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THE INDEX OF A CRITICAL POINT FOR DENSELY DEFINED OPERATORS OF TYPE $(S_+)_L$ IN BANACH SPACES

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Abstract. The purpose of this paper is to demonstrate that it is possible to define and compute the index of an isolated critical point for densely defined operators of type $(S_+)_L$ acting from a real, reflexive and separable Banach space X into X^*. This index is defined via a degree theory for such operators which has been recently developed by the authors. The calculation of the index is achieved by the introduction of a special linearization of the nonlinear operator at the critical point. This linearization is a new tool even for continuous everywhere defined operators which are not necessarily Fréchet differentiable. Various cases of operators are considered: unbounded nonlinear operators with unbounded linearization, bounded nonlinear operators with bounded linearization, and operators in Hilbert spaces. Examples and counterexamples are given in l^p, $p > 2$, illustrating the main results. The associated bifurcation problem for a pair of operators is also considered. The main results of the paper are substantial extensions and improvements of the classical results of Leray and Schauder (for continuous operators of Leray-Schauder type) as well as the results of Skrypnik (for bounded demicontinuous mappings of type (S_+)). Applications to nonlinear Dirichlet problems have appeared elsewhere.

1. Introduction and preliminaries

The concept of the index of a critical point for densely defined operators of type $(S_+)_L$ is introduced. The definition of this index is based on the concept of the degree for operators of this type which was established by the authors in [3]. It plays a key role in problems of solvability, estimates of number of solutions and branching of solutions of nonlinear operator equations and nonlinear boundary value problems for partial differential equations (see, e.g., [7] about applications of Leray-Schauder type and [2], [11] about applications for everywhere defined (S_+)-operators).

Let X be a real separable reflexive Banach space with dual space X^*. The norm of the space X (X^*) will be denoted by $\|\cdot\|$ $(\|\cdot\|_*)$. We let $\mathcal{R}^n$ denote the Euclidean space of dimension n and set $\mathcal{R} = \mathcal{R}^1$. For $x_0 \in X$ and $r > 0$, we let $B_r(x_0)$ denote the open ball $\{x \in X : \|x - x_0\| < r\}$. Unless otherwise stated, N is the set of natural numbers. An operator $A : X \supset D(A) \to X^*$ is "bounded" if it maps bounded subsets of its domain onto bounded sets in X^*. It is "compact" if it is strongly continuous and maps bounded subsets of $D(A)$ onto relatively compact sets in X^*. In what follows, the single term "continuous" means "strongly continuous". We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. We consider

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an operator $A : X \supset D(A) \to X^*$ with domain $D(A)$ dense in some open set $D_0 \subset X$. We assume that there exists a subspace L of the space X such that

$$D_0 \cap L \subset D(A), \quad \overline{L} = X. \tag{1.1}$$

Let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L.

Definition 1.1. We say that the operator A satisfies Condition $(S_+)_{0,L}$ if for every sequence $\{u_j\} \subset D(A)$ with

$$u_j \rightharpoonup u_0, \quad \limsup_{j\to\infty} \langle Au_j, u_j\rangle \le 0, \quad \lim_{j\to\infty} \langle Au_j, v\rangle = 0, \tag{1.2}$$

for some $u_0 \in X$ and any $v \in L$, we have

$$u_j \to u_0, \quad u_0 \in D(A), \quad Au_0 = 0. \tag{1.3}$$

In (1.2), and the sequel, $\langle h, u\rangle$ denotes the value of the functional $h \in X^*$ at the element $u \in X$.

Definition 1.2. We say that the operator A satisfies Condition $(S_+)_L$ if the operator $A_h : D(A) \to X^*$, defined by $A_h u = Au - h$ satisfies Condition $(S_+)_{0,L}$ for any $h \in X^*$.

In the paper [3] we introduced the degree function $Deg(A, D, 0)$ of the operator A with respect to an arbitrary bounded open set D of the space X, provided that

$$Au \neq 0, \quad \text{for } u \in D(A) \cap \partial D, \ \overline{D} \subset D_0 \tag{1.4}$$

and the operator A satisfies the following conditions:

A_1) there exists a subspace L of X satisfying (1.1) and such that the operator A satisfies Condition $(S_+)_{0,L}$;

A_2) for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $a(F,v) : D_0 \cap F \to \mathcal{R}$, defined by $(a(F,v))(u) = \langle Au, v\rangle$, is continuous.

Note. Actually, the degree theory in [3] was developed under the assumption that $D_0 = X$, which implies that $L \subset D(A)$ and $D(A)$ is dense in X. It is easy to see that all that we need there is our current assumption: $D_0 \cap L \subset D(A)$.

We are now ready to define the index of a critical point of the operator A satisfying Conditions A_1), A_2).

Definition 1.3. A point $u_0 \in D(A) \cap D_0$ is called a "critical point" of the operator A if $Au_0 = 0$. A critical point $u_0 \in D(A) \cap D_0$ is an "isolated critical point" of the operator A if there exists a ball $B_r(u_0) \subset D_0$ which contains no other critical point of the operator A.

As in the proof of Theorem 2.1 in [3], we can show that

$$Deg(A, B_{r'}(u_0), 0) = Deg(A, B_r(u_0), 0),$$

for every $r' \in (0, r]$.

Using this definition we have

Definition 1.4. The number

$$\lim_{\rho\to 0} Deg(A, B_\rho(u_0), 0) \tag{1.5}$$

is called the "index" of the isolated critical point u_0 of the operator A and is denoted by $Ind(A, u_0)$.

The purpose of this paper is the calculation of the index $Ind(A, u_0)$ by using a linearization of the nonlinear operator A at the critical point. In the known results, for Leray-Schauder operators [7, Theorem 4.7] and bounded demicontinuous operators of type (S_+) [10, Theorem 5.2], this linearization is given by means of Fréchet or Gateaux derivatives at the critical points of the nonlinear operators under consideration. We may further assume that $u_0 = 0$.

We now recall the assumptions for the calculation of the index in [11]. These assumptions are given in a form that can be used later for the relevant unbounded linear operators.

Let $A : X \supset B_r(0) \to X^*$ be a nonlinear operator which satisfies Condition (S_+) and $A(0) = 0$. Assume that A has the Fréchet derivative $A' : X \to X^*$ at zero. Let

$$Z_\varepsilon = \bigcup_{t\in[0,1]} \{u \ : \ tAu + (1-t)A'u = 0, \ 0 < \|u\| \le \varepsilon\}. \tag{1.6}$$

A') The equation $A'u = 0$ has only the zero solution. There exists a compact linear operator $\Gamma : X \to X^*$ such that

$$\begin{aligned} &\langle (A' + \Gamma)u, u\rangle > 0, \quad \text{for } u \in D(A'), \ u \neq 0, \\ &\langle (A' + \Gamma)^* v, v\rangle > 0, \quad \text{for } v \in D((A')^*), \ v \neq 0, \end{aligned} \tag{1.7}$$

and the operator $T = (A' + \Gamma)^{-1}\Gamma : X \to X$ is well defined and compact;

C) the weak closure of the set

$$\sigma_\varepsilon = \left\{ v = \frac{u}{\|u\|} \ : \ u \in Z_\varepsilon \right\} \tag{1.8}$$

does not contain zero for some sufficiently small $\varepsilon > 0$.

In (1.7) $(A')^*$ is the adjoint of the operator A'. We note that in [11] $D(A') = X$ and the second inequality in (1.7) follows from the first. By the assumptions A'), C) in [11], the value of $Ind(A, 0)$ is calculated in terms of the multiplicities of the characteristic values of the operator T. We also note that in [10] there is an example demonstrating the fact that it is generally impossible to calculate $Ind(A, 0)$ without Condition C).

A natural question arises now: how do we introduce a workable concept of linearization for a densely defined operator? Before we formulate our new linearization concept, we introduce the auxiliary operator $A_0 : X \supset D(A_0) \to X^*$ which satisfies the following condition:

A_0) A_0 is a bounded nonlinear operator which satisfies Conditions $(S_+)_L$, A_2) and is such that $D_0 \cap L \subset D(A_0)$, $A_0(0) = 0$ and

$$\lim_{\substack{u\to 0\\ u\in D(A_0)}} \frac{\|A_0 u\|_*}{\|u\|} = 0.$$

We solve the problem of linearizing for the nonlinear operator A, satisfying Conditions A_1), A_2), in the following way. We assume that there exist a nonlinear operator A_0 satisfying A_0) and a linear operator $A' : X \supset D(A') \to X^*$ such that $D(A) \subset D(A')$ and the next condition holds:

ω) for the operator $\omega : D(A) \to X^*$, defined by $\omega(u) = Au - A'u$, we have

$$\frac{\omega(u)}{\|u\|} \to 0 \quad \text{as } u \to 0, \ u \in Z'_\varepsilon,$$

for some $\varepsilon > 0$, where

$$Z'_\varepsilon = \bigcup_{t\in[0,1]} \left\{ u \in D(A_t^{(1)}) \ : \ A_t^{(1)} u = 0,\ 0 < \|u\| \leq \varepsilon \right\}, \tag{1.9}$$
$$A_t^{(1)} u = tAu + (1-t)\left[A_0 u + A'u\right].$$

We remark that Condition ω) is weaker that the conditions in terms of derivatives in [11]. Using Condition ω), it is possible to evaluate the indices of the critical points even for operators which are defined everywhere, but not differentiable in the usual sense. We shall formulate the relevant assertions in Section 2.

We shall assume that the operator A' satisfies Condition $(S')_L$ which is given in the following definition.

Definition 1.5. We say that the operator A' satisfies Condition $(S')_L$ if for every sequence $\{u_j\} \subset D(A')$ such that

$$u_j \rightharpoonup u_0, \quad \limsup_{j\to\infty} \langle A'u_j - h, u_j\rangle \leq 0, \quad \lim_{j\to\infty} \langle A'u_j - h, v\rangle = 0, \tag{1.10}$$

for some $u_0 \in X,\ h \in X^*$ and any $v \in L$, it follows that

$$u_0 \in D(A'), \quad A'u_0 = h, \quad \lim_{j\to\infty} \langle A'u_j, u_0\rangle = \langle h, u_0\rangle. \tag{1.11}$$

The main result of this paper is the evaluation of the index $Ind(A,0)$ under the conditions A'), $(S')_L,\ \omega$) and C) (the last condition is satisfied with a special choice of a set Z_ε). We are going to show, under some additional conditions, that zero is an isolated critical point of the operator A and

$$Ind(A,0) = (-1)^\nu, \tag{1.12}$$

where ν is the sum of the multiplicities of the characteristic values of the operator T lying in the interval $(0,1)$.

The exact formulation of the results concerning the value of the index of the critical point is given in Section 3. In Theorem 2.1 we give a result of the general situation of an unbounded operator A and an unbounded linearization operator A'. More specific cases are given in the subsequent theorems of Section 2. In Theorem 2.2 we consider the case of a bounded operator A of type $(S_+)_{0,L}$ with bounded linearization operator. The evaluation of the index for a bounded operator A satisfying Condition $(S_+)_L$ is given in Theorem 2.3. Finally, the case of operators in Hilbert spaces is considered in Theorem 2.4.

In Section 3 we construct examples and counterexamples in the space l^p demonstrating the possibilities for applications of the index formula and pointing out the necessity of conditions on the operators in the results of Section 2. The first example is connected with the evaluation of the index for unbounded operators with unbounded linearization. The third example contains the construction of a bounded operator of class $(S_+)_{0,L}$ such that the complement of its domain is dense in the space X. The evaluation of the index of its critical point is also contained therein. The second and fourth (counter)examples show that the main conditions 1) and 2) of Theorem 2.1 are necessary.

The proof of Theorem 2.1 is given in Section 4. In Section 5 we discuss a problem of bifurcation points for densely defined operators. We consider only the case of unbounded operators $A,\ A'$.

The results of this work open the possibility of studying problems of branching of solutions and the evaluation of the number of solutions for nonlinear elliptic problems in Sobolev spaces with strong coefficient growth. These problems can be reduced to operator equations with unbounded densely defined operators, and cannot be studied by the methods contained in the monograph [11]. Such a problem has been studied by the authors in [6]. Namely, we consider there the Dirichlet problem

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ [e^u + a(x)] \frac{\partial u}{\partial x_i} \right\} - \lambda u = 0, \quad x \in \Omega, \tag{1.13}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.14}$$

where $a(x)$ is a positive, bounded and measurable function, and Ω is a bounded open set in $\mathcal{R}^n$ with boundary $\partial\Omega \in \mathcal{C}^2$. We showed in [6] that every eigenvalue of odd multiplicity of the linear equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ [1 + a(x)] \frac{\partial u}{\partial x_i} \right\} - \lambda u = 0, \quad x \in \Omega,$$

with the boundary condition (1.14) is a point of bifurcation for the problem ((1.13), (1.14)).

The reader is referred to [4] for recent results on the solvability of nonlinear eigenvalue problems involving monotonicity conditions, and [5] for results on fully nonlinear parabolic problems involving Condition (S_+).

2. Formulation of the main results

Let X be a real separable reflexive Banach space satisfying the following conditions:

X_1) there exists a bounded demicontinuous operator $J : \overline{B_r(0)} \to X^*$, with $J(0) = 0$, satisfying Condition (S_+) for some $r > 0$;

X_2) there exists a bounded linear operator $K : X \to X^*$ such that $\langle Kx, x\rangle > 0$, for $x \neq 0$.

Condition (S_+) in X_1) coincides with Condition $(S_+)_L$ when $L = X$. We also note that the condition X_1) is satisfied, e.g., if $X,\ X^*$ are uniformly convex. In this case we can choose the operator J as in [11]. Condition X_2) is satisfied if the space X is included in some real Hilbert space H and the embedding operator $X \to H$ is continuous.

Let $A : X \supset D(A) \to X^*$ be an operator which satisfies Conditions A_1), A_2) and is such that

$$\langle Au, u - v\rangle \geq -C(v) \tag{2.1}$$

holds for $u,\ v \in L,\ \|u\| \leq r_0$, where $r_0 > 0$ is a constant and $C(v)$ depends only on v.

In order to formulate the main results of the paper we introduce certain subspaces of the spaces $X,\ X^*$ connected with the operators $A' + \Gamma,\ T$ which are defined in Condition A'). We first define two invariant subspaces of the compact operator $T : X \to X$. Denote by F the direct sum of all invariant subspaces of the operator T corresponding to the characteristic values of this operator lying in the interval $(0, 1)$. Let R be the closure of the direct sum of all those invariant subspaces of the operator T not included in F. Then F and R are invariant subspaces of the operator T and the splitting

$$X = F + R \tag{2.2}$$

holds in the sense of a direct sum. F is a finite-dimensional subspace of X and

$$dim F = \nu, \tag{2.3}$$

where ν is the same number as in (1.12).

We introduce a projection $\Pi : X \to F$ corresponding to the splitting (2.2):

$$\Pi(f + r) = f, \quad \text{for } f \in F,\ r \in R. \tag{2.4}$$

We define, for small enough $\varepsilon > 0$, the sets

$$\begin{aligned} &Z_\varepsilon = Z'_\varepsilon \cup Z''_\varepsilon, \\ &Z''_\varepsilon = \bigcup_{t\in[0,1]} \left\{ u \in D(\overline{A}_t) \ : \ \overline{A}_t u = 0,\ 0 < \|u\| \le \varepsilon \right\}, \end{aligned} \tag{2.5}$$

where $\overline{A}_t u = tA_0 u + A'u$, the operators $A_0,\ A'$ are defined according to the Condition ω) and the set Z'_ε is introduced in (1.6).

Theorem 2.1. *Let $A : X \supset D(A) \to X^*$ satisfy Conditions $A_1),\ A_2)$, the inequality (2.1) and be such that $0 \in D(A) \cap D_0$ and $A(0) = 0$. Assume that there exist operators $A_0 : X \supset D(A_0) \to X^*,\ A' : X \supset D(A') \to X^*$ satisfying Conditions $A_0),\ A'),\ (S')_L$ and ω), and such that the operator $A + qA' : X \supset D(A) \to X^*$ satisfies $(S_+)_L$, for any number $q > 0$. Suppose that the following conditions are satisfied:*

1) the operator $\Pi(A' + \Gamma)^{-1} : X^* \supset (A' + \Gamma)D(A') \to X$ is bounded, where the operators $\Pi,\ \Gamma$ are defined by (2.4) and A'), respectively;

2) Condition C) is satisfied with the set Z_ε defined by (2.5).

Then zero is an isolated critical point of the operator A and its index is equal to $(-1)^\nu$, where ν is the sum of the multiplicities of the characteristic values of the operator T lying in the interval $(0, 1)$.

We formulate below some particular cases of Theorem 2.1. In Theorem 2.2 we assume that the operators $A,\ A'$ are bounded. Thus, Conditions $(S')_L$, (2.1) are automatically satisfied. Furthermore, by changing some arguments in the proof of Theorem 2.1 we can establish an analogous result without assuming Condition $(S_+)_L$ for the operator $A + qA'$. We also note that in this case it suffices to assume only the first of (1.7) in Condition A').

Theorem 2.2. *Let $A : X \supset D(A) \to X^*$ be bounded and satisfy Conditions $A_1),\ A_2)$. Assume that $0 \in D(A) \cap D_0$ and $A(0) = 0$. Let there exist bounded operators $A_0 : X \supset D(A_0) \to X^*,\ A' : X \supset D(A') \to X^*$ satisfying Conditions $A_0),\ A')$ and ω). Suppose that Conditions 1) and 2) of Theorem 2.1 are satisfied. Then zero is an isolated critical point of the operator A and its index equals $(-1)^\nu$, with the same number ν as in Theorem 2.1.*

In the next theorem we assume that the operator A satisfies Condition $(S_+)_L$ and both operators $A,\ A'$ are bounded. Then we can pick the operator A_0 as the operator $\|u\|^2 Au$. In this case we can assume Condition ω) with a set

$$\widetilde{Z}_\varepsilon = \bigcup_{t\in[0,1]} \left\{ u \in D(\widetilde{A}_t) \ : \ \widetilde{A}_t u = 0,\ 0 < \|u\| \le \varepsilon \right\}, \tag{2.6}$$

where $\widetilde{A}_t u = tAu + (1 - t)A'u$ and in Condition C) we can take $Z_\varepsilon = \widetilde{Z}_\varepsilon$.

Theorem 2.3. *Let L be a subspace of the space X satisfying (1.1), and let $A : X \supset D(A) \to X^*$ be a bounded operator satisfying Conditions $(S_+)_L,\ A_2)$ and such that $0 \in D(A) \cap A_0,\ A(0) = 0$. Assume that there exists a bounded operator $A' : X \supset D(A') \to X^*$ satisfying Condition A') and such that Conditions ω) and C) are also satisfied with the set $\widetilde{Z}_\varepsilon$ defined by (2.6). Suppose that Condition 1) of Theorem 2.1 is satisfied. Then zero is an isolated critical point of the operator A and its index equals $(-1)^\nu$, where ν is as in Theorem 2.1.*

Finally, we state one result for the special case of a Hilbert space H in place of X. For simplicity, we consider only bounded operators $A,\ A'$. We use the following assumption: there exists a positive constant c such that

$$\langle (A' + \Gamma)u, u\rangle \geq c\|u\|^2 \tag{2.7}$$

holds for all $u \in H$, where the brackets denote now the scalar product of the space H.

Theorem 2.4. *Let H be a real separable Hilbert space and $A : H \supset D(A) \to H$ a bounded operator satisfying Conditions $(S_+)_L,\ A_2),\ 0 \in D(A) \cap D_0$ and $A(0) = 0$. Assume that there exist a bounded linear operator $A' : H \to H$ and a compact linear operator $\Gamma : H \to H$ such that the inequality (2.7) holds. Assume, further, that Condition ω) is satisfied with $Z'_\varepsilon = \widetilde{Z}_\varepsilon$, where $\widetilde{Z}_\varepsilon$ is defined by (2.6). Suppose that the equation $A'u = 0$ has only the zero solution. Then zero is an isolated critical point of the operator A and its index equals $(-1)^\nu$, where ν is as in Theorem 2.1.*

Let us verify that all the assumptions of Theorem 2.3 are satisfied for operators satisfying the conditions of Theorem 2.4. In fact, from (2.7) we obtain that the operator $(A' + \Gamma)^{-1}$ is bounded and that Condition 1) of Theorem 2.1 holds. In order to check the validity of Condition C), we prove that for some sufficiently small $\varepsilon > 0$ the set $\widetilde{Z}_\varepsilon$ is empty. Suppose that this is not true. Then there exist sequences $\{u_j\} \subset D(A),\ \{t_j\} \subset [0, 1]$ such that $u_j \to 0$ and

$$t_j A u_j + (1 - t_j)A' u_j = 0. \tag{2.8}$$

Then, by virtue of Condition ω), we obtain

$$\langle A' v_j, v_j\rangle \to 0 \quad \text{as } j \to \infty,\ \text{where } v_j = \frac{u_j}{\|u_j\|}. \tag{2.9}$$

From (2.7) and (2.9) follows that the weak limit v_0 of a weakly convergent subsequence of v_j cannot equal zero. From Condition ω) and (2.8) we obtain $A'v_0 = 0$, which contradicts the conditions of Theorem 2.4.

Remark 2.1. It is easy to verify that in the case of a bounded operator A', as in Theorems 2.2-2.4, we can assume instead of Condition ω) a weaker condition: in Condition ω) we replace $\|u\|^{-1}\omega(u) \to 0$ by $\|u\|^{-1}\omega(u) \rightharpoonup 0$.

Remark 2.2. Theorems 2.3, 2.4 are also new even for operators which satisfy Condition (S_+), are defined everywhere in a neighborhood of the critical point and have no derivatives in the usual sense. Examples can be constructed for these cases analogous to those of Section 3.

3. Examples and counterexamples

In this section we present some examples about our main results. We also give some counterexamples in cases where certain main assumptions of our results do not hold true.

We consider operators in the space $X = l^p$ consisting of all real sequences $u = \{c_n\}$ with finite norm

$$\|u\| = \left\{ \sum_{n=1}^{\infty} |c_n|^p \right\}^{\frac{1}{p}}, \quad p > 2.$$

Example 3.1. Let us define an operator $A_1 : X \supset D(A_1) \to X^*$ by

$$\langle A_1 u, v \rangle = \sum_{n=1}^{\infty} \left\{ n^{2p-1} |c_n|^{p-1} + |c_n|^{p-2} + n - \frac{N}{n} \right\} c_n d_n, \tag{3.1}$$

where $u = \{c_n\},\ v = \{d_n\}$, and N is a positive number such that $\sqrt{N}$ is not an integer. We define the domain $D(A_1)$ of the operator A_1 to be the set of all $u \in X$ such that the right-hand side of (3.1) generates a functional in X^*. Analogously, we define the domains of other operators that will appear in this section. In particular, we have

$$D(A_1) = \left\{ u = \{c_n\} \in X \ : \ \sum_{n=1}^{\infty} \left[n^{2p-1} |c_n|^p \right]^{\frac{p}{p-1}} < \infty \right\}. \tag{3.2}$$

In accordance with Section 2, we introduce the operators $A_{1,0} : X \supset D(A_{1,0}) \to X^*,\ A_1' : X \supset D(A_1') \to X^*$ by

$$\langle A_{1,0} u, v \rangle = \sum_{n=1}^{\infty} |c_n|^{p-2} c_n d_n, \quad \langle A_1' u, v \rangle = \sum_{n=1}^{\infty} \left(n - \frac{N}{n} \right) c_n d_n, \tag{3.3}$$

where $u,\ v$ are the same as in (3.1). We note that $D(A_{1,0}) = X$.

Proposition 3.1. *The operators $A_1,\ A_{1,0},\ A_1'$ satisfy all the assumptions of Theorem 2.1, respectively, zero is an isolated critical point of the operator A_1 and*

$$Ind(A_1, 0) = (-1)^{\nu_1}, \tag{3.4}$$

where ν_1 is the number $[\sqrt{N}]$. Here, $[\cdot]$ denotes the greatest integer function.

Proof. We introduce the finite-dimensional subspaces F_j of the space X by

$$F_j = \{ u = \{c_n\} \subset X \ : \ c_n = 0, \text{ for } n \geq j+1 \}$$

and let $L = \bigcup_{j=1}^{\infty} F_j$. We first verify that the operator A_1 satisfies Condition $(S_+)_L$. Let $\{u_j\} \subset D(A_1)$ be such that $u_j \rightharpoonup u_0$,

$$\limsup_{j \to \infty} \langle A_1 u_j - h, u_j \rangle \leq 0, \quad \lim_{j \to \infty} \langle A_1 u_j - h, v \rangle = 0, \tag{3.5}$$

for some $u_0 \in X,\ h \in X^*$ and any $v \in L$. Let $u_j = \{c_{n,j}\},\ u_0 = \{c_{n,0}\}$. From $u_j \rightharpoonup u_0$ we obtain

$$\sum_{n=1}^{\infty} |c_{n,j}|^p \leq R_1, \quad \lim_{j \to \infty} c_{n,j} = c_{n,0}, \text{ for } n = 1, 2, \dots, \tag{3.6}$$

for some positive number R_1. Using the first of (3.5) we get the estimate

$$\sum_{n=1}^{\infty} n^{2p-1}|c_{n,j}|^{p+1} \le R_2, \quad \text{for } j = 1, 2, \dots, \tag{3.7}$$

where the constant $R_2 > 0$ is independent of j. From (3.6) and (3.7) we obtain

$$\sum_{n=1}^{\infty} n^{2p-1}|c_{n,0}|^{p+1} \le \liminf_{j\to\infty} \sum_{n=1}^{\infty} n^{2p-1}|c_{n,j}|^{p+1}. \tag{3.8}$$

An analogous inequality can be obtained for

$$\sum_{n=1}^{\infty} n|c_{n,0}|^2.$$

Substituting $\{e_k\}$ in place of v in (3.5) (where $e_k = \{e_{k,n}\}$ with $e_{k,n} = 1$ for $k = n$ and $e_{k,n} = 0$ for $k \ne n$) we obtain from (3.6)

$$\left\{k^{2p-1}|c_{k,0}|^{p-1} + |c_{k,0}|^{p-2} + k - \frac{N}{k}\right\} c_{k,0} = h_k, \quad k = 1, 2, \dots, \tag{3.9}$$

with $h = \{h_n\} \in l^{p'}$, $p' = \frac{p}{p-1}$. From (3.9) it follows immediately that $u_0 \in D(A_1)$ and $A_1 u_0 = h$.

Let us now prove the strong convergence of u_j to u_0. From (3.1) we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\{|c_{n,j}|^{p-2}c_{n,j} - |c_{n,0}|^{p-2}\cdot c_{n,0}\}(c_{n,j} - c_{n,0}) \\
&= \langle A_1 u_j, u_j\rangle - \sum_{n=1}^{\infty}\left\{n^{2p-1}|c_{n,j}|^{p+1} + \left(n - \frac{N}{n}\right)|c_{n,j}|^2\right\} \\
&\quad - \sum_{n=1}^{\infty}|c_{n,j}|^{p-2}c_{n,j}c_{n,0} - \sum_{n=1}^{\infty}|c_{n,0}|^{p-2}c_{n,0}(c_{n,j} - c_{n,0}).
\end{aligned}$$

Evaluating the left-hand side of the last equality and passing to the limit as $j \to \infty$ by virtue of (3.5), (3.6) and (3.8), we get, for some positive μ_1,

$$\mu_1\|u_j - u_0\|^p \le \langle h, u_0\rangle - \sum_{n=1}^{\infty}\left\{n^{2p-1}|c_{n,0}|^{p+1} + |c_{n,0}|^p + \left(n - \frac{N}{n}\right)|c_{n,0}|^2\right\} = 0.$$

The equality is established by virtue of (3.9), and the proof of Condition $(S_+)_L$ is complete.

The fact that Condition A_0) holds for the operator $A_{1,0}$ is now clear. It is also easy to check the validity of Condition A') for the operator A'_1). We choose an operator Γ_1 by the formula

$$\langle \Gamma_1 u, v\rangle = \sum_{n=1}^{\infty} \frac{N}{n} c_n d_n. \tag{3.10}$$

Then

$$T_1 u = \left\{\frac{N}{n^2} c_n\right\}, \quad \text{for } u = \{c_n\},$$

and the operators Γ_1, T_1 are compact. Inequalities (1.7) are clearly true. We can check the validity of Condition $(S')_L$ for the operator A'_1 in a manner analogous to that of the proof of Condition $(S_+)_L$ for the operator A_1.

We verify that condition ω) is satisfied. If u belongs to the set Z'_ε, defined by (1.9), we have

$$\left\{ tn^{2p-1}|c_n|^{p-1} + |c_n|^{p-2} + \left(n - \frac{N}{n}\right)\right\} c_n = 0, \quad \text{for } n = 1, 2, \dots, \tag{3.11}$$

which implies that $c_n = 0$ for $n > \sqrt{N}$. This means that $Z'_\varepsilon \subset F_{\nu_1}$, where ν_1 is the same number as in (3.4). A simple calculation finishes now the proof of the validity of Condition ω).

In order to check Condition 2) of Theorem 2.1, we note that the inclusion $Z''_\varepsilon \subset F_{\nu_1}$ follows as in the case of Z'_ε, which shows that Condition C) holds. The rest of the assumptions of Theorem 2.1 can be easily shown to hold. This completes the proof of Proposition 3.1. □

Example 3.2. Let us define an operator $A_2 : X \supset D(A_2) \to X^*$ by

$$\langle A_2 u, v\rangle = \sum_{n=1}^{\infty} \left\{ \left[n|c_n|^{p-1} + |c_n|^{p-2} + \frac{1}{n^{p-2}}\right] c_n - \frac{3f(nc_n)}{n^{p-1}} \right\} d_n, \tag{3.12}$$

where the function $f : \mathcal{R} \to \mathcal{R}$ is defined as follows: $f(t) = 0$ for $|t-1| \geq 1/2,\ f(t) = 2t - 1$ for $1/2 \leq t \leq 1$, and $f(t) = -2t + 3$ for $1 \leq t \leq 3/2$.

We introduce the operators $A_{2,0} : D(A_{2,0}) = X \to X^*,\ A'_2 : D(A'_2) = X \to X^*$ as follows:

$$\langle A_{2,0} u, v\rangle = \sum_{n=1}^{\infty} |c_n|^{p-2} c_n d_n, \quad \langle A'_2 u, v\rangle = \sum_{n=1}^{\infty} \frac{1}{n^{p-2}} c_n d_n. \tag{3.13}$$

Here and in (3.12) $u,\ v$ are as in (3.1).

Proposition 3.2. *The operators $A_2,\ A_{2,0}$ and A'_2 satisfy all the assumptions of Theorem 2.1, respectively, except Condition 2). However, the assertion of Theorem 2.1 is false: zero is not an isolated critical point of A_2.*

Proof. We can check all the conditions for $A_2,\ A_{2,0},\ A'_2$, except Conditions ω) and 2), as we did in the proof of Proposition 3.1. We can also choose $\Gamma_2 = 0$ when we check the condition A').

Let us prove that ω) is satisfied for the operators under consideration. Let $u = \{c_n\}$ belong to the corresponding set Z'_ε defined by (1.9). Then, for $n = 1, 2, \dots,$ we have

$$\left[tn|c_n|^{p-1} + |c_n|^{p-2} + \frac{1}{n^{p-2}}\right] c_n = \frac{3tf(nc_n)}{n^{p-1}}, \tag{3.14}$$

for some $t \in [0, 1]$ and $0 < \|u\| \leq 1$. We may choose $\varepsilon = 1$. By the definition of the function f the right-hand side of (3.14) is not zero if and only if $t \neq 0,\ 1/2 < nc_n < 3/2$, and we have two possibilities for the value of c_n : either $c_n = 0$, or

$$\frac{1}{2} < nc_n < \frac{3}{2}. \tag{3.15}$$

Let us consider a sequence $\{u_j\}$ such that $u_j = \{c_{n,j}\} \in Z'_1$ and $u_j \to 0$. For each j we denote by $n(j)$ the integer with the property that $c_{n(j),j} \neq 0$ and $c_{n,j} = 0$ for $n < n(j)$. For $c_{n(j),j}$ the inequality (3.15) holds and, consequently, we have

$$\|u_j\| > \frac{1}{2}\frac{1}{n(j)} \text{ and } n(j) \to \infty \text{ as } j \to \infty. \tag{3.16}$$

It is sufficient to prove the following convergence:

$$\frac{\|\omega(u_j)\|_*}{\|u_j\|} \to 0, \quad \text{where } \omega(u) = A_2 u - A_2' u \tag{3.17}$$

and $\|\cdot\|_*$ is the norm in the space X^*. We estimate both factors in the left-hand side of (3.17). Using (3.15) we have

$$\|u_j\|^p \geq \left(\frac{1}{2}\right)^p \sum_n^{(j)} \left(\frac{1}{n}\right)^p, \tag{3.18}$$

where $\sum_n^{(j)}$ denotes summing over all n such that $c_{n,j} \neq 0$.

Taking into account that $X^* = l^{p'}$, $p' = \frac{p}{p-1}$, and using (3.15) we have the estimate

(3.19)

$$\|\omega(u_j)\|_*^{p'} \leq \mu_2 \sum_n^{(j)} \left\{ n|c_{n,j}|^p + |c_{n,j}|^{p-1} + \left(\frac{1}{n}\right)^{p-1} \right\}^{p'} \leq \mu_3 \sum_n^{(j)} \left(\frac{1}{n}\right)^p,$$

where $\mu_2,\ \mu_3$ are positive numbers depending only on p.

From (3.18), (3.19) and the definition of $n(j)$ we have

$$\frac{\|\omega(u_j)\|_*}{\|u_j\|} \leq \mu_4 \left\{ \sum_n^{(j)} \left(\frac{1}{n}\right)^p \right\}^{\frac{1}{p'} - \frac{1}{p}} \leq \mu_4 \left\{ \sum_{n=n(j)}^{\infty} \left(\frac{1}{n}\right)^p \right\}^{1-\frac{2}{p}},$$

where the right-hand side of the last inequality tends to zero by virtue of (3.16). This completes the proof of the fact that Condition ω) is satisfied for our operators.

Condition 2) of Theorem 2.1 is not satisfied because $A_2 \overline{u}_j = 0,\ j = 1, 2, \dots,$ where $\overline{u}_j = \{\overline{c}_{n,j}\}$ with $\overline{c}_{n,j} = 1/n$ for $n = j$ and $\overline{c}_{n,j} = 0$ for $n \neq j$. The sequence $\{\overline{u}_j\}$ belongs to Z_1' and the weak limit of the sequence $\{\overline{u}_j/\|\overline{u}_j\|\}$ is zero. The assertion of Theorem 2.1 is not true for the operators under consideration because $\{\overline{u}_j\}$ is a sequence of critical points of the operator A_2 which converges to zero. This completes the proof of Proposition 3.2. □

Example 3. Let us define an operator $A_3 : X \supset D(A_3) \to X^*$ by

$$\langle A_3 u, v\rangle = \sum_{n=1}^{\infty} \left\{ |c_n|^{p-2} + 1 - \frac{N}{n} + \delta \sum_{m=1}^{\infty} \frac{1}{2^m} \sin\left(\frac{1}{\|u - \tilde{u}_m\|}\right) \right\} c_n d_n, \tag{3.20}$$

where $u = \{c_n\},\ v = \{d_n\},\ N$ is a positive non-integer number and δ is a positive number satisfying

$$\delta < \min\left\{ \frac{N+1}{\nu_3 + 1} - 1, 1 - \frac{N}{\nu_3 + 1} \right\}, \tag{3.21}$$

and $\nu_3 = [N]$. The sequence $\{\tilde{u}_m\}$ is such that the closure of $\{\tilde{u}_m\}$ coincides with X and

$$\tilde{u}_n \notin L, \quad \tilde{u}_n - \tilde{u}_m \notin L, \quad \text{for } n,\ m = 1, 2, \dots,\ n \neq m, \tag{3.22}$$

where L is as in the proof of Proposition 3.2.

We introduce the operators $A_{3,0} : D(A_{3,0}) = X \to X^*,\ A_3' : D(A_3') = X \to X^*$ by

$$\begin{aligned} \langle A_{3,0}u, v\rangle &= \sum_{n=1}^{\infty} |c_n|^{p-2} c_n d_n, \\ \langle A_3' u, v\rangle &= \sum_{n=1}^{\infty} \left(1 - \frac{N}{n} + \delta \sum_{m=1}^{\infty} \frac{1}{2^m} \sin\left(\frac{1}{\|\tilde{u}_m\|}\right)\right) c_n d_n. \end{aligned} \tag{3.23}$$

Proposition 3.3. *The operators $A_3,\ A_{3,0},\ A_3'$ satisfy all the assumptions of Theorem 2.2, respectively, zero is an isolated critical point of the operator A_3 and*

$$Ind(A_3, 0) = (-1)^{\nu_3}, \tag{3.24}$$

where $\nu_3 = [N]$.

Proof. We shall check only Conditions $(S_+)_{0,L}$ and $\omega)$ for the operators under consideration because all the other conditions follow as in the proof of Proposition 3.1.

Let $\{u_j\}$ be an arbitrary sequence such that

$$u_j \in D(A_3), \quad u_j \rightharpoonup u_0, \quad \limsup_{j\to\infty} \langle A_3 u_j, u_j\rangle \le 0, \quad \lim_{j\to\infty} \langle A_3 u_j, v\rangle = 0, \tag{3.25}$$

for some $u_0 \in X$ and any $v \in L$. We may assume, via a diagonal process and passing to a subsequence if necessary, that

$$\sin\left(\frac{1}{\|u_j - \tilde{u}_m\|}\right) \to d_m \quad \text{as } j \to \infty, \text{ for } m = 1, 2, \dots, \tag{3.26}$$

for some numbers d_m. As in the proof of Proposition 3.1, we establish the strong convergence of u_j to $u_0 = \{c_{n,0}\}$ and

$$\left\{ |c_{k,0}|^{p-2} + 1 - \frac{N}{k} + \delta \sum_{m=1}^{\infty} \frac{d_m}{2^m} \right\} c_{k,0} = 0. \tag{3.27}$$

From (3.21) and (3.27) we have $c_{k,0} = 0$ for $k > N$ and, consequently, $u_0 \in F_{\nu_3} \subset D(A_3)$. Now, we can check that $d_m = \sin\left(\|u_0 - \tilde{u}_m\|^{-1}\right)$ and $A_3 u_0 = 0$ follows from (3.27). We have shown that Condition $(S_+)_{0,L}$ is satisfied for the operator A_3.

Let us prove that Condition $\omega)$ is also true for our operators. We consider a sequence $\{u_j\}$ such that $u_j \in Z_1',\ u_j \to 0$, where Z_1' is defined by (1.9) with $\varepsilon = 1$.

We evaluate the norm of $\omega(u_j) = A_3 u_j - A_3' u_j$ in X^*:

$$\|\omega(u_j)\|_*^{p'} \le \mu_5 \left\{ \|u_j\|^p + \sigma_j^{p'} \|u_j\|^{p'} \right\}, \tag{3.28}$$

where μ_5 is a positive constant independent of j and

$$\sigma_j = \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \sin\left(\frac{1}{\|u_j - \tilde{u}_m\|}\right) - \sin\left(\frac{1}{\|\tilde{u}_m\|}\right) \right|.$$

It is easy to show that $\sigma_j \to 0$ as $j \to \infty$. Thus, (3.28) implies $\|u_j\|^{-1}\|\omega(u_j)\|_* \to 0$ as $j \to \infty$ and, consequently, Condition $\omega)$ is satisfied. The proof of Proposition 3.3 is complete. ∎

We introduce the operators $\widetilde{A}'_3 : D(\widetilde{A}'_3) = X \to X^*$ by

$$\langle \widetilde{A}'_3 u, v\rangle = \sum_{n=1}^{\infty} \left(1 - \frac{N}{n}\right) c_n d_n. \tag{3.29}$$

Proposition 3.4. *Assume that $0 < N < 1$. Then the operators $A_3,\ A_{3,0}$ and $\widetilde{A}'_3$ satisfy all the assumptions of Theorem 2.1, respectively.*

Proof. In order to prove this proposition it is sufficient to note that for the operators under consideration the set Z'_ε defined by (1.9) is empty. The rest of the conditions can be proved as in Proposition 3.3. □

Remark 3.1. Propositions 3.3 and 3.4 indicate that the operator A' is not uniquely defined by the conditions of Theorem 2.1.

Remark 3.2. It is easy to verify that it is impossible to extend the operator A_3) outside the set $D(A_3) = X \setminus \bigcup_{m=1}^{\infty}\{\tilde{u}_m\}$ so that its extension is hemicontinuous.

Example 3.4. This example shows that Condition 1) of Theorem 2.1 is necessary for the use of the splitting argument in the proof of the index formula. It involves the linear operator A' which gives us a linear approximation to the nonlinear operator A in the sense of Condition ω).

We define the linear operators $A'_4 : D(A'_4) = X \to X^*,\ \Gamma_4 : X \to X^*$ by

$$\begin{aligned} \langle A'_4 u, v\rangle &= \left[-c_1 + \sum_{n=2}^{\infty} \frac{c_n}{n^2}\right] d_1 + \sum_{n=2}^{\infty} \frac{c_n d_n}{n^2}, \\ \langle \Gamma_4 u, v\rangle &= 2c_1 d_1, \quad \text{for } u = \{c_n\},\ v = \{d_n\}. \end{aligned} \tag{3.30}$$

Proposition 3.5. *The operators A'_4, Γ_4 satisfy all the assumptions of Theorem 2.1 for the operators A', Γ, respectively, except Condition 1). However, the assertion of Lemma 4.2 for these operators is false.*

Proof. It is easy to check that the operator $T_4 = (A'_4 + \Gamma_4)^{-1}\Gamma_4 : X \to X$ is defined now by $T_4 u = 2c_1$, for $u = \{c_n\}$. From the definition of $\Gamma_4,\ T_4$ we know that these operators are compact. It is also easy to verify that Conditions $(S')_L,\ A')$ are satisfied by the operators under consideration, respectively.

The invariant subspaces F_4 and R_4 of the operator T_4 are defined by

$$\begin{aligned} F_4 &= \{u = \{c_n\} \in X \ : \ c_n = 0, \text{ for } n \geq 2\}, \\ R_4 &= \{u = \{c_n\} \in X \ : \ c_1 = 0\}, \end{aligned}$$

and the splitting $X = F_4 + R_4$ corresponds to the splitting given by (2.2).

The operator $\Pi_4(A'_4 + \Gamma_4)^{-1}$ is defined now for $h = \{h_n\} \in (A'_4 + \Gamma_4)X$ by the formula

$$\Pi_4(A'_4 + \Gamma_4)^{-1} h = h_1 - \sum_{n=2}^{\infty} h_n$$

and it is clear that it is an unbounded operator.

We shall prove that

$$X^* = \overline{(A'_4 + \Gamma_4)R_4}. \tag{3.31}$$

In fact, if (3.31) is not true, there exists $w = \{w_n\} \in X$ such that $w \neq 0$ and

$$\langle (A'_4 + \Gamma_4) r, w\rangle = 0 \quad \text{for all } r = \{r_n\} \in R_4. \tag{3.32}$$

This says

$$\sum_{n=2}^{\infty}\frac{r_n}{n^2}\cdot w_1+\sum_{n=2}^{\infty}\frac{r_n w_n}{n^2}=0. \tag{3.33}$$

Since r is an arbitrary element of R_4, we obtain from (3.33)

$$w_1+w_n=0, \quad \text{for } n=2,3,\dots . \tag{3.34}$$

If $w_1=0$ then we have from (3.34) $w_n=0$ for $n=2,3,\dots$, which contradicts the assumption that $w\neq 0$. If $w_1\neq 0$, then we have from (3.34) $w_n=-w_1$ for all $n=2,3,\dots$. In this case $w\notin X$, i.e., a contradiction.

Since (3.31) is true, the sum

$$(A_4'+\Gamma_4)F_4+\overline{(A_4'+\Gamma_4)R_4}$$

is not direct. It follows that the assertion of Lemma 4.2 for $A_4',\ \Gamma_4$ is false. This finishes the proof of Proposition 3.5. □

4. The proof of Theorem 2.1

We show first that zero is an isolated critical point of the operator A. Assume that the contrary is true: there exists a sequence $\{u_j\}$ such that

$$u_j\in D(A), \quad Au_j=0, \quad u_j\neq 0, \quad u_j\to 0. \tag{4.1}$$

By the definition of the set Z_ε' in (1.9) we obtain that $u_j\in Z_\varepsilon'$ for all large j. Then from the condition $C)$ it follows that the weak closure of the set $\{v_j'=u_j/\|u_j\|\}$ does not contain zero. We may assume that $v_j'\rightharpoonup v_0',\ v_0'\neq 0$.

Using the condition $\omega)$ and (4.1) we have

$$\lim_{j\to\infty}\langle A'v_j',v_j'\rangle=0, \quad \lim_{j\to\infty}\langle A'v_j',v\rangle=0,$$

for an arbitrary $v\in L$. From the last equalities above and $(S')_L$ we get $A'v_0'=0$, which contradicts Condition $A')$ of the theorem. Consequently, the first conclusion of the theorem is now established.

At this point we need to recall some properties of the degree of densely defined operators. Choose a sequence $\{F_j\},\ j\in N$, such that, for each $j\in N$,

$$F_j\in\mathcal{F}(L), \quad F_j\subset F_{j+1}, \quad dimF_j=j, \quad \overline{L\{F_j\}}=X, \tag{4.2}$$

where L is the subspace of X such that $\overline{L}=X$ and $\mathcal{F}(L)$ is the family of all finite-dimensional subspaces of L. Moreover,

$$L\{F_j\}=\bigcup_{j=1}^{\infty}F_j.$$

We let $\{v_j\}$ be a sequence in X such that F_j is the span of $\{v_1,\dots,v_j\}$. We define, for every j, the finite-dimensional approximation A_j of the operator A by the formula

$$A_ju=\sum_{i=1}^{j}\langle Au,v_i\rangle v_i, \quad \text{for } u\in F_j\cap D_0, \tag{4.3}$$

and assume that the operator A satisfies the conditions of Theorem 2.1.

The degree $Deg(A, D, 0)$ is defined in [3] for the operator A w.r.t. an arbitrary open subset D of X provided that

$$Au \neq 0, \text{ for } u \in D(A) \cap \partial D, \quad \overline{D} \subset D_0. \tag{4.4}$$

This degree is defined by

$$Deg(A, D, 0) = \lim_{j\to\infty} deg(A_j, D_j, 0), \tag{4.5}$$

where $deg(A_j, D_j, 0)$ is the degree of the finite-dimensional mapping A_j defined by (4.3) and $D_j = D \cap F_j$. It has been shown in [3] that $deg(A_j, D_j, 0)$ is well defined, and that the limit (4.5) exists and does not depend on the choice of the sequences $\{F_j\}$, $\{v_j\}$. We need a homotopy invariance property for this degree.

Definition 4.1. Let $A_t : X \supset D(A_t) \to X^*$, $t \in [0, 1]$, be a family of nonlinear operators such that $D_0 \cap L \subset D(A_t)$, for $t \in [0, 1]$. We say that the family $\{A_t\}$ satisfies Condition $(S_+)^{(t)}_{0,L}$ w.r.t. the open set D, $\overline{D} \subset D_0$, if there exists a sequence of subspaces $\{F_j\}$ satisfying Condition (4.2) and such that for any sequences $\{u_j\} \subset L\{F_j\} \cap \partial D$ and $\{t_j\} \subset [0, 1]$ the assumptions $u_j \rightharpoonup u_0$, $t_j \to t_0$,

$$\lim_{j\to\infty} \langle A_{t_j}(u_j), u_j\rangle = 0, \quad \lim_{j\to\infty} \langle A_{t_j}(u_j), v\rangle = 0, \tag{4.6}$$

for some $u_0 \in X$, $t_0 \in [0, 1]$ and any $v \in L\{F_j\}$ imply the strong convergence of $\{u_j\}$ and $A_{t_0}(u_0) = 0$.

Definition 4.2. Let $A^{(i)} : X \supset D(A^{(i)}) \to X^*$, $i = 0, 1$, satisfy Conditions A_1), A_2) with a common space L and a common set D_0. The operators $A^{(0)}$, $A^{(1)}$ are called "homotopic" w.r.t. the open set D, $\overline{D} \subset D_0$, if there exists a one-parameter family $A_t : X \supset D(A_t) \to X^*$, $t \in [0, 1]$, satisfying Condition $(S_+)^{(t)}_{0,L}$ w.r.t. D and such that:

1) $A^{(0)} = A_0$, $A^{(1)} = A_1$ and

$$A_t(u) \neq 0, \quad \text{for } u \in D(A_t) \cap \partial D,\ t \in [0, 1]; \tag{4.7}$$

2) for every space $F \subset L\{F_j\}$ and every $v \in L\{F_j\}$ the mapping $\tilde{a}(F, v) : F \times [0, 1] \to \mathcal{R}$, defined by

$$\tilde{a}(F, v)(u, t) = \langle A_t u, v\rangle,$$

is continuous.

The following homotopy invariance property was shown by the authors in [3]: if $A^{(0)}$, $A^{(1)}$ satisfy the conditions of Definition 4.2, then

$$Deg(A^{(0)}, D, 0) = Deg(A^{(1)}, D, 0). \tag{4.8}$$

We shall say that the family $\{A_t\}$ is a homotopy realization of the operators $A^{(0)}$, $A^{(1)}$ if all the conditions of Definition 4.2 are satisfied.

The proof of the index formula is based on the following lemmas involving auxiliary homotopies.

Lemma 4.1. *Assume that the conditions of Theorem 2.1 are satisfied. Then there exists a positive number r_1 such that:*

1) for $t \in [0, 1]$, $u \in D(A^{(1)}_t)$, $0 < \|u\| \leq r_1$, we have $A^{(1)}_t(u) \neq 0$, where

$$A^{(1)}_t(u) = tAu + (1 - t)\,[A_0 u + A' u]\,; \tag{4.9}$$

2) the operator $A_0 + A'$ satisfies Condition $(S_+)_L$;

3) the family $\{A_t^{(1)}\}$ *is a homotopy realization of the operators* A *and* $A_0 + A'$ *w.r.t.* $B_r(0)$, *where* r *is any number from the interval* $(0, r_1]$.

Proof. The proof of the first assertion of this lemma is analogous to the proof of the fact that zero is an isolated critical point. We only need to mention that here we also make use of the property A_0).

Let us prove the second assertion of the lemma. Let $\{u_j\}$ be a sequence in $D(A_0) \cap D(A')$ such that

$$u_j \rightharpoonup u_0, \ \limsup_{j\to\infty}\langle A_0u_j + A'u_j - h, u_j\rangle \le 0, \ \lim_{j\to\infty}\langle A_0u_j + A'u_j - h, v\rangle = 0, \tag{4.10}$$

for some $u_0 \in X, \ h \in X^*$ and any $v \in L$. We may also assume that

$$A_0u_j \rightharpoonup h_0, \quad \langle A_0u_j, u_j\rangle \to a_0, \tag{4.11}$$

for some $h_0 \in X^*, \ a_0 \in \mathcal{R}$.

We first establish the inequality

$$\langle h_0, u_0\rangle \le a_0. \tag{4.12}$$

If (4.12) is not true then we have

$$\lim_{j\to\infty}\langle A_0u_j - h_0, u_j\rangle < 0, \quad \lim_{j\to\infty}\langle A_0u_j - h_0, v\rangle = 0, \tag{4.13}$$

for any $v \in L$, and the strong convergence of the sequence $\{u_j\}$ follows from (4.13) by Condition $(S_+)_L$ for the operator A_0. Thus, the first limit in (4.13) must be equal to zero. This contradiction proves the inequality (4.12).

Using (4.12) we derive from (4.10)

$$\limsup_{j\to\infty}\langle A'u_j - h', u_j\rangle \le 0, \quad \lim_{j\to\infty}\langle A'u_j - h', v\rangle = 0 \tag{4.14}$$

with $h' = h - h_0$ and any $v \in L$. By virtue of (4.14) and Condition $(S')_L$ for the operator A' we obtain

$$u_0 \in D(A'), \quad A'u_0 = h - h_0, \quad \lim_{j\to\infty}\langle A'u_j, u_0\rangle = \langle h', u_0\rangle. \tag{4.15}$$

In order to prove the strong convergence of $\{u_j\}$ we first note that

$$\begin{aligned}\langle A_0u_j - h_0, u_j\rangle &= \langle A_0u_j + A'u_j, u_j\rangle - \langle (A' + \Gamma)(u_j - u_0), u_j - u_0\rangle \\ &\quad + \langle \Gamma(u_j - u_0), u_j - u_0\rangle - \langle A'u_0, u_j - u_0\rangle \\ &\quad - \langle A'u_j, u_0\rangle - \langle h_0, u_j\rangle.\end{aligned}$$

Using (1.7), (4.10), (4.15) and the compactness of the operator Γ we obtain from the last equality above

$$\limsup_{j\to\infty}\langle A_0u_j - h_0, u_j\rangle \le 0. \tag{4.16}$$

The last equality, (4.10), (4.11) and Condition $(S_+)_L$ for the operator A_0 imply

$$u_j \to u_0, \quad u_0 \in D(A_0), \quad A_0u_0 = h_0. \tag{4.17}$$

Now, the second assertion of the lemma follows from (4.15) and (4.17).

Let us prove the third assertion of the lemma. Taking into account the first assertion and the properties of the operators $A, \ A_0$ we need to prove only Condition $(S_+)_{0,L}^{(t)}$ for the family $\{A_t^{(1)}\}$ and the ball $B_r(0), \ r \in (0, r_1]$. We choose an arbitrary sequence of subspaces $\{F_j\}$ satisfying the condition (4.2) and let $\{u_j\},\ \{t_j\}$ be such that

$$u_j \in L\{F_j\} \cap \partial B_r(0), \quad t_j \in [0,1], \quad u_j \rightharpoonup u_0, \quad t_j \to t_0$$

and

$$\lim_{j\to\infty} \langle A^{(1)}_{t_j}(u_j), u_j\rangle = 0, \quad \lim_{j\to\infty} \langle A^{(1)}_{t_j}(u_j), v\rangle = 0, \tag{4.18}$$

for some $u_0 \in X$ and any $v \in L$. We may also assume that

$$A_0 u_j \rightharpoonup h_0, \quad \langle A_0 u_j, u_j\rangle \to a_0, \tag{4.19}$$

for some $h_0 \in X^*,\ a_0 \in \mathcal{R}$.

We can show that

$$\langle h_0, u_0\rangle \le a_0 \tag{4.20}$$

as we did in the case of (4.12). Using (4.20) we derive from (4.18)

$$\begin{aligned} &\lim_{j\to\infty} \langle t_j A u_j + (1-t_j)\left[A' u_j + h_0\right], u_j\rangle \le 0, \\ &\lim_{j\to\infty} \langle t_j A u_j + (1-t_j)\left[A' u_j + h_0\right], v\rangle = 0, \end{aligned} \tag{4.21}$$

for any $v \in L$.

By conditions (2.1) and (1.7) we have

$$\langle A u_j, u_j - v\rangle \ge -C'(v), \quad \langle A' u_j, u_j - v\rangle \ge -C'(v), \tag{4.22}$$

where the positive number $C'(v)$ depends only on v. From (4.21) and (4.22) we obtain the estimates

$$\begin{aligned} t_j|\langle A u_j, u_j\rangle| \le c_1, \quad (1-t_j)|\langle A' u_j, u_j\rangle| \le c_1, \\ t_j|\langle A u_j, v_j\rangle| \le c_2(v), \quad (1-t_j)|\langle A' u_j, v\rangle| \le c_2(v), \end{aligned} \tag{4.23}$$

with a positive constant c_1 and a positive number $c_2(v)$ depending only on v.

We now consider separately three possible cases: a) $0 < t_0 < 1$; b) $t_0 = 0$; c) $t_0 = 1$. In the case a) we obtain from (4.21) and (4.23)

$$\begin{aligned} &\lim_{j\to\infty} \langle t_0 A u_j + (1-t_0) A' u_j + (1-t_0) h_0, u_j\rangle \le 0, \\ &\lim_{j\to\infty} \langle t_0 A u_j + (1-t_0) A' u_j + (1-t_0) h_0, v\rangle = 0. \end{aligned} \tag{4.24}$$

Using now Condition $(S_+)_L$ for the operator $A + [(1-t_0)/t_0]A'$ we get

$$u_j \to u_0, \quad t_0 A u_0 + (1-t_0) A' u_0 + (1-t_0) h_0 = 0. \tag{4.25}$$

The strong convergence of $\{u_j\}$ implies

$$\lim_{j\to\infty} \langle A_0 u_j - h_0, u_j\rangle = 0, \quad \lim_{j\to\infty} \langle A_0 u_j - h_0, v\rangle = 0,$$

for any $v \in L$. By Condition $(S_+)_L$ for the operator A_0 we obtain $h_0 = A_0 u_0$. Consequently, from (4.25) it follows that $A^{(1)}_{t_0}(u_0) = 0$, with $u_0 \in \partial B_r(0)$, which contradicts the first assertion of the lemma. We have established that case a) is impossible.

Let us consider now case b), $t_0 = 0$. From (4.21) we derive

$$\lim_{j\to\infty} \langle t_j A u_j + (1-t_j)\left[A' u_j + h_0\right], u_j - v\rangle \le 0, \tag{4.26}$$

for any $v \in L$. By virtue of (4.22) and (4.23) we get

$$\lim_{j\to\infty} \langle A'u_j + h_0, u_j - v\rangle \le 0, \tag{4.27}$$

for any $v \in L$. Since (4.27) is true for any $v \in L$, we obtain immediately that

$$\lim_{j\to\infty} \langle A'u_j + h_0, v\rangle = 0, \tag{4.28}$$

for any $v \in L$. From (4.27), (4.28) and Condition $(S')_L$ for the operator A' we conclude that

$$u_0 \in D(A'), \quad A'u_0 + h_0 = 0, \quad \lim_{j\to\infty} \langle A'u_j, u_0\rangle = -\langle h_0, u_0\rangle. \tag{4.29}$$

We now check that $u_j \to u_0, u_0 \in D(A_0)$ and $A_0u_0 = h_0$. This follows exactly as (4.17) was established from (4.15) and (4.16). Thus we have

$$A_0u_0 + A'u_0 = 0, \quad u_0 \in D(A_0) \cap D(A'), \quad \|u_0\| = r,$$

which contradicts the first assertion of the lemma. We have established the impossibility of the case b).

Let us now consider the last case c), $t_0 = 1$. From (4.22), (4.23) and (4.26) we derive

$$\lim_{j\to\infty} \langle Au_j, u_j - v\rangle \le 0, \quad \text{for } v \in L. \tag{4.30}$$

Since v is arbitrary in L this inequality is possible only if

$$\lim_{j\to\infty} \langle Au_j, v\rangle = 0, \quad \text{for } v \in L. \tag{4.31}$$

From (4.30), (4.31) and Condition $(S_+)_L$ for the operator A we conclude that $u_j \to u_0$ and $Au_0 = 0$. Thus, we have reached a contradiction with the first assertion of the lemma. This is the end of the proof of Lemma 4.1. □

We define the subspaces $F^*,\ R^*$ of the space X^* by

$$F^* = (A' + \Gamma)F, \quad R^* = \overline{(A' + \Gamma)(D(A') \cap R)}, \tag{4.32}$$

where $F,\ R$ are the subspaces of X from (2.2). It is easy to see that $F \subset D(A')$.

Lemma 4.2. *We have the splitting*

$$X^* = F^* + R^*. \tag{4.33}$$

The sum in (4.33) is direct.

Proof. At first we prove (4.33). If $X^* \ne F^* + R^*$ then there exists an element $w \in X,\ w \ne 0$, such that

$$\langle (A' + \Gamma)u, w\rangle = 0, \quad \text{for all } u \in D(A').$$

This means that $w \in D((A' + \Gamma)^*)$ and $(A' + \Gamma)^*w = 0$, but this is impossible by virtue of (1.7). This contradiction proves (4.33).

To show that the sum in (4.33) is direct, assume the contrary. Then there exist $f \in F,\ f \ne 0$, and a sequence $\{r_j\} \subset D(A') \cap R$ such that

$$\lim_{j\to\infty} (A' + \Gamma)(f + r_j) = 0. \tag{4.34}$$

Evaluating the operator $\Pi(A' + \Gamma)^{-1}$ on $(A' + \Gamma)(f + r_j)$ and using (4.34) and Condition 1) of Theorem 2.1 we obtain $f = 0$, i.e., a contradiction to the assumption $f \ne 0$. The proof of Lemma 4.2 is complete. □

According to (4.33), every $h \in X^*$ can be written as f^*+r^*, where $f^* \in F^*,\ r^* \in R^*$ are uniquely determined. Consequently, we can define a bounded linear operator $P^* : X^* \to F^*$ by

$$P^*h = f^*, \quad \text{for } h = f^* + r^*,\ f^* \in F^*,\ r^* \in R^*. \tag{4.35}$$

We also define the functional $\delta : D(A_0) \to \mathcal{R}$ by

$$\delta(u) = \max\{0, c \min_{0\le\tau\le1} \langle (I - P^*)A_0u, (I - \tau T)u\rangle, \tag{4.36}$$

where c is a positive number to be defined below. In particular, we assume that

$$\delta(u) \le 1, \quad \text{for } u \in D(A),\ \|u\| \le r_1, \tag{4.37}$$

where r_1 is the number from Lemma 2.1.

Lemma 4.3. *Assume that the conditions of Theorem 2.1 are satisfied. Then there exists a positive number r_2 such that $r_2 \le r_1$ and:*

1) $A_t^{(2)}(u) \neq 0$ holds for $t \in [0,1],\ u \in D(A_t^{(2)}),\ 0 < \|u\| \le r_2$, where

$$A_t^{(2)}(u) = [t + (1-t)\delta(u)]A_0u + A'u;$$

2) the operator $A_0^{(2)},\ A_0^{(2)}(u) = \delta(u)A_0u + A'u$, satisfies Condition $(S_+)_{0,L}$;

3) the family $\{A_t^{(2)}\}$ is a homotopy realization of the operators $A_0 + A'$ and $A_0^{(2)}$ w.r.t. each ball $B_r(0),\ r \in [0, r_2]$.

Proof. Taking into account Condition 2) of Theorem 2.1 we can prove the first assertion of the lemma as we did in the proof of the part of that theorem concerning the fact that zero is an isolated critical point.

Let us prove the second assertion of the lemma. Let $\{u_j\} \subset D(A_0) \cap D(A')$ be such that $u_j \rightharpoonup u_0$ and

$$\limsup_{j\to\infty} \langle \delta(u_j)A_0u_j + A'u_j, u_j\rangle \le 0, \quad \lim_{j\to\infty} \langle \delta(u_j)A_0u_j + A'u_j, v\rangle = 0, \tag{4.38}$$

for some $u_0 \in X$ and every $v \in L$. We may also assume that

$$A_0u_j \rightharpoonup h_0, \quad \delta(u_j) \to \delta_0, \quad \langle A_0u_j, u_j\rangle \to a_0. \tag{4.39}$$

As in the proof of Lemma 4.1, we can establish (4.12) and

$$u_0 \in D(A'), \quad A'u_0 = -\delta_0h_0, \quad \lim_{j\to\infty} \langle A'u_j, u_0\rangle = -\delta_0\langle h_0, u_0\rangle. \tag{4.40}$$

We shall consider, separately, two cases: a) $\delta_0 > 0$; b) $\delta_0 = 0$. In the first case we can work as in the proof of assertion (4.17) in Lemma 4.1 to establish that

$$u_j \to u_0, \quad u_0 \in D(A_0), \quad A_0u_0 = h_0. \tag{4.41}$$

From (4.39), (4.41) we get $\delta_0 = \delta(u_0)$ and from (4.40), (4.41) we see that $A_0^{(2)}(u_0) = 0$. Thus, we have verified the validity of Condition $(S_+)_{0,L}$ for the operator $A_0^{(2)}$ if $\delta_0 > 0$.

In the case b) we show first that either $u_0 \neq 0$, or $u_j \to 0$. From the definition of $\delta(u)$ we obtain the existence of a sequence $\tau_j \in [0,1]$ such that

$$\lim_{j\to\infty} \langle (I - P^*)A_0u_j, u_j - \tau_jTu_j\rangle \le 0.$$

If $u_0 = 0$, then the last inequality implies

$$\lim_{j\to\infty} \langle A_0u_j - h_0, u_j\rangle \le 0$$

and, by virtue of Condition $(S_+)_L$ for the operator A_0, we obtain $u_j \to 0$.

Since the case $u_j \to 0$ is trivial, we need to consider only the case $u_0 \neq 0$. Then from (4.40) we have $A'u_0 = 0$, which contradicts our assumption $A')$. This completes the proof of the second assertion of the lemma.

The proof of the third assertion of the lemma is similar to the proof of the second assertion. The lemma has been proved. □

Lemma 4.4. *Assume that the conditions of Theorem 2.1 are satisfied. Then there exists a positive number* $\overline{c}_1$ *such that for* c *in (4.36) with* $0 < c \leq \overline{c}_1$ *the following statements are true, where the number* r_2 *is as in Lemma 4.3.*

1) $A_t^{(3)}(u) \neq 0$, *for* $t \in [0,1]$, $u \in D(A_0) \cap D(A')$, $0 < \|u\| \leq r_2$, *where*

$$A_t^{(3)}(u) = \delta(u)A_0u + tA'u + (1-t)\{-(A'+\Gamma)\Pi u + (A'+\Gamma)(I-\Pi)u\}, \tag{4.42}$$

with Π *defined in (2.4).*

2) the operator $A_0^{(3)}$, *defined by (4.42) for* $t=0$, *satisfies Condition* $(S_+)_{0,L}$;

3) the family $\{A_t^{(3)}\}$ *is a homotopy realization of the operators* $A_0^{(2)}$ *and* $A_0^{(3)}$ *w.r.t. each ball* $B_r(0)$, *for* $r \in (0, r_2]$.

Proof. We shall prove the first assertion of the lemma by contradiction. Assume that there exist $u_0 \in D(A) \cap D(A')$, $t_0 \in [0,1]$, such that

$$A_{t_0}^{(3)}(u_0) = 0, \quad 0 < \|u_0\| \leq r_2. \tag{4.43}$$

Let $f_0 = \Pi u_0$, $r_0 = (I-\Pi)u_0$. Noting that

$$A'u = (A'+\Gamma)(I-T)u, \quad \text{for } u \in D(A'), \tag{4.44}$$

we can rewrite (4.43) in the form

$$\delta(u_0)A_0u_0 + (A'+\Gamma)\{(2t_0-1)f_0 - t_0Tf_0 + r_0 - t_0Tr_0\} = 0. \tag{4.45}$$

Using the invariance property of F and R w.r.t. the operator T we have from (4.45)

$$\begin{aligned} &\delta(u_0)P^*A_0u_0 + (A'+\Gamma)\{(2t_0-1)f_0 - t_0Tf_0\} = 0, \\ &\delta(u_0)(I-P^*)A_0u_0 + (A'+\Gamma)\{r_0 - t_0Tr_0\} = 0, \end{aligned} \tag{4.46}$$

where P^* is the operator defined by (4.35).

We consider first the case $\delta(u_0) = 0$. From (4.46) and (1.7) we obtain

$$(2t_0-1)f_0 - t_0Tf_0 = 0, \quad r_0 - t_0Tr_0 = 0. \tag{4.47}$$

Using the definitions of the subspaces F and R we obtain from (4.47) $f_0 = 0$, $r_0 = 0$, which say that $u_0 = f_0 + r_0 = 0$, i.e., a contradiction to (4.43). Thus, the case $\delta(u_0) = 0$ is impossible.

Let us now assume that $\delta(u_0) \neq 0$. Noting that the equation

$$(A'+\Gamma)\{(2t-1)f - tTf\} = 0, \quad f \in F,$$

has only the zero solution in the finite-dimensional space F, we can prove the inequality

$$\|f\| \leq c_1 \cdot \min_{0 \leq t \leq 1} \|(A'+\Gamma)\{(2t-1)f - tTf\}\|, \tag{4.48}$$

for every $f \in F$ and some positive constant c_1 independent of f. Using (4.48) we obtain from (4.46)

$$\|f_0\| \leq c_1c_2\|P^*\|\delta(u_0), \tag{4.49}$$

where $c_2 = \sup\{\|A_0u\| : u \in D(A_0),\ \|u\| \le r_2\}$. From (1.7) and the second equality in (4.46) we have

(4.50)
$$\langle (I - P^*)A_0u_0, r_0 - t_0Tr_0\rangle = -\frac{1}{\delta(u_0)}\langle (A' + \Gamma)(r_0 - t_0Tr_0), r_0 - t_0Tr_0\rangle \le 0.$$

Using (4.49), (4.50) and the definition of $\delta(u)$ from (4.36) we deduce

$$\begin{aligned}\delta(u_0) &\le c\langle (I - P^*)A_0u_0, u_0 - t_0Tu_0\rangle \\ &\le c\langle (I - P^*)A_0u_0, f_0 - t_0Tf_0\rangle \\ &\le cc_1c_2^2\|P^*\|\|I - P^*\|(1 + \|T\|)\delta(u_0).\end{aligned}$$

This leads to a contradiction if we set $c \le \overline{c}_1$, where

$$\overline{c}_1 = \frac{1}{2}\{c_1c_2^2\|P^*\|\|I - P^*\|(1 + \|T\|)\}^{-1}. \tag{4.51}$$

The proof of the first assertion of the lemma is now complete.

To prove the second assertion of the lemma, let $\{u_j\}$ be a sequence such that $u_j \in D(A_0) \cap D(A'),\ u_j \rightharpoonup u_0$,

$$\lim_{j\to\infty}\langle A_0^{(3)}(u_j), u_j\rangle \le 0, \quad \lim_{j\to\infty}\langle A_0^{(3)}(u_j), v\rangle = 0, \tag{4.52}$$

for some $u_0 \in X$ and any $v \in L$. We may also assume that (4.39) is true. As in the proof of Lemma 4.1, we can establish the inequality (4.12) and

$$\begin{gathered} u_0 \in D(A'), \quad A'u_0 = -\delta_0h_0 + 2(1 - t_0)(A' + \Gamma)\Pi u_0 - (1 - t_0)\Gamma u_0, \\ \lim_{j\to\infty}\langle A'u_j, u_0\rangle = \langle A'u_0, u_0\rangle. \end{gathered} \tag{4.53}$$

If $\delta_0 \ne 0$ we get

$$u_j \to u_0, \quad u_0 \in D(A_0), \quad A_0u_0 = h_0,$$

as in the proof of assertion (4.17) of Lemma 4.1. Using (4.53) we complete the proof of the second assertion for $\delta_0 \ne 0$.

If $\delta_0 = 0$ then, as in the proof of Lemma 4.3, we have that either $u_0 \ne 0$ or $u_j \to 0$. The case $u_j \to 0$ is trivial. If $u_0 \ne 0$, then we obtain from (4.53)

$$t_0A'u_0 + (1 - t_0)\{-(A' + \Gamma)\Pi u_0 + (A' + \Gamma)(I - \Pi)u_0\} = 0. \tag{4.54}$$

From this follows that $u_0 = 0$ as in the proof of the first assertion of the lemma. We now have a contradiction, and the proof of Condition $(S_+)_{0,L}$ for the operator $A_0^{(3)}$ is complete.

The proof of the third assertion of the lemma is similar to the proof of the second assertion. This completes the proof of Lemma 4.4. □

The next homotopy will reduce the calculation of the index of the critical point to the corresponding problem for operators defined everywhere on some neighborhood of the critical point. This reduction will be connected with a reconstruction of the operator K which was introduced at the beginning of Section 2.

Let $\{f_1, \dots, f_\nu\}$ be a basis of the linear space F from (2.2). Then the action of the operator P^*, defined by (4.35), is given by

$$P^*h = \sum_{i=1}^{\nu}\langle h, w_i\rangle(A' + \Gamma)f_i, \tag{4.55}$$

where $w_i \in X,\ i = 1, \dots, \nu$, satisfy the following conditions:

$$\langle (A' + \Gamma) f_j, w_i \rangle = \delta_{ij}, \quad \langle (A' + \Gamma) r, w_i \rangle = 0, \tag{4.56}$$

for $i,\ j = 1, \dots, \nu,\ r \in D(A') \cap R$. Here, $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

The operator Π, given by (2.4), may be written in the form

$$\Pi u = \sum_{i=1}^{\nu} \langle h_i, u \rangle f_i, \tag{4.57}$$

where $h_i \in X^*,\ i = 1, \dots, \nu$, satisfy the conditions:

$$\langle h_i, f_j \rangle = \delta_{ij}, \quad \langle h_i, r \rangle = 0, \quad \text{for } i,\ j = 1, \dots, \nu,\ r \in R. \tag{4.58}$$

Consider the matrix $\mathcal{D}$ with entries

$$d_{ij} = \langle h_i, w_j \rangle, \quad i,\ j = 1, \dots, \nu.$$

We show that its determinant is not zero. Assume that the contrary is true. Then we can find

$$\widetilde{w} = \sum_{j=1}^{\nu} \tilde{c}_j w_j \neq 0 \text{ such that } \langle h_i, \widetilde{w} \rangle = 0, \quad i = 1, \dots, \nu. \tag{4.59}$$

We shall prove that $\widetilde{w} \in D((A' + \Gamma)^*)$ and

$$(A' + \Gamma)^* \widetilde{w} = \tilde{h}, \quad \text{where } \tilde{h} = \sum_{j=1}^{\nu} \tilde{c}_j h_j. \tag{4.60}$$

It is necessary to establish the equality

$$\langle (A' + \Gamma) u, \widetilde{w} \rangle = \langle \tilde{h}, u \rangle, \tag{4.61}$$

for an arbitrary $u \in D(A')$. If $u \in D(A') \cap R$ then (4.61) follows from the second equalities in (4.56) and (4.58). If $u = f_i$ we obtain (4.61) from the first equalities in (4.56) and (4.58) and the formulas for $\widetilde{w},\ \tilde{h}$. We have thus shown (4.60) and from (4.59) we get $\langle (A' + \Gamma)^* \widetilde{w}, \widetilde{w} \rangle = 0$, which is a contradiction in view of (1.7). Consequently, the matrix $\mathcal{D}$ is invertible. We denote by c_{ij} the entries of the matrix $\mathcal{D}^{-1}$. We have

$$\sum_{i=1}^{\nu} c_{ki} \langle h_i, w_j \rangle = \delta_{kj}, \quad \text{for } k,\ j = 1, \dots, \nu. \tag{4.62}$$

We introduce the operator $\widetilde{K} : X \to X^*$ by

$$\widetilde{K} u = K u - \sum_{k,i=1}^{\nu} c_{ki} \langle K u, w_k \rangle h_i, \tag{4.63}$$

where K is the operator in Condition X_2) at the beginning of Section 2. We have the following properties of the operator $\widetilde{K}$:

$$P^* \widetilde{K} X = \{0\}, \quad \langle \widetilde{K} r, r \rangle > 0, \quad \text{for } r \in R,\ r \neq 0. \tag{4.64}$$

The first of (4.64) follows immediately from (4.55) and (4.62). The second follows directly from (4.58) and the positiveness property of the operator K.

Define the functional $\tilde{\delta} : D(\widetilde{A}_t) \times [0, 1] \to \mathcal{R}$ by

$$\tilde{\delta}(u, t) = \max\{0, \tilde{c} \min_{0 \le \tau \le 1} \langle (I - P^*) \widetilde{A}_t u, (I - \tau T) u \rangle\}, \tag{4.65}$$

where $\widetilde{A}_t u = (1-t)Ju + tA_0 u$. Here, J is the operator of Condition $X_1)$ at the beginning of Section 2 and $\tilde{c}$ is a positive number to be chosen below.

Lemma 4.5. *Assume that the conditions of Theorem 2.1 are satisfied. Then there exists a positive number $\tilde{c}_2$ such that for $\tilde{c}$ in (4.65) with $0 < \tilde{c} \le \overline{c}_2$ the following statements are true with $r_3 = \min\{r, r_2\}$, where $r,\ r_2$ are the numbers in Condition $X_1)$ and Lemma 4.3, respectively.*

1) $A_t^{(4)}(u) \neq 0$, for $t \in [0,1],\ u \in D(A_t^{(4)}),\ 0 < \|u\| \le r_3$, where

$$A_t^{(4)}(u) = \tilde{\delta}(u,t)\{tA_0 u + (1-t)Ju\} - (A' + \Gamma)\Pi u + t(A' + \Gamma)(I - \Pi)u + (1-t)\widetilde{K}(I - \Pi)u, \tag{4.66}$$

with the operators $\Pi,\ \widetilde{K},\ J$ *defined by (2.4), (4.63) and Condition* $X_1)$, *respectively;*

2) the operator $A_0^{(4)}$, defined by (4.66) for $t = 0$, satisfies Condition $(S_+)_{0,X}$;

3) the family $\{A_t^{(4)}\}$ is a homotopy realization of the operators $A_0^{(3)}$ and $A_0^{(4)}$ w.r.t. every ball $B_r(0)$, for $r \in (0, r_3]$.

Proof. The proof of the first assertion of the lemma is analogous to the proof of the first assertion of Lemma 4.4. We only need to note that we now should use, in addition, the properties (4.64) of the operator $\widetilde{K}$. The number $\overline{c}_2$ is defined by

$$\overline{c}_2 = \frac{1}{2}\{c_1 \tilde{c}_2^2 \|P^*\| \|I - P^*\|\}^{-1},$$

where

$$\tilde{c}_2 = \sup\{\|\widetilde{A}_t u\| \ :\ u \in D(\widetilde{A}_t),\ \|u\| \le r_3,\ t \in [0,1]\}$$

and c_1 is the constant from (4.48).

We can prove Condition $(S_+)_{0,X}$ for the operator $A_0^{(4)}$ as in the proof of the second assertion of Lemma 4.4. We only remark here that the property $(S')_X$ for the operator $\widetilde{K}$ is clear.

Let us prove the third assertion of the lemma. We only have to verify the validity of Condition $(S_+)_{0,L}^{(t)}$ for the family $\{A_t^{(4)}\}$ w.r.t. the ball $B_r(0),\ 0 < r \le r_3$. Let $\{u_j\},\ \{t_j\}$ be such that

$$u_j \in L\{F_j\}, \quad \|u_j\| = r, \quad t_j \in [0,1], \quad u_j \rightharpoonup u_0, \quad t_j \to t_0, \quad \lim_{j\to\infty} \langle A_{t_j}^{(4)}(u_j), u_j\rangle = 0, \quad \lim_{j\to\infty} \langle A_{t_j}^{(4)}(u_j), v\rangle = 0, \tag{4.67}$$

for some $u_0 \in X,\ t_0 \in [0,1]$ and every $v \in L\{F_j\}$, where $\{F_j\}$ is a fixed sequence of subspaces of the space L satisfying the condition (4.2). We may also assume that the following are true with some $h_0,\ \tilde{h} \in X^*,\ a_0,\ \tilde{a},\ \tilde{\delta}_0,\ k_0 \in \mathcal{R}$:

$$A_0 u_j \rightharpoonup h_0, \quad Ju_j \rightharpoonup \overline{h}, \quad \langle A_0 u_j, u_j\rangle \to a_0, \quad \langle Ju_j, u_j\rangle \to \overline{a}, \quad \tilde{\delta}(u_j, t_j) \to \tilde{\delta}_0, \quad \langle \widetilde{K}u_j, u_j\rangle \to k_0. \tag{4.68}$$

As in the proof of (4.12) we establish the inequalities

$$\langle h_0, u_0\rangle \le a_0, \quad \langle \overline{h}, u_0\rangle \le \overline{a}. \tag{4.69}$$

We note that from (4.64) we obtain

$$\langle \widetilde{K}u_0, u_0\rangle \le k_0. \tag{4.70}$$

Using (4.69) and (4.70) we derive from (4.67)

$$\lim_{j\to\infty}\langle t_jA'u_j+h',u_j\rangle\le 0,\quad \lim_{j\to\infty}\langle t_jA'u_j+h',v\rangle=0, \tag{4.71}$$

where

$$h'=\tilde\delta_0[t_0h_0+(1-t_0)\overline{h}]-(A'+\Gamma)\Pi u_0-t_0A'\Pi u_0$$
$$+t_0\Gamma(I-\Pi)u_0+(1-t_0)\widetilde{K}(I-\Pi)u_0.$$

We now apply Condition $(S')_L$ for the operator A' and the sequence $\widetilde{u}_j=t_ju_j$, and we obtain

$$t_0u_0\in D(A'),\quad A'(t_0u_0)=-h',\quad \lim_{j\to\infty}\langle t_jA'u_j,t_0u_0\rangle=-t_0\langle h',u_0\rangle. \tag{4.72}$$

We consider three possibilities:

$$\text{a) } t_0\ne 0,\ \tilde\delta\ne 0;\quad \text{b) } \tilde\delta=0;\quad \text{c) } t_0=0,\ \tilde\delta\ne 0.$$

In the case a) we obtain from (1.7), (4.67) and (4.72)

$$\lim_{j\to\infty}\langle\tilde\delta[t_0A_0u_j+(1-t_0)Ju_j-t_0h_0-(1-t_0)\overline{h}],u_j\rangle\le 0 \tag{4.73}$$

as in the proof of the inequality (4.16). From (4.68) and (4.73) we conclude that at least one of the following inequalities holds:

$$\lim_{j\to\infty}\langle A_0u_j-h_0,u_j\rangle\le 0,\quad \lim_{j\to\infty}\langle Ju_j-\overline{h},u_j\rangle\le 0.$$

Thus, using Condition $(S_+)_L$ for the operators $A_0,\ J$ we arrive at the strong convergence of $\{u_j\}$ to u_0 and the relations $h_0=A_0u_0,\ \overline{h}=Ju_0,\ \tilde\delta_0=\tilde\delta(u_0,t_0)$. Inequality (4.72) implies now $A^{(4)}_{t_0}(u_0)=0$ and the proof of Condition $(S_+)^{(t)}_{0,L}$ for the family $\{A^{(4)}_t\}$ is complete for the case a).

Case b). We first show that $u_0\ne 0$. From the definition of $\tilde\delta(u,t)$ we obtain the existence of a sequence $\tau_j\in[0,1]$ such that

$$\lim_{j\to\infty}\langle(I-P^*)[(1-t_0)Ju_j+t_0A_0u_j],u_j-\tau_jTu_j\rangle\le 0.$$

If $u_0=0$ the last inequality implies

$$\lim_{j\to\infty}\langle(1-t_0)[Ju_j-\overline{h}]+t_0[A_0u_j-h_0],u_j\rangle\le 0$$

and, as in Case a), we obtain from (4.68) and the last inequality the strong convergence of $\{u_j\}$ to u_0. This is impossible because $\|u_j\|=r$.

From (4.72) we obtain, in the current case b),

$$-(A'+\Gamma)\Pi u_0+t_0(A'+\Gamma)(I-\Pi)u_0+(1-t_0)\widetilde{K}(I-\Pi)u_0=0,$$

from which we have $u_0=0$ as in the proof of the first assertion of Lemma 4.4. This is a contradiction. Consequently, Case b) is also excluded.

Case c), $t_0=0,\ \tilde\delta\ne 0$. From (4.67) and (4.70) we derive

$$\lim_{j\to\infty}\langle\tilde\delta_0Ju_j+t_jA'u_j+\widetilde{K}(I-\Pi)u_0-(A'+\Gamma)\Pi u_0,u_j-v\rangle\le 0, \tag{4.74}$$

for every $v\in L\{F_j\}$. From (4.74) and (4.22) we get

$$\lim_{j\to\infty}\langle\tilde\delta_0Ju_j+\widetilde{K}(I-\Pi)u_0-(A'+\Gamma)\Pi u_0,u_j-v\rangle\le 0. \tag{4.75}$$

Since this is true for all $v \in L\{F_j\}$, we obtain immediately

$$\lim_{j\to\infty} \langle \tilde{\delta}_0 J u_j + \widetilde{K}(I-\Pi)u_0 - (A'+\Gamma)\Pi u_0, v\rangle = 0, \tag{4.76}$$

for all $v \in L\{F_j\}$. From (4.75), (4.76) and Condition (S_+) for the operator J we conclude that

$$u_j \to u_0, \quad \tilde{\delta}(u_0,0)Ju_0 + \widetilde{K}(I-\Pi)u_0 - (A'+\Gamma)\Pi u_0 = 0$$

and establish Condition $(S_+)^{(t)}_{0,L}$ for the family $\{A^{(4)}_t\}$ in the case c). This ends the proof of Lemma 4.5. □

We are now ready to calculate the index of the operator A at the critical point zero by using the previous lemmas and the fact that the degree is invariant under homotopies. We fix the value of $\tilde{c}$ in (4.65): $\tilde{c} = \overline{c}_2$, where $\overline{c}_2$ is defined in Lemma 4.5.

From Definition 1.4, (4.8) and Lemmas 4.1, 4.3-4.5 we obtain immediately

Lemma 4.6. *Assume that the conditions of Theorem 2.1 are satisfied. Then*

$$Ind(A,0) = Deg(A^{(4)}_0, B_r(0), 0), \tag{4.77}$$

where $A^{(4)}_0$ *is the operator introduced in Lemma 4.5 and* r *is an arbitrary number from the interval* $(0, r_3]$.

We note that the operator $A^{(4)}_0$ is defined everywhere on the ball $\overline{B_r(0)}$. We can now choose a sequence $\{F_j\}$ which satisfies condition (4.2) for $L = X$. We select this sequence in such a way that

$$F_\nu = PX, \quad F \subset F_{2\nu}, \tag{4.78}$$

where P is the adjoint of P^* defined by (4.55) and F is the subspace from the splitting (2.2). We select a complete system $\{v_i\},\ i = 1, 2, \dots$, in such a way that each subspace F_j is a linear combination of the elements $v_1, \dots, v_j$. We may assume that $v_i = w_i$, for $i \le \nu$, where w_i are the same as in (4.55).

We define a finite-dimensional approximation $A^{(4)}_{0,j}$ of the operator $A^{(4)}_0$ according to the formula (4.3):

$$A^{(4)}_{0,j}(u) = \sum_{i=1}^{j} \langle A^{(4)}_0(u), v_i\rangle v_i, \quad \text{for } u \in F_j \cap B_{r_3}(0). \tag{4.79}$$

By the definition of the degree mapping by (4.5), we may choose, for a given number $r \in (0, r_3]$, a number $j(r)$ such that

$$Deg(A^{(4)}_0, B_r(0), 0) = deg(A^{(4)}_{0,j}, B_{r,j}(0), 0), \quad \text{for } j \ge j(r), \tag{4.80}$$

where $B_{r,j} = B_r(0) \cap F_j$.

Lemma 4.7. *Assume that the conditions of Theorem 2.1 are satisfied. Let* r_0 *be a fixed number from the interval* $(0, r_3]$ *and* $j_0 = 2\nu + j(r_0)$. *Then the equation*

$$A^{(4)}_{0,j_0}(u) = 0 \tag{4.81}$$

has only the zero solution in $B_{r_0,j_0}(0)$.

Proof. Assume that the contrary is true: there exists a solution u of (4.81) in $B_{r_0,j_0}(0)$ such that $\overline{u} \neq 0$. We have from (4.66), (4.79) and (4.81)

$$\langle \tilde{\delta}(\overline{u}, 0)J\overline{u} - (A' + \Gamma)\overline{f} + \widetilde{K}\overline{r}, v_i\rangle = 0, \quad i = 1, \dots, j_0, \tag{4.82}$$

where $\overline{f} = \Pi\overline{u}$, $\overline{r} = (I - \Pi)\overline{u}$. Using our choice of $v_1, \dots, v_\nu$, the definition of the operator P^* in (4.55) and the equality in (4.64) we obtain from (4.82)

$$\tilde{\delta}(\overline{u}, 0)P^* J\overline{u} = (A' + \Gamma)\overline{f}, \tag{4.83}$$

from where, using (4.48), the estimate

$$\|\overline{f}\| \le c_1\tilde{c}_2\|P^*\|\tilde{\delta}(\overline{u}, 0) \tag{4.84}$$

follows with the same $\overline{c}_2$ as that in the proof of Lemma 4.5.

From (4.82) and (4.83) we derive

$$\langle \tilde{\delta}(\overline{u}, 0)(I - P^*)J\overline{u} + \widetilde{K}\overline{r}, v\rangle = 0, \tag{4.85}$$

for every $v \in F_{j_0}$. Using the fact that $F \subset F_{2\nu}$ from (4.78) we have $\overline{f} \in F_{j_0}$ and, consequently, $\overline{r} \in F_{j_0}$. Thus, from (4.85), with $v = \overline{r}$, we obtain

$$\tilde{\delta}(\overline{u}, 0)\langle (I - P^*)J\overline{u}, \overline{r}\rangle = -\langle \widetilde{K}\overline{r}, \overline{r}\rangle \le 0. \tag{4.86}$$

However, (4.84) and (4.86) imply $\tilde{\delta}(\overline{u}, 0) \neq 0$, since otherwise $\overline{f} = \overline{r} = 0$, which contradicts our choice of $\overline{u}$. Using (4.65), (4.84), (4.86) and our choice of $\tilde{c}$, we have

$$\begin{aligned}\langle (I - P^*)J\overline{u}, \overline{u}\rangle &\le c_1\tilde{c}_2^2\|P^*\|\|I - P^*\|\tilde{\delta}(\overline{u}, 0)\\ &\le \tilde{c}c_1\tilde{c}_2^2\|P^*\|\|I - P^*\|\langle (I - P^*)J\overline{u}, \overline{u}\rangle\\ &\le \frac{1}{2}\langle (I - P^*)J\overline{u}, \overline{u}\rangle\end{aligned}$$

and the contradiction follows because $\langle (I - P^*)J\overline{u}, \overline{u}\rangle > 0$. The last inequality in the last display follows from $\tilde{\delta}(\overline{u}, 0) > 0$. The proof of Lemma 4.7 is now complete. □

The next lemma is the last step in the proof of Theorem 2.1.

Lemma 4.8. *Assume that the conditions of Theorem 2.1 are satisfied. Then*

$$Ind(A, 0) = (-1)^\nu, \tag{4.87}$$

where ν is the same as in Theorem 2.1.

Proof. From (4.77), (4.80), Lemma 4.7 and the properties of the degree mapping for finite-dimensional spaces we deduce

$$Ind(A, 0) = deg(A^{(4)}_{0,j_0}, B_{\rho,j_0}(0), 0), \tag{4.88}$$

for every $\rho \in (0, r_0]$. It is easy to verify that for ρ sufficiently small the mapping $A^{(4)}_{0,j_0}$ is homotopic on $\overline{B_{\rho,j_0}(0)}$ to the mapping

$$A^{(5)}_{j_0} = \sum_{i=1}^{j_0} \langle -(A' + \Gamma)\Pi u + \widetilde{K}(I - \Pi)u, v_i\rangle v_i.$$

Thus, $Ind(A, 0) = deg(A^{(5)}_{j_0}, B_{\rho,j_0}(0), 0)$.

The degree of the mapping $A^{(5)}_{j_0}$ equals $(-1)^\nu$ and can be computed by the well-known formula for the degree for linear finite-dimensional mappings (see [11]). This completes the proofs of Lemma 4.8 and Theorem 2.1 as well. □

5. Branching of solutions of equations with densely defined operators

In this section we present an application of the previous results to the bifurcation problem. In what follows, D_0 is an open set containing the origin in a separable reflexive Banach space X. We consider a nonlinear operator $A : X \supset D(A) \to X^*$ satisfying Conditions $(S_+)_L,\ A_2)$, for some subspace L of X such that $D_0 \cap L \subset D(A),\ \overline{L} = X$. Let $C : D_0 \to X^*$ be a nonlinear compact operator. Assume further that $A(0) = C(0) = 0$. We can easily verify that the operator $A + \lambda C$ satisfies Condition $(S_+)_L$ for any real $\lambda > 0$.

We consider the bifurcation problem for the pair of operators $A,\ C$.

Definition 5.1. A real number λ_0 is called a "bifurcation point" of the operators $A,\ C$ if for every $\varepsilon > 0$ there exist $u_\varepsilon \in D(A)$ and $\lambda_\varepsilon \in \mathcal{R}$ such that

$$Au_\varepsilon + \lambda_\varepsilon C u_\varepsilon = 0, \quad |\lambda_\varepsilon - \lambda_0| < \varepsilon, \quad 0 < \|u_\varepsilon\| < \varepsilon. \tag{5.1}$$

We study necessary and sufficient conditions that λ_0 be a bifurcation point. For this, we may assume that there is some $\delta > 0$ such that zero is an isolated critical point of the operator $A + \lambda C$, for each λ from the interval $|\lambda - \lambda_0| < \delta$, since otherwise λ_0 itself would be a bifurcation point. Thus, the index $Ind(A + \lambda C, 0)$ of the operator $A + \lambda C$ at 0 is defined for $|\lambda - \lambda_0| < \delta$ according to Definition 1.4.

Let

$$\overline{i}_\pm(\lambda_0) = \limsup_{\lambda \to \lambda_0 \pm} Ind(A + \lambda C, 0), \quad \underline{i}^\pm(\lambda_0) = \liminf_{\lambda \to \lambda_0 \pm} Ind(A + \lambda C, 0). \tag{5.2}$$

Theorem 5.1. *Let $A : X \supset D(A) \to X^*$ be a nonlinear operator satisfying Conditions $(S_+)_L,\ A_2)$ and let $C : D_0 \to X^*$ be a nonlinear compact operator. Assume that $A(0) = C(0) = 0$ and that at least two of the numbers*

$$\underline{i}^-(\lambda_0),\ \underline{i}^+(\lambda_0), \overline{i}_-(\lambda_0),\ \overline{i}_+(\lambda_0),\ Ind(A + \lambda_0 C, 0) \tag{5.3}$$

are distinct. Then λ_0 is a bifurcation point of the pair $A,\ C$.

Proof. Let us assume that the first and the second number in (5.3) are distinct. Then for every $\varepsilon > 0$ we can find numbers $\lambda_\varepsilon^{(1)},\ \lambda_\varepsilon^{(2)}$ such that $0 < \lambda_0 - \lambda_\varepsilon^{(1)} < \varepsilon,\ 0 < \lambda_\varepsilon^{(2)} - \lambda_\varepsilon < \varepsilon$ and

$$Ind(A + \lambda_\varepsilon^{(1)} C, 0) \neq Ind(A + \lambda_\varepsilon^{(2)} C, 0). \tag{5.4}$$

We choose $\rho \in (0, \varepsilon)$ such that

$$Deg(A + \lambda_\varepsilon^{(i)} C, B_\rho(0), 0) = Ind(A + \lambda_\varepsilon^{(i)} C, 0), \quad i = 1, 2. \tag{5.5}$$

Then from (5.4) and (5.5) we have

$$Deg(A + \lambda_\varepsilon^{(1)} C, B_\rho(0), 0) \neq Deg(A + \lambda_\varepsilon^{(2)} C, B_\rho(0), 0) \tag{5.6}$$

and, consequently, the operators $A + \lambda_\varepsilon^{(1)} C,\ A + \lambda_\varepsilon^{2)} C$ are nonhomotopic w.r.t. the ball $B_\rho(0)$.

Consider the family of nonlinear operators $A_t : X \supset D(A) \to X^*$ defined by

$$A_t u = Au + [t\lambda_\varepsilon^{(1)} + (1 - t)\lambda_\varepsilon^{(2)}] Cu, \quad t \in [0, 1].$$

It is easy to verify that Condition $(S_+)_{0,L}^{(t)}$ holds for $A_t,\ t \in [0, 1]$, as well as all the other properties of a homotopic family in Definition 4.2 except, possibly, (4.7). However, (4.7) cannot be satisfied now because the operators $A+\lambda_\varepsilon^{(1)}C$ and $A+\lambda_\varepsilon^{(2)}C$ are nonhomotopic.

It follows that there exist $u_\varepsilon \in D(A)\cap \partial B_\rho(0)$ and $t_\varepsilon \in [0,1]$ such that

$$Au_\varepsilon + [t_\varepsilon \lambda_\varepsilon^{(1)} + (1-t_\varepsilon)\lambda_\varepsilon^{(2)}]Cu_\varepsilon = 0. \tag{5.7}$$

This proves that λ_0 is a bifurcation point of the pair $A,\ C$. All the other possible pairs of distinct numbers in (5.3) can be handled in a similar fashion. The proof of Theorem 5.1 is complete. □

Now, we are going to establish necessary conditions for a number λ_0 to be a bifurcation point. We need to state new forms of Conditions $\omega),\ C)$ from Section 1 so that they can be used in both necessary and sufficient conditions.

Assume that the operator C has Fréchet derivative at zero denoted by C'. The operator C' is compact [7]. We assume that there exist a nonlinear operator A_0 satisfying Condition $A_0)$ and a linear operator $A' : X \supset D(A') \to X^*$ such that $D(A)\subset D(A')$ and the condition

$\overline{\omega})$ for the operator $\omega : D(A)\to X^*$, defined by $\omega(u) = Au - A'u$, we have

$$\frac{\omega(u)}{\|u\|} \to 0, \quad \text{as } u\to 0,\ u\in Z'_{\varepsilon,\Lambda},$$

holds for every $\Lambda>0$ and some $\varepsilon>0$ depending on Λ, where

$$Z'_{\varepsilon,\Lambda} = \bigcup_{\substack{t\in[0,1]\\ |\lambda|\le\Lambda}} \left\{ u\in D(A^{(1)}_{t,\lambda}) \ :\ A^{(1)}_{t,\lambda}(u)=0,\ 0<\|u\|\le\varepsilon \right\}. \tag{5.8}$$

Here,

$$A^{(1)}_{t,\lambda}(u) = t(Au+\lambda Cu) + (1-t)(A_0u + A'u + \lambda C'u).$$

Define the sets $Z_{\varepsilon,\Lambda} = Z'_{\varepsilon,\Lambda}\cup Z''_{\varepsilon,\Lambda}$ with

$$Z''_{\varepsilon,\Lambda} = \bigcup_{\substack{t\in[0,1]\\ |\lambda|\le\Lambda}} \left\{ u\in D(A^{(2)}_{t,\lambda}) \ :\ A^{(2)}_{t,\lambda}(u)=0,\ 0<\|u\|\le\varepsilon \right\}, \tag{5.9}$$

where

$$A^{(2)}_{t,\lambda}(u) = tA_0u + A'u + \lambda C'u.$$

We also introduce the condition

$\overline{C})$ the weak closure of the set

$$\sigma_{\varepsilon,\Lambda} = \left\{ v = \frac{u}{\|u\|} \ :\ u\in Z_{\varepsilon,\lambda} \right\} \tag{5.10}$$

does not contain zero for any $\Lambda>0$ and all sufficiently small positive ε depending on Λ.

Theorem 5.2. *Let $A,\ C$ satisfy the conditions of Theorem 5.1 and let C' be the Fréchet derivative of the operator C at zero. Assume that there exist an operator $A_0 : X\supset D(A_0)\to X^*$ and a linear operator $A' : X\supset D(A')\to X^*$ satisfying Conditions $A_0)$ and $(S')_L$, respectively, as well as Conditions $\overline{\omega})$ and $\overline{C})$. Then a necessary condition that λ_0 be a bifurcation point of the pair $A,\ C$ is that the* *equation*

$$A'u + \lambda_0 C'u = 0 \tag{5.11}$$

has a nonzero solution.

Proof. Let λ_0 be a bifurcation point of the pair $A,\ C$ and let $\{\lambda_j\},\ \{u_j\}$ be such that

$$Au_j + \lambda_j Cu_j = 0, \quad u_j \in D(A),\ |\lambda_j - \lambda| < \frac{1}{j},\ 0 < \|u_j\| < \frac{1}{j}.$$

Condition $\overline{C}$) implies that the weak limit v_0 of the sequence $v_j = u_j/\|u_j\|$ is different from zero. Passing to the limit in

$$\frac{Au_j + \lambda_j Cu_j}{\|u_j\|} = 0$$

we see by virtue of Conditions $\overline{\omega}$) and $(S')_L$ that v_0 is a solution of (5.11). This completes the proof. □

A sufficient condition that λ_0 be a bifurcation point is given in the following theorem.

Theorem 5.3. *Assume that X is a real reflexive separable Banach space satisfying Conditions X_1) and X_2) of Section 2. Let $A,\ C$ satisfy all the assumptions of Theorem 5.2, respectively, and be such that (2.1) is satisfied and $A + qA'$ satisfies Condition $(S_+)_L$ for every number $q > 0$. Suppose that $\langle A'u, u\rangle > 0$ for all $u \in D(A')$ with $u \neq 0$ and that the operator $T = -(A')^{-1}C' : X \to X$ is well defined, compact and Condition 1) of Theorem 2.1 is satisfied with $\Gamma = 0$. Then each characteristic value of odd multiplicity of the operator T is a bifurcation point of the pair $A,\ C$.*

Proof. Let λ_0 be a characteristic value of odd multiplicity of the operator T. Choose a positive number δ_0 such that the interval $(\lambda_0 - \delta_0, \lambda_0 + \delta_0)$ contains only one characteristic value of T. For $\lambda \in (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$ the operator $\widetilde{A} = A + \lambda C$ satisfies all the assumptions of Theorem 2.1 with $\widetilde{A}' = A' + \lambda C',\ \widetilde{A}_0 = A_0,\ \widetilde{\Gamma} = -\lambda C'$. Applying this theorem we obtain, for $\lambda' \in (\lambda_0 - \delta_0, \lambda_0),\ \lambda'' \in (\lambda_0, \lambda_0 + \delta_0)$,

$$Ind(A + \lambda'' C, 0) = -Ind(A + \lambda' C, 0).$$

The conclusion of Theorem 5.3 follows now from Theorem 5.2. □

Remark 5.1. For a bounded operator A, or in the case of a Hilbert space X, the restrictions on the operator A in Theorems 5.2 and 5.3 can be weakened according to Theorem 2.4.

Remark 5.2. The conditions of Theorem 5.2 guarantee that the set of bifurcation points of the pair $A,\ C$ is discrete. In general, this set can contain entire intervals of the real line. Relevant examples can be constructed similarly to those of Section 3. For an operator A defined everywhere, such an example can be found in [11, p. 63].

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