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AN ESTIMATE FOR WEIGHTED HILBERT TRANSFORM VIA SQUARE FUNCTIONS

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Abstract. We show that the norm of the Hilbert transform as an operator on the weighted space $L^2(w)$ is bounded by a constant multiple of the 3/2 power of the A_2 constant of w, in other words by $c\sup_I(\langle\omega\rangle_I\langle\omega^{-1}\rangle_I)^{3/2}$. We also give a short proof for sharp upper and lower bounds for the dyadic square function.

1. Introduction

The question of finding sharp estimates for the Hilbert transform, the square function and a uniform bound for martingales on weighted L^2 spaces in terms of the A_2 constant of the weight has attracted considerable interest in recent years. S. Buckley proved in [1] that the norm of the square function is bounded by $Q_2(\omega)^{3/2}$ and that the Hilbert transform is bounded by $Q_2(\omega)^2$. More recently, S. Hukovic, S. Treil and A. Volberg proved in [3] the linear bound for the square function. An alternative proof by J. Wittwer can be found in [7].

We improve Buckley's bound for the Hilbert transform to $Q_2(\omega)^{3/2}$. Our proof uses a certain averaging technique introduced by the first author in [5]. The new bound for the Hilbert transform follows from upper and lower bounds for the square function in just *one* line.

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2. Formulation of result

We consider the space $L^2_{\mathbb{R}}(\omega)$ where ω is a positive L^1_{loc} function, called a weight. Let dx be Lebesgue measure on $\mathbb{R}$. The norm of $f\in L^2_{\mathbb{R}}(\omega)$ is $\left(\int_{\mathbb{R}}|f(x)|^2\omega(x)dx\right)^{1/2}$ and denoted by $\|f\|_\omega$. We are concerned with a special class of weights, called A_2. We say that $\omega\in A_2$, if

$$Q_2(\omega):=\sup_I\langle\omega\rangle_I\langle\omega^{-1}\rangle_I<\infty, \tag{2.1}$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$. The notation $\langle\omega\rangle_I$ stands for the average of the function ω over I.

Let $\mathcal{D}$ denote the collection of all dyadic intervals in $\mathbb{R}$. We call $\mathcal{D}$ the standard dyadic grid in $\mathbb{R}$. For each $\alpha\in\mathbb{R}, r>0$, let $\mathcal{D}^{\alpha,r}$ be the dyadic grid $\{\alpha+rI : I\in\mathcal{D}\}$.

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If we restrict the supremum in (2.1) to dyadic intervals of a certain grid, we will denote the class by $A_2^{\mathcal{D}^{\alpha,r}}$ and the corresponding supremum by $Q_2^{\mathcal{D}^{\alpha,r}}(\omega)$.

The symbol H stands for the Hilbert transform on $\mathbb{R}$, which is defined as

$$Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

Here is the main result of this paper:

Theorem 2.1. *$H : L^2_{\mathbb{R}}(\omega) \to L^2_{\mathbb{R}}(\omega)$ has operator norm $\|H\| \le c\, Q_2(\omega)^{3/2}$, where $c > 0$ is an absolute constant.*

We will reduce the problem to upper and lower bounds of certain square functions, using the averaging technique from [5].

3. Sharp lower and upper bounds for the dyadic square function

The following considerations hold for all dyadic grids $\mathcal{D}$, so we omit indices α, r.

Recall that the dyadic square function S is defined by

$$Sf(t) = \sqrt{\int_{\Sigma} |(T_\varepsilon f)(t)|^2 d\varepsilon} = \sqrt{\sum_{I \in \mathcal{D}} |(f, h_I)|^2 \frac{\chi_I(t)}{|I|}},$$

where Σ denotes the space $\{-1,1\}^{\mathcal{D}}$ equipped with the natural product measure $d\varepsilon$, which assigns equal measure 2^{-k} to every cylindrical subset of $\{-1,1\}^{\mathcal{D}}$ of length 2^k. T_ε is the martingale transform $f = \sum_I (f, h_I) h_I \mapsto \sum_I \varepsilon(I)(f, h_I) h_I$ associated to the sequence $\varepsilon(I) \in \{-1,1\}^{\mathcal{D}}$.

We first prove a lower bound for the square function.

Theorem 3.1. *There exists $c > 0$ so that for all $f \in L^2(\omega)$,*

$$\|f\|_\omega \le c\, Q_2^{\mathcal{D}}(\omega)^{1/2} \|Sf\|_\omega \,.$$

Proof. We have

$$\|Sf\|^2_\omega = \sum_I \langle \omega \rangle_I |(f, h_I)|^2 = (D_\omega f, f),$$

where D_ω stands for 'discrete multiplication' by ω and denotes the possibly unbounded operator which is densely defined on L^2 by $h_I \mapsto \langle \omega \rangle_I h_I$. Let M_ω denote the ordinary multiplication operator with ω. Of course, $\|f\|^2_\omega = (M_\omega f, f)$. We need to show that

$$M_\omega \le c\, Q_2^{\mathcal{D}}(\omega) D_\omega \,. \tag{3.1}$$

Here, the inequality is understood as an operator inequality.

Approximating ω by ω_n, where $\omega_n(x) = \max\{\min\{\omega(x), n\}, 1/n\}$, we can assume that M_ω and D_ω are bounded and invertible. Taking inverses, equation (3.1) becomes

$$D_\omega^{-1} \le c\, Q_2^{\mathcal{D}}(\omega) M_\omega^{-1}, \tag{3.2}$$

where D_ω^{-1} is defined by $h_I \mapsto \langle \omega \rangle_I^{-1} h_I$, and $M_\omega^{-1} = M_{\omega^{-1}}$. So we need to prove that

$$\sum_I \frac{1}{\langle \omega \rangle_I} |(f, h_I)|^2 \le c\, Q_2^{\mathcal{D}}(\omega) \|f\|^2_{\omega^{-1}} \,.$$

We switch to the system of disbalanced Haar functions h_I^ω that is orthonormal in L^2_ω, as done in [3]. For this, we define h_I^ω as $h_I = \delta_\omega^I h_I^\omega + \gamma_\omega^I \chi_I$, where

$$\delta_\omega^I = \sqrt{\frac{\langle\omega\rangle_{I_+}\langle\omega\rangle_{I_-}}{\langle\omega\rangle_I}} \text{ and } \gamma_\omega^I = \frac{(\omega, h_I)}{|I|\langle\omega\rangle_I}.$$

Furthermore, we write $\Delta_I\omega$ for $\langle\omega\rangle_{I_-} - \langle\omega\rangle_{I_+} = |I|^{-1/2}(\omega, h_I)$.

We now split the sum into three parts:

$$\sum_I \frac{1}{\langle\omega\rangle_I}|(f,h_I)|^2 = \sum_I \frac{1}{\langle\omega\rangle_I}|\delta_\omega^I|^2|(f,h_I^\omega)|^2 + 2\sum_I \frac{1}{\langle\omega\rangle_I}|\delta_\omega^I||\gamma_\omega^I||(f,h_I^\omega)||(f,\chi_I)| + \sum_I \frac{1}{\langle\omega\rangle_I}|\gamma_\omega^I|^2|(f,\chi_I)|^2 . \tag{3.3}$$

The first sum. Note that $\frac{(\delta_\omega^I)^2}{\langle\omega\rangle_I} \le 1$, so

$$\sum_I \frac{1}{\langle\omega\rangle_I}|\delta_\omega^I|^2|(f,h_I^\omega)|^2 \le \sum_I |(f,h_I^\omega)|^2 = \sum_I |(\omega^{-1}f, h_I^\omega)_\omega|^2 = \|\omega^{-1}f\|_\omega^2 = \|f\|_{\omega^{-1}}^2 . \tag{3.4}$$

The second sum.

$$\sum_I \frac{1}{\langle\omega\rangle_I}|\delta_\omega^I||\gamma_\omega^I||(f,h_I^\omega)||(f,\chi_I)| \le \sqrt{\sum_I \frac{1}{\langle\omega\rangle_I}(\delta_\omega^I)^2|(f,h_I^\omega)|^2}\sqrt{\sum_I \frac{1}{\langle\omega\rangle_I}(\gamma_\omega^I)^2|(f,\chi_I)|^2} , \tag{3.5}$$

where the first part can be estimated by $\|f\|_{\omega^{-1}}$ as above. The second term is exactly the square root of the third sum and will be estimated below.

The third sum.

$$\sum_I \frac{1}{\langle\omega\rangle_I}|\gamma_\omega^I|^2|(f,\chi_I)|^2 = \sum_I |I|\frac{|\Delta_I\omega|^2}{\langle\omega\rangle_I^3}\langle f\rangle_I^2. \tag{3.6}$$

We will apply the weighted Carleson Imbedding theorem to control (3.6). According to [4], it suffices to check (3.6) for test functions, in the sense that any sequence $\alpha_I \ge 0$ satisfying

$$\frac{1}{|J|}\sum_{I\subset J}\langle\omega\rangle_I^2\alpha_I \le C\langle\omega\rangle_J \text{ for all dyadic } J$$

also satisfies

$$\sum_I \langle f\rangle_I^2\alpha_I \le 4C\|f\|_{\omega^{-1}}^2 .$$

for all $f \in L^2(w^{-1})$.

We apply this to $\alpha_I = |I|\frac{|\Delta_I\omega|^2}{\langle\omega\rangle_I^3}$, and $C = cQ_2^{\mathcal{D}}$. So it suffices to check that for all dyadic J,

$$\frac{1}{|J|}\sum_{I\subset J}|I|\frac{|\Delta_I\omega|^2}{\langle\omega\rangle_I} \le cQ_2^{\mathcal{D}}\langle\omega\rangle_J .$$

This has been proven in [7] and can also be shown by a Bellman function argument. □

Corollary 3.2. *There exists $c > 0$ such that for all $f \in L^2(\omega)$ and for all weights ω, $\|Sf\|_\omega \le c\, Q_2^{\mathcal{D}}(\omega)\|f\|_\omega$.*

Proof. Using the same notation as before, we have to show that $D_\omega \le cQ_2^{\mathcal{D}}(\omega)^2 M_\omega$. By definiton of $Q_2^{\mathcal{D}}(\omega)$, we have $D_\omega \le Q_2^{\mathcal{D}}(\omega)(D_{\omega^{-1}})^{-1}$, and by equation (3.2) applied to ω^{-1} we obtain $D_\omega \le cQ_2^{\mathcal{D}}(\omega)^2 M_\omega$. □

Remark. This corollary was proven in [3] using Bellman function technique. The paper [7] also contains a short proof of the fact that the lower bound in Theorem 3.1 implies the linear upper bound for S of Theorem 3.2, which itself is sharp (see [1] and [3]). In particular, this argument shows that the lower bound in Theorem 3.1 is sharp.

4. The cubic bound for the Hilbert transform

By [5], H lies in the closed convex hull of operators densely defined by

$$\text{Ш}^{\alpha,r} h_I = \frac{1}{\sqrt{2}}(h_{I_-} - h_{I_+}).$$

We will refer to these operators as dyadic shifts. The indices α and r indicate that we have to consider translates and dilates of the standard dyadic grid as described above. The square function does not 'see' the dyadic shift:

Proposition 4.1. *$(S\text{Ш}f)(x) = (Sf)(x)$ for all x.*

Proof.

$$\begin{aligned} S\text{Ш}f(x)^2 &= \int_\Sigma |(T_\varepsilon \text{Ш}f)(x)|^2 d\varepsilon = \int_\Sigma |\sum_I \varepsilon(I)(\text{Ш}f, h_I)h_I(x)|^2 d\varepsilon \\ &= \int_\Sigma |\sum_I (f, h_I)(\varepsilon(I_-)h_{I_-} - \varepsilon(I_+)h_{I_+})|^2 d\varepsilon \\ &\overset{(\star)}{=} \int_\Sigma |\sum_I \varepsilon(I)(f, h_I)h_I(x)|^2 d\varepsilon = Sf(x)^2. \end{aligned} \tag{4.1}$$

Here, $(\star)$ is an effect of the averaging over sequences of signs $\varepsilon(I)$ and the fact that for each fixed x there exists a sequence of signs $\tilde{\varepsilon}(I)$ so that $\sqrt{2}h_I(x) = \tilde{\varepsilon}(I)(\varepsilon(I_-)h_{I_-} - \varepsilon(I_+)h_{I_+})(x)$. □

Now it is easy to prove Theorem 2.1:

Proof. Dyadic shifts with respect to all translates and dilates of the standard dyadic grid have cubic bound, indeed,

$$\begin{aligned} \|\text{Ш}^{\alpha,r} f\|_\omega &\overset{(1)}{\le} c\, Q_2^{\mathcal{D}^{\alpha,r}}(\omega)^{1/2} \|S\text{Ш}^{\alpha,r} f\|_\omega \overset{(2)}{=} c\, Q_2^{\mathcal{D}^{\alpha,r}}(\omega)^{1/2}\|Sf\|_\omega \\ &\overset{(3)}{\le} c\, Q_2^{\mathcal{D}^{\alpha,r}}(\omega)^{3/2}\|f\|_\omega, \end{aligned}$$

where (1) holds by Theorem 3.1, (2) by Proposition 4.1 and (3) by Corollary 3.2.

By convexity, we now obtain the desired bound for the Hilbert transform:

$$\|H\|_{L^2(\omega)\to L^2(\omega)} \leq c \sup_{\alpha,r} \|\text{Ш}^{\alpha,r}\|_{L^2(\omega)\to L^2(\omega)} \leq c \sup_{\alpha,r} Q_2^{\mathcal{D}^{\alpha,r}}(\omega)^{3/2} \leq c\, Q_2(\omega)^{3/2}.$$

This finishes the proof of the main result. □

Remark. After this paper was submitted, the first author improved the bound to $Q_2(\omega)$, which is sharp [6]. However, the proof is much more involved than the proof of the $Q_2(\omega)^{3/2}$ bound, which we present in this paper.

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