COLORING $\mathbb{R}^n$

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Abstract. If $1 \leq m \leq n$ and $A \subseteq \mathbb{R}$, then define the graph $G(A, m, n)$ to be the graph whose vertex set is $\mathbb{R}^n$ with two vertices $x, y \in \mathbb{R}^n$ being adjacent iff there are distinct $u, v \in A^m$ such that $\|x - y\| = \|u - v\|$. For various $m$ and $n$ and various $A$, typically $A = \mathbb{Q}$ or $A = \mathbb{Z}$, the graph $G(A, m, n)$ can be properly colored with $\omega$ colors. It is shown that in some cases a coloring $\varphi : \mathbb{R}^n \to \omega$ can also have the additional property that if $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ is an isometric embedding, then the restriction of $\varphi$ to $\alpha(A^m)$ is a bijection onto $\omega$.

Erdős [1] proved that there is a function $\varphi : \mathbb{R}^2 \to \omega$ such that whenever $x, y \in \mathbb{R}^2$ are distinct and the distance between them is rational (that is, $\|x - y\| \in \mathbb{Q}$), then $\varphi(x) \neq \varphi(y)$. There have been various generalizations of this result, including extensions to higher dimensions — to $\mathbb{R}^3$ by Erdős & Komjáth [2] and then to arbitrary $\mathbb{R}^n$ by Komjáth [3]. Another proof of Komjáth’s theorem, as well as proofs of some other similar theorems, can be found in [7]. In another direction, there is the recent improvement by Komjáth [4] who showed that the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ could, in addition, be required to satisfy the following interesting condition: if $\ell \subseteq \mathbb{R}^2$ is a line and $a \in \ell$, then $\varphi$ maps $\{x \in \ell : \|x - a\| \in \mathbb{Q}\}$ onto $\omega$. In this paper, Komjáth’s improvement is extended to arbitrary $\mathbb{R}^n$.

Theorem 1. There is a function $\varphi : \mathbb{R}^n \to \omega$ such that for any line $\ell \subseteq \mathbb{R}^n$ and $a \in \ell$, the restriction of $\varphi$ to $\{x \in \ell : \|x - a\| \in \mathbb{Q}\}$ is a bijection onto $\omega$.

Komjáth [4] proved some similar types of theorems related to sets having the Steinhaus property. A subset $B \subseteq \mathbb{R}^2$ is said to have the Steinhaus property if, for any isometry $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, there is exactly one lattice point in $\alpha(B)$ or, in other words, $|\alpha(B) \cap \mathbb{Z}^2| = 1$. In a very recent preprint, Jackson & Mauldin [5] settle a long-standing open problem by proving the existence of a set having the Steinhaus property. Earlier, Komjáth [4] had proved that there is a subset $B \subseteq \mathbb{R}^2$ such that for any isometry $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, there is exactly one point in $\alpha(B) \cap \mathbb{Q}^2$. We improve the Komjáth result by showing that $\mathbb{R}^2$ can be partitioned into countably many sets each having this property. Moreover, we will prove the following $n$-dimensional extension of the Komjáth result.

Theorem 2. There is a function $\varphi : \mathbb{R}^n \to \omega$ such that for any isometry $\alpha : \mathbb{R}^n \to \mathbb{R}^n$, the restriction of $\varphi$ to $\alpha(\mathbb{Q}^n)$ is a bijection onto $\omega$.

Notice that Theorem 1 can be rephrased in a manner similar to the way that Theorem 2 is phrased. We will often consider isometric embeddings $\alpha : \mathbb{R}^m \to \mathbb{R}^n$, ...
but we will refer to them as isometries, even when \( m < n \). Thus, the image of an
isometry \( \alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is just an \( m \)-dimensional hyperplane. Theorem 1 asserts
that there is a function \( \varphi : \mathbb{R}^n \rightarrow \omega \) such that for any isometry \( \alpha : \mathbb{R} \rightarrow \mathbb{R}^n \),
the restriction of \( \varphi \) to \( \alpha(\mathbb{Q}) \) is a bijection onto \( \omega \). Both Theorems 1 and 2 are
consequences of the more general Theorem 3.

Suppose \( 1 \leq m \leq n \) and \( A \subseteq \mathbb{R} \). Then we define \( G(A, m, n) \) to be the graph
having vertex set \( \mathbb{R}^n \) in which two distinct vertices \( x, y \) are adjacent iff \( \{x, y\} \subseteq \alpha(A^m) \) for some isometry \( \alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n \). We will sometimes refer to the elements
of \( \omega \) as colors. A function \( \varphi : D \rightarrow \omega \), where \( D \subseteq \mathbb{R}^n \), will be referred to as a
coloring, and it is proper if \( \varphi(x) \neq \varphi(y) \) whenever \( x, y \in D \) are adjacent. The
graph associated with Theorem 1 is \( G(\mathbb{Q}, 1, n) \). Komjáth’s theorem in [3] asserts
that this graph has chromatic number \( \aleph_0 \).

Whenever we have \( 1 \leq m \leq n \) and \( A \subseteq \mathbb{R} \), it will be understood that any
reference to a graph is to the graph \( G(A, m, n) \).

**Theorem 3.** Let \( 1 \leq m \leq n \) and \( A \subseteq \mathbb{R} \) be such that the following two conditions hold:

1. \( A \) is a countable subring of \( \mathbb{R} \) and \( 1 \in A \);
2. for any finite \( F \subseteq \mathbb{R}^n \) and isometry \( \alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n \), there is \( z \in \alpha(A^m) \setminus F \)
   which is not adjacent to any \( y \in F \setminus \alpha(A^m) \).

Then there is a coloring \( \varphi : \mathbb{R}^n \rightarrow \omega \) such that for any isometry \( \alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n \),
the restriction of \( \varphi \) to \( \alpha(A^m) \) is a bijection onto \( \omega \).

The proof of Theorem 3 will be presented in §1. In §2 we show how Theorem 3
implies Theorems 1 and 2. Another consequence of Theorem 3 is also given in that
section. Finally, we make a connection with sets having the Steinhaus property,
which concerns the graph \( G(\mathbb{Z}, 2, 2) \).

1. The Proof of Theorem 3

In this section we give a proof of Theorem 3. The proof will rely heavily on the
proof of Komjáth’s theorem that the chromatic number of \( \mathbb{R}^n \) is \( \aleph_0 \) as given in [7].
We present a summary of that proof in a form suitable for our needs here.

We will think of \( \mathbb{R} \) as an ordered field. Since \( A \) is countable, we can find a
countable real-closed field \( \mathbb{F} \subseteq \mathbb{R} \) such that \( A \subseteq \mathbb{F} \). We will take \( \mathbb{F} \) to be fixed for the
remainder of this proof. Notice that if \( x, y \) are adjacent, then \( \|x - y\| \in \mathbb{F} \). If
\( X \subseteq \mathbb{R} \) and \( R \subseteq \mathbb{R}^k \) for some \( k < \omega \), then we say that \( R \) is \( X \)-definable if it is
definable in the ordered field \( \mathbb{R} \) by a formula in which parameters from \( X \cup \mathbb{F} \)
are allowed. We say that \( a \in \mathbb{R}^k \) is \( X \)-definable if \( \{a\} \) is \( X \)-definable.

Let \( T \) be a transcendence basis for \( \mathbb{R} \) over \( \mathbb{F} \) which is to be fixed for the remainder
of this proof. (Note that the existence of \( T \) cannot be proved without some use of
the Axiom of Choice.) Then each \( a \in \mathbb{R}^n \) is \( T \)-definable. In fact there is a unique
smallest finite subset \( S \subseteq T \) such that \( a \) is \( S \)-definable; we will refer to this set as
the support of \( a \), and denote it by \( supp(a) \). When it is convenient, we will consider
\( supp(a) \) to be an ordered set: thus, if \( supp(a) = \{t_0, t_1, t_2, \ldots, t_s-1\} \), where \( t_0 < t_1 < \cdots < t_{s-1} \), then we will sometimes let \( supp(a) = \{t_0, t_1, t_2, \ldots, t_{s-1}\} \).
For any subset \( X \subseteq \mathbb{R}^n \), let \( supp(X) = \bigcup \{supp(a) : a \in X\} \). Let \( b_0 = (0, 0, \ldots, 0) \in \mathbb{R}^m \),
and for \( 1 \leq j \leq m \) let \( b_j = (0, 0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^m \), which has its unique
1 preceded by \( j - 1 \) 0’s. It follows from (1) that whenever \( \alpha, \beta : \mathbb{R}^m \rightarrow \mathbb{R}^n \) are
isometries \( \{\alpha(b_0), \alpha(b_1), \ldots, \alpha(b_m)\} \subseteq \beta(A^m) \), then \( \alpha(A^m) = \beta(A^m) \). Thus, for
each isometry $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$, each element of $\alpha(A^m)$ is \{ $\alpha(b_0), \alpha(b_1), \ldots, \alpha(b_m)$ \}-definable and therefore, $\text{supp}(\alpha(A))$ is finite. In fact, $\text{supp}(\alpha(A^m)) = \text{supp}(\alpha(b_0)) \cup \text{supp}(\alpha(b_1)) \cup \cdots \cup \text{supp}(\alpha(b_m))$.

For $s < \omega$, a subset $B \subseteq \mathbb{R}^n$ is a \textbf{special s-box} if there are rationals $p_0 < q_0 < p_1 < q_1 < \cdots < q_{s-1} < q_s = 1$ such that $B = (p_0, q_0) \times (p_1, q_1) \times \cdots \times (p_{s-1}, q_{s-1})$. Each of the intervals $(p_i, q_i)$ is a \textbf{factor} of $B$. Let $(f_r : r < \omega)$ be a list of all $\emptyset$-definable analytic functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $B$ is a special s-box for some $s < \omega$, and let $B_r$ be the domain of $f_r$.

The following lemma, which is Lemma 7 of \cite{7}, is a key fact which is used repeatedly.

**Lemma 1.1.** Suppose that $B$ is a special s-box and $g : B \rightarrow \mathbb{R}$ is a $\emptyset$-definable analytic function such that $g(\bar{t}) = 0$ for some $\bar{t} \in B \cap T^n$. Then $g(\bar{x}) = 0$ for every $\bar{x} \in B$.

Associate with each $x \in \mathbb{R}^n$ the set $\Psi(x)$ of colors, where $r \in \Psi(x)$ iff $\text{supp}(x) = \langle t_0, t_1, \ldots, t_{s-1} \rangle$ and $x = f_r(t_0, t_1, \ldots, t_{s-1})$. The crucial facts about the sets $\Psi(x)$ are contained in the next two lemmas. The first follows from the Implicit Function Theorem and the Tarski-Seidenberg Theorem on the elimination of quantifiers in $\mathbb{R}$. The second can be deduced from Lemma 1.1.

**Lemma 1.2.** If $x \in \mathbb{R}^n$, then $\Psi(x) \neq \emptyset$. \hfill $\Box$

**Lemma 1.3.** If $x, y \in \mathbb{R}^n$ are adjacent in $G(A, m, n)$ (or even if $0 < \|x - y\| \in \mathbb{F}$), then $\Psi(x) \cap \Psi(y) = \emptyset$. \hfill $\Box$

By Lemma 1.2, there is a coloring $\psi : \mathbb{R}^n \rightarrow \omega$ such that $\psi(x) \in \Psi(x)$ for each $x \in \mathbb{R}^n$, and from Lemma 1.3 we get that any such $\psi$ is proper.

The coloring $\varphi$ will be constructed inductively; that is, we will construct an increasing sequence $\langle \varphi_k : k < \omega \rangle$ of functions, and then let $\varphi$ be its union. This sequence of functions will be defined from two sequences $d_0, d_1, d_2, \ldots$ and $e_0, e_1, e_2, \ldots$ of colors. For each $k < \omega$, we let

$$D_k = \{ x \in \mathbb{R}^n : \Psi(x) \cap \{d_0, d_1, \ldots, d_{k-1}\} \neq \emptyset \},$$

and then let $\varphi_k : D_k \rightarrow \omega$ be such that if $x \in D_k$ then $\varphi_k(x) = e_m$, where $m < k$ is the least for which $d_m \in \Psi(x)$. Whenever we have $d_0, d_1, \ldots, d_{k-1}$, we will assume that $D_k$ has been defined in this way; and if, in addition, we have $e_0, e_1, \ldots, e_{k-1}$, then we also assume that $\varphi_k$ has been defined. Of course, for each $k$ we must have that $\varphi_k$ is a proper coloring of $D_k$; we will say that the finite sequence $d_0, d_1, \ldots, d_{k-1}, e_0, e_1, e_2, \ldots, e_{k-1}$ is \textbf{acceptable} if $\varphi_k$ is a proper coloring.

At the beginning of stage $k$, we have $d_0, d_1, \ldots, d_{k-1}$ and $e_0, e_1, \ldots, e_{k-1}$, and thus also $D_k$ and $\varphi_k$. Then, at stage $k$, we will obtain $d_k$, $e_k$, $D_{k+1}$ and $\varphi_{k+1}$. There are two requirements which must be taken care of in this construction: the domain of $\varphi$ should be $\mathbb{R}^n$; and for each isometry $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and color $r$, there should be some $z \in \alpha(A^m)$ such that $\varphi(z) = r$. The first of these requirements is easily handled by the following lemma.

**Lemma 1.4.** If $d_0, d_1, \ldots, d_{k-1}, e_0, e_1, \ldots, e_{k-1}$ is acceptable and if $d_k$ is any color, then there is a color $e_k$ such that $d_0, d_1, \ldots, d_{k-1}, d_k, e_0, e_1, \ldots, e_{k-1}, e_k$ is acceptable.

\textbf{Proof.} By Lemma 1.3, we can choose any $e_k \notin \{e_0, e_1, \ldots, e_{k-1}\}$. \hfill $\Box$
We now turn to taking care of the second requirement.

**Lemma 1.5.** Suppose that $d_0, d_1, \ldots, d_{k-1}$ are colors and that $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an isometry. Then there is $z \in \alpha(A^m) \setminus D_k$ such that $z$ is not adjacent to any $y \in D_k \setminus \alpha(A^m)$ and $\text{supp}(z) = \text{supp}(\alpha(A^m))$.

**Proof.** We begin this proof by showing that condition (2) of Theorem 3 can be improved to the following:

(2') for any finite $F \subseteq \mathbb{R}^n$ and isometry $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$, there is $z \in \alpha(A^m) \setminus F$ which is not adjacent to any $y \in F \setminus \alpha(A^m)$ and is such that $\text{supp}(z) = \text{supp}(\alpha(A^m))$.

For each $\mathbf{T}$, which is properly contained in $\text{supp}(\alpha(A^m))$, the set of elements in $\alpha(A^m)$ having support contained in $\mathbf{T}$ lie in some $(m-1)$-dimensional hyperplane of $\mathbb{R}^n$. (Otherwise, we would have that $\text{supp}(\alpha(A^m)) \subseteq \mathbf{T}$.) Therefore, $\{a \in A^m : \text{supp}(\alpha(a)) \subseteq \mathbf{T}\}$ is contained in an $(m-1)$-dimensional hyperplane of $\mathbb{R}^n$. Thus, the set $S$ of elements in $a \in A^m$ for which $\text{supp}(\alpha(a))$ is different from $\text{supp}(\alpha(A^m))$ is contained in the union of finitely many $(m-1)$-dimensional hyperplanes. Clearly, there are finitely many $v_0, v_1, \ldots, v_p \in A^m$ such that $A^m \subseteq (v_0 + (A^m \setminus S)) \cup (v_1 + (A^m \setminus S)) \cup \cdots \cup (v_p + (A^m \setminus S))$. Let $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the isometry defined by $\beta(x) = x + (\alpha(v_i) - \alpha(0))$. Then $\beta(\alpha(A^m)) = \alpha(A^m)$ for each $i \leq p$, and $\text{supp}(\alpha(A^m)) \subseteq \beta_0(\alpha(A^m \setminus S)) \cup \beta_1(\alpha(A^m \setminus S)) \cup \cdots \cup \beta_p(\alpha(A^m \setminus S))$. By (2) we let $x \in \alpha(A^m) \setminus (\beta_0(F) \cup \beta_1(F) \cup \cdots \cup \beta_p(F))$ be such that it is not adjacent to any $y \in (\beta_0(F) \cup \beta_1(F) \cup \cdots \cup \beta_p(F)) \setminus \alpha(A^m)$. There is $i \leq p$ such that $x \in \beta_i(\alpha(A^m \setminus S))$. Then $x \notin \beta_i(F)$ and $x$ is not adjacent to any point in $\beta_i(F) \setminus \alpha(A^m)$. Therefore, $z = \beta_i^{-1}(x)$ is as required.

We now return to the proof of the lemma. Consider an equivalence relation on $D_k \setminus \alpha(A^m)$ obtained in the following way. The points $y, y' \in D_k \setminus \alpha(A^m)$ are equivalent if $\Phi(y) \cap \{d_0, d_1, \ldots, d_{k-1}\} = \Phi(y') \cap \{d_0, d_1, \ldots, d_{k-1}\}$ and their supports are equivalent over $\text{supp}(\alpha(A^m))$ in the following sense: if $\text{supp}(y) = \{t_0, t_1, \ldots, t_s-1\}$, $\text{supp}(y') = \{t_0', t_1', \ldots, t_{s'}-1\}$, $u \in \text{supp}(\alpha(A^m))$ and $j < s$, then $t_j < u$ iff $t_j' < u$ and $u < t_j$ iff $u < t_j'$. Clearly, there are only finitely many equivalence classes.

We show that if $y$ and $y'$ are equivalent and $x \in \alpha(A^m)$, then $y$ is adjacent to $x$ iff $y'$ is adjacent to $x$; in fact, we will show that if $y$ is adjacent to $x$, then $\|y - x\| = \|y' - x\|$. So, suppose that $y$ and $y'$ are equivalent and $y$ is adjacent to $x \in \alpha(A^m)$. Then let $\mathbf{T}, \mathbf{T}'$ be their supports, so that $y = f_\mathbf{T}(\mathbf{T})$ and $y' = f_\mathbf{T}'(\mathbf{T}')$. Let $B$ be a special box for which $\text{supp}(y) \subseteq \text{supp}(\alpha(A^m)), \text{supp}(y') \subseteq \text{supp}(\alpha(A^m)) \in B$ and on which there is a $\emptyset$-definable analytic function $g$ for which $g(\text{supp}(y), \text{supp}(\alpha(A^m))) = \|y - x\|^2$ and $g(\text{supp}(y'), \text{supp}(\alpha(A^m))) = \|y' - x\|^2$. Since this $\|y - x\|^2 \in \mathbb{F}$, it follows from Lemma 1.1 that $g$ is constant on $B$, so that $\|y - x\| = \|y' - x\|$.

Now let $Y \subseteq D_k \setminus \alpha(A^m)$ be a finite set which meets every equivalence class. Then, by (2'), we can choose $z \in \alpha(A^m) \setminus D_k$ such that $\text{supp}(z) = \text{supp}(\alpha(A^m))$ and $z$ is not adjacent to any $y \in Y$. Then $z$ is not adjacent to any $y \in D_k \setminus \alpha(A^m)$, thereby proving the lemma.

We say that an isometry $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ has **type** $\tau = \langle i_0, i_1, \ldots, i_m \rangle$ if the following hold for each $j \leq m$:

- $i_j \in \Psi(\alpha(b_j));$
- $B_{i_j} = B_{b_j}$ for each $j \leq m$;
- $f_{i_j}(\text{supp}(\alpha(A^m))) = \alpha(b_j)$. 

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We will call the box $B_0$ the domain of $\tau$. It is possible for an isometry not to have a type, and it is also possible that an isometry have more than one type.

**Lemma 1.6.** For every isometry $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ there is an isometry $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $\gamma(A^m) = \alpha(A^m)$ and $\gamma$ has a type.

**Proof.** Using an argument like the one at the beginning of the proof of Lemma 1.5, we see that there is a point $v \in A^m$ such that $\text{supp}(\alpha(v + b_0)) = \text{supp}(\alpha(v + b_1)) = \cdots = \text{supp}(\alpha(v + b_m))$. Let $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be such that $\gamma(x) = \alpha(v + x)$.

If $C_0, C_1, \ldots, C_{k-1}, B$ are special boxes, then we say that $B$ is **refining** over $C_0, C_1, \ldots, C_{k-1}$ if, whenever $J$ is a factor of some $C_j$ and $I$ is a factor of $B$, then either $J \cap I = \emptyset$ or $J \supseteq I$.

**Lemma 1.7.** Let $\tau$ be the type of an isometry, and let $C_0, C_1, \ldots, C_{k-1}$ be special boxes. Then there are types $\tau_0, \tau_1, \ldots, \tau_p$ with domains $C_k, C_{k+1}, \ldots, C_{k+p}$ respectively such that the following hold:
- for $j \leq p$, $C_{k+j}$ is refining over $C_0, C_1, \ldots, C_{k+j-1}$;
- for any isometry $\alpha$ of type $\tau$, there is some $j \leq p$ such that $\alpha$ has a type $\tau_j$.

**Proof.** Let $\tau = \langle i_0, i_1, \ldots, i_m \rangle$ and let $B$ be the domain of $\tau$. Let $Q$ be the finite set of rationals which are the endpoints of the factors of the special boxes $C_0, C_1, \ldots, C_{k-1}$ and $B$. Let $\mathcal{B}$ be the finite set of all special boxes whose factors have endpoints in $Q$. Then let $C_k, C_{k+1}, \ldots, C_{k+p}$ be those special boxes which are minimal (with respect to inclusion) in $\mathcal{B}$ and which are included in $B$. For each $j \leq p$, let $\tau_j = \langle i_{0j}, i_{1j}, \ldots, i_{mj} \rangle$, where each $i_{rj}$ is such that $f_{i_{rj}} = f_{i_r}|_{C_{k+j}}$. It is clear that the conditions in the lemma are met.

**Lemma 1.8.** Suppose that $d_0, d_1, \ldots, d_{k-1}, e_0, e_1, \ldots, e_{k-1}$ is acceptable and $e_k$ is a color. Suppose that $\alpha$ is an isometry of type $\tau = \langle i_0, i_1, \ldots, i_m \rangle$ such that $\varphi_k(z) \neq e_k$ for all $z \in \alpha(A^m)$. Suppose that $B$, the domain of $\tau$, is refining over $B_{d_0}, B_{d_1}, \ldots, B_{d_{k-1}}$. Then there is a color $d_k$ such that $B_{d_k} = B$, $d_0, d_1, \ldots, d_{k-1}, d_k, e_0, e_1, \ldots, e_{k-1}, e_k$ is acceptable, and for any isometry $\beta : \mathbb{R}^m \rightarrow \mathbb{R}^n$ of type $\tau$, there is $w \in \beta(A^m)$ such that $\varphi_{k+1}(w) = e_k$.

**Proof.** By Lemma 1.5, let $z \in \alpha(A^m) \setminus D_k$ be such that $\text{supp}(z) = \text{supp}(\alpha(A^m))$ and $z$ is not adjacent to any $y \in D_k \setminus \alpha(A^m)$. Let $\text{supp}(\alpha(A^m)) = \mathbf{7}$ and let $a = \alpha^{-1}(z)$. Then $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$. Let $a_0 = 1 - (a_1 + a_2 + \cdots + a_m)$. We now let $d_k$ be such that $f_{d_k} : B \rightarrow \mathbb{R}^n$ is the analytic function defined by

$$f_{d_k}(\mathbf{x}) = a_0 f_{i_0}(\mathbf{x}) + a_1 f_{i_1}(\mathbf{x}) + \cdots + a_m f_{i_m}(\mathbf{x}).$$

Then $B_{d_k} = B$. Note that $f_{d_k}(\mathbf{7}) = z$ since

$$f_{d_k}(\mathbf{7}) = a_0 f_{i_0}(\mathbf{7}) + a_1 f_{i_1}(\mathbf{7}) + \cdots + a_m f_{i_m}(\mathbf{7}) = a_0 \alpha(b_0) + a_1 \alpha(b_1) + \cdots + a_m \alpha(b_m) = \alpha(a_0 b_0 + a_1 b_1 + \cdots + a_m b_m) = \alpha(a) = z.$$

It is clear that $\varphi_{k+1}(z) = e_k$. We will show that for every isometry $\beta$ having type $\tau$ there is $w \in \beta(A^m)$ for which $\varphi_{k+1}(w) = e_k$. Consider $\beta$ having type $\tau$, and let $\mathbf{7} = \text{supp}(\beta(A^m))$. Then let $w = \beta(a) = f_{d_k}(\mathbf{7})$.

Clearly, $w \in \beta(A^m)$. To show that $\varphi_{k+1}(w) = e_k$, it suffices to show that $w \in D_{k+1} \setminus D_k$. 
We show that $w \in D_{k+1}$ by showing that $d_k \in \Psi(w)$. Since $w = f_{d_k}(\mathfrak{S})$, we need only show that $\text{supp}(w) = \mathfrak{S}$. If not, then there is color $p$ such that $f_p(\mathfrak{S}) = w$ and $\mathfrak{S}$ is properly contained in $\mathfrak{S}$. Without loss of generality, we can assume that $s_0$ is the unique real in $\mathfrak{S}$ but not in $\mathfrak{S}$. Thus, we can let $w = f_p(\mathfrak{S}) = f_{d_k}(\mathfrak{S}, s_0)$. Since $s_0$ is not $\mathfrak{S}$-definable, it follows that for some open neighborhood $U$ of $s_0$, if $s \in U$, then $f_p(\mathfrak{S}) = f_{d_k}(\mathfrak{S}, s)$. Let $r \in U$ be a rational, and then $f_{d_k}(\mathfrak{S}, s_0) = f_{d_k}(\mathfrak{S}, r)$. It then easily follows from Lemma 1.3 that $\text{supp}(z) \neq \mathfrak{S}$, which is a contradiction.

Next, we must show that $w \notin D_k$. For a contradiction, suppose that $m < k$ and $d_m \notin \Psi(w)$. Thus $w = f_{d_m}(\mathfrak{S})$. Since $B_{d_k}$ is refining, $B_{d_k} \subseteq B_{d_m}$, so it follows from Lemma 1.1 that $f_{d_m}$ and $f_{d_k}$ agree on $B_{d_k}$. Therefore, $z = f_{d_m}(\mathfrak{S})$, contradicting that $z \notin D_k$.

It remains to prove that $\varphi_{k+1}$ is a proper coloring. Clearly, there is no $w \in D_{k+1}$ adjacent to $z$ such that $\varphi_{k+1}(w) = \varphi_{k+1}(z)$. Consider arbitrary $z' \in D_{k+1} \setminus D_k$ and some $w' \in D_{k+1}$ adjacent to it, with the intent of showing that $\varphi_{k+1}(w') \neq \varphi_{k+1}(z')$. Then $z' = f_{d_k}(\mathfrak{S})$ for some $\mathfrak{S}$. Let $m' \leq k$ be minimal such that $w' = f_{d_m}(\mathfrak{S})$. Since $B_{d_k}$ is refining, we can find a special box $C$ such that $(\mathfrak{S}, \mathfrak{S}'), (\mathfrak{S}, \mathfrak{S}) \in C$ and then let $g : C \rightarrow \mathbb{R}$ be the $\theta$-definable analytic function such that $g(\mathfrak{S}, \mathfrak{S}'') = \|f_{d_k}(\mathfrak{S}) - f_{d_m}(\mathfrak{S})\|^2$. Then $g(\mathfrak{S}, \mathfrak{S}') = \|z' - w'\|^2 \leq \mathfrak{S}$, so it follows from Lemma 1.1 that $g$ is constant. We can find $\mathfrak{S'}$ such that $(\mathfrak{S'}, \mathfrak{S'}) \in C$. Then $g(\mathfrak{S}, \mathfrak{S'}) = \|z' - w'\|^2 \leq \mathfrak{S}$. Let $v = f_{m'}(\mathfrak{S})$. Then $v \in D_{k+1}$, and $v$ and $z$ are adjacent. Therefore, $\varphi_{k+1}(v) \neq \varphi_{k+1}(z)$, so to complete the proof it suffices to show that $\varphi_{k+1}(w') = \varphi_{k+1}(v)$.

Suppose $\varphi_{k+1}(w') \neq \varphi_{k+1}(v)$. Then there is $m < m'$ such that $d_m \in \Psi(v)$, so that $f_m(\mathfrak{S}) = f_{m'}(\mathfrak{S})$. It follows from Lemma 1.1, that $f_m(\mathfrak{S'}) = f_{m'}(\mathfrak{S'}) = w'$, which contradicts the minimality of $m'$.

We finish off the proof of Theorem 3. We are constructing the two sequences $d_0, d_1, d_2, \ldots$ and $e_0, e_1, e_2, \ldots$. At each stage $k$ we have the first $k$ terms of each sequence, and $d_0, d_1, \ldots, d_{k-1}, e_0, e_1, \ldots, e_{k-1}$ is acceptable. There are the two requirements mentioned just before Lemma 1.4.

For the first of these, by Lemma 1.2, it suffices that $\omega = \{d_0, d_1, d_2, \ldots\}$. So at some stage $k$ we are concerned that $d$ gets into this sequence. By Lemma 1.4, we can let $d_k = d$ and then get $e_k$ such that $d_0, d_1, \ldots, d_{k-1}, d_k, e_0, e_1, \ldots, e_{k-1}, e_k$ is acceptable.

To meet the second requirement, it suffices by Lemma 1.6 to show that for every type $\tau$ and color $r$, if $\alpha$ has type $\tau$, then there is $z \in \alpha(A^m)$ such that $\varphi(z) = r$. So at some stage $k$ we will consider $\tau$ and $r$. Let $C_0, C_1, \ldots, C_{k-1}$ be the special boxes $B_{d_0}, B_{d_1}, \ldots, B_{d_{k-1}}$. Apply Lemma 1.7 to get types $\tau_0, \tau_1, \ldots, \tau_p$ with domains $C_k, C_{k+1}, \ldots, C_{k+p}$. Now apply Lemma 1.8 $p + 1$ times, at the $j$th time using $\tau_j$, to get acceptable $d_0, d_1, \ldots, d_{k+p}, e_0, e_1, \ldots, e_{k+p}$. Clearly, the second requirement will be met, completing the proof of Theorem 3.

2. The Consequences

To derive Theorem 1 from Theorem 3, it suffices to show that when $A = \mathbb{Q}$ conditions (1) and (2) of Theorem 3 hold. Condition (1) is obvious. Condition (2) follows from the following lemma which is from Komjáth [6]. The proof presented here is a little different from the one in [6].

Lemma 2.1. Let $F \subseteq \mathbb{R}^n$ be a finite set of points and $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n$ be an isometry. Then there is $x \in \alpha(\mathbb{Q}) \setminus F$ such that $\|x - y\| \notin \mathbb{Q}$ for all $y \in F \setminus \alpha(\mathbb{Q})$. 


Proof. Without loss of generality we can assume that $n = 2$ and $\alpha$ is such that $\alpha(x) = (x, 0)$ for all $x \in \mathbb{R}$. If $(a, b) \in F$ and for two distinct rationals $q$ and $r$, both $\|(a, b) - (q, 0)\|$ and $\|(a, b) - (r, 0)\|$ are rational, then $a, b^2 \in \mathbb{Q}$. Thus, by appropriate scaling and translating, we can assume that if $(a, b) \in F$ and $\|(a, b) - (q, 0)\|$ is rational, where $0 < q \in \mathbb{Q}$, then $a$ and $b^2$ are integers. Let $c$ be a positive integer such that $c > a + b^2$ whenever $(a, b) \in F$ and let $x = (c, 0)$. To see that $x$ is as required, let $y = (a, b) \in F \setminus \alpha(\mathbb{Q})$. Then $b \neq 0$ and $d^2 = (c - a)^2 + b^2 = \|x - y\|^2$ is an integer, so if $d$ is rational, it also must be an integer. But $c - a < d < c - a + 1$, so $d$ is not an integer.

To derive Theorem 2 from Theorem 3, it suffices to show that (1) and (2) hold when $A = \mathbb{Q}$. Again, (1) is trivial. The following lemma shows that (2) holds.

Lemma 2.2. Let $F \subseteq \mathbb{R}^n$ be a finite set of points. Then there is $x \in \mathbb{Q}^n \setminus F$ such that $\|x - y\|^2 \not\in \mathbb{Q}$ for all $y \in F \setminus \mathbb{Q}^n$.

Proof. Let $m = \|F\|$. Let $B \subseteq \mathbb{Q}$ be such that $|B| = m + 1$ and $B^n \cap F = \emptyset$. Consider some $y = (y_0, y_1, \ldots, y_{n-1}) \in F \setminus \mathbb{Q}^n$. Then there is $j < n$ such that $y_j \notin \mathbb{Q}$. Hence, if $u, v \in \mathbb{Q}^n$ agree except at the $j$-th coordinate, then not both $\|u - y\|^2 \in \mathbb{Q}$ and $\|v - y\|^2 \in \mathbb{Q}$. Therefore, for every $y \in F \setminus \mathbb{Q}^n$, there are at most $(m + 1)^{n-1}$ points $u \in B^n$ such that $\|u - y\|^2 \in \mathbb{Q}$. It follows that there are at most $m(m + 1)^{n-1} \leq |B^n|$ points $u \in B^n$ such that $\|u - y\|^2 \in \mathbb{Q}$ for some $y \in F \setminus \mathbb{Q}^n$. Therefore, there is $x \in B$ such that $\|x - y\|^2 \not\in \mathbb{Q}$ for every $y \in F \setminus \mathbb{Q}^n$.

The preceding lemma remains true if $\mathbb{Q}$ is replaced by any countable subfield $F \subseteq \mathbb{R}$. Thus, we get the following corollary extending Theorem 2.

Corollary 2.3. Let $F \subseteq \mathbb{R}$ be any countable subfield. Then there is a coloring $\varphi : \mathbb{R}^n \rightarrow \omega$ such that for any isometry $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the restriction of $\varphi$ to $\alpha(\mathbb{R}^n)$ is a bijection onto $\omega$.

Komjáth [H] proved that there is a subset $B \subseteq \mathbb{R}^2$ such that whenever $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an isometry, then $\alpha(B) \cap \mathbb{Z} = 1$. In fact, we can partition $\mathbb{R}^2$ into countably many such sets since Theorem 3 applies when $A = \mathbb{Z}$, $m = 1$ and $n = 2$. This can be extended to all $n$ using the following lemma.

Lemma 2.4. Let $F \subseteq \mathbb{R}^n$ be a finite set of points and $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n$ an isometry. Then there is $x \in \alpha(\mathbb{Z}) \setminus F$ such that $\|x - y\| \not\in \mathbb{Z}$ for all $y \in F \setminus \alpha(\mathbb{Z})$.

Proof. If there is $x \in \alpha(\mathbb{R}) \cap (F \setminus \alpha(\mathbb{Z}))$ then choose that point. Otherwise, let $x$ be as in Lemma 2.1.

Corollary 2.5. There is a coloring $\varphi : \mathbb{R}^n \rightarrow \omega$ such that for any isometry $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n$, the restriction of $\varphi$ to $\alpha(\mathbb{Z})$ is a bijection onto $\omega$.

The question of whether there is such a result for the graph $G(\mathbb{Z}, 2, 2)$ appears to be open. A positive answer would result in a partition of $\mathbb{R}^2$ into countably many sets each having the Steinhaus property. Theorem 3 cannot be used to get such a partition since the set $F = \{(\frac{k}{n}, k + \frac{1}{2}) : k = 0, 1, \ldots, 4\}$ is a counterexample to (2) (for $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ being the identity isometry).

There is also the question concerning the graphs $G(\mathbb{Q}, m, n)$ when $2 \leq m < n$. For $m = 2, 3$, this question also appears to be open. However, if $4 \leq m < n$, then there is no such result since for any isometry $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\alpha(\mathbb{Q}^m)$ is not a maximal clique of $G(\mathbb{Q}, m, n)$ by Lagrange’s Theorem on sums of 4 squares.
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