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BLOCK REPRESENTATION TYPE OF REDUCED ENVELOPING ALGEBRAS

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Abstract. Let K be an algebraically closed field of characteristic p, G a connected, reductive K-group, $\mathfrak{g} = \mathrm{Lie}(G)$, $\chi \in \mathfrak{g}^*$ and $U_\chi(\mathfrak{g})$ the reduced enveloping algebra of $\mathfrak{g}$ associated with χ. Assume that $G^{(1)}$ is simply-connected, p is good for G and $\mathfrak{g}$ has a non-degenerate G-invariant bilinear form. All blocks of $U_\chi(\mathfrak{g})$ having finite and tame representation type are determined.

1. Introduction

Let G be a connected, reductive algebraic group over an algebraically closed field of characteristic p and $\mathfrak{g} = \mathrm{Lie}(G)$. The Lie algebra $\mathfrak{g}$ carries a natural restriction map $x \mapsto x^{[p]}$. We assume that the derived group $G^{(1)}$ of G is simply-connected, p is a good prime for the root system of G, and $\mathfrak{g}$ has a non-degenerate G-invariant bilinear form. Given a linear function $\chi \in \mathfrak{g}^*$, we denote by $U_\chi(\mathfrak{g})$ the reduced enveloping algebra of $\mathfrak{g}$ associated with χ. Let $\chi = \chi_s + \chi_n$ be the Jordan decomposition of χ. We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and, without loss of generality, assume that χ vanishes on $\mathfrak{n}_+$ and χ_s vanishes on $\mathfrak{n}_\pm$. Then the blocks $\mathcal{B}_{\chi,\lambda}$ of $U_\chi(\mathfrak{g})$ are parametrised by the set $W\Lambda_{\chi_s}/W$, where $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ is the Weyl group of $\mathfrak{g}$ and Λ_{χ_s} is the set of all $\lambda \in \mathfrak{h}^*$ satisfying $\lambda(h)^p - \lambda(h^{[p]}) = \chi_s(h)^p$ for all $h \in \mathfrak{h}$, [3]. Given $\lambda \in \Lambda_{\chi_s}$, we denote the stabiliser of λ by $W(\lambda)$.

In this paper, we show that the rank variety of $\mathcal{B}_{\chi,\lambda}$ coincides with the intersection of the rank variety of $\mathcal{B}_{\chi_s,\lambda}$ with the p-nilpotent cone of the coadjoint stabiliser $\mathfrak{z}_\mathfrak{g}(\chi)$, see Theorem 4.2. This result implies that the rank variety of $U_\chi(\mathfrak{g})$ is equal to the p-nilpotent cone of $\mathfrak{z}_\mathfrak{g}(\chi)$, hence generalises in our class of Lie algebras the main theorem of [38]. In proving Theorem 4.2 we use a deformation argument and some tools from [38]. In particular, we employ the Mil'ner map $\beta : U(\mathfrak{g}) \to S(\mathfrak{g})$ and its analogue $\beta_\chi : U_\chi(\mathfrak{g}) \to S_\chi(\mathfrak{g})$ constructed in [38].

According to [9], [11] the dimension of the rank variety of a finite dimensional $\mathcal{B}_{\chi,\lambda}$-module M equals the rate of growth of a minimal projective resolution of M. So Theorem 4.2 enables us to compute the maximum value of the rate of growth of minimal projective resolutions in the category of finite dimensional $\mathcal{B}_{\chi,\lambda}$-modules, hence provides valuable information on the homological complexity of the blocks of $U_\chi(\mathfrak{g})$, see our analysis in Section 8. Combined with the results of Section 3 on block degeneration and our determination of partial coinvariant algebras of tame

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representation type in Section 7, this information turns out to be *almost* sufficient for detecting all blocks of $U_\chi(\mathfrak{g})$ having finite and tame representation type. For G almost simple and χ nilpotent the only case where we have to look closely at the basic algebra of $\mathcal{B}_{\chi,\lambda}$, apart from the relatively easy regular case, is the case where χ is subregular, G has type A_n and the stabiliser of λ in W has type A_{n-1} as a Coxeter group. These subregular blocks, denoted $\mathcal{B}$, are studied in detail in Section 9. It is shown there that any such $\mathcal{B}$ is Morita equivalent to a quiver with relations of "special biserial type", hence tame. Notably, the quiver of any subregular tame block has the same shape as in the $\mathfrak{sl}_2$-case, and n, the rank of $G^{(1)}$, appears in the relations only.

The classification of all blocks $\mathcal{B}_{\chi,\lambda}$ of finite and tame representation type is given in Theorems 5.2 and 5.3, which should be combined with Proposition 2.7. The particular case when G is simple is straightforward to present. The general case is similar, but there is one further example of a block of tame representation type.

Theorem. *In addition to the underlying hypotheses, assume G is simple. Let $\chi \in \mathcal{N}$ and $\lambda \in \Lambda/W$.*
(i) The block $\mathcal{B}_{\chi,\lambda}$ has finite representation type if and only if one of the following occurs:

1. *$W = W(\lambda)$;*
2. *χ is regular and one of the following holds:*
   (a) *W is of type A_n and $W(\lambda)$ is of type A_{n-1};*
   (b) *W is of type B_n (or C_n) and $W(\lambda)$ is of type B_{n-1} (or C_{n-1});*
   (c) *W is of type G_2 and $W(\lambda)$ is of type A_1.*

(ii) The block $\mathcal{B}_{\chi,\lambda}$ has tame representation type if and only if one of the following occurs:

1. *χ is regular and one of the following holds:*
   (a) *W has rank 2;*
   (b) *W is of type A_3 and $W(\lambda)$ is of type $A_1 \times A_1$;*
   (c) *W is of type B_3 (or C_3) and $W(\lambda)$ is of type A_2;*
   (d) *W is of type D_n and $W(\lambda)$ is of type D_{n-1};*
2. *χ is subregular, W is of type A_n and $W(\lambda)$ is of type A_{n-1}.*

The classification of the representation type of reduced enveloping algebras and their blocks was begun, in the case $\chi = 0$, by Pollack, [33], and explicit calculations in the $\mathfrak{sl}_2$ case were made by Fischer, Rudakov and Drozd, proving tameness, [8], [42] and [5]. Since 0 is the subregular element of $\mathfrak{sl}_2$, this is consistent with the final part of the above theorem. For a general character, partial results on the representation type of reduced enveloping algebras were discovered by the second author, using rank varieties, [38]. In Theorem 5.4 we give a precise description of the representation type of reduced enveloping algebras, refining this. Later, blocks of finite representation type were classified for characters of standard Levi type by Nakano and Pollack, [32]. This was generalised to arbitrary characters by Brown and the first author, [3], giving Part (i) of the above theorem.

It is known that any indecomposable non-projective $U_0(\mathfrak{sl}_2)$-module is up to isomorphism either a Weyl module or a dual Weyl module or a maximal submodule of a Weyl module, see [34]. It would be interesting to obtain a purely Lie theoretic description of all indecomposable representations of the subregular tame blocks.

2. Generalities

2.1. Let G be a connected, reductive algebraic group over K, an algebraically closed field of characteristic p, and let $\mathfrak{g} = \mathrm{Lie}(G)$. We assume the following hypotheses are satisfied:

(A) the derived group $G^{(1)}$ of G is simply-connected;
(B) p is a good prime for G;
(C) $\mathfrak{g}$ has a non-degenerate G-invariant bilinear form.

We will denote the bilinear form on $\mathfrak{g}$ by

$$B(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow K.$$

Let T be a maximal torus of G and let $\mathfrak{h} = \mathrm{Lie}(T)$. Let Φ be the root system of G with respect to T. For each $\alpha \in \Phi$ let U_α denote the corresponding root subgroup of G and let $\mathfrak{g}_\alpha = \mathrm{Lie}(U_\alpha)$ be its Lie algebra, a root subspace of $\mathfrak{g}$. We will abuse notation by considering $\alpha \in \mathfrak{h}^*$ rather than its proper designation $d\alpha$. Choose a system Φ^+ of positive roots and set $\mathfrak{n}^+$ equal to the sum of all $\mathfrak{g}_\alpha$ with $\alpha > 0$. The subalgebra $\mathfrak{n}^-$ is similarly defined on Φ^-, the negative roots. We have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$, the Lie algebra of a Borel subgroup of G containing T. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots associated with the choice of positive roots Φ^+.

Let $X = X(T)$ be the character group of T. This contains the root lattice, $Q = \mathbb{Z}\Phi$, as a subgroup. For $\alpha \in \Phi^+$, there exists $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\lambda(h_\alpha) \equiv \langle \lambda, \alpha^\vee \rangle (p)$ for all $\lambda \in X(T)$. Let W be the Weyl group of G. Then W is generated by the simple reflections s_α for all $\alpha \in \Delta$. There is an action of W on both $X(T)$ and $\mathfrak{h}^*$, given by $s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$.

2.2. We write $g.x$ for the adjoint action of an element $g \in G$ on an element $x \in \mathfrak{g}$. Similarly we will write $g.\chi$ for the coadjoint action of G on $\mathfrak{g}^*$, defined by $g.\chi(x) = \chi(g^{-1}.x)$. Let $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ send $x \in \mathfrak{g}$ to the functional $\theta(x)$ defined by $\theta(x)(y) = B(x, y)$ for all $y \in \mathfrak{g}$. By Hypothesis (C), θ is a G-equivariant isomorphism.

Recall there is a Jordan decomposition in $\mathfrak{g}$: each element $x \in \mathfrak{g}$ can be written uniquely as $x = x_s + x_n$ with x_s semisimple, x_n nilpotent and $[x_s, x_n] = 0$. Given $x \in \mathfrak{g}$, we can always find $g \in G$ such that $g.x \in \mathfrak{b}^+$, [2, Proposition 14.25]. Let $\mathfrak{z}_\mathfrak{g}(x) = \{y \in \mathfrak{g} : [x, y] = 0\}$ and $Z_G(x) = \{g \in G : g.x = x\}$. If x is semisimple, then $Z_G(x)$ is a connected, reductive algebraic group satisfying Hypotheses (A), (B) and (C), and $\mathrm{Lie}(Z_G(x)) = \mathfrak{z}_\mathfrak{g}(x)$, [19, Theorem 3.10], [44, II.3.19] and [24, 6.5].

Using θ, we can transfer the Jordan decomposition to $\mathfrak{g}^*$. In particular, any element of $\mathfrak{g}^*$ is conjugate to $\chi \in \mathfrak{g}^*$ such that $\chi(\mathfrak{n}^+) = 0$. For $\chi \in \mathfrak{g}^*$ let $\mathfrak{z}_\mathfrak{g}(\chi) = \{y \in \mathfrak{g} : \chi([\mathfrak{g}, y]) = 0\}$ and $Z_G(\chi) = \{g \in G : g.\chi = \chi\}$. Then $Z_G(x) = Z_G(\theta(x))$ and $\mathfrak{z}_\mathfrak{g}(x) = \mathfrak{z}_\mathfrak{g}(\theta(x))$.

2.3. Since $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}$ is a restricted Lie algebra with restriction $x \longmapsto x^{[p]}$. The *p-centre* $Z_p = K[x^p - x^{[p]} : x \in \mathfrak{g}]$ is a central subalgebra of $U = U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$. By the PBW theorem Z_p is a polynomial ring in $\dim \mathfrak{g}$ variables and $U(\mathfrak{g})$ is free over Z_p of rank $p^{\dim \mathfrak{g}}$.

Given $\chi \in \mathfrak{g}^*$, define I_χ as the ideal of $U(\mathfrak{g})$ generated by the elements $x^p - x^{[p]} - \chi(x)^p$ for $x \in \mathfrak{g}$. Set $U_\chi = U_\chi(\mathfrak{g}) = U(\mathfrak{g})/I_\chi$, a *reduced enveloping algebra.* This is an algebra of dimension $p^{\dim \mathfrak{g}}$. As the isomorphism class of U_χ depends only on the G-orbit of $\chi \in \mathfrak{g}^*$, [24, 2.9], it suffices, by Section 2.2, to look at χ satisfying $\chi(\mathfrak{n}^+) = 0$. In this case if $\chi = \chi_s + \chi_n$ is the Jordan decomposition, then, possibly after further conjugation, we also have $\chi_s(\mathfrak{n}^-) = 0$ and $\chi_n(\mathfrak{h}) = 0$, and so in particular we can consider $\chi_s \in \mathfrak{h}^*$.

2.4. Let $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and set

$$\Lambda_{\chi_s} = \{\lambda \in \mathfrak{h}^* : \lambda(h)^p - \lambda(h^{[p]}) = \chi_s(h)^p \text{ for all } h \in \mathfrak{h}\}.$$

By Hypothesis (A) the elements h_α for α simple are linearly independent, so, since $h_\alpha^{[p]} = h_\alpha$, we can find $\rho \in \Lambda_0$ such that $\rho(h_\alpha) = 1$ for all simple α. Fix once and for all such a ρ. Note that W acts on Λ_0. For each $\lambda \in \Lambda_0$ let $W(\lambda) = \{w \in W : w(\lambda) = \lambda\}$, a parabolic subgroup, [30, Lemma 7]. In general we will only be interested in λ up to W-conjugacy, so we may assume without loss of generality that $W(\lambda)$ is a standard parabolic subgroup; that is, it is generated by simple reflections.

2.5. For each $\lambda \in \Lambda_{\chi_s}$ one defines a *baby Verma module*

$$Z_\chi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}^+)} K_{\lambda-\rho},$$

where $\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$ and $K_{\lambda-\rho}$ is the one dimensional $U_\chi(\mathfrak{b}^+)$-module defined by $\lambda - \rho$. Every irreducible $U_\chi(\mathfrak{g})$-module is a factor of a baby Verma module, although in general a baby Verma module can have several irreducible images, and different choices of λ can yield the same module [24, 6.7, 6.9].

2.6. Let $Z = Z(U(\mathfrak{g}))$, $Z_1 = U(\mathfrak{g})^G \subset Z$. By [30, Theorem 2] and [3, Theorem 3.5] there is a K-algebra isomorphism

$$Z \cong Z_1 \otimes_{Z_p \cap Z_1} Z_p. \tag{1}$$

We have a natural map $Z \otimes_{Z_p} K_{\chi_s} \longrightarrow U_\chi$ sending $z \otimes 1$ to $z + I_\chi$. By [30, Theorem 10] the primary components of $Z \otimes_{Z_p} K_{\chi_s}$ are labelled by elements of the orbits of W on the set $W\Lambda_{\chi_s}$, denoted by $W\Lambda_{\chi_s}/W$. Thanks to (1) we have an isomorphism

$$Z \otimes_{Z_p} K_{\chi_s} \cong Z_1 \otimes_{Z_p \cap Z_1} K_{\chi_s}. \tag{2}$$

Let $\{e_{\tilde{\lambda}} \in Z_1 : \tilde{\lambda} \in W\Lambda_{\chi_s}/W\}$ be a set of elements such that $\{e_{\tilde{\lambda}} \otimes 1 : \tilde{\lambda} \in W\Lambda_{\chi_s}/W\}$ correspond to a set of primitive idempotents of $Z \otimes_{Z_p} K_{\chi_s}$ under (2). By [3, Theorem 3.18] the elements $e_{\tilde{\lambda}} + I_\chi \in U_\chi$ give a complete list of central primitive idempotents of U_χ as $\tilde{\lambda}$ runs through $W\Lambda_{\chi_s}/W$. In other words we have a block decomposition of $U_\chi(\mathfrak{g})$:

$$U_\chi(\mathfrak{g}) = \bigoplus_{\tilde{\lambda} \in W\Lambda_{\chi_s}/W} \mathcal{B}_{\chi,\tilde{\lambda}}(\mathfrak{g}), \tag{3}$$

where $\mathcal{B}_{\chi,\tilde{\lambda}}(\mathfrak{g}) = e_{\tilde{\lambda}} U_\chi(\mathfrak{g})$. By [24, 10.11] the baby Verma module $Z_\chi(\lambda)$ belongs to $\mathcal{B}_{\chi,\tilde{\lambda}}(\mathfrak{g})$, where $\tilde{\lambda}$ is a representative for the orbit of λ.

Remarks. 1. The reference [3, Theorem 3.5] requires the assumption $p \neq 2$. If $p = 2$ then the block decomposition of $U_\chi(\mathfrak{g})$ remains valid thanks to [16, Theorem 3.6]. Moreover, if $G = SL_n(K)$ or $GL_n(K)$, then [3, Theorem 3.5] continues to hold and the results of this section go through. Thanks to Proposition 6.5 this is sufficient for our applications.

2. If no confusion can occur, we will write $\mathcal{B}_{\chi,\lambda}$ when really we mean $\mathcal{B}_{\chi,\tilde{\lambda}}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$ and some representative $\tilde{\lambda}$ of the orbit of $\lambda \in \Lambda_{\chi_s}$.

2.7. The following is an easy consequence of [46, Theorem 2] and [11, Theorem 3.2].

Proposition. *Let $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$ and let W' be the Weyl group of $Z_G(\chi_s)$. Then there is a bijection*

$$\pi : W\Lambda_{\chi_s}/W \longrightarrow \Lambda_0/W',$$

such that for any $\lambda \in W\Lambda_{\chi_s}/W$ we have an algebra isomorphism

$$\mathcal{B}_{\chi,\lambda}(\mathfrak{g}) \cong \mathrm{Mat}_{p^d}\left(\mathcal{B}_{\chi_n,\pi(\lambda)}(\mathfrak{z}_{\mathfrak{g}}(\chi_s))\right),$$

where $d = \frac{1}{2}(\dim G.\chi_s)$.

Proof. We may assume that $\chi(\mathfrak{n}^+) = 0$. Let $\Phi_\chi = \{\alpha \in \Phi : \chi_s(h_\alpha) = 0\}$. Then $\mathfrak{z}_{\mathfrak{g}}(\chi_s)$ is the algebra generated by $\mathfrak{h}$ and the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_\chi$, [24, 7.4].

Choose an element $\nu \in \Lambda_{\chi_s}$ such that $\nu(h_\alpha) = 0$ for all $\alpha \in \Phi_\chi$. We have a W'-equivariant bijection between Λ_0 and Λ_{χ_s} sending μ to $\mu + \nu$. Thus Λ_0/W' is isomorphic to $W'\Lambda_{\chi_s}/W'$. Moreover, the inclusion $\Lambda_{\chi_s} \longrightarrow W\Lambda_{\chi_s}$ induces an isomorphism between $W'\Lambda_{\chi_s}/W'$ and $W\Lambda_{\chi_s}/W$. Combining these isomorphisms yields π^{-1}.

By [46, Theorem 2] and [11, Theorem 3.2], there is an isomorphism

$$U_\chi(\mathfrak{g}) \cong \mathrm{Mat}_{p^d}\left(U_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_s))\right).$$

Let σ_ν be the *winding automorphism* of $U(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$ which sends h to $h - \nu(h)$ for $h \in \mathfrak{h}$ and is the identity on the root spaces of $\mathfrak{z}_{\mathfrak{g}}(\chi_s)$. Then σ_ν induces an isomorphism between $U_{\chi_n}(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$ and $U_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$ which sends the baby Verma module $Z_{\chi_n}(\mu)$ to $Z_\chi(\mu + \nu)$. The proposition follows. □

Remarks. 1. Let $\mathcal{N} = \{x \in \mathfrak{g} : x \text{ nilpotent}\}$ be the nilpotent cone of $\mathfrak{g}$. The image of $\mathcal{N}$ under θ is the set $\{\chi \in \mathfrak{g}^* : \chi \text{ nilpotent}\}$. We will denote this by $\mathcal{N}$ also. Proposition 2.7 shows that it is sufficient to consider only blocks $\mathcal{B}_{\chi,\lambda}$ with $\chi \in \mathcal{N}$ and $\lambda \in \Lambda_0/W$.
2. To ease notation we write Λ for the $\mathbb{F}_p$-space Λ_0. That is,

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \lambda(h)^p - \lambda(h^{[p]}) = 0 \text{ for all } \lambda \in \mathfrak{h}\}.$$

It is clear that Λ is W-invariant.

3. Block degeneration

3.1. We intend to study the behaviour of $\mathcal{B}_{\chi,\lambda}$ as we vary $\chi \in \mathcal{N}$. Let $\pi_\Lambda : \mathcal{N} \times \Lambda \longrightarrow \Lambda$ be the projection map.

Lemma. *The function $\phi : \mathcal{N} \times \Lambda \longrightarrow \mathbb{N}$ defined by sending (χ, λ) to $\dim \mathcal{B}_{\chi,\lambda}$ is constant on the fibres of π_Λ.*

Proof. Let $\chi(x) = B(x, e)$ for some $e \in \mathcal{N}$. There exists $e' \in \mathcal{N}$ such that $e_\tau = e + \tau e' \in \mathcal{N}$ for all $\tau \in K$ and such that $e_1 = e + e'$ is regular nilpotent, [45, Section 5]. Let $\chi_\tau \in \mathfrak{g}^*$ be defined by $\chi_\tau(x) = B(x, e_\tau)$. For $\lambda \in \Lambda/W$ the function

$$A_\lambda : K \longrightarrow \mathrm{Mat}_d(K)$$

sending τ to the matrix of left multiplication in U_{χ_τ} by e_λ is, by construction, a morphism of varieties. Thus the set

$$\mathcal{O}_\lambda = \{\tau \in K : \operatorname{rank} A_\lambda(\tau) \geq \operatorname{rank} A_\lambda(0)\}$$

is open and dense in K. Let $\mathcal{O} = \bigcap_{\lambda \in \Lambda/W} \mathcal{O}_\lambda$. By (3) the rank of $A_\lambda(\tau)$ equals the dimension of $\mathcal{B}_{\chi_\tau,\lambda}$. We deduce that

$$\mathcal{O} = \{\tau \in K : \dim \mathcal{B}_{\chi_\tau,\lambda} = \dim \mathcal{B}_{\chi,\lambda} \text{ for all } \lambda \in \Lambda/W\}.$$

Since the set of regular nilpotent elements, $\mathcal{N}_{\mathrm{reg}}$, is dense in $\mathcal{N}$, we deduce that $\mathcal{O} \cap \mathcal{N}_{\mathrm{reg}}$ is dense in $\mathcal{O}$. The lemma now follows since the restriction of ϕ to $\mathcal{N}_{\mathrm{reg}} \times \Lambda$ is constant along the fibres of π_Λ, see for example [3, Proposition 3.16]. □

3.2. We recall the following definitions from the theory of finite dimensional algebras, [13] and [27, Chapter II]. Let

$$\mathrm{Bil}(n) = \{\text{bilinear maps } m : K^n \times K^n \longrightarrow K^n\} \cong \mathbb{A}^{n^3},$$

and

$$\mathrm{Alg}(n) = \{\text{associative, bilinear } m \text{ which have an identity}\} \subseteq \mathrm{Bil}(n).$$

As discussed in [13], $\mathrm{Alg}(n)$ is an affine variety, locally closed in $\mathrm{Bil}(n)$. The group $GL(n)$ acts on $\mathrm{Alg}(n)$, the orbits being isomorphism classes of n-dimensional algebras. We let $\mathcal{O}_A$ denote the orbit in $\mathrm{Alg}(n)$ of algebras isomorphic to A. We say that A' *is a degeneration of* A if $\mathcal{O}_{A'} \subseteq \overline{\mathcal{O}_A}$, the closure of $\mathcal{O}_A$.

Theorem. *Let $\chi, \chi' \in \mathcal{N}$ be such that χ is in the closure of the orbit $G.\chi'$. Then for any $\lambda \in \Lambda/W$ the algebra $\mathcal{B}_{\chi,\lambda}$ is a degeneration of $\mathcal{B}_{\chi',\lambda}$.*

Proof. Let $d = p^{\dim \mathfrak{g}}$ and let $w_1, \ldots, w_d$ be a free basis of the Z_p-module U. Then the cosets $w_i + I_\zeta$ form a K-basis of U_ζ for all $\zeta \in \mathfrak{g}^*$.

Fix $\lambda \in \Lambda/W$. Let $u_1, \ldots, u_r \in U$ be such that $\{e_\lambda u_i + I_\chi : 1 \leq i \leq r\}$ is a basis of $\mathcal{B}_{\chi,\lambda}$. We have a morphism

$$\overline{G.\chi'} \longrightarrow \mathrm{Mat}_{r\times d}(K),$$

sending ζ to M_ζ, the matrix expressing the elements $e_\lambda u_i + I_\zeta$ in terms of the basis elements $\{w_i + I_\zeta\}$. The set

$$\mathcal{O}_\lambda = \{\zeta \in \overline{G.\chi'} : \operatorname{rank} M_\zeta = r\}$$

is a dense open subset of $\overline{G.\chi'}$ and includes χ. Now Lemma 2.1 says that for all $\zeta \in \mathcal{O}_\lambda$ the block $\mathcal{B}_{\zeta,\lambda}$ has a basis $\{e_\lambda u_i + I_\zeta : 1 \leq i \leq r\}$, and so, by definition, we have a morphism

$$\mathcal{O}_\lambda \longrightarrow \mathrm{Alg}(r)$$

sending ζ to $\mathcal{B}_{\zeta,\lambda}$. Since $G.\chi'$ is open in its closure, the theorem follows. □

3.3. A finite dimensional K-algebra A is said to have *wild representation type*, or be *wild* for short, if for any finite dimensional K-algebra B, there is a representation embedding of the module category of B into that of A, so that A-mod contains B-mod although not necessarily as a full subcategory. The algebra A has *finite representation type* or has *finite type* if it has finitely many isomorphism classes of indecomposable modules. Finally, A has *tame representation type* or is *tame* if it is neither wild nor of finite type.

Thanks to Theorem 3.2 we can compare the representation type of two blocks.

Corollary. *Let $\chi, \chi' \in \mathcal{N}$ be such that χ is in the closure of the orbit $G.\chi'$. Let $\lambda \in \Lambda/W$.*

1. *If $\mathcal{B}_{\chi,\lambda}$ has finite representation type then $\mathcal{B}_{\chi',\lambda}$ has finite representation type.*
2. *If $\mathcal{B}_{\chi,\lambda}$ has tame representation type then $\mathcal{B}_{\chi',\lambda}$ has either finite or tame representation type.*

Proof. By Theorem 3.2 $\mathcal{B}_{\chi,\lambda}$ is a degeneration of $\mathcal{B}_{\chi',\lambda}$. The first assertion follows since the algebras in $\mathrm{Alg}(d)$ having finite representation type form an open set, [13, Theorem 4.2]. The second assertion is a consequence of the main result in [14], which states that a degeneration of a wild algebra is wild. □

4. Rank Varieties

4.1. Let $\mathcal{L}$ be a finite dimensional restricted Lie algebra over K and define

$$\mathcal{N}_p(\mathcal{L}) = \{x \in \mathcal{L} : x^{[p]} = 0\},$$

a Zariski closed, conical subset of $\mathcal{L}$, called the *p-nilpotent cone of $\mathcal{L}$*. Let M be a finite dimensional $U_\zeta(\mathcal{L})$-module. We define the *rank variety of M* to be

$$\mathcal{V}_{\mathcal{L}}(M) = \{x \in \mathcal{N}_p(\mathcal{L}) : M|_{\langle x\rangle} \text{ is not free}\},$$

where $\langle x\rangle$ denotes the subalgebra of $U_\zeta(\mathcal{L})$ generated by x.

Let $\chi \in \mathfrak{g}^*$ and $\lambda \in W\Lambda_{\chi_s}/W$. Let $S_1, \ldots, S_r$ be a list of the simple $\mathcal{B}_{\chi,\lambda}$-modules (up to isomorphism). We define the rank variety of the block $\mathcal{B}_{\chi,\lambda}$ to be

$$\mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = \bigcup_{i=1}^{r} \mathcal{V}_{\mathfrak{g}}(S_i).$$

By [11, Section 7] the rank variety of any $\mathcal{B}_{\chi,\lambda}$-module is contained in $\mathcal{V}_{\mathfrak{g}}(\chi,\lambda)$.

Remark. For $G^{(1)}$ of type A_n the variety $\mathcal{V}_{\mathfrak{g}}(0,\lambda)$ is described in [23, Proposition 2.6]. This was recently generalised to all types in [31, Theorem 6.2.1] only under the assumption that p is good.

4.2. The rest of this section is devoted to a proof of the following result.

Theorem. *Let $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$ and $\lambda \in W\Lambda_{\chi_s}/W$. Then*

$$\mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = \mathcal{V}_{\mathfrak{g}}(\chi_s,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(\chi).$$

Remark. If $S_1, \ldots, S_t$ is a complete list of the simple U_χ-modules up to isomorphism, we define

$$\mathcal{V}_{\mathfrak{g}}(\chi) = \bigcup_{i=1}^{t} \mathcal{V}_{\mathfrak{g}}(S_i).$$

Suppose $\chi \in \mathcal{N}$. By [21, Satz 2.14] we have $V_{\mathfrak{g}}(0) = \mathcal{N}_p(\mathfrak{g})$, so we deduce the principal result of [38], namely

$$\mathcal{V}_{\mathfrak{g}}(\chi) = \mathcal{N}_p(\mathfrak{z}_{\mathfrak{g}}(\chi)).$$

Our class of Lie algebras, however, is smaller than that considered in [38] (the latter includes for example $\mathfrak{sl}_{np}(K)$).

4.3. We recall the Mil'ner map, introduced in [29]. The algebra U has a natural increasing filtration $\{U^k\}$, where U^k denotes the span of all products of at most k elements of $\mathfrak{g}$.

Given a vector space V, we let $S(V)$ be the symmetric algebra of V. The algebra $S(V)$ also has a natural increasing filtration $\{S^{\leq k}(V)\}$, where $S^{\leq k}(V)$ denotes the sum of the homogeneous components $S^i(V)$ with $i \leq k$.

Let $\underline{x} = (x_1, \dots, x_r) \in \mathfrak{g}$. For $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, r\}$ we set $x_I = x_{i_1} \dots x_{i_k} \in U$. Let

$$\phi_r(\underline{x}) = \sum x_{I_1} \dots x_{I_k} \in S(U),$$

where the summation runs through all decompositions

$$I_1 \cup \dots \cup I_k = \{1, \dots, r\}$$

of $\{1, \dots, r\}$ into non-empty disjoint subsets. By [10, Section 1.1] there exists a unique linear map $\phi : U \longrightarrow S(U)$ such that $\phi(1) = 1$ and $\phi(x_1 \dots x_r) = \phi_r((x_1, \dots, x_r))$ for all $x_1, \dots, x_r \in \mathfrak{g}$. Let $\pi : U \twoheadrightarrow \mathfrak{g}$ be an $\operatorname{ad}\mathfrak{g}$-equivariant projection, which exists by [38, Section 3.3]. This induces a map $S(\pi) : S(U) \longrightarrow S(\mathfrak{g})$. We let $\beta = S(\pi) \circ \phi : U \longrightarrow S(\mathfrak{g})$.

Given $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$, let J_χ be the two-sided ideal of $S(\mathfrak{g})$ generated by the elements $x^p - \chi(x)^p$ for all $x \in \mathfrak{g}$. We let $S_\chi = S(\mathfrak{g})/J_\chi$. There is a natural action of $\mathfrak{g}$ on S_χ induced by the adjoint action on $\mathfrak{g} \subset S_\chi$. The filtration $\{U^k\}$ (respectively $\{S^{\leq k}(\mathfrak{g})\}$) induces a natural increasing filtration on U_χ (respectively S_χ).

Lemma. [38, Lemma 3.2] *The map $\beta : U \longrightarrow S(\mathfrak{g})$ induces a $\mathfrak{g}$-equivariant filtration-preserving isomorphism $\beta_\chi : U_\chi \longrightarrow S_\chi$.*

4.4. We define the map

$$\gamma_\chi : S_\chi \longrightarrow S_{\chi_s}$$

by sending $x \in \mathfrak{g} \subset S_\chi$ to $x + \chi_n(x) \in S_{\chi_s}$, and extending algebraically. Since $\mathfrak{z}_\mathfrak{g}(\chi) \subseteq \mathfrak{z}_\mathfrak{g}(\chi_s)$, the map γ_χ is a $\mathfrak{z}_\mathfrak{g}(\chi)$-equivariant isomorphism. This allows us to construct a $\mathfrak{z}_\mathfrak{g}(\chi)$-equivariant isomorphism

$$\phi_\chi = \beta_{\chi_s}^{-1} \circ \gamma_\chi \circ \beta_\chi : U_\chi \longrightarrow U_{\chi_s}.$$

4.5. It will be important for us to vary χ. To this end we let t be a indeterminate. We define $\mathfrak{g}_t = \mathfrak{g} \otimes_K K[t]$. We can consider the algebras $U_{\chi_s + t\chi_n}(\mathfrak{g}_t), U_{\chi_s}(\mathfrak{g}_t)$, $S_{\chi_s + t\chi_n}(\mathfrak{g}_t)$ and $S_{\chi_s}(\mathfrak{g}_t)$, all of which are free $K[t]$-modules of rank $p^{\dim \mathfrak{g}}$. We define a map

$$\beta_{\chi_s + t\chi_n} : U_{\chi_s + t\chi_n}(\mathfrak{g}_t) \longrightarrow S_{\chi_s + t\chi_n}(\mathfrak{g}_t),$$

which, for $x_1, \dots, x_r \in \mathfrak{g} \subset \mathfrak{g}_t$, sends $x_1 \dots x_r$ to $\sum \pi(x_{I_1}) \dots \pi(x_{I_k})$, where the summation runs through all decompositions $I_1 \cup \dots \cup I_k$ of $\{1, \dots, r\}$ into non-empty disjoint subsets, and is extended to $U_{\chi_s + t\chi_n}(\mathfrak{g}_t)$ by $K[t]$-linearity. As before, it can be shown that this is a $\mathfrak{g}_t$-equivariant isomorphism of $K[t]$-modules. Moreover, $\beta_{\chi_s + t\chi_n}$ preserves the natural filtrations on $U_{\chi_s + t\chi_n}(\mathfrak{g}_t)$ and $S_{\chi_s + t\chi_n}(\mathfrak{g}_t)$.

4.6. Similarly we have a $\mathfrak{z}_{\mathfrak{g}_t}(\chi)$-equivariant isomorphism of $K[t]$-algebras

$$\gamma_{\chi_s+t\chi_n} : S_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow S_{\chi_s}(\mathfrak{g}_t),$$

sending $x \in \mathfrak{g} \subset \mathfrak{g}_t$ to $x + t\chi_n(x) \in S_{\chi_s}(\mathfrak{g}_t)$. This allows us to construct a $\mathfrak{z}_{\mathfrak{g}_t}(\chi)$-equivariant isomorphism of $K[t]$-modules

$$\phi_{\chi_s+t\chi_n} = (\beta_{\chi_s})_t^{-1} \circ \gamma_{\chi_s+t\chi_n} \circ \beta_{\chi_s+t\chi_n} : U_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow U_{\chi_s}(\mathfrak{g}_t).$$

4.7. By definition we have that, as elements of $S_{\chi_s}(\mathfrak{g}_t)$,

$$\begin{aligned}(\gamma_{\chi_s+t\chi_n} \circ \beta_{\chi_s+t\chi_n})(x_1 \dots x_r) \\ = \sum (\pi(x_{I_1}) + t\chi_n(\pi(x_{I_1}))) \dots (\pi(x_{I_k}) + t\chi_n(\pi(x_{I_k}))),\end{aligned}$$

for $x_1, \dots, x_r \in \mathfrak{g} \subset \mathfrak{g}_t$. This expression is equal to

$$\beta_{\chi_s}(x_1 \dots x_r) + t \cdot (\text{linear combination of } y_1 \dots y_d \text{ with } y_i \in \mathfrak{g}_t \text{ and } d < r).$$

Since the Mil'ner map $(\beta_{\chi_s})_t$ is a filtration-preserving isomorphism of $K[t]$-modules, we see that all $y_1 \dots y_d$'s are contained in $(\beta_{\chi_s})_t(U_{\chi_s}(\mathfrak{g}_t)^d)$. We deduce that the map $\phi_{\chi_s+t\chi_n} : U_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow U_{\chi_s}(\mathfrak{g}_t)$ is such that

$$\phi_{\chi_s+t\chi_n}(x_1 \dots x_r) \equiv x_1 \dots x_r \mod t(U_{\chi_s}(\mathfrak{g}_t)^{r-1}).$$

4.8. It follows from [35, Theorem 2.5] that one can find a one-dimensional torus $h(\tau)$ in $Z_G(\chi_s)$ such that $h(\tau).\chi_n = \tau^2\chi_n$ for all $\tau \in K^*$. Since the elements e_λ are G-invariant, the isomorphism between U_χ and $U_{g.\chi}$ for $g \in G$ restricts to an isomorphism between $\mathcal{B}_{\chi,\lambda}$ and $\mathcal{B}_{g.\chi,\lambda}$. In particular we can find weight vectors for $h(\tau)$, say $u_1, \dots, u_r \in U$, such that $\{e_\lambda u_i + I_{\chi_s} : 1 \leq i \leq r\}$ is a basis for $\mathcal{B}_{\chi_s,\lambda}$. Arguing as in the proof of Theorem 3.2, we find a dense open subset $\mathcal{O}$ of K such that $\{e_\lambda u_i + I_{\chi_s+v\chi_n} : 1 \leq i \leq r\}$ is a basis of $\mathcal{B}_{\chi_s+v\chi_n,\lambda}$ for all $v \in \mathcal{O}$. Conjugating by $h(\tau)$ shows that the set $\{e_\lambda u_i + I_\chi : 1 \leq i \leq r\}$ is a basis of $\mathcal{B}_{\chi,\lambda}$.

4.9. For $1 \leq i \leq r$ we have

$$\phi_{\chi_s+t\chi_n}(e_\lambda u_i) = e_\lambda u_i + tm_{1,i} + t^2 m_{2,i} + \dots,$$

where $m_{j,i} \in U_{\chi_s}(\mathfrak{g}) \subset U_{\chi_s}(\mathfrak{g}_t)$. Let $\pi_\lambda : U_{\chi_s}(\mathfrak{g}) \twoheadrightarrow \mathcal{B}_{\chi_s,\lambda}$ be projection, a $\mathfrak{g}$-invariant homomorphism. We find that

$$\pi_\lambda \circ \phi_{\chi_s+t\chi_n}(e_\lambda u_i) = e_\lambda u_i + tn_{1,i} + t^2 n_{2,i} + \dots,$$

where $n_{j,i} = \pi_\lambda(m_{j,i})$. It follows that the matrix expressing $\pi_\lambda \circ \phi_{\chi_s+t\chi_n}$ in terms of the basis $\{e_\lambda u_i : 1 \leq i \leq r\}$ has the form

$$C_t = \begin{pmatrix} 1+ta_{11} & ta_{12} & \dots & ta_{1r} \\ ta_{21} & 1+ta_{22} & \dots & ta_{2r} \\ \vdots & \vdots & & \vdots \\ ta_{r1} & ta_{r2} & \dots & 1+ta_{rr} \end{pmatrix}$$

for some $a_{ij} \in K[t]$. Thus, for almost all $v \in K$, we have that $\det(C_v)$ is nonzero and, from 4.8, that $\{e_\lambda u_i + I_{\chi_s+v\chi_n} : 1 \leq i \leq r\}$ is a basis of $\mathcal{B}_{\chi_s+v\chi_n,\lambda}$. Thus, for almost all $v \in K$, restriction of $\phi_{\chi_s+v\chi_n}$ yields a $\mathfrak{z}_{\mathfrak{g}}(\chi_s + v\chi_n)$-equivariant isomorphism

$$\mathcal{B}_{\chi_s+v\chi_n,\lambda} \longrightarrow \mathcal{B}_{\chi_s,\lambda}.$$

In particular, for almost all $v \in K^*$, we have an equality

$$\mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi_s+v\chi_n)}(\mathcal{B}_{\chi_s,\lambda}) = \mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi_s+v\chi_n)}(\mathcal{B}_{\chi_s+v\chi_n,\lambda}), \tag{4}$$

where $\mathcal{B}_{\chi_s,\lambda}$ and $\mathcal{B}_{\chi_s+v\chi_n,\lambda}$ are modules for the adjoint action of the stabiliser $\mathfrak{z}_\mathfrak{g}(\chi_s+v\chi_n)=\mathfrak{z}_\mathfrak{g}(\chi_s)\cap\mathfrak{z}_\mathfrak{g}(v\chi_n)=\mathfrak{z}_\mathfrak{g}(\chi)$.

4.10. Let $L=Z_G(\chi_s)$, a Levi subgroup of G, and $N_L(\mathfrak{z}_\mathfrak{g}(\chi))=\{g\in L : g.z\in\mathfrak{z}_\mathfrak{g}(\chi)$ for all $z\in\mathfrak{z}_\mathfrak{g}(\chi)\}$. Let N be the identity component of $N_L(\mathfrak{z}_\mathfrak{g}(\chi))$. Recall that L, and hence N, acts on U_{χ_s}. Moreover, if we decompose U_{χ_s} as a direct sum of $\operatorname{ad}\mathfrak{z}_\mathfrak{g}(\chi)$-modules, say $\bigoplus M_i$, then, by the Krull–Remak–Schmidt theorem, N permutes the isomorphism classes of the modules M_i. Since, however, N is the minimal normal subgroup of finite index in $N_L(\mathfrak{z}_\mathfrak{g}(\chi))$, we deduce that N fixes the isomorphism class of the modules M_i. This implies that the rank variety of M_i is N-invariant for any i.

It can be deduced from [35, Theorem 2.5] that, given $v\in K^*$, there exists $g\in N$ such that $g.\chi=\chi_s+v\chi_n$. As a result we deduce from the above that

$$\mathcal{V}_{\mathfrak{z}_\mathfrak{g}(\chi)}(\mathcal{B}_{\chi,\lambda})=Ad(g)\Big(\mathcal{V}_{\mathfrak{z}_\mathfrak{g}(\chi)}(\mathcal{B}_{\chi,\lambda})\Big)=\mathcal{V}_{\mathfrak{z}_\mathfrak{g}(\chi_s+v\chi_n)}(\mathcal{B}_{\chi_s+v\chi_n,\lambda}). \tag{5}$$

4.11. By [37, Theorem 1.1] and [38, Theorem 2.4], we have that for any U_χ-module M, $\mathcal{V}_\mathfrak{g}(M)\subseteq\mathfrak{z}_\mathfrak{g}(\chi)$. Since $\mathcal{V}_{\mathfrak{z}_\mathfrak{g}(\chi)}(M)=\mathfrak{z}_\mathfrak{g}(\chi)\cap\mathcal{V}_\mathfrak{g}(M)$ for any U_χ-module M, we deduce that

$$\mathcal{V}_\mathfrak{g}(\chi,\lambda)\subseteq\mathfrak{z}_\mathfrak{g}(\chi). \tag{6}$$

4.12. Arguing as in [38, Proposition 2.2], we have

$$\mathcal{V}_\mathfrak{g}(\chi,\lambda)=\mathcal{V}_\mathfrak{g}(\mathcal{B}_{\chi,\lambda}). \tag{7}$$

The required equality $\mathcal{V}_\mathfrak{g}(\chi,\lambda)=\mathcal{V}_\mathfrak{g}(\chi_s,\lambda)\cap\mathfrak{z}_\mathfrak{g}(\chi)$ now follows by combining (7), (6), (5) and (4).

5. Representation type

5.1. We come to the description of the representation type of a block, $\mathcal{B}_{\chi,\lambda}$, of the reduced enveloping algebra U_χ. By Proposition 2.7 we can assume without loss of generality that χ is nilpotent and $\lambda\in\Lambda/W$. Let $G_1,\dots,G_m$ be the simple (simply-connected) normal subgroups of $G^{(1)}$ and let $\mathfrak{g}'=\operatorname{Lie}G^{(1)}$ and $\mathfrak{g}_j=\operatorname{Lie}G_j$ for $1\le j\le m$. Let $e=\theta^{-1}(\chi)$ be the nilpotent element in $\mathfrak{g}$ corresponding to χ. By [36, Section 2.9] $e\in\mathfrak{g}'$, so we can decompose $\chi=\chi_1+\ldots+\chi_m$, where $\chi_j\in\mathfrak{g}_j^*$ is the restriction of χ to $\mathfrak{g}_j$. The Weyl group W can be identified with $W_1\times\cdots\times W_m$, where W_i is the Weyl group of G_i. Under this identification $W(\lambda)$ decomposes as $W_1(\lambda)\times\cdots\times W_m(\lambda)$ for any $\lambda\in\Lambda$. We will retain this notation throughout this section.

5.2. We first classify the blocks of finite representation type, completing work begun in [32].

Theorem. *Recall the notation of 5.1. Let $\chi\in\mathcal{N}$ and $\lambda\in\Lambda/W$. Then the block $\mathcal{B}_{\chi,\lambda}$ has finite representation type if and only if one of the following occurs.*

1. $W(\lambda)=W$.
2. *There exists an integer j, $1\le j\le m$, such that χ_j is regular, $W_i(\lambda)=W_i$ for $i\neq j$, and one of the following conditions holds:*
   (a) *W_j is of type A_n and $W_j(\lambda)$ is of type A_{n-1};*
   (b) *W_j is of type B_n (or C_n) and $W_j(\lambda)$ is of type B_{n-1} (or C_{n-1});*
   (c) *W_j is of type G_2 and $W_j(\lambda)$ is of type A_1.*

(Here we take $A_0=\emptyset$ and $B_1=A_1$.)

5.3. We come to the classification of blocks of tame representation type.

Theorem. *Recall the notation of 5.1. Let $\chi \in \mathcal{N}$ and $\lambda \in \Lambda/W$. The block $\mathcal{B}_{\chi,\lambda}$ has tame representation type if and only if one of the following occurs.*

1. *There exists j, $1 \leq j \leq m$, such that χ_j is regular, $W_i(\lambda) = W_i$ for $i \neq j$, and one of the following holds:*
   (a) *W_j has rank 2;*
   (b) *W_j is of type A_3 and $W_j(\lambda)$ is of type $A_1 \times A_1$;*
   (c) *W_j is of type B_3 (or C_3) and $W_j(\lambda)$ has type A_2;*
   (d) *W_j is of type D_n and $W_j(\lambda)$ is of type D_{n-1}.*
2. *There exists j, $1 \leq j \leq m$, such that χ_j is subregular, $W_i(\lambda) = W_i$ for $i \neq j$, W_j is of type A_n and $W_j(\lambda)$ is of type A_{n-1}.*
3. *There exist j_1, j_2, $1 \leq j_1 < j_2 \leq m$, such that both χ_{j_1} and χ_{j_2} are regular, $W_i(\lambda) = W_i$ for $i \neq j_1, j_2$, and $W_{j_1} \times W_{j_2}$ is of type $A_1 \times A_1$ whilst $W_{j_1}(\lambda) \times W_{j_2}(\lambda)$ is trivial.*

(Here we take $A_0 = \emptyset$ and $D_3 = A_3$.)

5.4. Finally, we describe the representation type of a reduced enveloping algebra U_χ.

Theorem. *Let $\chi \in \mathcal{N}$.*

1. *The algebra U_χ has finite representation type if χ is regular and one of the following holds:*
   (a) *W is trivial;*
   (b) *W has type A_1;*
   (c) *$p = 2$ and W has type A_2;*
   (d) *$p = 3$ and W has type B_2 (or C_2);*
   (e) *$p = 5$ and W has type G_2.*
2. *The algebra U_χ has tame representation type if one of the following holds:*
   (a) *$\chi = 0$ and W has type A_1;*
   (b) *χ is regular and W has type $A_1 \times A_1$;*
   (c) *$p \neq 2$, χ is regular and W has type A_2;*
   (d) *$p \neq 3$, χ is regular and W has type B_2 (or C_2);*
   (e) *$p \neq 5$, χ is regular and W has type G_2;*
   (f) *$p = 2$, χ is subregular and W has type A_2;*
   (g) *$p = 2$, χ is regular and W has type A_3.*
3. *In all other cases U_χ has wild representation type.*

Remark. In [38, Proposition 5.2] a list of *possible* tame U_χ was given. This list, however, was incomplete and should have also included case 2(f) above. (Indeed the proof of [38, Proposition 5.2] should be adjusted on p.278, line 7 from *"If the number of blocks is > 1 ..."* to *"If the number of blocks of size > 1 is > 1 ..."*. Then one must consider A_2 and A_3 with χ subregular and $p = 2$. A calculation shows that the support variety in the A_3 case is three-dimensional, implying wildness.) Of course, Part 2 of the above theorem refines the (corrected) list in [38, Proposition 5.2].

5.5. Theorem 5.2 will be proved in 8.16 and Theorem 5.4 in 8.18. Most of the rest of the paper is concerned with proving Theorem 5.3.

6. A Reduction

6.1. Let $G^{(1)}$ be the derived subgroup of G and let $\mathfrak{g}' = \mathrm{Lie}(G^{(1)})$. Then $\mathfrak{g}_\alpha \subset \mathfrak{g}'$ for all $\alpha \in \Phi$. Let $G_1, \dots, G_m$ be the simple (simply-connected) normal subgroups of $G^{(1)}$ and let $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ for $1 \le i \le m$. Then $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s = \mathfrak{g}' \subseteq \mathfrak{g}$, and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

For $1 \le i \le m$, define $\tilde{G}_i$ by setting

$$\tilde{G}_i = \begin{cases} GL(V_i), & \text{if } G_i \cong SL(V_i) \text{ and } p \mid \dim V_i, \\ G_i, & \text{otherwise.} \end{cases}$$

Put $\tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_m$, $\tilde{\mathfrak{g}}_i = \mathrm{Lie}(\tilde{G}_i)$ and $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{G})$. We have $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \cdots \oplus \tilde{\mathfrak{g}}_m$, and either $\tilde{\mathfrak{g}}_i = \mathfrak{g}_i$, or $\mathfrak{g}_i \cong \mathfrak{sl}(V_i)$ and $\tilde{\mathfrak{g}}_i \cong \mathfrak{gl}(V_i)$ and $p \mid \dim V_i$. We identify each $\mathfrak{g}_i$ with an ideal of codimension at most one in $\tilde{\mathfrak{g}}_i$.

6.2. Recall that a restricted Lie algebra is called *toral* if it is abelian and has a basis consisting of toral elements, that is, a basis consisting of elements which satisfy $t^{[p]} = t$. Thanks to [38, Lemma 4.1] there is a toral Lie algebra $\mathfrak{t}_0$ and an embedding of restricted Lie algebras

$$\psi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \oplus \mathfrak{t}_0$$

such that $\psi(\mathfrak{g}_i) = \mathfrak{g}_i \subseteq \tilde{\mathfrak{g}}$ and $\psi(\mathfrak{h}) \subseteq \tilde{\mathfrak{h}} \oplus \mathfrak{t}_0$, where $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$ satisfying $\mathfrak{h} \cap \mathfrak{g}' = \tilde{\mathfrak{h}} \cap \mathfrak{g}'$. Set $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathfrak{t}_0$ and $\hat{\mathfrak{h}} = \tilde{\mathfrak{h}} \oplus \mathfrak{t}_0$.

Lemma. *There exists a toral subalgebra $\mathfrak{t}_1 \subset \hat{\mathfrak{h}}$ such that $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}_1$ and $[\mathfrak{t}_1, \hat{\mathfrak{g}}] = 0$.*

Proof. Let $\mathfrak{c}$ denote the centraliser of $\mathfrak{g}'$ in $\hat{\mathfrak{g}}$. Since $\mathfrak{g}'$ is invariant under the adjoint action of $G^{(1)}$, so is $\mathfrak{c}$. Now $\mathfrak{g}_\alpha \subset \mathfrak{g}'$ for any root α, and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \ni h_\alpha \neq 0$. Therefore the maximal torus $T \cap G^{(1)}$ of $G^{(1)}$ acts trivially on $\mathfrak{c}$, implying $\mathfrak{c} \subseteq \hat{\mathfrak{h}}$. Since $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} + \mathfrak{g}'$ and $\hat{\mathfrak{h}}$ is abelian, we deduce that $\mathfrak{c}$ is a central toral subalgebra of $\hat{\mathfrak{g}}$. Let $\mathfrak{c}^{\mathrm{tor}}$ denote the $\mathbb{F}_p$-subspace of toral elements of $\mathfrak{c}$.

Identify $\mathfrak{h}$ with $\psi(\mathfrak{h}) \subseteq \hat{\mathfrak{h}}$. Let $\hat{h} \in \hat{\mathfrak{h}}$ and let $a_i = \alpha_i(\hat{h})$, where $1 \le i \le n$. Let $e_{\pm\alpha_i}$ be root vectors such that $[e_{\alpha_i}, e_{-\alpha_i}] = h_{\alpha_i}$. It follows from the T-invariance of B that $B(e_{\alpha_i}, e_{-\alpha_i}) \neq 0$. Since $h_{\alpha_1}, \dots, h_{\alpha_n} \in \mathfrak{h}$ are linearly independent and the restriction of B to $\mathfrak{h}$ is non-degenerate, there is $h \in \mathfrak{h}$ such that $B(h_{\alpha_i}, h) = a_i B(e_{\alpha_i}, e_{-\alpha_i})$ for all i. By the $\mathfrak{g}$-invariance of B,

$$a_i B(e_{\alpha_i}, e_{-\alpha_i}) = B([e_{\alpha_i}, e_{-\alpha_i}], h) = B(e_{-\alpha_i}, [h, e_{\alpha_i}]) = \alpha_i(h) B(e_{\alpha_i}, B_{-\alpha_i}),$$

yielding $\alpha_i(h) = a_i$ for all i. As a consequence, $\hat{h} - h$ centralises the Lie subalgebra generated by all $e_{\pm\alpha_i}$. The latter coincides with $\mathfrak{g}'$, because $G^{(1)}$ is simply-connected, see [26, 1.2] for example. Thus for any $\hat{h} \in \hat{\mathfrak{h}}$ there is $h \in \mathfrak{h}$ such that $\hat{h} - h \in \mathfrak{c}$. In other words, $\hat{\mathfrak{g}} = \mathfrak{c} + \mathfrak{g}$. Now $\mathfrak{c} \cap \mathfrak{g}$ is a restricted subalgebra of $\mathfrak{c}$, and hence is spanned by its $\mathbb{F}_p$-subspace $(\mathfrak{c} \cap \mathfrak{g})^{\mathrm{tor}}$ of toral elements. The latter has a complement in $\mathfrak{c}^{\mathrm{tor}}$, say $\mathfrak{t}_1^{\mathrm{tor}}$. The K-span $\mathfrak{t}_1$ of $\mathfrak{t}_1^{\mathrm{tor}}$ is a central toral subalgebra of $\hat{\mathfrak{g}}$ satisfying $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}_1$. □

6.3. Let T_0 be an algebraic torus over K such that $\mathfrak{t}_0 = \mathrm{Lie}(T_0)$ and let $\hat{T} = \tilde{T} \times T_0$ be a maximal torus of $\tilde{G} \times T_0$ such that $\hat{\mathfrak{h}} = \mathrm{Lie}(\hat{T})$. There is a decomposition $\tilde{T} = T_1 \times \cdots \times T_m$, where T_j is a maximal torus of $\tilde{G}_j$ for $1 \le j \le m$. Let $\mathfrak{h}_j = \mathrm{Lie}(T_j)$ for $1 \le j \le m$. Define

$$\hat{\Lambda} = \{\lambda \in \hat{\mathfrak{h}}^* : \lambda(h)^p - \lambda(h^{[p]}) = 0 \text{ for all } h \in \hat{\mathfrak{h}}\}.$$

Any element of $\hat{\Lambda}$ can be uniquely decomposed into $(\lambda_1, \dots, \lambda_m, \lambda_0) \in \mathfrak{h}_1^* \oplus \dots \oplus \mathfrak{h}_m^* \oplus \mathfrak{t}_0^*$. There is an isomorphism between W, the Weyl group of G with respect to T or of $\tilde{G} \times T_0$ with respect to $\hat{T}$, and $W_1 \times \cdots \times W_m$, where W_i is the Weyl group of $\tilde{G}_j$ with respect to T_j for $1 \leq j \leq m$. Under this isomorphism $W(\lambda)$ is identified with $W_1(\lambda_1) \times \cdots \times W_m(\lambda_m)$.

6.4. Since $\psi : \mathfrak{h} \longrightarrow \hat{\mathfrak{h}}$ is an embedding of restricted Lie algebras, the induced map $\psi^* : \hat{\mathfrak{h}}^* \longrightarrow \mathfrak{h}^*$ is surjective and sends $\hat{\Lambda}$ onto Λ. Recall that $\mathfrak{g}'$ contains all root spaces of $\mathfrak{g}$. Therefore we can consider any root α as an element of $\mathfrak{h}^*$ and as an element of $\hat{\mathfrak{h}}^*$. We have

$$\psi^*(s_\alpha \lambda) = \psi^*(\lambda - \lambda(\psi(h_\alpha))\alpha) = \psi^*(\lambda) - \psi^*(\lambda)(h_\alpha)\alpha = s_\alpha(\psi^* \lambda),$$

showing that ψ^* is W-equivariant. Hence we have a surjective map

$$\pi : \hat{\Lambda}/W \longrightarrow \Lambda/W.$$

Given $\xi \in \mathfrak{g}^*$, we denote by $\hat{\xi}$ the functional on $\hat{\mathfrak{g}}$ whose restriction to $\mathfrak{g}$ (respectively to $\mathfrak{t}_1$) equals ξ (respectively 0). If ξ vanishes on $\mathfrak{b}^+$ then $\hat{\xi}$ vanishes on $\hat{\mathfrak{h}} + \mathfrak{b}^+$. In particular, $\hat{\xi}$ is nilpotent in this case.

Lemma. *Keep the above notation and suppose that χ vanishes on $\mathfrak{b}^+$. Let $\lambda \in \hat{\Lambda}/W$. Then there is an isomorphism of algebras*

$$\mathcal{B}_{\hat{\chi},\lambda}(\hat{\mathfrak{g}}) \cong \mathcal{B}_{\chi,\pi(\lambda)}(\mathfrak{g}). \tag{8}$$

Proof. Since $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{t}_1$, each $\mu \in \hat{\Lambda}$ decomposes into $(\psi^*\mu, \mu_1) \in \mathfrak{h}^* \oplus \mathfrak{t}_1^*$. Clearly $\lambda_1(h)^p = \lambda_1(h^{[p]})$ for all $h \in \mathfrak{t}_1$. Let K_{λ_1} denote the one-dimensional $U_0(\mathfrak{t}_1)$-module corresponding to λ_1. Thanks to Lemma 6.2 and the PBW theorem, there exists an algebra isomorphism

$$\phi :\ U_{\hat{\chi}}(\hat{\mathfrak{g}}) \xrightarrow{\sim} U_\chi(\mathfrak{g}) \otimes U_0(\mathfrak{t}_1)$$

such that $\phi(u) = u \otimes 1$ for any $u \in U_\chi(\mathfrak{g}) \subset U_{\hat{\chi}}(\hat{\mathfrak{g}})$. Since $U_0(\mathfrak{t}_1)$ is a commutative semisimple algebra, there is a primitive idempotent $e \in U_0(\mathfrak{t}_1)$ such that $e(K_{\lambda_1}) \neq 0$ and

$$\phi\big(\mathcal{B}_{\hat{\chi},\lambda}(\hat{\mathfrak{g}})\big) \ =\ \mathcal{B}_{\chi,\mu}(\mathfrak{g}) \otimes e \cong \mathcal{B}_{\chi,\pi(\nu)}(\mathfrak{g})$$

for some $\nu \in \hat{\Lambda}/W$. It is straightforward to see that ϕ sends the $U_\chi(\mathfrak{g}) \otimes U_0(\mathfrak{t}_1)$-module $Z_\chi(\psi^*\lambda) \otimes K_{\lambda_1}$ to the baby Verma module $Z_{\hat{\chi}}(\lambda)$. It follows that $\mathcal{B}_{\chi,\pi(\nu)}$ acts non-trivially on $Z_\chi(\psi^*\lambda)$. Therefore $\pi(\nu) = \pi(\lambda)$, as required. □

6.5. Let χ be as above, and define $\chi_i = \hat{\chi}|_{\tilde{\mathfrak{g}}_i}$. Since $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \dots \oplus \tilde{\mathfrak{g}}_m \oplus \mathfrak{t}_0$, we have

$$\mathcal{B}_{\hat{\chi},\lambda}(\hat{\mathfrak{g}}) \cong \mathcal{B}_{\chi_1,\lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \cdots \otimes \mathcal{B}_{\chi_m,\lambda_m}(\tilde{\mathfrak{g}}_m) \otimes \mathcal{B}_{0,\lambda_0}(\mathfrak{t}_0).$$

Since $\mathfrak{t}_0$ is toral, $\mathcal{B}_{0,\lambda_0}(\mathfrak{t}_0) \cong K$. Combining this with 6.3 and Lemma 6.4, we have the following result.

Proposition. *Keep the above notation. There is an isomorphism of algebras*

$$\mathcal{B}_{\chi_1,\lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \cdots \otimes \mathcal{B}_{\chi_m,\lambda_m}(\tilde{\mathfrak{g}}_m) \cong \mathcal{B}_{\chi,\pi(\lambda)}(\mathfrak{g})$$

such that $\chi = \chi_1 + \dots + \chi_m$ (considered as functionals on $\mathfrak{g}'$) and $W(\lambda) = W_1(\lambda_1) \times \cdots \times W_m(\lambda_m)$.

7. Regular algebras

7.1. Recall from 6.1 the definition of $\tilde{G}$. We will assume throughout this section that $G = \tilde{G}$ and $G^{(1)}$ is simple. We determine the representation type of the blocks $\mathcal{B}_{\chi,\lambda}$ for $\chi \in \mathfrak{g}^*$ regular nilpotent and $\lambda \in \Lambda/W$. Thanks to [30, Theorem 12] and [3, Proposition 3.16], the block $\mathcal{B}_{\chi,\lambda}$ is Morita equivalent to the partial coinvariant algebra

$$C_\lambda = S(\mathfrak{h})^{W(\lambda)} \otimes_{S(\mathfrak{h})^W} K, \tag{9}$$

where $W(\lambda)$ is defined as in 2.4. So we only need to calculate the representation type of C_λ. If $p = 2$ then $\mathfrak{g} = \mathfrak{sl}(V)$ (respectively $\mathfrak{g} = \mathfrak{gl}(V)$) if $\dim V$ is odd (respectively even). It is straightforward to check that in these situations the same analysis applies.

7.2. The partial coinvariant algebras of finite representation type are described in [3, Corollary 3.19], see also [32, Theorem 4.2]. They are as follows.

Theorem. *Let $\lambda \in \Lambda/W$. The algebra C_λ has finite representation type if and only if one of the following cases occurs:*

1. *$W(\lambda) = W$;*
2. *W is of type A_n and $W(\lambda)$ is of type A_{n-1};*
3. *W is of type B_n (or C_n) and $W(\lambda)$ is of type B_{n-1} (or C_{n-1}) ;*
4. *W is of type G_2 and $W(\lambda)$ is of type A_1.*

In all these cases $C_\lambda \cong K[X]/(X^r)$, where $r = [W : W(\lambda)]$.

7.3. We spend the rest of the section proving the following result.

Theorem. *Let $\lambda \in \Lambda/W$. The algebra C_λ has tame representation type if and only if one of the following cases occurs:*

1. *W has rank 2 and $W(\lambda) = 1$;*
2. *W is of type A_3 and $W(\lambda)$ is of type $A_1 \times A_1$;*
3. *W is of type B_3 (or C_3) and $W(\lambda)$ is of type A_2;*
4. *W is of type D_n and $W(\lambda)$ is of type D_{n-1}.*

We prove this by case-by-case analysis after making several general observations and simplifications.

7.4. Let $\lambda \in \Lambda/W$. By [3, 3.8] the partial coinvariant algebra C_λ is a local, symmetric, commutative algebra of dimension $[W : W(\lambda)]$. Let W_λ be the subset of W consisting of minimal length coset representatives for the subgroup $W(\lambda)$ in W, [18, 1.10]. It is shown in [3, Lemma 3.19] that C_λ admits a $\mathbb{N}$-grading such that its Poincaré series is

$$P(C_\lambda, t) = \sum_{w \in W_\lambda} t^{\ell(w)}. \tag{10}$$

7.5. Tame, local, symmetric, commutative algebras are classified in [6, Theorem III.1] (the proof follows the ideas in [40]). They have the form $K[X,Y]/I$, where I is an ideal of the following type:

1. $I = (X^m - Y^n, XY)$, where $m \geq n \geq 2$ and $m + n > 4$;
2. $I = (X^2, Y^2)$;
3. $I = (X^2, Y^2 - XY)$, where $\operatorname{char} K = 2$.

In particular any algebra with minimal number of generators greater than two is wild. Note that the first type of algebra has dimension $m + n > 4$.

7.6. As a consequence of 7.4 and 7.5, if $\lambda \in \Lambda/W$ is such that C_λ is tame, then the rank of W and the rank of $W(\lambda)$ as Coxeter groups must differ by either one or two. Indeed, if the ranks differ by more than two then C_λ has at least three generators by (10), implying wildness.

7.7. The following lemma gives a useful criterion for the wildness of a partial coinvariant algebra.

Lemma. *Suppose $\lambda \in \Lambda/W$ is such that*

$$P(C_\lambda, t) = (1 + t + \ldots + t^i) + 2(t^{i+1} + t^{i+2}) + t^{i+1}\mathbb{N}[t]$$

for some $i \geq 1$, or

$$P(C_\lambda, t) = 1 + t + 3t^2 + t\mathbb{N}[t].$$

Then C_λ is wild.

Proof. Since C_λ is local, it admits a minimal generating set consisting of elements homogeneous with respect to the grading of 7.4.

We consider the first Poincaré series. We have a generator X in degree one. If Y is a new generator in degree $j \leq i$, then $X^j = 0$. Therefore the algebra generated by X and Y has only one-dimensional homogeneous components. Since $\dim(C_\lambda)_{i+1} \geq 2$, we deduce that C_λ requires a third generator, implying wildness. Similarly, if $X^{i+1} = 0$ or $X^{i+2} = 0$, then C_λ requires at least three generators, so is wild.

Assume $X^{i+2} \neq 0$. Let Y be any element in $(C_\lambda)_{i+1}$, linearly independent from X^{i+1}. Then, since $i \geq 1$, either C_λ requires a third generator in degree $i+2$, implying wildness, or X^{i+2} and XY are linearly independent. In this last situation the non-vanishing of XY implies that C_λ cannot be an algebra of Type 7.5.1. Since $\dim C_\lambda > 4$, we deduce from 7.5 that C_λ is indeed wild.

For the second Poincaré series we need a generator in degree one and at least two new generators in degree two, implying wildness. □

7.8. We also have a sufficient criterion for the tameness of a partial coinvariant algebra.

Lemma. *Let $\lambda \in \Lambda/W$ be such that C_λ is generated in degree one with Poincaré series*

$$P(C_\lambda, t) = 1 + 2(t + t^2 + \ldots + t^{r-1}) + t^r,$$

for some $r \geq 3$. Assume that the unique quadratic relation in C_λ is square-free. Then C_λ is tame.

Proof. Since the quadratic relation in C_λ is square-free, we can find linearly independent elements $X, Y \in (C_\lambda)_1$ such that $XY = 0$. By hypothesis these elements generate C_λ. We deduce from the Poincaré series that the elements X^i and Y^i for $1 \leq i \leq r-1$ are linearly independent. Since $\dim(C_\lambda)_r = 1$, we see that X^r and Y^r are linearly dependent. Therefore either exactly one of X^r, Y^r is zero, or $0 \neq X^r = cY^r$ for some $c \in K^*$. If $X^r = 0$ (respectively $Y^r = 0$), then both X^{r-1}

(respectively Y^{r-1}) and Y^r (respectively X^r) belong to the socle of C_λ, contradicting symmetry. So, possibly after rescaling, we have an algebra isomorphism

$$C_\lambda \cong \frac{K[X,Y]}{(XY, X^r - Y^r)},$$

implying that C_λ has tame representation type, by 7.5. □

7.9. Let $\lambda \in \Lambda/W$ be such that the rank of $W(\lambda)$ is two less than the rank of W. Consider the following part of a Dynkin diagram:

$$\cdots \circ \text{———} \bullet \text{———} \bullet \cdots \quad .$$

Here (and for the rest of this section) the coloured nodes indicate the simple reflections of W not contained in $W(\lambda)$. Using the grading from (10), we see that C_λ has two elements of degree one, namely those corresponding to s_{i+1} and s_{i+2}, and at least three elements of degree two, corresponding to $s_i s_{i+1}$, $s_{i+1}s_{i+2}$ and $s_{i+2}s_{i+1}$. It follows from Lemma 7.7 that C_λ is wild in this case.

Now consider the following diagram:

$$\cdots \bullet \cdots\cdots \circ \text{———} \bullet \cdots \quad .$$

By (10) the algebra C_λ has two elements of degree one, corresponding to s_i and s_{j+1}, and at least three of degree two, corresponding to $s_i s_{j+1}$, $s_{i+1}s_i$ and $s_j s_{j+1}$. It follows from Lemma 7.7 that C_λ is wild.

7.10. So, in the case when $W(\lambda)$ has rank two less than W, the only possibility for C_λ to be tame is if W has rank two, so is one of A_2, B_2 or G_2. In case A_2 (respectively B_2 and G_2) a straightforward calculation shows that the hypotheses of Lemma 7.8 are satisfied with $r = 3$ (respectively $r = 4$ and $r = 6$), and hence C_λ is tame.

7.11. **Type A_n.** Assume that the rank of $W(\lambda)$ is one less than the rank of W. We know by Theorem 7.2.2 that if $W(\lambda)$ has type A_{n-1}, then C_λ has finite representation type. Suppose we have the following diagram:

$$\cdots \circ \text{———} \circ \text{———} \bullet \text{———} \circ \cdots \quad . \tag{11}$$

Then C_λ has a unique element of degree one corresponding to s_{i+2}, whilst there are two elements of degree two corresponding to $s_{i+1}s_{i+2}$ and $s_{i+3}s_{i+2}$. In degree three we have at least the elements corresponding to $s_i s_{i+1} s_{i+2}$ and $s_{i+1}s_{i+3}s_{i+2}$, so C_λ is wild by Lemma 7.7.

7.12. We have only the following case to consider in type A_3:

$$\circ \text{———} \bullet \text{———} \circ \quad .$$

Here C_λ is isomorphic to the algebra with generators $A = X_1 + X_2$, $B = X_3 + X_4$, $C = X_1X_2$ and $D = X_3X_4$ subject to the relations $A + B = 0$, $AB + C + D = 0$, $AD + BC = 0$ and $CD = 0$. This is a six-dimensional algebra. Calculation shows that this algebra is generated by $X = A$ and $Y = A^2 - 2C$, and that $XY = 0$ and $X^4 = Y^2$. We deduce that C_λ is isomorphic to the algebra $K[X,Y]/(X^4 - Y^2, XY)$ of Type 7.5.1, and hence tame.

7.13. **Types B_n and C_n.** We assume that $W(\lambda)$ has rank one less than W. If $W(\lambda)$ has type B_{n-1} (or C_{n-1}), then we know by Theorem 7.2.3 that C_λ has finite representation type. Consider the following diagram:

$$\cdots \circ \!-\!\!-\!\!-\! \bullet \cdots\cdots = \circ \quad . \tag{12}$$

Then C_λ has one element of degree one corresponding to s_{i+1}, two elements of degree two corresponding to $s_i s_{i+1}$ and $s_{i+1}s_{i+2}$, and elements of degree three corresponding to $s_{i+2}s_i s_{i+1}$ and at least one of $s_{i+3}s_{i+2}s_{i+1}$ or $s_{i+1}s_{i+2}s_{i+1}$ (which occurs if $i = n-2$). Therefore C_λ is wild by Lemma 7.7.

Consider the following diagram:

$$\cdots \circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \circ = \bullet \quad .$$

Then C_λ has one element of degree one corresponding to s_n, and one element of degree two corresponding to $s_{n-1}s_n$. In degree three we have the elements corresponding to $s_{n-2}s_{n-1}s_n$ and $s_n s_{n-1} s_n$, whilst in degree four we have the elements corresponding to $s_{n-3}s_{n-2}s_{n-1}s_n$ and $s_{n-2}s_n s_{n-1}s_n$. This implies that C_λ is wild by Lemma 7.7.

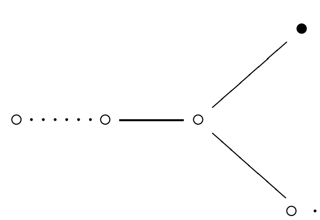
7.14. The only remaining case is the following diagram:

$$\circ \!-\!\!-\!\!-\! \circ = \bullet \quad .$$

In this case it can be checked that C_λ is the algebra generated by elements $A = X_1+X_2+X_3$, $B = X_1X_2+X_1X_3+X_2X_3$ and $C = X_1X_2X_3$ subject to the relations induced by $X_1^2+X_2^2+X_3^2 = 0$, $X_1^2X_2^2+X_1^2X_3^2+X_2^2X_3^2 = 0$ and $X_1^2X_2^2X_3^2 = 0$. This is an eight-dimensional algebra. Calculation shows that $A^2 = 2B$ and $B^2 = 2AC$ and $A^4 = 8AC$ and $C^2 = 0$. This implies that $A^7 = 0$. Since $\mathrm{char} K \neq 2$ (this is a bad prime), we deduce that C_λ is a quotient of $K[A,C]/(A^7, C^2, A^4 - 8AC)$. Making the substitution $X = \eta A$ and $Y = A^3 - 8C$, we find that C_λ is the algebra $K[X,Y]/(X^6 - Y^2, XY)$ of Type 7.5.1, and so tame.

7.15. **Type D_n.** We assume that the rank of $W(\lambda)$ is one less than the rank of W. Consider the following diagram:

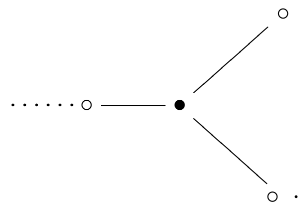
(13)



In degree one (respectively two) C_λ has a unique element corresponding to s_n (respectively $s_{n-2}s_n$). In degree three it has elements corresponding to $s_{n-1}s_{n-2}s_n$ and $s_{n-3}s_{n-2}s_n$; in degree four it has elements corresponding to $s_{n-3}s_{n-1}s_{n-2}s_n$ and $s_{n-4}s_{n-3}s_{n-2}s_n$. This implies that C_λ is wild by Lemma 7.7.

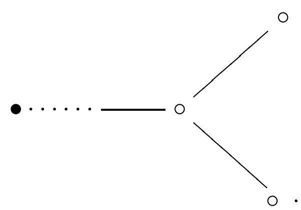
Consider the following diagram:

(14)



Then C_λ has an element in degree one corresponding to s_{n-2} whilst in degree two we have elements corresponding to $s_n s_{n-2}$, $s_{n-1}s_{n-2}$ and $s_{n-3}s_{n-2}$. Therefore C_λ is wild by Lemma 7.7.

7.16. Thanks to (11) the only remaining case is the following diagram:

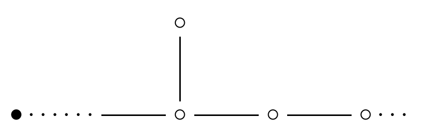


Let $\sigma_i \in K[X_1, \dots X_n]$ (respectively $\sigma'_i \in K[X_2, \dots, X_n]$) be the i^{th} elementary symmetric polynomial in n (respectively $n-1$) variables. Then the algebra C_λ is generated by the polynomials X_1, $\sigma'_i(X_2^2, \dots, X_n^2)$ for $1 \le i \le n-2$, and $X_2 \dots X_n$, subject to the relations induced by $\sigma_i(X_1^2, \dots, X_n^2) = 0$ for $1 \le i \le n-1$ and $X_1 \dots X_n = 0$. This is a $2n$-dimensional algebra. Let $A = X_1$ and $B = X_2 \dots X_n$. The formula

$$\sigma_i(X_1^2, \dots, X_n^2) = X_1^2 \sigma'_{i-1}(X_2^2, \dots, X_n^2) + \sigma'_i(X_2^2, \dots, X_n^2)$$

shows by induction that A and B generate this algebra. In fact we find that $\sigma'_i = (-A^2)^i$ for $1 \le i \le n-1$. It follows that $B^2 = (-A^2)^{n-1}$, and it is clear that $AB = 0$. Thus C_λ is a tame algebra of Type 7.5.1.

7.17. **Type E.** We assume that the rank of $W(\lambda)$ is one less than the rank of W. Consider the following diagram:



Arguing as in the previous subsections, we find that C_λ has a Poincaré series of the form described in Lemma 7.7, and is therefore wild. Other possibilities for diagrams are dealt with in (11), (13) and (14).

7.18. **Type F_4.** Suppose the rank of $W(\lambda)$ is three. Consider the following diagram:

Then in degrees one, two and three respectively C_λ has elements corresponding to s_1, s_2s_1 and $s_3s_2s_1$ respectively, whilst in degree four we have elements corresponding to $s_2s_3s_2s_1$ and $s_4s_3s_2s_1$. In degree five C_λ has elements corresponding to $s_1s_2s_3s_2s_1$ and $s_2s_4s_3s_2s_1$. It follows from Lemma 7.7 that C_λ is wild. The case (12) finishes type F_4, and therefore the proof of Theorem 7.3.

8. Complexity

8.1. The purpose of this section is to find a necessary condition on $\chi \in \mathfrak{g}^*$ and $\lambda \in \Lambda/W$ for $\mathcal{B}_{\chi,\lambda}$ to be tame, and to prove Theorems 5.2 and 5.4 (we continue assuming that χ vanishes on $\mathfrak{b}^+$). Our principal tools will be the results of Sections 3 and 4.

8.2. We assume until 8.14 that $G = \tilde{G}$ and that $G^{(1)}$ is simple. In the next lemma we do not assume that the parabolic subgroup $W(\lambda)$ is standard in W.

Lemma. *Let $\lambda \in \Lambda/W$.*

1. *Suppose that $W(\lambda)$ does not contain simple reflections $s_{i(1)}, \ldots, s_{i(d)}$ such that $i(j)$ and $i(k)$ are not adjacent on the Dynkin diagram of Δ for $j \neq k$. Then $G.(e_{\alpha_{i(1)}} + \ldots + e_{\alpha_{i(d)}}) \subseteq \mathcal{V}_\mathfrak{g}(0, \lambda)$.*
2. *If $W(\lambda) \neq W$, then $G.e_\alpha \subseteq \mathcal{V}_\mathfrak{g}(0, \lambda)$ for some short root $\alpha \in \Delta$.*

Proof. 1. There exists $\nu \in X(T)$ such that $0 < \langle \nu, \alpha_i^\vee \rangle \leq p$ for all i, and $d\nu = \lambda$. Let V denote the Weyl module for G with highest weight $\nu - \rho$. Differentiating the action of G on V gives V a natural $\mathcal{B}_{0,\lambda}$-module structure. Let G_0 denote the subgroup of G generated by all $U_{\pm\alpha_{i(k)}}$, and $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$. Obviously $e := e_{\alpha_{i(1)}} + \ldots + e_{\alpha_{i(d)}} \in \mathfrak{g}_0$. Let V_0 denote the subspace of V spanned by all weight spaces V_μ with $\nu - \rho - \mu \in \mathbb{Z}\alpha_{i(1)} \oplus \ldots \oplus \mathbb{Z}\alpha_{i(d)}$. By construction V_0 is a direct summand of the $\mathfrak{g}_0$-module V. It is well-known (and easy to see) that $\dim V_0 = \prod_k \nu(h_{\alpha_{i(k)}})$. Since $s_{i(k)} \notin W(\lambda)$, we have that $\nu(h_{\alpha_{i(k)}}) \leq p - 1$ for all k. Therefore $p \nmid \dim V_0$. In particular, V_0 is not a free $\langle e \rangle$-module. Since all $e_{\alpha_{i(k)}}$ commute, we also have that $e \in \mathcal{N}_p(\mathfrak{g})$. Using our discussion in 4.1, we now deduce that $e \in \mathcal{V}_\mathfrak{g}(0, \lambda)$.

2. Since the short roots in Φ span the root lattice Q and p is a good prime for G, there is a short root $\beta \in \Phi$ such that $\langle \lambda, \beta^\vee \rangle \not\equiv 0\,(p)$, for otherwise $W(\lambda) = W$, contradicting our assumption. Since $G^{(1)}$ is simple, all short roots in Φ are conjugate under W. Replacing λ by its W-conjugate, we can assume that there exists a short root $\alpha \in \Delta$ such that $\langle \lambda, \alpha^\vee \rangle \not\equiv 0\,(p)$. Then $s_\alpha \notin W(\lambda)$. Applying the first part of this lemma (with $d = 1$), we now deduce that $e_\alpha \in \mathcal{V}_\mathfrak{g}(0, \lambda)$. Since this set is G-invariant, the result follows. ∎

8.3. In the following subsections we give lower bounds for $\dim \mathcal{V}_\mathfrak{g}(\chi, \lambda)$ in a number of important cases. We let $e = \theta^{-1}(\chi) \in \mathcal{N}$ be the element of $\mathfrak{g}$ corresponding to χ. Recall that by 2.2 and Theorem 4.2, $\dim \mathcal{V}_\mathfrak{g}(\chi, \lambda) = \dim \mathcal{V}_\mathfrak{g}(0, \lambda) \cap \mathfrak{z}_\mathfrak{g}(e)$.

Given a subset $J \subseteq \Delta$, we denote by $\mathfrak{u}_J$ the Lie algebra of the unipotent radical of the standard parabolic of type J in G.

8.4. **Type A_n.** For $\alpha = \alpha_i + \ldots + \alpha_j \in \Phi$ we choose as a root vector e_α the matrix $E_{i,j+1}$ whose $(i, j+1)^{\text{th}}$ entry equals 1 and all other entries are zero, and we set $e_{-\alpha} = {}^tE_{i,j+1}$, the transpose of $E_{i,j+1}$. Assume that $W(\lambda) \neq W$. First we consider the subregular case.

Assume $n \geq 2$. Let $e = e_{\alpha_1} + \ldots + e_{\alpha_{n-1}}$ and $J = \{2, \ldots, n\}$. The orbit $G.e_{\alpha_1}$ is minimal in $\mathcal{N}$, and so is the Richardson class corresponding to $\mathfrak{u}_J$; that is, it intersects densely with $\mathfrak{u}_J$. Thus $\overline{G.e_{\alpha_1}} = G.\mathfrak{u}_J$. We have

$$Ke_{\alpha_1+\ldots+\alpha_{n-1}} + Ke_{\alpha_1+\ldots+\alpha_n} \subseteq G.\mathfrak{u}_J \cap \mathfrak{z}_{\mathfrak{g}}(e),$$

so it follows that $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 2$.

Now assume $n = 1$. Then zero is the subregular orbit, and the closure of the minimal orbit is the nilpotent cone. It is immediate that $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) = 2$ in this situation.

8.5. The closure ordering of conjugacy classes in type A_n is given by the dominanace ordering on partitions of n, [19, 7.19]. It can be checked that for $n \geq 2$ there is a unique maximal non-regular, non-subregular class. If $n \geq 3$ this corresponds to the partition $(n-1, 2)$. Let $e = e_{\alpha_1} + \ldots + e_{\alpha_{n-2}} + e_{\alpha_n}$ be a representative of this orbit. The element e is centralised by the set

$$Ke_{\alpha_1+\ldots+\ldots\alpha_{n-2}} + Ke_{\alpha_1+\ldots+\alpha_n} + Ke_{-\alpha_{n-1}} + Ke_{\alpha_n} \subset \mathcal{N}_p.$$

The orbit $G.e_{\alpha_1}$ is characterised as those matrices of rank one whose square is zero. We deduce that the set

$$\{x_1 e_{\alpha_1+\ldots+\alpha_{n-2}} + x_2 e_{\alpha_1+\ldots+\alpha_n} + x_3 e_{-\alpha_{n-1}} + x_4 e_{\alpha_n} : x_1x_4 - x_2x_3 = 0\}$$

is contained in $\overline{G.e_{\alpha_1}} \cap \mathfrak{z}_{\mathfrak{g}}(e)$. Therefore $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$.

Now assume $n = 2$. We have to consider the zero orbit. We have $\overline{G.e_{\alpha_2}} = G.\mathfrak{u}_{\{2\}}$. Since any element in $\mathfrak{u}_{\{2\}}$ has square zero, we deduce that $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(0) \geq \dim G.\mathfrak{u}_{\{2\}} = 4$.

8.6. Suppose $n = 3$ and $W(\lambda) = \Sigma_2 \times \Sigma_2$. Adopt the notation of [23, Proposition 2.6]. It is easy to see that the subsystem R_λ of Φ has type $A_1 \times A_1$ and the partition ${}^t\pi(\lambda)$ of $n+1 = 4$ introduced in [23, 2.6] equals $(2,2)$. Combining [23, Proposition 2.6] with our discussion in 4.1, we derive that $e_{\alpha_1} + e_{\alpha_3} \in \mathcal{V}_{\mathfrak{g}}(0,\lambda)$. As a consequence, $\mathcal{V}_{\mathfrak{g}}(0,\lambda)$ contains all matrices in $\mathfrak{g}$ of rank two whose square is zero. Let $e = e_{\alpha_1} + e_{\alpha_2}$, a subregular nilpotent element in $\mathfrak{g}$. The element e is centralised by $e_{\alpha_1} + e_{\alpha_2}$, $e_{\alpha_1+\alpha_2}$, $e_{\alpha_1+\alpha_2+\alpha_3}$ and $e_{-\alpha_3}$. The set

$$\{x_1(e_{\alpha_1} + e_{\alpha_2}) + x_2 e_{\alpha_1+\alpha_2} + x_3 e_{\alpha_1+\alpha_2+\alpha_3} + x_4 e_{-\alpha_3} : x_1^2 + x_3x_4 = 0\}$$

is a three-dimensional subvariety of $\overline{G.(e_{\alpha_1} + e_{\alpha_3})} \cap \mathfrak{z}_{\mathfrak{g}}(e)$. It follows that

$$\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3.$$

8.7. Suppose that $n = 2$ and $W(\lambda) = 1$. Then λ is a regular weight. Hence p is greater than or equal to 3, the Coxeter number, and $\mathcal{N}_p = \mathcal{N}$. Moreover, $\mathcal{V}_{\mathfrak{g}}(0,\lambda) = \mathcal{N}_p$, by [23, Proposition 2.6]. Let $e = e_{\alpha_1}$, a subregular element. We have

$$Ke_{\alpha_1} + Ke_{\alpha_1+\alpha_2} + Ke_{-\alpha_2} \subseteq \mathcal{N} \cap \mathfrak{z}_{\mathfrak{g}}(e),$$

showing that $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$.

8.8. **Type B_n.** We begin by identifying the orbit $G.e_{\alpha_n}$. Since α_n is a short root, the closure of $G.e_{\alpha_n}$ contains the minimal nilpotent orbit strictly, so we deduce that $\dim G.e_{\alpha_n} \geq 4(n-1) + 2$. Let $J = \{2, \ldots, n\}$. We have $G.\mathfrak{u}_J \supseteq G.e_{\alpha_n}$. Moreover, by [19, Theorem 5.3], $\dim G.\mathfrak{u}_J = 2\dim \mathfrak{u}_J = 2(n^2 - (n-1)^2) = 4n - 2$. We deduce that $\overline{G.e_{\alpha_n}} = G.\mathfrak{u}_J$, the closure of the Richardson class of $\mathfrak{g}$ corresponding to J. Since $e_{\alpha_n}^{[p]} = 0$, we must have $x^{[p]} = 0$ for all $x \in \overline{G.e_{\alpha_n}}$. In particular, $\mathfrak{u}_J^{[p]} = 0$.

Now let $e = e_{\alpha_2} + \ldots + e_{\alpha_n}$, a subregular element [26, 3.2]. We mention for further reference that e is conjugate under $N_G(T)$ to $e_{\alpha_1} + \ldots + e_{\alpha_{n-2}} + e_{\alpha_{n-1}+\alpha_n}$.

The centraliser of e includes the elements $e_{-\alpha_1}$, $e_{\alpha_1+2\alpha_2+\ldots+2\alpha_n}$ and $e_{\alpha_2+2\alpha_3+\ldots+2\alpha_n}$ (if $n=2$ take $e_{-\alpha_1}$, e_{α_2} and $e_{\alpha_1+2\alpha_2}$). Applying a preimage of s_1 in $N_G(T)$ to these elements yields non-zero multiples of e_{α_1}, $e_{\alpha_1+2\alpha_2+\ldots+2\alpha_n}$ and $e_{\alpha_1+\alpha_2+2\alpha_3+\ldots+2\alpha_n}$ (respectively $e_{\alpha_1}, e_{\alpha_1+\alpha_2}$ and $e_{\alpha_1+2\alpha_2}$), all of which lie in $\mathfrak{u}_J$. Therefore

$$\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda)\cap \mathfrak{z}_{\mathfrak{g}}(e)\geq 3.$$

8.9. **Type C_n.** We can assume that $n\geq 3$. Let us describe $G.e_{\alpha_{n-1}}$. To this end let

$$\beta = s_2 s_3 \ldots s_{n-1} s_1 s_2 \ldots s_{n-2} s_n(\alpha_{n-1}),$$

a short root. We have

$$\langle \alpha_i, \beta^\vee\rangle = \begin{cases} 1 & \text{if } i=2,\\ 0 & \text{otherwise.}\end{cases}$$

By [44, IV.2.2.8] the dimension of the orbit is independent of p. Using the grading on $\mathfrak{g}$ induced by the above, we therefore calculate that $\dim G.e_\beta = 4n-2$. Let $J=\{2,\ldots,n\}$. Then, by [19, Theorem 5.3], $\dim G.\mathfrak{u}_J = 2\dim \mathfrak{u}_J = 4n-2$. Since $e_\beta\in\mathfrak{u}_J$ we deduce that $\overline{G.e_\beta} = G.\mathfrak{u}_J$, the closure of the Richardson class corresponding to J.

Let $e = e_{\alpha_2}+\ldots+e_{\alpha_n}+e_{2\alpha_1+\ldots 2\alpha_{n-1}+\alpha_n}$. It is easily seen that e is conjugate under $N_G(T)$ to $e_{\alpha_1}+\ldots+e_{\alpha_{n-2}}+e_{\alpha_n}+e_{2\alpha_{n-1}+\alpha_n}$, a Richardson element in $\mathfrak{u}_{\{n-1\}}$. It follows that e is a subregular nilpotent element of $\mathfrak{g}$. The centraliser of e includes the elements $e_{2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n}, e_{\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n}$ and $e_{2\alpha_1+\ldots+2\alpha_{n-1}+\alpha_n}$. It is straightforward to check that the set

$$U_{-\alpha_1}.(Ke_{\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n}+Ke_{2\alpha_1+\ldots+2\alpha_{n-1}+\alpha_n})\subseteq G.\mathfrak{u}_J\cap\mathfrak{z}_{\mathfrak{g}}(e),$$

is three-dimensional. We deduce that $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda)\cap\mathfrak{z}_{\mathfrak{g}}(e)\geq 3$.

8.10. **Type D_n.** The present assumption on G implies that $G=G^{(1)}$. Assume that $W(\lambda)$ is a standard parabolic subgroup of W of type D_{n-1}. Then $\lambda = r\varpi_1$ for some $r\in\mathbb{F}_p^*$. Adopt Bourbaki's notation for the root system of type D_n. Since $\varpi_1=\epsilon_1$, we can find $w\in W$ such that $w\lambda = r\epsilon_{n-1}$. Set $\mu = r\epsilon_{n-1}$. For $k=n-1,n$ we have

$$\mu(h_{\alpha_k}) = \frac{2r(\epsilon_{n-1}|\epsilon_{n-1}\pm\epsilon_n)}{(\epsilon_{n-1}\pm\epsilon_n|\epsilon_{n-1}\pm\epsilon_n)} = r\neq 0.$$

Therefore we can assume in what follows that λ is such that $s_{n-1}, s_n\notin W(\lambda)$. Then, by Lemma 8.2, $G.(e_{\alpha_{n-1}}+e_{\alpha_n})\subset\mathcal{V}_{\mathfrak{g}}(0,\lambda)$. The present assumption on G implies that $G=G^{(1)}$.

8.11. Let σ denote the outer involution of the algebraic group G induced by the nontrivial symmetry of Δ. It induces an automorphism of $\mathfrak{g}$, also denoted by σ, which swaps $e_{\pm\alpha_{n-1}}$ and $e_{\pm\alpha_n}$ and fixes $e_{\pm\alpha_i}$ for $i\leq n-2$. Let G^0 denote the connected component of the fixed point group G^σ, and $\mathfrak{g}^\sigma$ the fixed point subalgebra of σ. It is well-known (and not hard to see) that $\mathfrak{g}^\sigma = \mathrm{Lie}(G^0)$ has type B_{n-1} and is generated by $e_{\pm\alpha_1},\ldots,e_{\pm\alpha_{n-2}}$ and $e_{\pm\alpha_{n-1}}+e_{\pm\alpha_n}$. Moreover, the elements e_{α_i} with $i\leq n-2$ together with $e_{\alpha_{n-1}}+e_{\alpha_n}$ generate a maximal nilpotent subalgebra of $\mathfrak{g}^\sigma$ and can be viewed as simple root vectors for $\mathfrak{g}^\sigma$ with respect to $\mathfrak{h}^\sigma=\mathfrak{g}^\sigma\cap\mathfrak{h}$. Let $\beta_1,\ldots,\beta_{n-1}$ denote the corresponding simple roots, so that $e_{\beta_i}=e_{\alpha_i}$ for $i\leq n-2$ and $e_{\beta_{n-1}} = e_{\alpha_{n-1}}+e_{\alpha_n}$. Note that β_{n-1} is a *short* root of $\mathfrak{g}^\sigma$.

8.12. The element $e = e_{\alpha_1} + \ldots + e_{\alpha_{n-3}} + e_{\alpha_{n-3}+\alpha_{n-2}} + e_{\alpha_{n-3}+\alpha_{n-1}} + e_{\alpha_{n-3}+\alpha_n}$, fixed by σ, is Richardson in $\mathfrak{u}_{\{n-2\}}$, hence subregular in $\mathfrak{g}$. There exist root vectors in $\mathfrak{g}^\sigma$ such that $e = e_{\beta_1} + \ldots + e_{\beta_{n-3}} + e_{\beta_{n-3}+\beta_{n-2}} + e_{\beta_{n-2}+\beta_{n-1}}$. This is conjugate under $U_{\beta_{n-2}} = U_{\alpha_{n-2}}$ to $e_{\beta_1} + \ldots + e_{\beta_{n-3}} + e_{\beta_{n-2}+\beta_{n-1}}$, a subregular nilpotent element of $\mathfrak{g}^\sigma$ (see 8.8). Since $e_{\beta_{n-1}}$ is a short root vector in $\mathfrak{g}^\sigma$ we have, by 8.8, that

$$\dim \overline{G^0.\, e_{\beta_{n-1}}} \cap \mathfrak{z}_{\mathfrak{g}^\sigma}(e) \geq 3.$$

Since $\mathfrak{z}_{\mathfrak{g}^\sigma}(e) \subseteq \mathfrak{z}_{\mathfrak{g}}(e)$ and $e_{\beta_{n-1}} = e_{\alpha_{n-1}} + e_{\alpha_n}$ our discussion in 8.10 yields

$$\dim V_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3.$$

8.13. **Type G_2.** The element $e = e_{\alpha_2} + e_{\alpha_1+2\alpha_2}$ is conjugate under $N_G(T)$ to $e_{\alpha_1+\alpha_2} + e_{\alpha_1+2\alpha_2}$, a Richardson element in $\mathfrak{u}_{\{2\}}$. Hence e is subregular in $\mathfrak{g}$. Then $\mathfrak{z}_{\mathfrak{g}}(e)$ has a basis consisting of $e_{-\alpha_1}$, $e_{\alpha_1+3\alpha_2}$, $e_{2\alpha_1+3\alpha_2}$ and e. It is easily seen that $U_{\alpha_1+2\alpha_2}U_{-\alpha_1-\alpha_2}(Ke_{\alpha_2})$ is a dense subset of $Ke_{\alpha_2} \bigoplus Ke_{-\alpha_1} \bigoplus Ke_{\alpha_1+3\alpha_2}$. From this it is immediate that the Zariski closure Y of $U_{\alpha_1+\alpha_2}U_{\alpha_1+2\alpha_2}U_{-\alpha_1-\alpha_2}(Ke_{\alpha_2})$ is an irreducible, conical hypersurface in

$$X := Ke_{\alpha_2} \oplus Ke_{-\alpha_1} \oplus Ke_{\alpha_1+2\alpha_2} \oplus K_{\alpha_1+3\alpha_2} \oplus Ke_{2\alpha_1+3\alpha_2} \cong \mathbb{A}^5.$$

Since $\mathfrak{z}_{\mathfrak{g}}(e)$ is a hyperplane in X, all irreducible components of $Y \cap \mathfrak{z}_{\mathfrak{g}}(e)$ must be at least three-dimensional. As a consequence, $\dim \mathcal{V}_{\mathfrak{g}}(0,\lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$.

8.14. Let us return to general G satisfying the hypotheses of 2.1. Recall the *complexity* of a module over a finite dimensional algebra is the rate of growth of its minimal projective resolution. If we consider a $U_\chi(\mathfrak{g})$-module M, then the complexity of M equals $\dim \mathcal{V}_{\mathfrak{g}}(M)$, see [9, Proposition 3.2] and [11, Section 6]. Thus the following lemma provides the link between rank varieties and representation type.

Lemma. *Let $\chi \in \mathcal{N}$ and $\lambda \in \Lambda/W$.*

1. *The algebra $\mathcal{B}_{\chi,\lambda}$ is simple if and only if $\mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = 0$.*
2. [7, Theorem 3.2] *The algebra $\mathcal{B}_{\chi,\lambda}$ has finite representation type if and only if $\dim \mathcal{V}_{\mathfrak{g}}(\chi,\lambda) \leq 1$.*
3. [39, Theorem 2] *If $\mathcal{B}_{\chi,\lambda}$ has tame representation type, then $\mathcal{V}_{\mathfrak{g}}(\chi,\lambda)$ is two-dimensional.*

8.15. Thanks to Proposition 6.5, in order to prove the results of Section 5, we may assume that $G = \tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_m$, $\mathfrak{g} = \tilde{\mathfrak{g}}_1 \oplus \ldots \oplus \tilde{\mathfrak{g}}_m$, $\chi = \chi_1 + \ldots + \chi_m$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$. In this situation

$$\mathcal{B}_{\chi,\lambda}(\mathfrak{g}) \cong \mathcal{B}_{\chi_1,\lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \cdots \otimes \mathcal{B}_{\chi_m,\lambda_m}(\tilde{\mathfrak{g}}_m).$$

As in [37, p.242], we have $\mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = \mathcal{V}_{\tilde{\mathfrak{g}}_1}(\chi_1,\lambda_1) \times \cdots \times \mathcal{V}_{\tilde{\mathfrak{g}}_m}(\chi_m,\lambda_m)$. In particular,

$$\dim \mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = \sum_{j=1}^{m} \dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j,\lambda_j). \tag{15}$$

Let us remark that, thanks to [12, Theorem 4.2], if $W_i(\lambda_i) = W_i$, then $\mathcal{B}_{\chi_i,\lambda}(\tilde{\mathfrak{g}}_i)$ is simple (this also follows from Theorem 4.2 combined with the irreducibility and projectivity of the restricted Steinberg module).

8.16. **Proof of Theorem 5.2.** It follows from Theorem 7.2 that the blocks considered in Theorem 5.2 have finite representation type.

Thanks to 8.14.2, $\mathcal{B}_{\chi,\lambda}$ has finite representation type if and only if $\mathcal{V}_{\mathfrak{g}}(\chi,\lambda)$ has dimension at most 1. So by (15) $\mathcal{B}_{\chi,\lambda}$ has finite representation type if and only if there exists j, $1 \leq j \leq m$, such that $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i,\lambda_i) = 0$ for $i \neq j$ and $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j,\lambda_j) \leq 1$.

Suppose $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(0,\lambda)$ is non-zero. Since it is a closed $\tilde{G}_i$-invariant variety, it contains $e_{\tilde{\alpha}}$, where $\tilde{\alpha}$ is the longest root of $\tilde{\mathfrak{g}}_i$. Since $\chi(\mathfrak{n}^+ \cap \tilde{\mathfrak{g}}_i) = 0$, we see that $e_{\tilde{\alpha}} \in \mathcal{V}_{\tilde{\mathfrak{g}}_i}(0,\lambda) \cap \mathfrak{z}_{\tilde{\mathfrak{g}}_i}(\chi_i)$. We deduce that $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(0,\lambda) = 0$ for all $i \neq j$, implying $W(\lambda_i) = W$ for all such i by Lemma 8.2.2. By Lemma 8.14.1 we have that $\mathcal{B}_{\chi_i,\lambda_i}(\tilde{\mathfrak{g}}_i)$ is a simple algebra for all such i.

Suppose $\mathcal{B}_{\chi,\lambda}$ is of a type not considered in Theorem 5.2. By Theorem 7.2 χ_j is not regular. If ξ_j is regular nilpotent, then $\mathcal{B}_{\xi_j,\lambda_j}(\tilde{\mathfrak{g}}_j)$ has finite representation type by Corollary 3.3. Thus the pair $(W_j, W_j(\lambda_j))$ must be of the type described in Theorem 7.2. Since χ_j is not regular, however, the calculations of 8.4, 8.8, 8.9 and 8.13 show that $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j,\lambda_j) \geq 2$, a contradiction.

8.17. We give a necessary condition for tame representation type.

Proposition. *Let $\chi \in \mathcal{N}$ and $\lambda \in \Lambda/W$. If $\mathcal{B}_{\chi,\lambda}$ has tame representation type, then χ and λ satisfy one of the conditions of Theorem 5.3.*

Proof. Suppose $\mathcal{B}_{\chi,\lambda}$ is tame. By Lemma 8.14.3 we must have $\dim \mathcal{V}_{\mathfrak{g}}(\chi,\lambda) = 2$. By (15) there are two cases to consider.

1. There exist j_1, j_2, $1 \leq j_1 < j_2 \leq m$, such that $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i,\lambda_i) = \delta_{ij_1} + \delta_{ij_2}$.
2. There exists j, $1 \leq j \leq m$, such that $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i,\lambda_i) = 2\delta_{ij}$.

In Case 1, arguing as in 8.16, the blocks $\mathcal{B}_{\chi_i,\lambda_i}(\tilde{\mathfrak{g}}_i)$ are simple and $W_i(\lambda_i) = W_i$ for $i \neq j_1, j_2$. By Lemma 8.14.2, Theorem 5.2 and Theorem 7.2 the block $\mathcal{B}_{\chi_{j_k},\lambda_{j_k}}(\tilde{\mathfrak{g}}_{j_k})$ is isomorphic to $K[X]/(X^{r_k})$, where $k = 1, 2$ and $r_k = [W_{j_k} : W_{j_k}(\lambda)]$. By 7.5 the tensor product of these algebras is tame if and only if

$$[W_{j_1} : W_{j_1}(\lambda)] = [W_{j_2} : W_{j_2}(\lambda)] = 2.$$

This is equivalent to Condition 3 in Theorem 5.3.

In Case 2 the blocks $\mathcal{B}_{\chi_i,\lambda_i}(\tilde{\mathfrak{g}}_i)$ are simple and $W(\lambda_i) = W_i$ for $i \neq j$. By Corollary 3.3, if ξ_j is regular then the block $\mathcal{B}_{\xi_j,\lambda_j}(\tilde{\mathfrak{g}}_j)$ has either finite or tame representation type. Thus the pair $(W_j, W_j(\lambda_j))$ belongs to one of the cases of Theorem 7.2 and Theorem 7.3. Suppose χ_j is itself not regular. Then, applying Corollary 3.3 once again, we deduce that $\mathcal{B}_{\zeta_j,\lambda_j}(\tilde{\mathfrak{g}}_j)$ has finite or tame representation type for subregular ζ_j. In particular, $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\zeta_j,\lambda_j) \leq 2$. The calculations in 8.6 - 8.13 show that W_j is of type of A_n and $W_j(\lambda_j)$ is of type A_{n-1}. If χ_j is not itself subregular, then 8.5 shows that $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j,\lambda_j) \geq 3$, contradicting tameness. □

As a consequence it only remains to show that Condition 2 of Theorem 5.3 yields a tame block. This will be done in Section 9.

8.18. **Proof of Theorem 5.4.** We assume that Theorem 5.3 has been proved. This is proved in [38, Proposition 5.3] that U_χ has finite representation type if and only if one of the conditions of Part 1 is satisfied. It is also a consequence of Theorem 5.2, see [32, 4.3].

We want to describe χ for which all blocks $\mathcal{B}_{\chi,\lambda}$ are tame. It is straightforward to check using Theorems 5.2 and 5.3 that all the algebras mentioned in Theorem 5.4.2

are tame. We need to discount all other possibilities. It follows from Theorems 5.2 and 5.3 that if $\mathcal{B}_{\chi,\lambda}$ is tame then χ must be regular or subregular.

Suppose first that χ_j is subregular, and assume $p \neq 2$. Then $\tilde{\mathfrak{g}}_j$ is of type A_n by Theorem 5.3.2, and $\operatorname{rank} W_j(\lambda_j) \geq \operatorname{rank} W_j - 1$ for all $\lambda \in \Lambda/W$. If $n \geq 3$, then the parabolic subgroup of W corresponding to any weight whose restriction to $T \cap G^{(1)}$ equals $\varpi_{n-1} + \varpi_n$ contains at least $\operatorname{rank} W_j - 2$ simple reflections. If $p = 2$ and $n \geq 3$ then the parabolic subgroup corresponding to 0 is reducible as a Coxeter group. But then U_χ has a wild block by Theorem 5.3. The only case remaining (apart from A_1) is A_2. Since the Coxeter number is 3, for $p \geq 3$ there are λ such that $W(\lambda) = 1$, by [22, II.6.2]. We deduce that $\tilde{\mathfrak{g}}_j$ is of type A_1 (any p) or A_2 ($p = 2$). In both cases $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j) = 2$ is achieved, and so by (15) $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i, \lambda_i) = 0$ for all λ_i and all $i \neq j$. This implies $W_i(\lambda_i) = W_i$ for all $i \neq j$ and all λ_i. Therefore W_i is trivial for all $i \neq j$. This yields cases 2(a) and 2(b).

Suppose that χ_j is regular for $1 \leq j \leq m$. If $\tilde{\mathfrak{g}}_j$ is of type A_n, then the argument of the previous paragraph shows that either $n \leq 2$ or $n = 3$ and $p = 2$ (and if $n = 1$ then we have finite representation type). If $\tilde{\mathfrak{g}}_j$ has type B_n or C_n, then $G_j = G_j^{(1)}$, and the parabolic subgroup of W corresponding to ϖ_n is of type A_{n-1}. Therefore $n \leq 3$. If $n = 3$ the Coxeter number is 6, so for $p \geq 7$ there exists λ such that $W_j(\lambda_j) = 1$. If $p = 5$, then the weight $\varpi_2 + \varpi_3$ provides a wild block, whilst if $p = 3$, the weight ϖ_2 yields a wild block. If $\tilde{\mathfrak{g}}_j$ has type D_n, then again $G_j = G_j^{(1)}$, and the weight ϖ_n yields a parabolic subgroup of type A_{n-1}, and so yields a wild block unless $n = 4$. In case D_4 the Coxeter number is 6, so we need to consider the cases $p = 3$ and $p = 5$ only. In both cases the weight ϖ_2 yields a wild block. By Theorem 5.3 there are no tame blocks in types E and F_4. If $G^{(1)}$ has more than two simple components, then (15) shows that $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \geq 3$ for some λ, implying wildness. If there are two simple components, then $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \geq 3$ for some λ unless the components belong to the finite representation list of Theorem 5.4. It follows from 7.5 and Theorem 7.2 that both components must be of type A_1. The theorem follows.

9. The subregular tame case

9.1. In this section we show that the blocks occurring in Theorem 5.3.2 have tame representation type. Thanks to Theorem 7.3 and Proposition 8.17, this completes the proof of Theorem 5.3. The proof relies heavily on the results of [26].

9.2. Let $G = SL_{n+1}(K)$ if $\operatorname{char} K = p$ does not divide $n+1$, and $G = GL_{n+1}(K)$ if $\operatorname{char} K = p$ divides $n+1$. Let $\mathfrak{g} = \operatorname{Lie}(G)$, so that $\mathfrak{g} = \mathfrak{sl}_{n+1}(K)$, respectively $\mathfrak{g} = \mathfrak{gl}_{n+1}(K)$. Let $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow K$ be the non-degenerate G-invariant bilinear form defined by $B(x, y) = \operatorname{tr}(xy)$ for $x, y \in \mathfrak{g}$. Choose root vectors e_α as in 8.4 and let $\chi \in \mathfrak{g}^*$ be the subregular nilpotent element defined by $\chi(x) = B(x, e)$, where $e = e_{\alpha_1} + \ldots + e_{\alpha_{n-1}}$.

Let $\lambda \in \Lambda/W$ be such that the group $W(\lambda)$ is the standard parabolic subgroup of W generated by the simple reflections $s_1, \ldots, s_{n-1}$.

9.3. We will be concerned with the category of finite dimensional U_χ-modules, U_χ-mod, or more specifically the subcategory of $\mathcal{B}_{\chi,\lambda}$-modules. These categories have graded analogues, which we introduce now.

If $G = SL_{n+1}(K)$ let

$$T_0 = \bigcap_{i=1}^{n-1} \ker(\alpha_i), \tag{16}$$

whilst if $G = GL_{n+1}(K)$ let

$$T_0 = \{\tau E_{1,1} + \ldots + \tau E_{n,n} + E_{n+1,n+1} : \tau \in K^*\}. \tag{17}$$

Thus, in either case, T_0 is a one-parameter subgroup of the torus $T \subset G$ such that $\chi(\mathrm{Ad}(t)x) = \chi(x)$ for all $x \in \mathfrak{g}$ and $t \in T_0$. As a result the adjoint action of T_0 on U passes to an action on the quotient U_χ.

We define a U_χ-T_0-module to be a finite dimensional vector space V over K that has a structure both as a U_χ-module and as a rational T_0-module such that the following compatibility conditions hold:

1. We have $t(xv) = (\mathrm{Ad}(t)x)tv$ for all $x \in \mathfrak{g}, t \in T_0$ and $v \in V$.
2. The restriction of the $\mathfrak{g}$-action on V to $\mathrm{Lie}(T_0) \subseteq \mathfrak{h}$ is equal to the derivative of the T_0-action on V.

We obtain the category, U_χ-T_0-mod, with objects the U_χ-T_0-modules and morphisms T_0-equivariant U_χ-module homomorphisms. Since $\mathcal{B}_{\chi,\lambda} = e_\lambda U_\chi$ for some G-invariant element e_λ, the full subcategory $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod of the category U_χ-T_0-mod is well-defined. Its objects are $\mathcal{B}_{\chi,\lambda}$-modules with a compatible rational T_0-action.

Remark. In [26] the case $G = SL_{n+1}(K)$ where p does not divide $n+1$ is considered, and the category U_χ-T_0-mod is studied where T_0 is as in (16). The results of [26, Section 2] continue to hold for $G = GL_{n+1}(K)$ where p divides $n+1$, if we choose T_0 as in (17). The proofs can be repeated almost verbatim. The only difference occurs in character formulae, where appearances of $(n+1)$ in [26] should be replaced by 1. This is due essentially to the following fact: if ϕ is the cocharacter of T corresponding to T_0 in (16) (respectively (17)), then $\langle \alpha_n, \phi\rangle = n+1$ (respectively $\langle \alpha_n, \phi\rangle = 1$). From now on we will use the results in [26] in both cases without further comment.

9.4. Let $F : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-mod}$ denote the functor which forgets the T_0-structure. The objects of U_χ-mod which are in the image of F are called *gradable*. Suppose M is gradable, that is, there exists a U_χ-T_0-module V such that $F(V) = M$. Then, by [25, Remark 1.5], we have $F(\operatorname{soc} V) = \operatorname{soc} M$ and $F(\operatorname{rad} V) = \operatorname{rad} M$.

It follows from [17, Corollary 3.4] and [25, Corollary 1.4.1] that the simple U_χ-modules and their projective covers are gradable. Moreover, any lift of a simple U_χ-module is simple in U_χ-T_0-mod and any lift of a projective indecomposable U_χ-module is projective indecomposable in U_χ-T_0-mod.

9.5. The category U_χ-T_0-mod has shift functors

$$[\pm 1] : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-}T_0\text{-mod}.$$

These send a given U_χ-T_0-module V to the object having the same U_χ-module structure but with $t \in T_0$ acting by $t.v = (\pm p\alpha_n)(t)tv$ for all $v \in V$. Given $i \in \mathbb{N}$, we write $[i]$ (respectively $[-i]$) for the i-fold composition of [1] (respectively $[-1]$).

Given U_χ-T_0-modules V_0 and V_1, we have a natural isomorphism

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{U_\chi\text{-}T_0}(V_0[i], V_1) \cong \mathrm{Hom}_{U_\chi}(F(V_0), F(V_1)). \tag{18}$$

Moreover, if $V_0 = V_1$, then the left hand side of (18) acquires an algebra structure through the identification

$$\mathrm{Hom}_{U_\chi\text{-}T_0}(V_0[i+j], V_1[j]) \cong \mathrm{Hom}_{U_\chi\text{-}T_0}(V_0[i], V_1),$$

for all $i, j \in \mathbb{Z}$. With this (18) becomes an isomorphism of algebras.

9.6. Thanks to [26, Theorem 2.6], there are exactly two non-isomorphic simple $\mathcal{B}_{\chi,\lambda}$-modules. We let S_0 and S_1 be lifts of these to $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod, and let $P(S_0)$ and $P(S_1)$ be lifts of their projective covers, projecting onto S_0 and S_1 respectively. By [17, Theorem 4.1] and [25, Theorem 1.4.2] we have that $\{S_0[i], S_1[i] : i \in \mathbb{Z}\}$ is a complete set of representatives of mutually non-isomorphic simple $\mathcal{B}_{\chi,\lambda}$-$T_0$-modules. Then $P(S_0)[i]$ (respectively $P(S_1)[i]$) is the projective cover of $S_0[i]$ (respectively $S_1[i]$) in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod.

9.7. The category U_χ-T_0-mod admits a contravariant self-equivalence, D, whose square is the identity functor. By [26, Proposition 2.16] we have for all $i \in \mathbb{Z}$

$$D(S_0[i]) \cong S_0[i], \quad D(S_1[i]) \cong S_1[i]. \tag{19}$$

Since U_χ is a symmetric algebra, [43] and [11, 1.2], $P(S_0)$ and $P(S_1)$ are both projective and injective. So we deduce from (19) that

$$D(P(S_0)) \cong P(S_0), \quad D(P(S_1)) \cong P(S_1). \tag{20}$$

9.8. Besides the simples and their projective covers there are two other families of distinguished objects in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod. The first we denote by $Z(S_0)$ and $Z(S_1)$. These are indecomposable $\mathcal{B}_{\chi,\lambda}$-$T_0$-modules whose underlying U_χ-modules $F(Z(S_0))$ and $F(Z(S_1))$ are the baby Verma modules in U_χ-mod with heads $F(S_0)$ and $F(S_1)$ respectively. We have, by [26, Theorem 2.6], short exact sequences in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod

$$0 \longrightarrow S_1 \longrightarrow Z(S_0) \longrightarrow S_0 \longrightarrow 0, \tag{21}$$

and

$$0 \longrightarrow S_0[-1] \longrightarrow Z(S_1) \longrightarrow S_1 \longrightarrow 0. \tag{22}$$

To define the second objects we need a little notation. Let $\Phi' \subset \Phi$ be the root system generated by $\alpha_1, \dots, \alpha_{n-1}$. Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{b}^+$ and all $\mathfrak{g}_\alpha$ for $\alpha \in \Phi'$. Then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the unipotent radical of $\mathfrak{p}$ and $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{g}$ which equals the direct sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ for $\alpha \in \Phi'$. Since $\chi(\mathfrak{u}) = 0$, any $U_\chi(\mathfrak{l})$-module M can be extended to a $U_\chi(\mathfrak{p})$-module by letting $\mathfrak{u}$ act trivially. Thus we obtain a $U_\chi(\mathfrak{g})$-module by *Harish-Chandra induction*,

$$\mathrm{Ind}(M) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} M.$$

Let $\mu \in X(T)$ and let $Z_\chi^{\mathfrak{l}}(\mu)$ be the corresponding baby Verma module over $\mathfrak{l}$. Let $Q_\chi^{\mathfrak{l}}(\mu)$ be the projective cover of $Z_\chi^{\mathfrak{l}}(\mu)$ as a $U_\chi(\mathfrak{l})$-module. Then $\mathrm{Ind}(Z_\chi^{\mathfrak{l}}(\mu)) = Z_\chi(\mu)$, and we write $\Delta_\chi(\mu)$ for $\mathrm{Ind}(Q_\chi^{\mathfrak{l}}(\mu))$. We furnish these modules with a compatible T_0-structure by concentrating $1 \otimes Z_\chi^{\mathfrak{l}}(\mu)$ (respectively $1 \otimes Q_\chi^{\mathfrak{l}}(\mu)$) in the weight space given by the restriction of μ to T_0 and letting T_0 act on $U_\chi(\mathfrak{g})$ via the adjoint action. This is well-defined, since the T_0-action on $U_\chi(\mathfrak{l})$ is by construction trivial. For further details, see [24, Sections 10 and 11] and [25, Section 2].

Using the above construction, we see that for $i = 1, 2$ there exists $\mu_i \in X(T)$ such that $Z(S_i) \cong \mathrm{Ind}(Z_\chi^{\mathfrak{l}}(\mu_i))$ as a $\mathcal{B}_{\chi,\lambda}$-$T_0$-module. We define $\Delta(S_i) := \Delta_\chi(\mu_i)$

as $\mathcal{B}_{\chi,\lambda}$-$T_0$-modules. By [24, 11.18] and [25, 2.9] we have $\Delta(S_0) \cong Z(S_0)$, whilst the module $\Delta(S_1)$ has a filtration in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod

$$0 = \Delta_0(S_1) \subset \Delta_1(S_1) \subset \cdots \subset \Delta_n(S_1) = \Delta(S_1), \tag{23}$$

such that $\Delta_i(S_1)/\Delta_{i-1}(S_1) \cong Z(S_1)$ for $1 \leq i \leq n$. By [24, Proposition 11.18] and [25, Proposition 2.9] we have short exact sequences

$$0 \longrightarrow \Delta(S_1)[1] \longrightarrow P(S_0) \longrightarrow \Delta(S_0) \longrightarrow 0, \tag{24}$$

and

$$0 \longrightarrow \Delta(S_0) \longrightarrow P(S_1) \longrightarrow \Delta(S_1) \longrightarrow 0. \tag{25}$$

In particular, both $\Delta(S_0)$ and $\Delta(S_1)$ are indecomposable with simple heads S_0 and S_1 respectively.

9.9. We are able to calculate some extension groups in U_χ-T_0-mod.

Lemma. *Let V be a simple U_χ-T_0-module. Then*

$$\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_0, V) = \begin{cases} K & \text{if } V \cong S_1, S_1[1] \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, V) = \begin{cases} K & \text{if } V \cong S_0, S_0[-1] \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Thanks to [26, Proposition 2.19] it only remains to prove that S_0 and S_1 have no self-extensions.

We have an exact sequence induced from (24)

$$\operatorname{Hom}_{U_\chi\text{-}T_0}(\Delta(S_1)[1], S_0) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(\Delta(S_0), S_0) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(P(S_0), S_0).$$

Since $P(S_0)$ is injective and since the head of $\Delta(S_1)[1]$ is $S_1[1]$, we deduce that $\operatorname{Ext}^1_{U_\chi\text{-}T_0}(\Delta(S_0), S_0) = 0$. We have an exact sequence from (21)

$$\operatorname{Hom}_{U_\chi\text{-}T_0}(S_1, S_0) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_0, S_0) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(\Delta(S_0), S_0).$$

It follows that $\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_0, S_0) = 0$.

To deal with S_1, consider the following exact sequence induced from (25):

$$\operatorname{Hom}_{U_\chi\text{-}T_0}(S_1, \Delta(S_1)) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, \Delta(S_0)) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, P(S_1)).$$

Since $P(S_1)$ is projective and the socle of $\Delta(S_1)$ is $S_0[-1]$, it follows that

$$\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, \Delta(S_0)) = 0.$$

The exact sequence induced from (21),

$$\operatorname{Hom}_{U_\chi\text{-}T_0}(S_1, S_0) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, S_1) \longrightarrow \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, \Delta(S_0)),$$

now shows that $\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_1, S_1) = 0$, as required. □

9.10. A module is called *uniserial* if it has a unique composition series. Given a U_χ-T_0-module V, we define the radical series of V as follows: set $\operatorname{rad}^0 V = V$ and for $i > 0$ let $\operatorname{rad}^i V = \operatorname{rad}(\operatorname{rad}^{i-1} V)$.

Lemma. *Suppose V is a U_χ-T_0-module such that $\operatorname{rad}^i V/\operatorname{rad}^{i+1} V$ is simple for all i. Then V is uniserial.*

Proof. It is sufficient to prove that any U_χ-T_0-submodule M of V equals $\operatorname{rad}^i V$ for some $i \geq 0$. We prove this by downward induction on the number of composition factors of M. Let M be a proper U_χ-T_0-submodule of V and let $M' \supsetneq M$ be such that M'/M is simple. By induction $M' = \operatorname{rad}^i V$ for some $i \geq 0$, and so $\operatorname{rad}^{i+1} V = \operatorname{rad}(M') \subseteq M$. Therefore M'/M is a factor of $\operatorname{rad}^i V/\operatorname{rad}^{i+1} V$, a simple module, implying equality. $\square$

9.11. The following lemma is the crucial result in this section.

Lemma. *The modules $\Delta(S_0)$ and $\Delta(S_1)$ are uniserial.*

Proof. The module $\Delta(S_0)$ is trivially uniserial. We will show that the module $\operatorname{rad}^i \Delta(S_1)/\operatorname{rad}^{i+1}\Delta(S_1)$ is simple in U_χ-T_0-mod for all $i \geq 0$, and then apply Lemma 9.10. Suppose for a contradiction that i is minimal such that the module $\operatorname{rad}^i \Delta(S_1)/\operatorname{rad}^{i+1}\Delta(S_1)$ is not simple. Since $\Delta(S_1)$ has a simple head we have $i \geq 1$. There is a short exact sequence

$$0 \longrightarrow \frac{\operatorname{rad}^i \Delta(S_1)}{\operatorname{rad}^{i+1}\Delta(S_1)} \longrightarrow \frac{\operatorname{rad}^{i-1} \Delta(S_1)}{\operatorname{rad}^{i+1}\Delta(S_1)} \longrightarrow \frac{\operatorname{rad}^{i-1} \Delta(S_1)}{\operatorname{rad}^{i}\Delta(S_1)} \longrightarrow 0. \tag{26}$$

By hypothesis the term on the right hand side of (26) is simple. The module $\Delta(S_1)$ has only two isomorphism classes of composition factors, S_1 and $S_0[-1]$. We assume that the right hand side of (26) is isomorphic to $S_0[-1]$. The other case is treated similarly.

The left hand side of (26) is semisimple, so isomorphic to a direct sum of copies of $S_0[-1]$ and S_1. By Lemma 9.9, $\operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_0[-1], S_0[-1]) = 0$, so we deduce that this direct sum can contain no copies of $S_0[-1]$.

Let us write $V = \operatorname{rad}^{i-1}\Delta(S_1)/\operatorname{rad}^{i+1}\Delta(S_1)$, a module with head isomorphic to $S_0[-1]$ and socle a number of copies of S_1. By [20, Proof of Theorem 9] we have that

$$[\operatorname{rad} P(S_0)[-1]/\operatorname{rad}^2 P(S_0)[-1] : S_1] = \dim \operatorname{Ext}^1_{U_\chi\text{-}T_0}(S_0[-1], S_1). \tag{27}$$

By Lemma 9.9 the right hand side of (27) equals one. We have a commutative diagram

$$\begin{tikzcd} P(S_0)[-1] \arrow[r] \arrow[dr, dashed, "f"'] & S_0[-1] \\ & V \arrow[u] \end{tikzcd}$$

where f exists by the projectivity of $P(S_0)[-1]$. This induces a homomorphism

$$\tilde{f} : \frac{\operatorname{rad} P(S_0)[-1]}{\operatorname{rad}^2 P(S_0)[-1]} \longrightarrow \operatorname{rad} V.$$

If $\tilde{f}$ is not surjective, then we have $V/\operatorname{im} f \neq 0$, and so there is a copy S_1 in the head of V, a contradiction. Therefore $\tilde{f}$ is surjective and so $\operatorname{rad} V = S_1$, as required. $\square$

9.12. Let A be a finite dimensional K-algebra. Given any A-module M with simple head, we define the *heart of* M to be

$$H(M) = \frac{\operatorname{rad} M}{\operatorname{soc} M}.$$

The algebra A is called *biserial* if every nonuniserial projective indecomposable left or right A-module P contains two uniserial submodules whose sum is the unique maximal submodule of P and whose intersection is either zero or simple. It is easily seen that if A is symmetric and the heart of any projective indecomposable A-module is a direct sum of two uniserial modules, then A is biserial. According to [4], any biserial algebra A is tame.

Theorem. *Let $\chi \in \mathfrak{g}^*$ and $\lambda \in \Lambda/W$ be as in 9.2. Then $\mathcal{B}_{\chi,\lambda}$ is a biserial algebra. In particular $\mathcal{B}_{\chi,\lambda}$ is tame.*

Proof. By 9.4 the radical and socle of a $\mathcal{B}_{\chi,\lambda}$-$T_0$-module V agree with the radical and socle of $F(V)$, so it is enough to prove that the heart of the projective indecomposables in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod are direct sums of two uniserial modules.

We claim that

$$H(P(S_0)) \cong \frac{\Delta(S_1)[1]}{S_0} \oplus S_1,$$

and

$$H(P(S_1)) \cong \operatorname{rad}\Delta(S_1) \oplus S_0.$$

The uniseriality of these summands follows from Lemma 9.11.

Thanks to (24), we have a short exact sequence

$$0 \longrightarrow \frac{\Delta(S_1)[1]}{S_0} \longrightarrow H(P(S_0)) \longrightarrow S_1 \longrightarrow 0.$$

Using (24), (19), (20) and (21), we also have a composition

$$0 \longrightarrow S_1 \cong D(S_1) \cong \frac{D(\Delta(S_0))}{D(S_0)} \longrightarrow \frac{D(P(S_0))}{D(S_0)} \cong \frac{P(S_0)}{S_0},$$

showing that there is a copy of S_1 in the head and the socle of $H(P(S_0))$. By (23) and (24) $S_1 = S_1[0]$ occurs only once as a composition factor of $P(S_0)$ in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod. So we deduce that S_1 is a direct summand of $H(P(S_0))$, proving the first half of the claim.

For the second half of the claim we have, thanks to (25) and Lemma 9.11, a short exact sequence

$$0 \longrightarrow S_0 \longrightarrow H(P(S_1)) \longrightarrow \operatorname{rad}\Delta(S_1) \longrightarrow 0.$$

By (25), (19) and (20) we have a composition of maps

$$\operatorname{rad} P(S_1) \cong \operatorname{rad} D(P(S_1)) \longrightarrow D(S_0) \cong S_0 \longrightarrow 0.$$

As above, we deduce that S_0 is a direct summand of $H(P(S_1))$, as required. $\square$

9.13. Quiver and relations for $\mathcal{B}_{\chi,\lambda}$. Using the results of the previous subsections, we will determine the quiver and relations for the algebra $\mathcal{B}_{\chi,\lambda}$. The basic algebra Morita equivalent to $\mathcal{B}_{\chi,\lambda}$ is simply $\mathrm{End}_{\mathcal{B}_{\chi,\lambda}}(F(P(S_0)) \oplus F(P(S_1)))$. To ease notation, set $P_0 = P(S_0)$ and $P_1 = P(S_1)$.

By (18) we have an algebra isomorphism

$$\mathrm{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1)) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{End}_{\mathcal{B}_{\chi,\lambda}\text{-}T_0}(P_0[i] \oplus P_1[i], P_0 \oplus P_1). \tag{28}$$

We construct four elements in the right hand side of (28), show that these generate the basic algebra, and then describe the relations between these elements.

Let A be the unique (up to scalars) map in $\mathcal{B}_{\chi,\lambda}$-$T_0$-mod of degree zero, defined as the composition $P_0 \twoheadrightarrow \Delta(S_0) \hookrightarrow P_1$. Similarly, let $\tilde{B} : P_1[1] \twoheadrightarrow \Delta(S_1)[1] \hookrightarrow P_0$ be the unique map of degree 1.

By the proof of Theorem 9.12, $\mathrm{rad}\, P_1$ has head isomorphic to $S_0 \oplus S_0[-1]$. So we have

$$\begin{tikzcd} P_0[-1] \arrow[r] \arrow[dr, dashed, "f"'] & S_0[-1] \\ & \mathrm{rad}\, P_1 \arrow[u] \end{tikzcd}$$

Composing f with the inclusion $\mathrm{rad}\, P_1 \subset P_1$, we obtain $\tilde{A} : P_0[-1] \longrightarrow P_1$, a map of degree -1. Similarly, using the proof of Theorem 9.12, we define B as

$$\begin{tikzcd} P_1 \arrow[r] \arrow[drrr, dashed, "B"'] & S_1 \arrow[r] & H(P_0) \arrow[r] & P_0/S_0 \\ & & & P_0 \arrow[u] \end{tikzcd}$$

By construction, B is a map of degree zero.

Lemma. *The elements $A, \tilde{A}, B$ and $\tilde{B}$ generate* $\mathrm{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$ *as an algebra.*

Proof. Consider the map induced by $\tilde{A}$,

$$\bar{A} : \frac{P_0[-1]}{\tilde{A}^{-1}(S_1)} \longrightarrow \frac{P_1}{S_1}.$$

Using Theorem 9.12 we see the image of $\bar{A}$ is $\mathrm{rad}\,\Delta(S_1)$. Since $S_1 = \mathrm{soc}\, P_1 \subseteq \ker \tilde{B}$, we deduce that

$$\mathrm{im}(\tilde{B} \circ \tilde{A}) = \mathrm{im}(\tilde{B} \circ \bar{A}) = \mathrm{rad}\,\Delta(S_1)[1]. \tag{29}$$

Write $X = \tilde{B}\tilde{A}$. By (29) X induces an endomorphism of $\Delta(S_1)[1] \subset P_0$ which sends the head of $\Delta(S_1)[1]$ onto $\mathrm{rad}^2\,\Delta(S_1)[1]$. Since $\Delta(S_1)[1]$ is uniserial, it follows that X sends $\mathrm{rad}^i\,\Delta(S_1)[1]$ to $\mathrm{rad}^{i+2}\,\Delta(S_1)[1]$. Hence the image of $X^n : P_0 \longrightarrow P_0$ equals $S_0 = \mathrm{soc}\, P_0$.

Let $Y = \tilde{A}\tilde{B}$. We have

$$\tilde{B} \circ Y^{n-1} : P_1 \xrightarrow{\ \tilde{B}\ } P_0[-1] \xrightarrow{(X[-1])^{n-1}} P_0[-1].$$

It follows from the definition of $\tilde{B}$ and the previous paragraph that the image of $\tilde{B} \circ Y^{n-1}$ equals $\mathrm{rad}^{2n-2}\,\Delta(S_1)$, a module with composition factors $S_0[-1]$ and S_0 by Lemma 9.11. Using (21), (22), (23), (24) and (25), we see that the kernel of $\tilde{A}$ has composition factors $S_0[-1]$ and $S_1[-1]$. It follows that Y^n has image $S_1 = \operatorname{soc} P_1$.

It is clear that $\operatorname{im}(B \circ A) = \operatorname{soc} P_0$ and $\operatorname{im}(A \circ B) = \operatorname{soc} P_1$. Thus, possibly after rescaling, $X^n = cB \circ A$ and $Y^n = A \circ B$, for some non-zero scalar c.

Let $\tilde{A}_i = Y^i \circ \tilde{A}$ and $\tilde{B}_i = X^i \circ \tilde{B}$. The elements $X^i, Y^i, \tilde{A}_i$ and $\tilde{B}_i$ are linearly independent in $\operatorname{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$ for $0 \leq i \leq n$. On the other hand, by (21), (22), (23), (24) and (25), we have

$$[F(P_0) : F(S_0)] = [F(P_0) : F(S_1)] = [F(P_1) : F(S_0)] = [F(P_1) : F(S_1)] = n + 1.$$

We deduce, by [1, Lemma 1.7.6], that

$$\dim \operatorname{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1)) = \sum_{i,j=0}^{1} [F(P_i) : F(S_j)] = 4(n+1),$$

as required. $\square$

9.14. It is now straightforward to obtain the quiver and relations (up to scalars).

Theorem. *Let χ be subregular and $\lambda \in \Lambda/W$ be such that $W(\lambda)$ has type A_{n-1}. Then $\mathcal{B}_{\chi,\lambda}$ is Morita equivalent to the quiver*

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, "A"] \arrow[r, bend left=20, "\tilde{A}"] & \bullet \arrow[l, bend left=20, "B"] \arrow[l, bend left=70, "\tilde{B}"] \end{tikzcd}$$

with relations $A\tilde{B} = \tilde{B}A = \tilde{A}B = B\tilde{A} = 0$, $AB = (\tilde{A}\tilde{B})^n$ and $BA = c(\tilde{B}\tilde{A})^n$, for some non-zero scalar $c \in K$.

Proof. We have shown in Lemma 9.13 that $A, \tilde{A}, B$ and $\tilde{B}$ generate the $4(n+1)$-dimensional algebra $\operatorname{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$. The first set of relations holds thanks to the grading on P_0 and P_1: the module P_0 (respectively P_1) has no composition factors isomorphic to $S_0[i]$ (respectively $S_1[i]$) for i non-zero. The second set of relations already appeared in the proof of Lemma 9.13. Therefore the endomorphism ring is a quotient of the above path algebra. The path algebra has a basis consisting of A, B, e_1, e_2 and alternating products of $\tilde{A}$ and $\tilde{B}$ of length no more than $2n$. Consequently the algebras are isomorphic. $\square$

Remark. In the case $n = 1$ there is a unique subregular nilpotent element, $\chi = 0$, and λ is a regular weight. The above description of the basic algebra of $\mathcal{B}_{0,\lambda}$ was given in [8], where it was shown that $c = 1$. For $n = 2$, the quiver and relations were found independently in [28], and it is shown that $c = \pm 1$ in this case.

Remark. It follows from Theorem 9.14 that the basic algebra of $\mathcal{B}_{\chi,\lambda}$ is *special* biserial (see [6, Definition II.1.1]). All indecomposable representations of such algebras are classified, [6, II.3]. The classification is based on [15] and [41]. It would be interesting to pull the information provided by [6, II.3] back to $\mathfrak{g}$ to obtain an *explicit* description of all indecomposable representations of the tame subregular blocks.

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