we have the following equality:

$$\begin{aligned}\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 &= \sum_{j=1}^{\infty} \|G(e_j) - F(e_j)\|^2 = \sum_{j=1}^{\infty} \|(G - W|F|)(e_j)\|^2 \\ &= \sum_{j=1}^{\infty} \|(G - W|F|)(h_j)\|^2 = \sum_{j=1}^{\infty} \|G(h_j) - \lambda_j W(h_j)\|^2 . \qquad (4)\end{aligned}$$

Since G is an isometry we have the following lower estimates for every $i \in \mathbb{N}$:

$$\begin{aligned}\|G(h_i) - \lambda_i W(h_i)\|^2 &= \|G(h_i)\|^2 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \|W(h_i)\|^2 \\ &= 1 - 2\lambda_i \mathrm{Re}\langle G(h_i), W(h_i)\rangle + \lambda_i^2 \\ &\geq 1 - 2\lambda_i + \lambda_i^2 = (1 - \lambda_i)^2 \qquad (5)\end{aligned}$$

since

$$\mathrm{Re}\langle G(h_i), W(h_i)\rangle \leq |\langle G(h_i), W(h_i)\rangle| \leq \|G(h_i)\| \|W(h_i)\| = 1 .$$

Therefore, we get the estimate

$$\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \sum_{j=1}^{\infty} (1 - \lambda_j)^2 = \sum_{j=1}^{\infty} \|(I - |F|)(h_j)\|^2 = \|(I - |F|)\|_{c_2}^2$$

which is valid for all orthonormal subsets $\{\mu_i\}_{i\in\mathbb{N}} \subset H$.

Now, suppose the finiteness of the sum $\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2$ for some orthonormal subset $\{\mu_i\}_{i\in\mathbb{N}}$ of H. Consider the operator $T : l_2 \to H$ defined by $T(e_i) = \mu_i - f_i$ for the fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of l_2. Then

$$\left\| T\left(\sum_j \alpha_j e_j\right)\right\| \leq \sum_j |\alpha_j| \cdot \|\mu_j - f_j\| \leq \|\{\alpha_j\}_j\| \sqrt{\sum_j \|\mu_j - f_j\|^2} < \infty$$

for any sequence $\{\alpha_i\}_i \in l_2$ and, hence, T can be approximated by finite rank operators in norm on l_2. So $T = G - F$ is compact for $G(e_i) = \mu_i$, $(i \in \mathbb{N})$. Counting $F^*F = I - T^*G - G^*T + T^*T$ we see that F^*F equals the identity operator minus a compact one and, hence, must be diagonalizable on l_2. So $|F|$ is diagonalizable, too.

Let $\{h_i\}_{i\in\mathbb{N}}$ be an orthonormal system of eigenvectors of $|F|$ with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$. Repeating the calculations (4) and (5) we arrive at

$$\infty > \sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \|(I - |F|)\|_{c_2}^2$$

and the operator $(I - |F|)$ is Hilbert-Schmidt. Its Hilbert-Schmidt norm satisfies the desired inequality. □

Example 3.5. We give another example demonstrating that even the condition on $(I-|F|)$ to be Hilbert-Schmidt does not guarantee that $\ker(|F|)=\{0\}$. Let $H=l_2$ with

$$f_1=e_1\,,\quad f_2=\frac{1}{\sqrt{2}}\,(e_1+e_2),$$
$$f_n=\left(\frac{1}{\sqrt{2}}\right)^{n-1}e_1-\sum_{j=2}^{n-1}\left(\frac{1}{\sqrt{2}}\right)^{n-j+1}e_j+\frac{1}{\sqrt{2}}\,e_n\ \text{ for } n\geq 3\,.$$

Obviously, the set $\{f_i\}_{i\in\mathbb{N}}$ is linearly independent in l_2 by construction. Representing F as an infinite matrix and counting the entries of the infinite matrix that represents F^*F as scalar products of column vectors we obtain

$$(I-|F|^2)(e_1)=-\sum_{j=2}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{j-1}e_j,\quad (I-|F|^2)(e_i)=-\left(\frac{1}{\sqrt{2}}\right)^{i-1}e_1\ \text{ for } i\geq 2\,.$$

Applying Lemma 1.2 for the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ we get a finite value for the Hilbert-Schmidt norm of the operator $(I-|F|^2)$: $\|(I-|F|^2)\|_{c_2}=\sqrt{2}$. Since

$$\|(I-|F|)\|_{c_2}^2=\sum_{j=1}^{\infty}(1-\lambda_j)^2\leq\sum_{j=1}^{\infty}(1-\lambda_j)^2(1+\lambda_j)^2=\|(I-|F|^2)\|_{c_2}^2$$

(where $\{\lambda_i\}_{i\in\mathbb{N}}$ are the eigenvalues of $|F|$) we conclude that the operator $(I-|F|)$ is Hilbert-Schmidt, too.

Consider the element $x=-e_1+\sum_{j=2}^{\infty}(\sqrt{2})^{-(j-1)}e_j$ that belongs to l_2 since its norm is $\|x\|=\sqrt{2}$. Counting the value of $|F|^2(x)$ we obtain $x\in\ker(|F|^2)$. The equality $\||F|(x)\|^2=\langle|F|(x),|F|(x)\rangle=\langle|F|^2(x),x\rangle=0$ forces $|F|(x)=0$.

The next theorem demonstrates that $\{f_i\}_{i\in\mathbb{N}}$ does not possess a symmetric orthogonalization.

Theorem 3.6. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a linearly independent set of a (separable) Hilbert space H such that the derived operator $(I-|F|)$ is Hilbert-Schmidt. Then there exists a symmetric orthogonalization $\{\nu_i\}_{i\in\mathbb{N}}$ of this set in H if and only if* $\dim((\mathrm{ran}F)^{\perp})\geq\dim(\ker F)$. *In this case, setting $\nu_i=(V+W)(e_i)$ for $i\in\mathbb{N}$, with an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of l_2 and any partial isometry $V:l_2\to H$ with initial space* $\ker|F|$ *and* $\mathrm{ran}V\perp\mathrm{ran}W=\mathrm{ran}F$ *yields a symmetric orthogonalization. Moreover, all symmetric orthogonalizations arise in this way.*

The set $\{f_i\}_{i\in\mathbb{N}}$ possesses a unique symmetric orthogonalization $\{\nu_i\}_{i\in\mathbb{N}}$ if and only if $\ker F=\{0\}$, *if and only if the sets $\{\nu_i\}_{i\in\mathbb{N}}$ and $\{f_i\}_{i\in\mathbb{N}}$ span the same Hilbert subspace of H. In this case, $\nu_i=W(e_i)$ for any $i\in\mathbb{N}$ and $V=0$. In other words, the linearly independent set $\{f_i\}_{i\in\mathbb{N}}$ is a frame (and, therefore, a Riesz basis) for the Hilbert subspace it spans if and only if* $\ker F=\{0\}$.

Proof. Let us first show that $\mathrm{ran}F=\mathrm{ran}W$, i.e. that $\mathrm{ran}F$ is actually closed. Since $(I-|F|)$ was supposed to be Hilbert-Schmidt the operator $|F|$ is Fredholm and its range $\mathrm{ran}|F|$ is closed by the definition of Fredholm operators. Let $y_n=F(x_n)$, $(n\in\mathbb{N})$, form a Cauchy sequence in $\mathrm{ran}F$, where $\{x_n\}_{n\in\mathbb{N}}\in l_2$. Then for every $\varepsilon>0$ there exists a number N such that $\|y_m-y_n\|<\varepsilon$ for all $m,n\geq N$. Therefore,

$$\varepsilon>\|y_m-y_n\|=\|F(x_m-x_n)\|=\|W|F|(x_m-x_n)\|=\||F|(x_m-x_n)\|$$

since $|F|(x_m-x_n)\in(\ker|F|)^{\perp}$ belongs to the initial space of W. So the sequence $\{|F|(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{ran}|F|$. However, $\mathrm{ran}|F|$ is closed and, hence, there exists a $z \in \mathrm{ran}|F|$ and an $x \in l_2$ such that $|F|(x) = z = \lim_n |F|(x_n)$. Then

$$\lim_n y_n = \lim_n F(x_n) = \lim_n W|F|(x_n) = W(\lim_n |F|(x_n)) = W|F|(x) = F(x)$$

and $\mathrm{ran}F$ is shown to be closed and to coincide with the range of the partial isometry W.

Suppose $\dim((\mathrm{ran}F)^\perp) \geq \dim(\ker F)$. If $\ker F = \{0\}$ then let $V = 0$ on l_2. If $\ker F \neq \{0\}$ then it has to be finite-dimensional since $|F|$ is Fredholm and $\ker|F| = \ker F$. Suppose, $\{\xi_i\}_1^n$ is an orthonormal basis of $\ker F$. Since $\dim((\mathrm{ran}F)^\perp) \geq n$ by assumption we can select another orthonormal set $\{\rho_i\}_1^n \subset (\mathrm{ran}F)^\perp$ and define $V(\xi_i) = \rho_i$ for $i = 1, ..., n$. Then V is an isometry for elements of $\ker|F|$. On $(\ker|F|)^\perp$ we set V to be the zero operator. Consequently, $\mathrm{ran}V \perp \mathrm{ran}W$, $\|(V+W)(x)\| = \|x\|$ for every $x \in l_2$ and the set $\{(V+W)(e_i)\}_{i \in \mathbb{N}}$ is an orthonormal set for the orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of l_2. Since $(V+W)|F|(x) = F(x)$ for every $x \in l_2$ we obtain the equality

$$\begin{aligned} \sum_{j=1}^{\infty} \|(V+W)(e_j) - f_j\|^2 &= \sum_{j=1}^{\infty} \|(V+W)(I - |F|)(e_j)\|^2 \\ &= \sum_{j=1}^{\infty} \|(I - |F|)(e_j)\|^2 = \|(I - |F|)\|_{c_2}^2 . \end{aligned}$$

By Proposition 3.4 the estimate

$$\sum_{j=1}^{\infty} \|\mu_j - f_j\|^2 \geq \|(I - |F|)\|_{c_2}^2 = \sum_{j=1}^{\infty} \|(V+W)(e_j) - f_j\|^2$$

is valid for any orthonormal subset $\{\mu_i\}_{i \in \mathbb{N}}$ of H. Therefore, we have found a symmetric orthogonalization of the linearly independent subset $\{f_i\}_{i \in \mathbb{N}} \subset H$ inside H.

To show the converse, assume $\dim((\mathrm{ran}F)^\perp) < \dim(\ker F)$ and the existence of a symmetric orthogonalization $\{\nu_i\}_{i \in \mathbb{N}}$ of the set $\{f_i\}_{i \in \mathbb{N}}$ in H. Setting $G(e_i) = \nu_i$ for $i \in \mathbb{N}$ we obtain an isometry $G : l_2 \to H$. Consequently,

$$\|G - F\|_{c_2}^2 = \sum_{j=1}^{\infty} \|(G - F)(e_j)\|^2 = \sum_{j=1}^{\infty} \|\nu_j - f_j\|^2 < \infty$$

by the definition of a symmetric orthogonalization. So the operator $(G - F)$ is compact and Hilbert-Schmidt. Since $(I - |F|)$ was assumed to be Hilbert-Schmidt the kernel of F is finite-dimensional, and so is $(\mathrm{ran}F)^\perp$ by assumption. Hence, F is Fredholm and $G = F + (G - F)$ is Fredholm, too, by the compactness of $(G - F)$. Moreover, the indices of G and F coincide and are greater than zero by assumption. However, $\mathrm{ind}(G) = \dim(\ker G) - \dim((\mathrm{ran}G)^\perp) = -\dim((\mathrm{ran}G)^\perp)$ since G is an isometry. So we arrive at a contradiction: $\dim((\mathrm{ran}G)^\perp) < 0$. The only possible conclusion is the non-existence of a symmetric orthogonalization in case $\dim((\mathrm{ran}F)^\perp) < \dim(\ker F)$.

We want to know more about the canonical form of symmetric orthogonalizations in case at least one exists. First of all, if $\{\nu_i\}_{i=1}^\infty$ is another symmetric orthogonalization of the linearly independent set $\{f_i\}_{i \in \mathbb{N}}$ beside the orthonormal set

$\{(V+W)(e_i)\}_{i\in\mathbb{N}}$ constructed above we have the relation

$$\|(I-|F|)\|_{c_2}^2 = \sum_{j=1}^{\infty} \|(V+W)(e_j)\|^2 \geq \sum_{j=1}^{\infty} \|\nu_j - f_j\|^2 = \|(I-|F|)\|_{c_2}^2$$

by Proposition 3.4. Let $G : l_2 \to H$ again be the isometry defined by $G(e_i) = \nu_i$ for $i \in \mathbb{N}$. The goal of the subsequent considerations is to establish that $(G-W)$ is actually a partial isometry with initial space $\ker|F|$ and, hence, $\nu_i = ((G-W)+W)(e_i)$ for $i \in \mathbb{N}$, where $(G-W) = V$.

Let $x \in \ker|F|$. Then $\|(G-W)(x)\| = \|G(x)\| = \|x\|$. So the operator $(G-W)$ acts on $\ker|F|$ as an isometry. We want to show that $G = W$ on $(\ker|F|)^\perp$. Let $\{h_i\}_{i\in\mathbb{N}}$ be an orthonormal set of eigenvectors of $|F|$, let $\{\lambda_i\}_{i\in\mathbb{N}}$ be the corresponding sequence of eigenvalues. In analogy to the considerations at (4) and (5) we conclude

$$\begin{aligned}
\|(I-|F|)\|_{c_2}^2 &= \sum_{j=1}^{\infty} \|\nu_j - f_j\|^2 = \sum_{j=1}^{\infty} \|(G-W|F|)(e_j)\|^2 \\
&= \sum_{j=1}^{\infty} \|(G-W|F|)(h_j)\|^2 = \sum_{j=1}^{\infty} \|G(h_j) - \lambda_j W(h_j)\|^2 \\
&\geq \sum_{j=1}^{\infty} (1-\lambda_j)^2 = \sum_{j=1}^{\infty} \|(I-|F|)(h_j)\|^2 \\
&= \|(I-|F|)\|_{c_2}^2 .
\end{aligned}$$

thus, $\sum_{j=1}^{\infty} \|(G-\lambda_j W)(h_j)\|^2 = \sum_{j=1}^{\infty}(1-\lambda_j)^2$ which implies $\mathrm{Re}(\langle G(h_j), W(h_j)\rangle) = 1$ and, consequently, $G(h_j) = W(h_j)$ for any eigenvector h_j with non-zero eigenvalue λ_j. Hence, $G = W$ on $(\ker|F|)^\perp$ since the eigenvectors of $|F|$ with non-zero eigenvalues form a basis of this subspace.

Finally, we verify that $\mathrm{ran}(G-W) \perp \mathrm{ran}W$. Let $z_1 \in \mathrm{ran}(G-W)$, $z_2 \in \mathrm{ran}W$. There exist $t_1, t_2 \in l_2$ such that $(G-W)(t_1) = z_1$, $W(t_2) = z_2$. Furthermore, $t_i = x_i + y_i$ with $x_i \in \ker|F|$, $y_i \in (\ker|F|)^\perp$ and $i = 1, 2$. Now

$$\begin{aligned}
\langle z_1, z_2\rangle &= \langle (G-W)(x_1+y_1), W(x_2+y_2)\rangle = \langle G(x_1), W(y_2)\rangle \\
&= \langle G(x_1), G(y_2)\rangle = \langle x_1, y_2\rangle = 0
\end{aligned}$$

since $W(x_1) = W(x_2) = 0$, $G(y_1) = W(y_1)$, $G(y_2) = W(y_2)$ by the location of x_i, y_i and the established properties of W, G. This shows the orthogonality of the ranges of $(G-W)$ and W.

To establish uniqueness conditions we first note that $\ker|F| = \{0\}$ implies $V = 0$ and, hence, the uniqueness of the symmetric orthogonalization $\{W(e_i)\}_{i\in\mathbb{N}}$. In this case the sets $\{\nu_i\}_{i\in\mathbb{N}}$ and $\{f_i\}_{i\in\mathbb{N}}$ span the same Hilbert subspace of H. Suppose now, $\dim(\ker|F|) = n > 0$. Let $\{\xi_i\}_1^n$ be an orthonormal basis of $\ker|F|$. Since the assumed existence of a symmetric orthogonalization of $\{f_i\}_{i\in\mathbb{N}}$ in H implies $\dim((\mathrm{ran}F)^\perp) \geq \dim(\ker F)$ we can choose an orthonormal set $\{\rho_i\}_1^n$ in $(\mathrm{ran}F)^\perp$. Define $V_1 : l_2 \to H$ by $V_1(\xi_i) = \rho_i$ for $i = 1, ..., n$ and $V_1 = 0$ on $(\ker|F|)^\perp$, and $V_2 : l_2 \to H$ by $V_2 = -V_1$. Then $\nu_i = (V_1 + W)(e_i)$ and $\nu_i' = (V_2 + W)(e_i)$, $(i \in \mathbb{N})$, are symmetric orthogonalizations of $\{f_i\}_{i\in\mathbb{N}}$. If we assume uniqueness of the symmetric orthogonalization, then the equalities $\nu_i = \nu_i'$ for $i \in \mathbb{N}$ lead to the contradiction $\rho_i = -\rho_i = 0$ for all $i \in \mathbb{N}$. This proves the theorem. □

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