TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 3, Pages 1265–1278
S 0002-9947(01)02943-9
Article electronically published on November 2, 2001

HAMBURGER AND STIELTJES MOMENT PROBLEMS IN SEVERAL VARIABLES

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ABSTRACT. In this paper we give solutions to the Hamburger and Stieltjes moment problems in several variables, in algebraic terms, via extended sequences. Some characterizations of the uniqueness of the solutions are also presented.

1. PRELIMINARIES

Let $\mathcal{R}$ be an algebra of complex-valued functions, defined on the Euclidean space $\mathbf{R}^n$, such that the constant function $1 \in \mathcal{R}$, and if $f \in \mathcal{R}$ then $\bar{f} \in \mathcal{R}$. A linear map $L : \mathcal{R} \to \mathbf{C}$ is said to be *positive semi-definite* if $L(f\bar{f}) \geq 0$ for all $f \in \mathcal{R}$. If L is positive semi-definite on $\mathcal{R}$, we shall always assume that $L(1) > 0$ (i.e., L is not degenerate).

Let $\mathcal{R}$ be an algebra as above, and let $L : \mathcal{R} \to \mathbf{C}$ be positive semi-definite. This pair can be associated, in a canonical way, with a certain pre-Hilbert space (this idea goes back to Gelfand and Naimark [GeNa]; see also [DuSc], [Fug], etc.). To recall this construction, let $\mathcal{N} = \{f \in \mathcal{R}; L(f\bar{f}) = 0\}$. Since L satisfies the Cauchy-Schwarz inequality, it follows that $\mathcal{N}$ is an ideal of $\mathcal{R}$. Moreover, the quotient $\mathcal{R}/\mathcal{N}$ is a pre-Hilbert space, whose inner product is given by

$$\langle f + \mathcal{N}, g + \mathcal{N} \rangle = L(f\bar{g}), \qquad f, g \in \mathcal{R}. \tag{1.1}$$

Note also that $\mathcal{R}/\mathcal{N}$ is an $\mathcal{R}$-module.

An arbitrary map $L : \mathcal{R} \to \mathbf{C}$ is said to be a *moment function* if there exists a positive measure μ on $\mathbf{R}^n$ such that $\mathcal{R} \subset L^2(\mu)$, and $L(f) = \int f d\mu, f \in \mathcal{R}$. In this case, the measure μ is said to be a *representing measure* for the (necessarily linear) map L. Clearly, every moment function is positive semi-definite. The *moment problem* on $\mathcal{R}$ is to characterize those positive semi-definite maps on $\mathcal{R}$ which are moment functions. A solution of a moment problem is said to be *determined* if the corresponding representing measure is uniquely determined.

In this paper we shall be particularly interested by the following case. Let us denote by $\mathbf{Z}_+^n$ the set of all multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, i.e., $\alpha_j \in \mathbf{Z}_+$ for all $j = 1, ..., n$. Let $\mathcal{P}_n$ be the algebra of all polynomial functions on $\mathbf{R}^n$, with complex coefficients. We shall denote by t^α the monomial $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $t = (t_1, ..., t_n)$ is the current variable in $\mathbf{R}^n$, and $\alpha \in \mathbf{Z}_+^n$.

An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ is said to be *positive semi-definite* if the *associated linear map* $L_\gamma : \mathcal{P}_n \to \mathbf{C}$ is positive semi-definite, where $L_\gamma(t^\alpha) = \gamma_\alpha, \alpha \in \mathbf{Z}_+^n$.

Received by the editors July 10, 2001.
2000 *Mathematics Subject Classification.* Primary 47A57, 44A60.

Similarly, an n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbf{Z}_+^n}$ is said to be a *moment sequence* if the associated map L_γ is a moment function. In this case, the representing measure of L_γ is also called a *representing measure* for γ.

The aim of this paper is to describe the solutions to the (determined) moment problem for the algebra $\mathcal{P}_n$, which is usually called the *Hamburger moment problem* (in several variables), via extended sequences. Let us explain what we mean by this in the case of the plane. We obtain the following result (as a particular case of Theorem 2.3):

A 2-sequence $\gamma = (\gamma_{m_1,m_2})_{m_1,m_2\in\mathbf{Z}_+}$ ($\gamma_{0,0} > 0$) is a moment 2-sequence if and only if there exists a positive semi-definite 4-sequence

$$\delta = (\delta_{m_1,m_2,m_3,m_4})_{m_1,m_2,m_3,m_4\in\mathbf{Z}_+}$$

with the following properties:

(1) $\gamma_{m_1,m_2} = \delta_{m_1,m_2,0,0}$;
(2′) $\delta_{m_1,m_2,m_3,m_4} = \delta_{(m_1,m_2,m_3+1,m_4)} + \delta_{m_1+2,m_2,m_3+1,m_4}$;
(2″) $\delta_{m_1,m_2,m_3,m_4} = \delta_{(m_1,m_2,m_3,m_4+1)} + \delta_{m_1,m_2+2,m_3,m_4+1}$ for all $m_1, m_2, m_3, m_4 \in \mathbf{Z}_+$.

The collection of all positive semi-definite 4-sequences δ having the properties $(2'), (2'')$ is a well-defined set of functions, and the 2-sequences γ have a representing measure if and only if they are restrictions of such 4-sequences δ (as part of the statement in Theorem 2.3).

The passage from a 2-sequence to an extended one is partially motivated by the fact that there are 2-sequences which are positive semi-definite and which are not moment sequences (see [BCR], [BeCh]). Therefore, some new parameters must be introduced. In addition, when the moment problem has several solutions, a parametrization of all solutions is also of interest. The existence of "optimal" choices for such parameters is still to be investigated.

When one seeks, in this context, representing measures whose support is concentrated in $\mathbf{R}_+^n$, then the corresponding moment problem is called the *Stieltjes moment problem* (in several variables). Solutions to Hamburger and Stieltjes moment problems, in several variables, by extended sequences, are provided by Theorems 2.3 and 2.6. For a thorough discussion concerning these problems in one variable, as well as for historical remarks, we refer to the monographs [Akh] and [ShTa] (see also [BCR], [Dev], [Esk], [Fug], [Hav], [KoMi], [Sch], etc. for various solutions in several variables). In the determined case, even for one-variable problems our statements (Corollaries 2.5 and 2.8, and Theorem 3.9) are seemingly new (see [Akh], [ShTa] for other characterizations).

The first two sections of this paper have circulated as a preprint since the Spring of 1998, under the same name. It has partially inspired the work [PuVa], where similar (but not identical) problems are treated. Let us also mention the independent paper [StSz], where solutions to some moment problems in the plane are given, via certain extended sequences, using different methods.

The third section, which is a natural continuation of the second one, has been subsequently added. It contains an operator theoretic characterization of the uniqueness of the representing measures (Theorem 3.4), as well as some related results. In particular, an explicit (topological) characterization of the strongly determined moment problems (Theorem 3.7), as defined in [Fug], using some results from [Vas2]. The case of determined moment problems on the real line is separately treated (Theorem 3.9), and we give a relatively simple explicit condition, similar to the classical one due to Hamburger (see [Dev]).

The author would like to thank R. Curto and M. Putinar for their interest in this work. Thanks are also due to J. d'Almeida for an idea related to the proof of Lemma 2.2. He is also indebted to the referee for some useful suggestions, in particular for a more appropriate terminology in Definition 3.1.

2. The main results

Let $\mathcal{H}$ be a complex Hilbert space, whose scalar product (resp. norm) will be denoted by $\langle *, * \rangle$ (resp. $\| * \|$).

By *operator* in $\mathcal{H}$ we mean a linear map S, defined a linear subspace $D(S) \subset \mathcal{H}$, with values in $\mathcal{H}$. The *range* of S will be denoted by $R(S)$.

Lemma 2.1. *Let S be a symmetric densely defined operator in $\mathcal{H}$. If the sets $R(S \pm i)$ are dense in $\mathcal{H}$, then the canonical closure of S is a selfadjoint operator.*

Proof. Let A be the canonical closure of S, which is also a symmetric operator. From the classical identity

$$\|(A \pm i)x\|^2 = \|Ax\|^2 + \|x\|^2, \qquad x \in D(A),$$

it follows that $R(A \pm i)$ are closed subspaces of $\mathcal{H}$. As we have $R(A \pm i) \supset R(S \pm i)$, our hypothesis implies $R(A \pm i) = \mathcal{H}$.

Let V be the Cayley transform of A (see [Rud, 13.17]). Since $D(V) = R(A+i), R(V) = R(A-i)$, the operator V is unitary, and so A must be self-adjoint (via [Rud, 13.19]). □

Lemma 2.2. *Set $\theta_j(t) = (1+t_j^2)^{-1}, 1 \le j \le n, t = (t_1, ..., t_n) \in \mathbf{R}^n$, and let $\mathcal{R}_\theta$ be the algebra of rational functions generated by $\mathcal{P}_n$ and $\theta = (\theta_j)_{1 \le j \le n}$. Let $\rho : \mathcal{P}_{2n} \to \mathcal{R}_\theta$ be given by $\rho : p(t,s) \to p(t, \theta(t))$. Then ρ is a surjective unital algebra homomorphism, whose kernel is the ideal generated by the polynomials $\sigma_j(t,s) = s_j(1+t_j^2) - 1, 1 \le j \le n$.*

Proof. That ρ is a surjective unital algebra homomorphism is obvious. We have only to determine the kernel of ρ.

Let $p \in \mathcal{P}_{2n}$ be a polynomial with the property $p(t, \theta(t)) = 0, t \in \mathbf{R}^n$. We write $p(t,s) = \sum_{\beta \in \mathbf{Z}_+^n} p_\beta(t) s^\beta$, with $p_\beta \in \mathcal{P}_n \setminus \{0\}$ only for a finite number of indices β. Then we have

$$p(t,s) = p(t,s) - p(t,\theta(t)) = \sum_{\beta \neq 0} p_\beta(t)(s^\beta - \theta(t)^\beta)$$

$$= \sum_{1 \le j \le n} (s_j - \theta_j(t)) \ell_j(t, s, \theta(t)),$$

where ℓ_j are polynomials.

Let $a_j = \max\{\beta_j ; p_\beta \neq 0\}, 1 \le j \le n$, and let

$$\tau(t) = \prod_{1 \le j \le n} (1 + t_j^2)^{a_j}.$$

Then, from the above calculation, we deduce the equation

$$\tau(t) p(t,s) = \sum_{1 \le j \le n} (s_j(1+t_j^2) - 1) q_j(t,s), \tag{2.1}$$

with $q_j \in \mathcal{P}_{2n}$ for all indices j.

If $a_j = 0$ for all j, then $p(t, s) = p_0(t) = p(t, \theta(t)) = 0$. Therefore, with no loss of generality, we may assume $a_j \neq 0$ for some indices j.

It is easily seen that the polynomials $\tau, \sigma_j, 1 \leq j \leq n$, have no common zero in $\mathbf{C}^{2n}$. By a special case of Hilbert's Nullstellensatz (see, for instance, [Van, Section 16.5]), there are polynomials $\tilde{\tau}, (\tilde{\sigma}_j)_{1 \leq j \leq n}$ in $\mathcal{P}_{2n}$ such that

$$\tau\tilde{\tau} + \sum_{1 \leq j \leq n} \sigma_j \tilde{\sigma}_j = 1. \tag{2.2}$$

If we multiply (2.2) by p, and use (2.1), we obtain the relation

$$p = \sum_{1 \leq j \leq n} \sigma_j (q_j \tilde{\tau} + \tilde{\sigma}_j p),$$

which is precisely our assertion.

We shall denote by $e_j \in \mathbf{Z}_+^n, j = 1, ..., n$, the multi-index whose coordinates are null except for the j^{th}-coordinate, which is equal to one.

A solution of the Hamburger moment problem in several variables is given by the following.

Theorem 2.3. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ ($\gamma_0 > 0$) is a moment n-sequence if and only if there exists a positive semi-definite $2n$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n}$ with the following properties:*

(1) *$\gamma_\alpha = \delta_{(\alpha,0)}$ for all $\alpha \in \mathbf{Z}_+^n$.*
(2) *$\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_j)} + \delta_{(\alpha+2e_j,\beta+e_j)}$ for all $\alpha, \beta \in \mathbf{Z}_+^n, 1 \leq j \leq n$.*

In this case, the n-sequence γ has a uniquely determined representing measure in $\mathbf{R}^n$ if and only if the $2n$-sequence δ is unique.

Proof. We prove first that conditions (1), (2) are necessary. Assume that the sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ has a representing measure μ. Define

$$\delta_{(\alpha,\beta)} = \int_{\mathbf{R}^n} t^\alpha \theta(t)^\beta \, d\mu(t), \qquad \alpha, \beta \in \mathbf{Z}_+^n,$$

where $\theta_j(t) = (1 + t_j^2)^{-1}, 1 \leq j \leq n, t = (t_1, ..., t_n) \in \mathbf{R}^n$, and $\theta = (\theta_j)_{1 \leq j \leq n}$. Clearly, $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n}$ is a positive semi-definite $2n$-sequence, satisfying (1). Moreover, since

$$\int_{\mathbf{R}^n} (\theta_j(t)(1 + t_j^2) - 1) t^\alpha \theta(t)^\beta \, d\mu(t) = 0$$

for all $\alpha, \beta \in \mathbf{Z}_+^n, 1 \leq j \leq n$, we also have (2).

Conversely, assume that the $2n$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n}$ exists. Let θ_j be as above, and let $\mathcal{R}_\theta$ be the algebra generated by $\mathcal{P}_n$, and $\theta = (\theta_j)_{1 \leq j \leq n}$. We shall define a positive semi-definite map Λ on $\mathcal{R}_\theta$, via the equality

$$\Lambda(r) = L_\delta(p), \qquad r \in \mathcal{R}_\theta,$$

where L_δ is the linear map associated with δ, and $p \in \mathcal{P}_{2n}$ satisfies $r(t) = p(t, \theta(t))$, $t \in \mathbf{R}^n$.

Notice first that Λ is correctly defined. Indeed, by virtue of Lemma 2.2, the algebra $\mathcal{R}_\theta$ is isomorphic to the quotient $\mathcal{P}_{2n}/\mathcal{I}_\sigma$, where $\mathcal{I}_\sigma$ is the ideal generated in $\mathcal{P}_{2n}$ by the polynomials $\sigma_j(t, s) = s_j(1 + t_j^2) - 1, 1 \leq j \leq n$. Note that condition (2) implies $L_\delta \mid \mathcal{I}_\sigma = 0$. Therefore, the map Λ, which can be identified with the map induced by L_δ on the quotient $\mathcal{P}_{2n}/\mathcal{I}_\sigma$, is correctly defined, and positive semi-definite as well, on $\mathcal{R}_\theta$. This allows us to define a sesquilinear form on $\mathcal{R}_\theta$ by the equation

$$\langle r_1, r_2\rangle_\theta = \Lambda(r_1\bar{r}_2), \qquad r_1, r_2 \in \mathcal{R}_\theta. \tag{2.3}$$

Let $\mathcal{N} = \{r \in \mathcal{R}_\theta; \Lambda(r\bar{r}) = 0\}$. Then (2.3) induces a scalar product $\langle *, *\rangle$ on the quotient $\mathcal{R}_\theta/\mathcal{N}$ (corresponding to (1.1)), and let $\mathcal{H}$ be the completion of the quotient $\mathcal{R}_\theta/\mathcal{N}$ with respect to this scalar product.

We define in $\mathcal{H}$ the operators

$$T_j(r + \mathcal{N}) = t_j r + \mathcal{N}, \qquad r \in \mathcal{R}_\theta, j = 1, ..., n, \tag{2.4}$$

which are symmetric and densely defined, with $D(T_j) = \mathcal{R}_\theta/\mathcal{N}$ for all j. We shall show that T_j satisfies the conditions of Lemma 2.1 for each j. Indeed, if $r \in \mathcal{R}_\theta$ is arbitrary, then the functions $u_\pm(t) = (t_j \mp i)\theta_j(t)r(t)$ are solutions in $\mathcal{R}_\theta$ of the equations $(t_j \pm i)u_\pm(t) = r(t)$. This implies the equalities $R(T_j \pm i) = D(T_j)$, and therefore Lemma 1.3 applies to T_j. Hence T_j is essentially selfadjoint, and let A_j be the canonical closure of T_j. We shall show that the operators $(i - A_1)^{-1}, ...,$ $(i - A_n)^{-1}$ mutually commute. Indeed, the previous argument shows that the map $(i - T_j)^{-1}$ is well defined on $D = D(T_j)$, and leaves this space invariant, for all j. Moreover, the maps $(i - T_1)^{-1}, ..., (i - T_n)^{-1}$ mutually commute on D. Since A_j extends T_j, we clearly have

$$(i - A_j)((i - A_j)^{-1} - (i - T_j)^{-1})\xi = 0, \qquad \xi \in D,$$

implying $(i - A_j)^{-1}|D = (i - T_j)^{-1}$. Therefore, for all $j, k, j \neq k$, we have

$$(i - A_j)^{-1}(i - A_k)^{-1}\xi = (i - T_j)^{-1}(i - T_k)^{-1}\xi$$

$$= (i - T_k)^{-1}(i - T_j)^{-1}\xi = (i - A_k)^{-1}(i - A_j)^{-1}\xi$$

where $\xi \in D$ is arbitrary. Since $(i - A_1)^{-1}, ..., (i - A_n)^{-1}$ are bounded and D is dense, this implies that they mutually commute. In particular, the selfadjoint operators $A_1, ..., A_n$ have a joint spectral measure (see, for instance, [Vas1]). If E is the joint spectral measure of $A_1, ..., A_n$, then $\mu(*) = \langle E(*)(1 + \mathcal{N}), 1 + \mathcal{N}\rangle$ is a representing measure for the given sequence γ. We shall prove that we have, in fact, the equality

$$\delta_{\alpha,\beta} = \int_{\mathbf{R}^n} t^\alpha \theta(t)^\beta d\langle E(t)(1 + \mathcal{N}), 1 + \mathcal{N}\rangle \tag{2.5}$$

for all multi-indices α, β. Indeed, if $r(T)$ is the linear map on D given by $r(T)(f + \mathcal{N}) = rf + \mathcal{N}$, for all $r, f \in \mathcal{R}_\theta$, then we have $\theta(A)^\beta \supset \theta(T)^\beta$, where $\theta(A)^\beta$ is given by the functional calculus of A. This follows (as above for T_j) from the obvious relations $\theta(A)^{-\beta} \supset \theta(T)^{-\beta}$, and $\theta(A)^{-\beta}(\theta(A)^\beta - \theta(T)^\beta) = 0$. Therefore:

$$\delta_{\alpha,\beta} = \langle t^\alpha \theta(t)^\beta 1, 1\rangle_\theta = \langle T^\alpha \theta(T)^\beta(1 + \mathcal{N}), 1 + \mathcal{N}\rangle = \langle A^\alpha \theta(A)^\beta(1 + \mathcal{N}), 1 + \mathcal{N}\rangle$$

$$= \int_{\mathbf{R}^n} t^\alpha \theta(t)^\beta d\langle E(t)(1 + \mathcal{N}), 1 + \mathcal{N}\rangle.$$

We have only to discuss the uniqueness of the representing measure of γ.

Assume first that the sequence δ is uniquely determined, and let ν be an arbitrary representing measure of γ. Then the space $\mathcal{H}$ can be identified with a subspace of $L^2(\nu)$. Indeed, as we must have $\delta_{\alpha,\beta} = \int t^\alpha \theta(t)^\beta d\nu(t)$ for all indices α, β, by the uniqueness of δ, it follows $\langle r_1, r_2\rangle_\theta = \int r_1\bar{r}_2\, d\nu$ for all $r_1, r_2 \in \mathcal{R}_\theta$. Therefore, as the functions from $\mathcal{N}$ are null ν-almost everywhere, the space $\mathcal{H}$ is identified with the closure of $\mathcal{R}_\theta$ in $L^2(\nu)$.

We proceed now as in [Fug, Theorem 7]. The operators $(H_j f)(t) = t_j f(t), t = (t_1, ..., t_n) \in \mathbf{R}^n, f \in D(H_j) = \{g \in L^2(\nu); t_j g \in L^2(\nu)\}, j = 1, ..., n$, are commuting selfadjoint in $L^2(\nu)$. Clearly, $H_j \supset T_j$, and so $H_j \supset A_j$ for all j. Therefore, since $(A_j + iu)^{-1} = (H_j + iu)^{-1}|\mathcal{H}$ for all $u \in \mathbf{R}$ (see a similar argument above), it follows that the spectral measure E_j of H_j leaves invariant the space $\mathcal{H}$, as a consequence of [DuSc, Theorem XII.2.10], for all j. If E_H is the joint spectral measure of $H = (H_1, ..., H_n)$, then we have $E_H(B_1 \times \cdots \times B_n) = E_1(B_1) \cdots E_n(B_n)$ for all Borel sets $B_1, ..., B_n$ in $\mathbf{R}$. This implies that the space $\mathcal{H}$ is invariant under E_H. Hence, $\chi_B = E_H(B)1 \in \mathcal{H}$ for all Borel subsets B of $\mathbf{R}^n$, where χ_B is the characteristic function of B. This shows that $L^2(\nu) = \mathcal{H}$, since the simple functions form a dense subspace of $L^2(\nu)$. In particular, we have the equalities $H_j = A_j, j = 1, ..., n$. Therefore, with μ and E as above, $\mu(B) = \langle E(B)1, 1\rangle = \langle E_H(B)1, 1\rangle = \int \chi_B d\nu$, for all Borel sets B. Consequently, $\mu = \nu$, showing that the representing measure is unique.

Conversely, if the representing measure μ of γ is unique, and if the sequences δ', δ'' satisfy (1),(2), then we have $\delta'_{\alpha,\beta} = \int t^\alpha \theta(t)^\beta d\mu(t) = \delta''_{\alpha,\beta}$ for all indices α, β, via (2.5), which completes the proof of the theorem.

Remark 2.4. Let $\mathcal{R}_\theta$ be as in the previous proof, and let $\Lambda : \mathcal{R}_\theta \to \mathbf{C}$ be an arbitrary positive semi-definite map. Then Λ has a uniquely determined representing measure, say μ, and the space $\mathcal{R}_\theta$ is dense in $L^2(\mu)$. The proof of this assertion is practically contained in the previous proof.

Corollary 2.5. *A sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+} (\gamma_0 > 0)$ is a moment sequence, and has a uniquely determined representing measure, if and only if there exists a uniquely determined positive semi-definite 2-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^2}$, with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbf{Z}_+$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2,\beta+1)}$ *for all* $\alpha, \beta \in \mathbf{Z}_+$.

The difficulties to find a reasonable characterization of the Stieltjes n-sequences are emphasized in [BeCh]. Nevertheless, the next result is a solution to the Stieltjes moment problem in several variables, vie extended sequences.

Theorem 2.6. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n} (\gamma_0 > 0)$ is a moment sequence, and has a representing measure in $\mathbf{R}_+^n$, if and only if there exists a positive semi-definite $2n$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbf{Z}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_j)} + \delta_{(\alpha+2e_j,\beta+e_j)}$ *for all* $\alpha, \beta \in \mathbf{Z}_+^n, 1 \le j \le n$.
(3) $(\delta_{(\alpha+e_j,\beta)})_{(\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n}$ *is a positive semi-definite $2n$-sequence for all* $j = 1, ..., n$.

In this case, the n-sequence γ has a uniquely determined representing measure in $\mathbf{R}_+^n$ if and only if the $2n$-sequence δ is unique.

Proof. Assuming that the n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ has a representing measure μ in $\mathbf{R}_+^n$, we define the $2n$-sequence $(\delta_{(\alpha,\beta)})_{\alpha,\beta \in \mathbf{Z}_+^n}$ as in the proof of Theorem 2.3. Then δ is a positive semi-definite $2n$-sequence, satisfying (1) and (2). Moreover, since

$$\int_{\mathbf{R}_+^n} t_j \mid p(t,\theta(t))|^2 \, d\mu(t) \geq 0$$

for all $p \in \mathcal{P}_{2n}, j = 1, ..., n$, we also have (3).

Conversely, conditions (1) and (2) insure the existence of selfadjoint extensions $A_1, ..., A_n$ of the operators $T_1, ..., T_n$ (given by (2.4)), respectively. Since A_j is the canonical closure of T_j, and T_j is positive by (3), then A_j is positive for all $j = 1, ..., n$. Therefore, the joint spectral measure of $A_1, ..., A_n$ is concentrated on $\mathbf{R}_+^n$.

The assertion concerning the uniqueness of the representing measure follows from Theorem 2.3. □

Remark 2.7. Let $\mathcal{R}_\theta$ be as in the proof of Theorem 2.3, and let $\Lambda : \mathcal{R}_\theta \to \mathbf{C}$ be an arbitrary positive semi-definite map. In addition, assume that $\Lambda(t_j|r|^2) \geq 0$ for all $r \in \mathcal{R}_\theta$ and $j = 1, ..., n$. Then Λ has a uniquely determined representing measure, say μ, whose support is concentrated in $\mathbf{R}_+^n$. Moreover, the space $\mathcal{R}_\theta$ is dense in $L^2(\mu)$. The proof of this assertion is obtained from the Remark 2.4 combined with some ideas from the proof of Theorem 2.6. We omit the details.

Corollary 2.8. *A sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+}(\gamma_0 > 0)$ is a moment sequence, and has a uniquely determined representing measure in $\mathbf{R}_+$, if and only if there exists a uniquely determined positive semi-definite 2-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbf{Z}_+}$, with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbf{Z}_+$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2,\beta+1)}$ *for all* $\alpha, \beta \in \mathbf{Z}_+$.
(3) *The sequence* $(\delta_{(\alpha+1,\beta)})_{\alpha,\beta\in\mathbf{Z}_+}$ *is positive semi-definite.*

Remark 2.9. We can state and prove operator versions of the previous results. See [Vas3] for details.

3. More about the uniqueness

The uniqueness of a representing measure of a moment n-sequence is characterized in Theorem 2.3 by the uniqueness of the associated $2n$-sequence. Using some of the previous assertions and techniques, we shall give in this section an operator theoretic characterization of the uniqueness of the representing measure, as well as some related results.

Definition 3.1. Let $S = (S_1, ..., S_n)$ be a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$. We say that S has a *smallest selfadjoint extension* if there exist a Hilbert space $\mathcal{K} \supset H$ and a tuple $A = (A_1, ..., A_n)$ consisting of commuting selfadjoint operators in $\mathcal{K}$ with the following properties:

(1) $A_j \supset S_j,\ j = 1, ..., n$;
(2) if $B = (B_1, ..., B_n)$ is a tuple consisting of commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset H$ such that $B_j \supset S_j,\ j = 1, ..., n$, then $\mathcal{L} \supset K$ and $B_j \supset A_j,\ j = 1, ..., n$.

Remarks 3.2. (i) In the previous definition, we write $\mathcal{K} \supset H$ when there exists a linear isometry from $\mathcal{H}$ into $\mathcal{K}$, which allows the identification of $\mathcal{H}$ with a closed subspace of $\mathcal{K}$. In particular, the smallest selfadjoint extension, when exists, is uniquely determined.

(ii) If $S = (S_1, ..., S_n)$ is a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$ such that the canonical closures $A_1, ..., A_n$ of $S_1, ..., S_n$ are commuting selfadjoint operators, then $A = (A_1, ..., A_n)$ is the smallest selfadjoint extension of S.

If $n = 1$, and the deficiency indices are equal, this condition is also necessary. Indeed, if $S = S_1$ is a closed symmetric operator whose deficiency indices are equal, then $D(S)$ equals the intersection of the domains of all selfadjoint extensions of S, as proved in the Appendix of [Dev]. Assuming that S has a smallest selfadjoint extension $A = A_1$, we infer readily that $S = A$.

For $n > 1$, the smallest selfadjoint extension, whose structure is not yet well understood, may have unexpected properties. For instance, it follows from Theorem 4.4 of [BeTh] (see also Theorem 3.4 below) that, for some tuples of symmetric operators, the smallest selfadjoint extension may exist in a Hilbert space strictly larger than the given one.

Theorem 3.3. *Let $S_1, ..., S_n$ be symmetric operators in a Hilbert space $\mathcal{H}$, such that $D = D(S_1) = \cdots = D(S_n)$, is invariant under $S_1, ..., S_n$. Let also $A_1, ..., A_n$ be commuting selfadjoint operators in a Hilbert space $\mathcal{K} \supset H$, with $A_j \supset S_j,\ j = 1, ..., n$. Let*

$$\mathcal{K}_0 = \{(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x; x \in D, m \in \mathbf{Z}_+\},$$

which is a linear subspace of $\mathcal{K}$ invariant under $A_1, ..., A_n$.

The tuple $A = (A_1, ..., A_n)$ is the smallest selfadjoint extension of the tuple $S = (S_1, ..., S_n)$ if and only if

(1) *the subspace $\mathcal{K}_0$ is dense in $\mathcal{K}$;*

(2) *if $B_1, ..., B_n$ are commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset H$, such that $B_j \supset S_j,\ j = 1, ..., n$, then*

$$\|(1 + B_1^2)^{-m} \cdots (1 + B_n^2)^{-m} x\| = \|(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x\|$$

for all $x \in D, m \in \mathbf{Z}_+$.

Proof. Since D is invariant under $S_1, ..., S_n$ and $A_j \supset S_j$ for all j, it is easily seen that $\mathcal{K}_0$ is a linear subspace of $\mathcal{K}$, invariant under $A_1, ..., A_n$.

Assume that A is the smallest selfadjoint extension of S. Let $\mathcal{G}$ be the closure of $\mathcal{K}_0$ in $\mathcal{K}$, and set $C_j = A_j|\mathcal{K}_0,\ j = 1, ..., n$. We shall show that the canonical closures of $C_1, ..., C_n$ are commuting selfadjoint operators in $\mathcal{G}$.

Put $r_m(A) = (1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m},\ m \in \mathbf{Z}_+$, and let $y = r_m(A)x,\ x \in D$, be fixed. For every index j we have

$$y = r_m(A)x = (C_j \pm i) r_{m+1}(A) \prod_{k \neq j} (1 + S_k^2)(S_j \mp i)x.$$

This shows that $R(C_j \pm i) = \mathcal{K}_0$. According to Lemma 2.1, the canonical closure $\bar{C}_j$ of the operator C_j is selfadjoint in $\mathcal{G}$. Clearly, $\bar{C}_j \subset A_j$, implying that $(i - \bar{C}_j)^{-1} \subset (i - A_j)^{-1}$ for all indices j. From commutation of $(i - A_j)^{-1}, (i - A_k)^{-1}$ we obtain the commutation of $(i - \bar{C}_j)^{-1}, (i - \bar{C}_k)^{-1}$ for all indices j, k.

The hypothesis on A implies that $\bar{C}_j = A_j$ for all j. Therefore, the canonical closure of $A_j|\mathcal{K}_0$ coincides with A_j for all j. In particular, the subspace $\mathcal{K}_0$ is dense in $\mathcal{K}$, which is condition (1).

If $B_1, ..., B_n$ are commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset H$ such that $B_j \supset S_j,\ j = 1, ..., n$, then, by the hypothesis on A, one must have $\mathcal{L} \supset K$ and

$B_j \supset A_j$ for all j. Hence $(1+B_1^2)^{-m}\cdots(1+B_n^2)^{-m}x = (1+A_1^2)^{-m}\cdots(1+A_n^2)^{-m}x$ for all $x \in D$ and $m \in \mathbf{Z}_+$, i.e., condition (2) also holds.

Conversely, suppose that conditions (1) and (2) are satified. We shall show that A is the smallest selfadjoint extension of S.

Let $B_1, ..., B_n$ be commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset H$ such that $B_j \supset S_j,\ j = 1, ..., n$. We may define a linear map from $\mathcal{K}_0$ into $\mathcal{L}$ via the formula

$$(*) \qquad \mathcal{K}_0 \ni (1+A_1^2)^{-m}\cdots(1+A_n^2)^{-m}x \to (1+B_1^2)^{-m}\cdots(1+B_n^2)^{-m}x \in \mathcal{L}.$$

Condition (2) shows that the map $(*)$ is well defined and isometric. Moreover, $\mathcal{K}_0$ is dense in $\mathcal{K}$ via condition (1). Therefore, the map $(*)$ extends to a linear isometry from $\mathcal{K}$ into $\mathcal{L}$, and we may identify $\mathcal{K}$ with a closed subspace of $\mathcal{L}$. Note that $A_j|\mathcal{K}_0 = B_j|\mathcal{K}_0$ for all j, via this identification. Let us show that the canonical closure of $A_j|\mathcal{K}_0$ is A_j. Indeed, assuming the existence of a pair $u \oplus A_j u$ in the graph of A_j orthogonal to all pairs $r_m(A)x \oplus A_j r_m(A)x$ with $x \in D$ and $m \geq 0$ arbitrary (see the notation above), we infer that $\langle u, (1+A_j^2)r_m(A)x\rangle = 0$. This implies $u = 0$ because of the equality $(1+A_j^2)\mathcal{K}_0 = \mathcal{K}_0$. Therefore, $B_j \supset A_j,\ j = 1, ..., n$, showing that A is the smallest selfadjoint extension of S.

Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ be a positive semi-definite n-sequence and let $L_\gamma : \mathcal{P}_n \to \mathbf{C}$ be the associated linear map, given by $L_\gamma(t^\alpha) = \gamma_\alpha,\ \alpha \in \mathbf{Z}_+^n$. Let also $\mathcal{N} = \{p \in \mathcal{P}_n; L_\gamma(p\bar{p}) = 0\}$. Then L_γ induces a scalar product $\langle *, * \rangle$ on the quotient $\mathcal{P}_n/\mathcal{N}$, and let $\mathcal{H} = \mathcal{H}_\gamma$ be the completion of the quotient $\mathcal{P}_n/\mathcal{N}$ with respect to this scalar product.

As in the proof of Theorem 2.3, we define in $\mathcal{H}$ the operators

$$(3.1) \qquad T_j(p+\mathcal{N}) = t_j p + \mathcal{N}, \qquad p \in \mathcal{P}_n, j = 1, ..., n,$$

which are symmetric and densely defined, with $D(T_j) = \mathcal{P}_n/\mathcal{N}$ for all j.

Theorem 3.4. *Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ be a moment n-sequence. The representing measure of γ is unique if and only if the tuple $T = (T_1, ..., T_n)$ has a smallest selfadjoint extension.*

Proof. Suppose that $T = (T_1, ..., T_n)$ has a smallest selfadjoint extension $A = (A_1, ..., A_n)$, acting in a Hilbert space $\mathcal{K} \supset H = H_\gamma$. If E_A is the spectral measure of A, then $\mu(*) = \langle E_A(*)(1+\mathcal{N}), 1+\mathcal{N}\rangle$ is a representing measure for γ (see the proof of Theorem 2.3).

Let ν be another representing measure for γ. Let $B_j f(t) = t_j f(t), t \in \mathbf{R}^n, f \in D(B_j) = \{g \in L^2(\nu); t_j g \in L^2(\nu)\}, j = 1, ..., n$. Since $\int |p|^2 d\mu = \int |p|^2 d\nu$ for all polynomials $p \in \mathcal{P}_n$, the space $\mathcal{H}$ may be regarded as a closed subspace of $L^2(\nu)$, and $B_j \supset T_j$ for all j. Moreover, $B_1, ..., B_n$ are commuting selfadjoint operators. The hypothesis implies that $L^2(\nu) \supset \mathcal{K}$ and $B_j \supset A_j,\ j = 1, ..., n$. Therefore, if E_B is the spectral measure of $B = (B_1, ..., B_n)$, then $E_A = E_B|\mathcal{K}$, and

$$\nu(*) = \langle E_B(*)1, 1\rangle = \langle E_A(*)(1+\mathcal{N}), 1+\mathcal{N}\rangle = \mu(*).$$

Conversely, suppose that γ has a unique representing measure μ. Then $\mathcal{P}_n \subset L^2(\mu)$, and let $A_j f(t) = t_j f(t), t \in \mathbf{R}^n, f \in D(A_j) = \{g \in L^2(\mu); t_j g \in L^2(\mu)\}, j = 1, ..., n$, and $A = (A_1, ..., A_n)$. We shall use Theorem 3.3 to prove that A is the smallest selfadjoint extension of T.

First of all, note that the space $\mathcal{K}_0$ from Theorem 3.3 is equal in this case to the space $\mathcal{R}_\theta$ (defined in Lemma 2.2). Since $\mathcal{R}_\theta$ is dense in $L^2(\mu)$ (via Remark 2.4), condition (1) from Theorem 3.3 is fulfilled.

Next, let $B = (B_1, ..., B_n)$ be a tuple consisting of commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset H$, such that $B_j \supset T_j,\ j = 1, ..., n$. If E_B is the spectral measure of B, then $\nu(*) = \langle E_B(*)1, 1\rangle$ is a representing measure for γ, and we must have $\nu = \mu$.

Let $r \in \mathcal{R}_\theta$ be arbitrary. We have

$$\|r(A)1\|^2 = \int |r(t)|^2 d\mu(t) = \int |r(t)|^2 d\langle E_B(t)1, 1\rangle = \|r(B)1\|^2.$$

Particularly, if $r_m(t) = (1+t_1^2)^{-m} \cdots (1+t_n^2)^{-m}$ and $p \in \mathcal{P}_n$, we obtain the equalities

$$\|r_m(A)p\| = \|r_m(A)p(A)1\| = \|r_m(B)p(B)1\| = \|r_m(B)p\|,$$

showing that condition (2) from Theorem 3.3 is also fulfilled. By virtue of this theorem, the tuple A is the smallest selfadjoint extension of T.

Corollary 3.5. *A positive semi-definite sequence $\gamma = (\gamma_k)_{k \in \mathbf{Z}_+}$ has a uniquely determined representing measure, say μ, if and only if the operator T given by* (3.1) *is essentially selfadjoint. In this case, the space of polynomial functions is dense in $L^2(\mu)$.*

Proof. The fact that T is essentially selfadjoint if and only if μ is unique is well-known (see, for instance, [Dev]). It can be obtained from Theorem 3.4, via Remark 3.2(ii) and the fact that the operator T commutes with the natural involution on $\mathcal{H}$, and so its deficiency indices are equal (see [DuSc, Theorem XII.4.18]). As for the last assertion, we identify the space $\mathcal{H}_\gamma$ with a closed subspace of $L^2(\mu)$. Since the canonical closure of T, say A, is selfadjoint in $\mathcal{H}_\gamma$, we obtain that $(1+A^2)^{-m}p \in \mathcal{H}_\gamma$ for all $p \in \mathcal{P}_1$ and all integers $m \geq 0$. But $(1 + A^2)^{-m}p = (1 + t^2)^{-m}p$, implying that $\mathcal{R}_\theta$ is in $\mathcal{H}_\gamma$. The density of $\mathcal{R}_\theta$ in $L^2(\mu)$ concludes the proof (see also [Fug]).

Remarks 3.6. (a) Let $\mathcal{S} = (S_1, S_2)$ be a pair of symmetric operators in $\mathcal{H}$. We define the matrix

$$Q(\mathcal{S}) = \begin{pmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{pmatrix}$$

on the subspace

$$D(Q(\mathcal{S})) = [D(S_1) \cap D(S_2)] \oplus [D(S_1) \cap D(S_2)] \subset \mathcal{H}^2.$$

Put also $Q(\mathcal{S})^\# = Q(\mathcal{S}^\#)$, where $\mathcal{S}^\# = (S_1, -S_2)$. If D is a subspace of $D(S_1) \cap D(S_2)$ and if S_1, S_2 commute on D, it is easily seen that $\|Q(\mathcal{S})x\|_2 = \|Q(\mathcal{S})^\# x\|_2,\ x \in D \oplus D$, where $\| * \|_2$ is the natural norm on $\mathcal{H}^2$.

We set

$$Q' = \begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix},$$

where $\sigma_0 = i\sqrt{2}/2$.

In the proof of the next result we shall use the following assertion:

Let $S_1, ..., S_n$ be symmetric operators in the Hilbert space $\mathcal{H}$. Suppose that there exists a linear subspace $D \subset D(S_1) \cap \cdots \cap D(S_n)$, D is dense in $\mathcal{H}$, with the following properties:

(1) $\|Q(\mathcal{S}_{jk})x\|_2 = \|Q(\mathcal{S}_{jk})^{\#}x\|_2, x \in D \oplus D$, *for all indices* j, k *with* $1 \leq j < k \leq n$, *where* $\mathcal{S}_{jk} = (S_j, S_k)$.

(2) *The set* $\{(Q(\mathcal{S}_{jk}) + Q')x; x \in D \oplus D\}$ *is dense in* $\mathcal{H}^2$ *for all indices* j, k *with* $1 \leq j < k \leq n$.

Then $S_1, ..., S_n$ *have commuting selfadjoint extensions.*

The proof of this result, as well as other details, can be found in [Vas2] (see especially Corollary 3.9 from [Vas2]).

(b) We recall that a given positive semi-definite n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ is said to be *strongly determined* [Fug], if the operators given by (3.1) are essentially selfadjoint and commute. In that case the sequence γ has a uniquely determined representing measure, by Theorem 3.4 and Remark 3.2(ii). Moreover, if μ is the unique representing measure of γ, then, as noticed in [Fug], the space of polynomial functions is dense in $L^2(\mu)$ (the argument for the corresponding assertion from Corollary 3.5 can be easily adapted to this case).

The next result provides an explicit characterization of the strongly determined n-sequences.

Theorem 3.7. *Let* $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ *be a positive semi-definite* n*-sequence. We define on* $\mathbf{R}^n$ *the polynomials*

$$P'_{\alpha,j,k}(t; a, b) = |t^\alpha + \sum_\xi a_\xi (t_j + \sigma_0) t^\xi + \sum_\eta b_\eta (t_k + \sigma_0) t^\eta|^2,$$

$$P''_{j,k}(t; a, b) = |\sum_\xi a_\xi (-t_k + \sigma_0) t^\xi + \sum_\eta b_\eta (t_j - \sigma_0) t^\eta|^2,$$

$$Q'_{\beta,j,k}(t; a, b) = |t^\beta + \sum_\xi a_\xi (-t_k + \sigma_0) t^\xi + \sum_\eta b_\eta (t_j - \sigma_0) t^\eta|^2,$$

$$Q''_{j,k}(t; a, b) = |\sum_\xi a_\xi (t_j + \sigma_0) t^\xi + \sum_\eta b_\eta (t_k + \sigma_0) t^\eta|^2,$$

for all finite sequences of complex numbers $a = (a_\xi)_{\xi \geq 0}$, $b = (b_\eta)_{\eta \geq 0}$, *where* $\alpha, \beta \in \mathbf{Z}_+^n$ *and* $1 \leq j < k \leq n$.

The n*-sequence* γ *is strongly determined if and only if*

$$\inf_{a,b} L_\gamma(P'_{\alpha,j,k}(t; a, b) + P''_{j,k}(t; a, b)) = 0, \tag{3.2'}$$

$$\inf_{a,b} L_\gamma(Q'_{\beta,j,k}(t; a, b) + Q''_{j,k}(t; a, b)) = 0, \tag{3.2''}$$

for all $\alpha, \beta \in \mathbf{Z}_+^n$ *and* $1 \leq j < k \leq n$.

Proof. If γ is strongly determined, then the canonical closures $A_1, ..., A_n$ of the operators $T_1, ..., T_n$ (respectively), given by (3.1), are commuting selfadjoint operators. Then the representing measure μ of γ is uniquely determined, via Theorem 3.4, and we may identify the Hilbert space $\mathcal{H}$ from (3.1) with $L^2(\mu)$. If $\mathcal{T}_{j,k} = (T_j, T_k)$, then the canonical closure of the operator $Q(\mathcal{T}_{j,k})$ (defined on $\mathcal{P}_n \oplus \mathcal{P}_n$) is the operator $Q(\mathcal{A}_{j,k})$ for all indices j, k with $1 \leq j < k \leq n$, where $\mathcal{A}_{j,k} = (A_j, A_k)$ (see Theorem 3.8 from [Vas2]). In addition, the operator $Q(\mathcal{A}_{jk}) + Q'$ is surjective (in fact, it has a bounded inverse) by Theorem 3.14 from [Vas2]. This implies that the set $R(Q(\mathcal{T}_{jk}) + Q')$ is dense in $L^2(\mu)^2$. In particular, the elements $t^\alpha \oplus 0$ and $0 \oplus t^\beta$ can be approximated with elements from $R(Q(\mathcal{T}_{jk})+Q')$, which is equivalent to $(3.2')$ and $(3,2'')$.

Conversely, conditions $(3.2')$, $(3.2'')$ imply that the closure of the set $\{(Q(\mathcal{T}_{jk})+Q')x; x \in D\oplus D\}$, where $\mathcal{T}_{jk}=(T_j,T_k)$ and $D=\mathcal{P}_n/\mathcal{N}$ (the notation is again related to (3.1)), contains the set $D\oplus D$ (which is generated by the elements of the form $(t^\alpha+\mathcal{N})\oplus 0$ and $0\oplus(t^\beta+\mathcal{N})$). Therefore, the set $\{(Q(\mathcal{T}_{jk})+Q')x; x\in D\oplus D\}$ is dense in $\mathcal{H}^2$ for all indices j,k with $1\le j<k\le n$. In other words, condition (2) in Remark 3.6(a) is fulfilled. As condition (1) is also fulfilled (see Remark 3.6(a)), it follows that the canonical closures of $T_1,...,T_n$ are commuting selfadjoint operators, by Corollary 3.9 from [Vas2].

Remark 3.8. Conditions $(3.2')$ and $(3.2'')$ depend only on the given sequence γ. In fact, choosing an algebraic basis $(t^\alpha+\mathcal{N})_{\alpha\in A}$ in $D=\mathcal{P}_n/\mathcal{N}$, for each pair of indices j,k $(1\le j<k\le n)$ the familiy $\{(Q(\mathcal{T}_{jk})+Q')((t^\xi+\mathcal{N})\oplus 0),(Q(\mathcal{T}_{jk})+Q')(0\oplus(t^\eta+\mathcal{N})),\ \xi,\eta\in A\}$ is an algebraic basis of $R(Q(\mathcal{T}_{jk})+Q')$ (due to the fact that $Q(\mathcal{T}_{jk})+Q'$ is injective [Vas2]). Ortogonalizing this family via the Gram procedure, we can express the distance from an element of the form $(t^\alpha+\mathcal{N})\oplus 0$ or of the form $0\oplus(t^\beta+\mathcal{N})$ to the space $R((Q(\mathcal{T}_{jk})+Q')$ only in terms of γ, which is equivalent to conditions $(3.2')$ and $(3.2'')$.

Theorem 3.9. *Let $\gamma=(\gamma_k)_{k\in\mathbf{Z}_+}$ be a positive semi-definite sequence. We define the function*

$$P(\gamma;a,b)=\gamma_0+2\sum_m(a_m\gamma_{m+1}-b_m\gamma_m)+\sum_{m,\ell}(a_ma_\ell+b_mb_\ell)(\gamma_{m+\ell}+\gamma_{m+\ell+2})$$

for all finite sequences of real numbers $a=(a_m)_{m\ge0}$, $b=(b_m)_{m\ge0}$.

The sequence γ has a uniquely determined representing measure if and only if

$$\inf_{a,b}P(\gamma;a,b)=0. \tag{3.3}$$

Proof. Note that

$$P(\gamma;a,b)=L_\gamma(|1+\sum_m(t+i)c_mt^m|^2),$$

where $c_m=a_m+ib_m$ for all $m\ge0$. Therefore, if (3.3) is fulfilled, then the vector $y_0=1+\mathcal{N}$ must be in the closure of the range of the operator $T+i$ (with $T=T_1$ given by (3.1) for $n=1$). If $(x_k)_{k\ge1}$ is a sequence from $\mathcal{P}_1/\mathcal{N}$ such that $y_0=\lim_k(T+i)x_k$, because the operator $(T+i)^{-1}:R(T+i)\to\mathcal{H}=H_\gamma$ is well defined and contractive, it follows that $x_k\to x_0\,(k\to\infty)$, with $x_0\in D(A)$, and $Tx_k\to Ax_0\,(k\to\infty)$, where A is the canonical closure of T. Therefore, $y_0=Ax_0+ix_0$. This equation implies, by recurrence, that $T^ky_0=(A+i)A^kx_0$ for all integers $k\ge1$. In other words, the range of $A+i$ contains the space $\mathcal{P}_1/\mathcal{N}$, which is dense in $\mathcal{H}$. Since the deficiency indices of T are equal (because T commutes with the natural involution of $\mathcal{H}$), it follows that A must be a selfadjoint operator, which implies the uniqueness of the representing measure of γ, by Corollary 3.5.

Conversely, if the representing measure of γ is unique, then T is essentially selfadjoint, and so the space $(T+i)(\mathcal{P}_1/\mathcal{N})$ must be dense in $\mathcal{H}$, implying, in particular, condition (3.3).

Remark 3.10. H. L. Hamburger showed that the uniqueness of the representing measure of a positive semi-definite sequence can be characterized by a condition similar to (3.3). Hamburger's condition is a consequence of an interesting and deep phenomenon, but the proof is rather intricate (see the Appendix of [Dev] for an operator theoretic approach). Our condition (3.3), whose proof is much simpler, is explicit enough to be used in applications. Indeed, with the notation of Theorem 3.9, and by identifying the equivalence classes with their representatives in the space $\mathcal{H} = H_\gamma$, we may assume that the family of monomials $(t^k)_{k\geq 0}$ is linearly independent in $\mathcal{H}$ (otherwise the space $\mathcal{H}$ is finite dimensional, the operator $T = T_1$ selfadjoint on $\mathcal{H}$, $T + i$ bijective, and so $1 = (T + i)p$ for a certain polynomial p). If we define $q_j = (T + i)t^{j-1},\ j \geq 1$, then it is clear that the family $(q_j)_{j\geq 1}$ is also linearly independent in $\mathcal{H}$. Using the Gram procedure, we may replace condition (3.3) by the equivalent condition

$$\lim_{m\to\infty} \frac{G(q_1, ..., q_m, 1)}{G(q_1, ..., q_m)} = 0,$$

where, with the notation $g_{j,k} = L_\gamma(q_k\bar{q}_j) = \gamma_{j+k} + \gamma_{j+k-2}, 1 \leq j, k \leq m, g_{j,m+1} = L_\gamma(\bar{q}_j) = \gamma_j - i\gamma_{j-1}, 1 \leq j \leq m, g_{m+1,j} = L_\gamma(q_j) = \gamma_j + i\gamma_{j-1}, 1 \leq j \leq m, g_{m+1,m+1} = L_\gamma(1) = \gamma_0$, $G(q_1, ..., q_m)$ is the determinant of the matrix $(g_{j,k})_{j,k=1}^m$, and $G(q_1, ..., q_m, 1)$ is the determinant of the matrix $(g_{j,k})_{j,k=1}^{m+1}$.

Corollary 3.11. *Let μ be a probability measure on the real line such that $\mathcal{P}_1 \subset L^2(\mu)$, and let $\mathcal{P}_{1,r}$ be the (real) subspace of $\mathcal{P}_1$ consisting of all polynomials with real coeficients. If*

$$\inf_{p_1,p_2\in\mathcal{P}_{1,r}} \int ((1 + (1 + t^2)(p_1(t)^2 + p_2(t)^2) + 2(tp_1(t) - p_2(t)))d\mu(t) = 0,$$

then $\mathcal{P}_1$ is dense in $L^2(\mu)$.

Proof. The assertion is a direct consequence of the previous theorem, via Corollary 3.5. □

References

[Akh] N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, Oliver and Boyd, Edinburgh and London, 1965. MR **32:**1518

[BCR] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic analysis on semigroups*, Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1984. MR **86b:**43001

[BeCh] C. Berg and J. P. R. Christensen, *Suites complètement définies positives, majoration de Schur et le problème des moments de Stieltjes en dimension k*, C. R. Acad. Sc. Paris, Série I, **297** (1983), 45-48. MR **84k:**44013

[BeTh] C. Berg and M. Thill, *Rotation invariant moment problem*, Acta Math. **167** (1991), 207-227. MR **92j:**44004

[CuFi1] R. E. Curto and L. A. Fialkow, *Solution of the truncated complex moment problem for flat data*, Memoirs of the A. M. S. Number 568, 1996. MR **96j:**47009

[CuFi2] R. E. Curto and L. A. Fialkow, *Flat extensions of positive moment matrices: Recursively generated relations*, Memoirs of the A. M. S. Number 648, 1998. MR **99d:**47015

[DuSc] N. Dunford and J.T. Schwartz, *Linear Operators, Part II*, Interscience Publishers, New York/London/Sydney, 1963. MR **90g:**47001b

[Dev] A. Devinatz, *Two parameter moment problems*, Duke Math. J. **24** (1957), 481-498. MR **19:**1047g

[Esk] G.I. Eskin, *A sufficient condition for the solvability of the moment problem in several dimensions*, Transl. from Dokl. Akad. Nauk SSSR, **133** (1960), 540-543. MR **22:**12394

[Fug] B. Fuglede, *The multidimensional moment problem*, Expo. Math. **1** (1983), 47-65. MR **85g:**44010

[GeNa] I.M. Gelfand and M. A. Naimark, *On the inclusion of a normed ring into the ring of operators in a Hilbert space (Russian)*, Matem. Sb. **12** (1943), 197-213. MR **5:**147

[Hav] E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension. II*, Amer. J. Math., **58** (1936), 164-168.

[KoMi] A. G. Kostyuchenko and B. S. Mityagin, *Multidimensional moment problem*, Transl. from Dokl. Akad. Nauk **131** (1960), 1249-1252. MR **22:**12392

[PuVa] M. Putinar and F.-H. Vasilescu, *Solving moment problems by dimensional extension*, Annals of Math. **149** (1999), 1087-1107. MR **2001c:**47023b

[Rud] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973. MR **51:**1315

[Sch] K. Schmüdgen, *Unbounded operator algebras and representation theory*, Birkhäuser Verlag, Basel/Boston/Berlin, 1990. MR **91f:**47062

[ShTa] J. Shohat and J. Tamarkin, *The Problem of Moments*, Math. Surveys I, Amer. Math. Soc., Providence, RI, 1943. MR **5:**5c

[StSz] J. Stochel and F.H. Szafraniec, *The complex moment problem and subnormality: A polar decomposition approach*, J. Functional Analysis, **159** (1998), 432-491. MR **2001c:**47023a

[Van] B. L. van der Waerden, *Algebra*, Vol. **2**, Frederick Ungar Publ. Co., New York, 1970. MR **41:**8187b

[Vas1] F.-H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, Editura Academiei, Bucharest, and D. Reidel Publ. Co., Dordrecht, 1982. MR **85b:**47016

[Vas2] F.-H. Vasilescu, *Quaternionic Cayley transform*, J. Functional Anal. **164** (1999), 134-162. MR **2000d:**47015

[Vas3] F.-H. Vasilescu, *Operator moment problems in unbounded sets*, Proceedings of the Memorial Conference for Bélà Szökefalvi-Nagy (in print).

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