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GAUSSIAN BOUNDS FOR DERIVATIVES OF CENTRAL GAUSSIAN SEMIGROUPS ON COMPACT GROUPS

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Abstract. For symmetric central Gaussian semigroups on compact connected groups, assuming the existence of a continuous density, we show that this density admits space derivatives of all orders in certain directions. Under some additional assumptions, we prove that these derivatives satisfy certain Gaussian bounds.

1. Introduction

Let G be a compact connected group equipped with its normalized Haar measure ν. Let $(\mu_t)_{t>0}$ be a weakly continuous convolution semigroup of probability measures on G. This means precisely that each μ_t, $t>0$, is a probability measure on G and that $(\mu_t)_{t>0}$ satisfies

(i) $\mu_t * \mu_s = \mu_{t+s}$, $t,s>0$;
(ii) $\mu_t \to \delta_e$ weakly as $t \to 0$.

Such a semigroup is called Gaussian if it also satisfies

(iii) $t^{-1}\mu_t(V^c) \to 0$ as $t \to 0$ for any neigborhood V of the identity element $e \in G$.

We say that $(\mu_t)_{t>0}$ is symmetric if $\mu_t(A) = \mu_t(A^{-1})$ for all $t>0$ and all Borel sets $A \subset G$. We say that $(\mu_t)_{t>0}$ is central if $\mu_t(a^{-1}Aa) = \mu_t(A)$ for all $t>0$, all $a \in G$, and any Borel subset $A \subset G$.

Given a Gaussian semigroup $(\mu_t)_{t>0}$, set

$$H_t f(x) = \int_G f(xy)d\mu_t(y). \tag{1.1}$$

The operators $(H_t)_{t>0}$ form a Markov semigroup. If μ_t is symmetric then H_t extends to $L^2(G,d\nu)$ as a semigroup of self-adjoint operators. One can then associate to $(\mu_t)_{t>0}$ its $L^2(G,d\nu)$-infinitesimal generator $(-L,\mathrm{Dom}(L))$ and its Dirichlet form $(\mathcal{E},\mathrm{Dom}(\mathcal{E}))$ so that

$$H_t = e^{-tL} \quad \text{on} \quad L^2(G,d\nu)$$

and

$$\mathcal{E}(f,g) = \langle L^{1/2}f, L^{1/2}g\rangle, \;\; f,g \in \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(L^{1/2}).$$

Definition 1.1. Consider a Gaussian semigroup $(\mu_t)_{t>0}$ on G.

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- We say that $(\mu_t)_{t>0}$ has property (CK) if, for all $t > 0$, μ_t is absolutely continuous with respect to the Haar measure ν on G and has a continuous density $x \mapsto \mu_t(x)$.
- We say that $(\mu_t)_{t>0}$ has property (CK$*$) if it has property (CK) and that

$$\lim_{t\to 0} t \log \mu_t(e) = 0.$$

It is known (see [9, 20]) that no Gaussian semigroup can have such properties if the group G is not locally connected or is not metrizable. In [5], we proved that any locally compact, connected, locally connected, metrizable group G admits many symmetric Gaussian semigroups satisfying (CK$*$). In [8], we proved that any compact, connected, locally connected, metrizable group G admits a host of symmetric central Gaussian semigroups satisfying (CK$*$), and even stronger properties of this type. In [6, 7], we obtained some Gaussian estimates for the density of symmetric Gaussian semigroups satisfying (CK$*$). This Gaussian estimates differ from the classical Gaussian estimates developed by Davies and others [16, 29] in that they do not use the so-called intrinsic distance. Indeed, as explained in [6, 7], there are many symmetric Gaussian semigroups satisfying (CK$*$) for which the associated intrinsic distance is infinite almost everywhere.

The aim of the present paper is to prove Gaussian estimates for the time and space derivatives of the density $(t, x) \mapsto \mu_t(x)$ under the hypothesis that $(\mu_t)_{t>0}$ is symmetric, central, and satisfies (CK$*$). In order to obtain Gaussian estimates on space derivatives, we will adapt a line of reasoning introduced by the second author in [26].

Such estimates are crucial for a number of further developments concerning symmetric central Gaussian semigroups on compact groups. This is illustrated in [10, 11]. In these two papers, we show that property (CK$*$) characterizes those symmetric central Gaussian semigroups whose infinitesimal generator is hypoelliptic. Our proof that property (CK$*$) implies hypoellipticity is adapted from the line of reasoning developed in [24, Section 8] for second order differential operators in $\mathbb{R}^n$ (the authors are grateful to D. Stroock for asking whether hypoellipticity could be studied by the method of [24, Section 8] in the present infinite dimensional setting). This approach makes essential use of the Gaussian estimates obtained below.

2. Background and notation

2.1. **Projective structure.** The following setup and notation will be in force throughout this article. Let G be a connected compact group with neutral element e. Such a group contains a descending family of compact normal subgroups K_α indexed by a suitable index set $\aleph$, such that $\bigcap_{\alpha\in\aleph} K_\alpha = \{e\}$ and, for each α, G/K_α is a Lie group. Consider the projection maps $\pi_{\alpha,\beta} : G_\beta \to G_\alpha$, $\beta \geq \alpha$. G is the projective limit of the projective system $(G_\alpha, \pi_{\beta,\alpha})_{\beta\geq\alpha}$. The Lie algebra $\mathfrak{G}$ of G is then defined to be the projective limit of the Lie algebras $\mathfrak{G}_\alpha$ of the groups G_α equipped with the projection maps $d\pi_{\beta,\alpha}$.

Throughout the paper we assume that G is compact, connected, locally connected and metrizable. The latter hypothesis is equivalent to saying that the topology of G is generated by a countable basis. See [21]. Under this hypothesis, the family K_α, $\alpha \in \aleph$, can be taken to be finite (if G is a Lie group) or countable and we will assume throughout the paper that the index set $\aleph$ is indeed at most countable so that G is the projective limit of the sequence of Lie groups (G_α). By results of Heyer and Siebert [20], the topological hypotheses that G is locally connected and metrizable are necessary for the existence of Gaussian semigroups which are absolutely continuous with respect to Haar measure.

For a compact Lie group N, denote by $\mathcal{C}^\infty(N)$ the set of all smooth functions on N. For any compact connected group G, set

$$\mathcal{B}(G) = \{f : G \to \mathbb{R}, f = \phi \circ \pi_\alpha \text{ for some } \alpha \in \aleph \text{ and } \phi \in \mathcal{C}^\infty(G_\alpha)\}. \tag{2.1}$$

The space $\mathcal{B}(G)$ is the space of Bruhat test functions introduced in [15]. We refer to [15] for a precise definition of its topology. Since G is metrizable, i.e., $\aleph$ is at most countable, $\mathcal{B}(G)$ is the inductive limit of the sequence of topological vector spaces $\mathcal{C}^\infty(G_\alpha)$ ([15, p. 46]). By [15, Lemme 1], $\mathcal{B}(G)$ is independent of the choice of the family K_α, $\alpha \in \aleph$.

By definition, a distribution on G is any continuous linear functional on $\mathcal{B}(G)$. This definition was introduced in [15] and such distributions are called Bruhat distributions.

Following [13], we consider the notion of projective family and projective basis.

Definition 2.1. A family $(Y_i)_{i\in I}$ of $\mathfrak{G}$ is a **projective family** of left-invariant vector fields (w.r.t. the family (K_α)) if it has the property that, for each $\alpha \in \aleph$, there is a finite subset $I_\alpha \subset I$ such that $d\pi_\alpha(Y_i) = 0$ if $i \notin I_\alpha$. A family $(Y_i)_{i\in I}$ of $\mathfrak{G}$ is a **projective basis** of $\mathfrak{G}$ (w.r.t. the family (K_α)) if it is projective and $(d\pi_\alpha(Y_i))_{i\in I_\alpha}$ is a basis of the Lie algebra $\mathfrak{G}_\alpha$.

By [13], $\mathfrak{G}$ does admit a projective basis. If $(Y_i)_{i\in I}$ is a projective basis, we can identify $\mathfrak{G}$ with $\mathbb{R}^I$ as a topological vector space: For any $Z \in \mathfrak{G}$, there exists a unique $a = (a_i)_{i\in I}$ such that for any $\alpha \in \aleph$, $d\pi_\alpha(Z) = \sum_{i\in I} a_i d\pi_\alpha(Y_i)$ and convergence in $\mathfrak{G}$ is equivalent to convergence coordinate by coordinate. Since the group G is assumed to be metrizable, projective families have at most a countable number of elements.

Given a projective basis Y, a homogeneous left-invariant differential operator of degree k on G is a sum

$$P = \sum_{\ell\in I^k} a_\ell Y^\ell, \quad a_\ell \in \mathbb{C},$$

where, for $\ell = (\ell_1, \dots, \ell_k) \in I^k$, $Y^\ell = Y_{\ell_1} \cdots Y_{\ell_k}$, $Y_{\ell_i} \in Y$ (this notion is in fact independent of Y). Such a P can be interpreted as a linear operator from $\mathcal{B}(G)$ to $\mathcal{B}(G)$, and also as a linear operator acting on Bruhat distributions. Indeed, if $f = \phi \circ \pi_\alpha \in \mathcal{B}(G)$, we have

$$Pf(x) = \sum_{\ell\in I^k} a_\ell Y^\ell f(x) = \sum_{(\ell_1,\ell_2,\dots,\ell_k)\in I_\alpha^k} a_\ell \left[d\pi_\alpha(Y_{\ell_1}) d\pi_\alpha(Y_{\ell_2}) \cdots d\pi_\alpha(Y_{\ell_k})\phi\right](\pi_\alpha(x))$$

where the sum on the right-hand side is a finite sum since I_α is finite for each $\alpha \in \aleph$.

2.2. **Gaussian semigroups and sums of squares.** Given a (finite or) countable set I, let $\mathbb{R}^{(I)}$ be the set of all $z = (z_i) \in \mathbb{R}^I$ with finitely many non-zero entries. Using [23] and the projective structure, Heyer and Born [20, 14] proved the following theorem.

Theorem 2.2. *Given a projective basis $(Y_i)_{i\in\mathcal{I}}$, the infinitesimal generators of symmetric Gaussian convolution semigroups on G are exactly the second order left-invariant differential operators of the form*

$$L = -\sum_{i,j\in\mathcal{I}} a_{i,j} Y_i Y_j$$

where $A = (a_{i,j})_{\mathcal{I}\times\mathcal{I}}$ is a real symmmetric non-negative matrix in the sense that $a_{i,j} = a_{j,i} \in \mathbb{R}$ and $\forall\, \xi \in \mathbb{R}^{(\mathcal{I})}$, $\sum a_{i,j}\xi_i\xi_j \geq 0$.

Given an infinite matrix $T = (t_{i,j})$ indexed by a countable set $\mathcal{I}$, define

$$j_T : \mathcal{I} \to \mathcal{I} \cup \{+\infty\}, \quad j_T(i) = \inf\{j : t_{i,j} \neq 0\}$$

with the usual convention that $\inf \emptyset = +\infty$. Set

$$I = \{i \in \mathcal{I} : j_T(i) < +\infty\}. \tag{2.2}$$

Note that, for a given matrix T, the property that j_T is increasing implies that T is upper-triangular. The following lemma is simple but important. The proof is left to the reader.

Lemma 2.3. *Let $A = (a_{i,j})$ be an infinite symmetric non-negative matrix indexed by a countable set $\mathcal{I}$. There exists an infinite matrix $T = (t_{i,j})_{\mathcal{I}\times\mathcal{I}}$ such that j_T is increasing and*

$$\forall\, \xi \in \mathbb{R}^{(\mathcal{I})}, \quad \sum_{i,j\in\mathcal{I}} a_{i,j}\xi_i\xi_j = \sum_{k\in I} \eta_k^2$$

where I is given by (2.2) *and*

$$\forall\, i \in I, \;\; \eta_i = \sum_{j\in\mathcal{I}} t_{i,j}\xi_j.$$

In other words, $A = T^t T$. The matrix A is positive, i.e.,

$$\forall \xi \in \mathbb{R}^{(\mathcal{I})} \setminus \{0\}, \quad \sum_{i,j} a_{i,j}\xi\xi_j > 0,$$

if and only if $t_{i,i} > 0$ for all $i \in \mathcal{I}$.

Theorem 2.4. *Fix a projective basis $Y = (Y_i)_{\mathcal{I}}$ and $L = -\sum_{i,j\in\mathcal{I}} a_{i,j}Y_iY_j$ with A symmetric non-negative. Let I be defined by* (2.2)*. Let $T = (t_{i,j})_{\mathcal{I}\times\mathcal{I}}$ be the matrix given by Lemma 2.3. The family $X = (X_i)_I$ given by*

$$X_i = \sum_j t_{i,j}Y_j, \;\; i \in I,$$

is a projective family of linearly independent vectors and yields a decomposition of $-L$ as a sum of squares:

$$\forall\, f \in \mathcal{B}(G), \;\; Lf = -\sum_I X_i^2 f.$$

Proof. The fact that T is upper-triangular and Y is a projective basis implies that X is a projective family. Hence, for any $f \in \mathcal{B}(G)$, the sum $\sum_I X_i^2 f$ reduces to a finite sum. Plugging the definition of the X_i's in terms of the Y_i's in $\sum_I X_i^2 f$ shows that this sum equals $-Lf$. ∎

Note that the family X of Theorem 2.4 is a projective basis if and only if $t_{i,i} > 0$ for all $i \in \mathcal{I}$. In this case $\mathcal{I} = I$. Note also that, for a given L, there are many decompositions of L as minus a sum of squares.

2.3. **The Hilbert space of good directions.** Define the field operator Γ to be the symmetric bilinear form

$$\Gamma(f,g) = \frac{1}{2}\left(-L(fg) + fLg + gLf\right) \tag{2.3}$$

on the space $\mathcal{B}(G)$ of Bruhat test functions. Computing Γ in a projective family (not necessarily a basis) $X = (X_i)_{i\in I}$ where $L = -\sum_{i\in I} X_i^2$ we find

$$\Gamma(f,g) = \sum_{i\in I} (X_i f)(X_i g). \tag{2.4}$$

The next definition plays a crucial role in this paper.

Definition 2.5. Given the generator $-L$ of a symmetric Gaussian semigroup on G, let $\mathcal{H}(L)$ be the vector space

$$\mathcal{H}(L) = \{Z \in \mathfrak{G} : \exists\, c(Z),\ \forall\, f \in \mathcal{B}(G),\ \ |Zf(e)|^2 \le c(Z)\Gamma(f,f)(e)\}$$

equipped with the norm

$$\|Z\|_L = \sup_{\substack{f\in\mathcal{B}(G)\\ \Gamma(f,f)(e)\le 1}} \{|Zf(e)|\}.$$

We now give a different description of $\mathcal{H}(L)$.

Lemma 2.6. *The space $\mathcal{H}(L)$ equipped with the norm $\|Z\|_L$ is a Hilbert space. In particular, for any projective family $X = (X_i)_{i\in I}$ of linearly independent vectors such that $L = -\sum_{i\in I} X_i^2$, we have*

$$\mathcal{H}(L) = \{Z = \sum_{i\in I} \zeta_i X_i : \sum_{i\in I} |\zeta_i|^2 < \infty\}$$

and, for all $Z = \sum_{i\in I} \zeta_i X_i$,

$$\|Z\|_L^2 = \sum_{i\in I} |\zeta_i|^2.$$

Proof. For $L = -\sum_{i\in I} X_i^2$, Γ is given by (2.4). Thus, if $Z = \sum_{i\in I} \zeta_i X_i$ with $\sum |\zeta_i|^2 < \infty$, $Z \in \mathcal{H}(L)$, and

$$\|Z\|_L^2 \le \left(\sum_{i\in I} |\zeta_i|^2\right).$$

Let us first assume that $X = (X_i)_I$ is a projective family extracted from a projective basis $(X_i)_{\mathcal{I}}$. Let $Z = \sum_{\mathcal{I}} \zeta_i X_i$ be an arbitrary left-invariant vector field. For any finite subset $J \subset \mathcal{I}$ and any sequence $\xi = (\xi_j)_J$, we can find $f_J^\xi \in \mathcal{B}(G)$ such $X_j f_J^\xi(e) = \xi_j$ if $j \in J$, $X_j f_J^\xi(e) = 0$ if $j \notin J$. Then we have

$$|Zf_J^\xi(e)|^2 = \left(\sum_{j\in J} \zeta_j \xi_j\right)^2 \quad \text{and} \quad \Gamma(f_J^\xi, f_J^\xi) = \sum_{j\in J\cap I} |\xi_j|^2.$$

Thus if $Z \in \mathcal{H}(L)$ then we must have $\zeta_i = 0$ for all $i \in \mathcal{I}\setminus I$ and also $\sum_I |\zeta_i|^2 < \infty$. Moreover,

$$\|Z\|_L^2 \ge \sup_\xi \frac{|Zf_J^\xi(e)|^2}{\Gamma(f_J^\xi, f_J^\xi)} = \sum_{j\in J} |\zeta_j|^2.$$

Since this holds for any finite subset $J \subset I$, we conclude that

$$\|Z\|_L^2 \geq \sum_{i \in I} |\zeta_i|^2$$

as desired. A simple Hilbert space argument then shows that any independent projective family $(X_i)_I$ such that $L = -\sum X_i^2$ must be a basis of $\mathcal{H}(L)$. □

Remark 2.7. The space $\mathcal{H}(L)$ must be interpreted as a space of good directions in $\mathfrak{G}$. It captures very important non-trivial information about L and necessarily plays a crucial role in any precise analysis of L and the associated Gaussian semigroup. For instance, the one parameter subgroups associated to directions in $\mathcal{H}(L)$ are rectifiable for the intrinsic distance. See, e.g., [6, 7] and Definition 4.7.

Example 2.8. Let $G = \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{R} = \mathbb{R}^\infty$ and $\mathbf{Z} = \mathbb{Z}^\infty$. Thus, $\mathbf{T}$ is the countable product of circle groups, each isomorphic to $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. However, for the following discussion it is important to observe that $\mathbf{T}$ is defined independently of the product structure. Writing $\mathbf{T}$ as an infinite product yields a projective basis of its Lie algebra $\mathbf{R} = \mathbb{R}^\infty$, say $Y = (Y_i)_1^\infty$, where $Y_i = \partial_i$ can be identified with partial differentiation in the i-th coordinate. Any symmetric Gaussian semigroup $(\mu_t)_{t>0}$ is determined by a matrix $A = (a_{i,j})$ as explained above. One usually says that $(\mu_t)_{t>0}$ is diagonal if A is a diagonal matrix with $a_{i,i} = a_i$ and quite a lot is known about the properties of $(\mu_t)_{t>0}$ in this case. See [2, 3, 6, 12]. In such a case, $\mathcal{H}(L)$ is the Hilbert space contained in $\mathbf{R}$ with orthonormal Hilbert basis

$$(a_i^{1/2}\partial_i)_{i \in I}, \quad I = \{i : a_i > 0\}.$$

Let us now look at two non-diagonal A's:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 2 & 2 & 2 & 2 & 2 & \cdot & \cdot \\ 1 & 2 & 3 & 3 & 3 & 3 & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 4 & 4 & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 5 & 5 & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \frac{1}{2} & \frac{5}{4} & \frac{2}{3} & 0 & 0 & 0 & \cdot & \cdot \\ 0 & \frac{2}{3} & \frac{13}{9} & \frac{3}{4} & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \frac{3}{4} & \frac{25}{16} & \frac{4}{5} & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \frac{4}{5} & \frac{41}{25} & \frac{5}{6} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{61}{36} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Thus, for A_1, $a_{i,j} = \min\{i,j\}$ whereas, for A_2, $a_{i,i} = 1 + (\frac{i-1}{i})^2$, $a_{i,i+1} = \frac{i}{i+1}$, $a_{i-1,i} = \frac{i-1}{i}$ and $a_{i,j} = 0$ if $|i-j| \geq 2$. A simple calculation shows that the corresponding Hilbert space $\mathcal{H}(L)$ has orthonormal basis $X = (X_i)_1^\infty$ given by

$$X_i = \begin{cases} \sum_{j \geq i} Y_i & \text{for } A_1, \\ Y_i + \frac{i}{i+1} Y_{i+1} & \text{for } A_2. \end{cases}$$

In both cases, the family $(X_i)_1^\infty$ is also a projective basis of the Lie algebra $\mathbf{R}$ of $\mathbf{T}$.

The case of A_1: Consider the "integer lattice"

$$\mathbf{Z}_X = \{Z = \sum_1^\infty z_i X_i : z_i \in \mathbb{Z}\} \subset \mathbf{R}$$

and observe that it coincides in the case of A_1 with the original integer lattice $\mathbf{Z} = \mathbf{Z}_Y$. Since the infinitesimal generator of $(\mu_t)_{t>0}$ is $\sum X_i^2$, this means that the Gaussian semigroup $(\mu_t)_{t>0}$ associated to A_1 is exactly the infinite product of identical standard Gaussian semigroups on the circles

$$[\mathbb{R}X_i]/[2\pi\mathbb{Z}X_i].$$

To illustrate what this says, observe that Kakutani's theorem implies easily that the measure μ_t is singular with respect to Haar measure for each $t > 0$.

The case of A_2: In this case, we cannot find a basis which "diagonalizes" $(\mu_t)_{t>0}$. One can ask what is a "good" basis to study $(\mu_t)_{t>0}$ but it seems hard to make this precise. For instance, one may want to try $X_i' = Y_i + Y_{i+1}$ since X_i tends to X_i' as i tends to infinity and the integer lattice $\mathbf{Z}_{X'}$ coincides with the original one. But, in $X' = (X_i')_1^\infty$, the matrix A' representing $(\mu_t)_{t>0}$ has

$$a'_{i,i} = 1 + \sum_2^i \frac{1}{j^2} \quad \text{and} \quad a'_{i,j} = (-1)^{j-i}\left(\frac{1}{i+1} + \sum_2^i \frac{1}{j^2}\right) \text{ for } j > i,$$

which is not easy at all to interpret. Developing a theory to study this kind of examples appears to be a real challenge. For instance, although we strongly suspect that the present Gaussian semigroup $(\mu_t)_{t>0}$ is singular with respect to Haar measure for all $t > 0$, we have no proof of this fact at the present writing.

3. Spaces of smooth functions

3.1. **The spaces $\mathcal{C}_X^k$.** Any left-invariant vector field $Z \in \mathfrak{G}$ generates a one parameter group $t \mapsto e^{tZ}$ in G. By definition, a function $f : G \mapsto \mathbb{R}$ has a derivative at x in the direction of Z if

$$Zf(x) = \lim_{t\to 0} \frac{f(xe^{tZ}) - f(x)}{t} = \frac{\partial}{\partial t} f(xe^{tZ})\Big|_{t=0}$$

exists. For $Z_i \in \mathfrak{G}$, $Z_1 \cdots Z_k f(x) = Z_1[Z_2 \cdots Z_k f](x)$ is defined inductively and we set

$$D_x^k f(Z_1, \dots, Z_k) = Z_1 \cdots Z_k f(x).$$

For instance, for all $x \in G$, any function f in $\mathcal{B}(G)$ has a derivative at x in any direction $Z \in \mathfrak{G}$ and $D_x^k f$ is a k-linear form on $\mathfrak{G}$.

The proof of the following classical statement is left to the reader.

Lemma 3.1. *Let u be a continuous function and $Z \in \mathfrak{G}$. Consider u as a Bruhat distribution and assume that the Bruhat distribution Zu can be represented by a continuous function v. Then u has a continuous derivative Zu in the direction of Z and $Zu = v$.*

Fix a projective family $X = (X_i)_{i\in I}$ of $\mathfrak{G}$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. For any $k \in \mathbb{N}$ and any $\ell \in I^k$, consider the seminorms on $\mathcal{B}(G)$ defined by

$$N_X^\ell(f) = \|X^\ell f\|_\infty = \sup_G |X^\ell f|, \quad \ell \in I^k. \tag{3.1}$$

Definition 3.2. Let $\mathcal{C}^0(G) = \mathcal{C}(G)$ be the set of all continuous functions on G and, for each $k = 1, 2, \dots$, let $\mathcal{C}_X^k$ be the linear space of all continuous functions $f : G \to \mathbb{R}$ such that, for each $\alpha \le k$ and each $i \in I^\alpha$, $X^i f = X_{i_1} \dots X_{i_\alpha} f$ exists and is a continuous function on G. The space $\mathcal{C}_X^k$ is equipped with the topology defined by the family of seminorms N_X^ℓ, $\ell \in I^m$, $m = 0, 1, \dots, k$. Set also

$$\mathcal{C}_X^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{C}_X^k$$

equipped with the seminorms N_X^ℓ, $\ell \in I^k$, $k = 0, 1, 2, \ldots$.

Recall that the left and right convolutions of a function $f \in \mathcal{C}(G)$ and a measure μ are defined by

$$\mu * f(x) = \int_G f(y^{-1}x)d\mu(y), \quad f * \mu(x) = \int_G f(xy^{-1})d\mu(y).$$

With this notation, the semigroup of operators $(H_t)_{t>0}$ associated to a Gaussian convolution semigroup $(\mu_t)_{t>0}$ on G is given by $H_t f = f * \check{\mu}_t$ where $\check{\mu}(B) = \mu(B^{-1})$ for any Borel set B and any Borel measure μ. If μ is central, i.e., $\mu(a^{-1}Ba) = \mu(B)$ for any Borel set B and any $a \in G$, then $f * \mu = \mu * f$. Thus, for any symmetric central Gaussian semigroup $(\mu_t)_{t>0}$, $H_t f = f * \mu_t = \mu_t * f$.

The following proposition gathers some properties of the spaces $\mathcal{C}_X^k$. See, e.g., [10].

Lemma 3.3. *Fix a projective family X and $k = 0, 1, 2, \ldots$.*

1. *For any Borel measure μ of total mass $\|\mu\|$,*
$$\forall f \in \mathcal{C}_X^k, \quad \|X^\ell(\mu * f)\|_\infty \le \|\mu\| \|X^\ell f\|_\infty.$$
2. *Let $\phi_n \in L^1(G)$, $\phi_n \to \delta_e$. Then, for any $f \in \mathcal{C}_X^k$, $f_n = \phi_n * f$ converges to f in $\mathcal{C}_X^k$.*
3. *$\mathcal{B}(G)$ is dense in $\mathcal{C}_X^k$.*
4. *$\mathcal{C}_X^k$ is an algebra for pointwise multiplication.*
5. *Let $E \subset \mathcal{C}(G)$ be such that, for any projective basis Y, $E \subset \mathcal{C}_Y^\infty$. Then $E \subset \mathcal{B}(G)$.*

3.2. **The spaces $\mathcal{S}_X^k$.** Fix a projective family $X = (X_i)$. For $f \in \mathcal{B}(G)$, set

$$|D_x^k f|_X = \left(\sum_{(\ell_1, \ell_2, \ldots, \ell_k) \in I^k} |D_x^k f(X_{\ell_1}, X_{\ell_2}, \ldots, X_{\ell_k})|^2 \right)^{1/2}.$$

Consider also the function $|D^k f|_X : G \to [0, +\infty]$ defined by

$$x \mapsto |D^k f|_X(x) = |D_x^k f|_X$$

and set

$$\||D^m f|_X\|_\infty = \sup_{x \in G} \{|D_x^m f|_X\}, \quad S_X^k(f) = \sup_{m \le k} \||D^m f|_X\|_\infty. \tag{3.2}$$

Definition 3.4. Given a projective family $X = (X_i)$, let $\mathcal{S}_X^k$ be the closure of $\mathcal{B}(G)$ for the norm $S_X^k(f)$. Let $\mathcal{S}_X^\infty$ be the space

$$\mathcal{S}_X^\infty = \bigcap_{k \in \mathbb{N}} \mathcal{S}_X^k$$

equipped with the topology defined by the family of seminorms S_X^k, $k = 0, 1, 2, \ldots$.

The spaces $\mathcal{S}_X^k$ have the followig simple but remarkable property.

Proposition 3.5. *Let $X = (X_i)_I$ and $Z = (Z_j)_J$ be two projective families such that $\sum_I X_i^2 = \sum_J Z_j^2$ on $\mathcal{B}(G)$. Then, for each k,*

$$\forall f \in \mathcal{B}(G), \quad |D^k f|_X = |D^k f|_Z.$$

In particular, $\mathcal{S}_X^k = \mathcal{S}_Z^k$.

Proof. It suffices to show that for any $f \in \mathcal{B}(G)$,

$$|D_e^k f|_X^2 = |D_e^k f|_Z^2.$$

We can assume without loss of generality that these two families X, Z, are indexed by the same countable set I. We can also assume that X is extracted from a projective basis $(X_i)_{i\in\mathcal{I}}$. Set $L = -\sum_I X_i^2 = -\sum_I Z_i^2$. By (2.4), we have

$$\forall\, f, g \in \mathcal{B}(G), \quad \Gamma(f,g) = \sum_I X_i f X_i g = \sum_I Z_i f Z_i g. \tag{3.3}$$

Moreover, each X_i, Z_i belongs to $\mathcal{H}(L)$ and, by Lemma 2.6, X is a basis of the Hilbert space $\mathcal{H}(L)$. Thus there are coefficients $b_{i,j}$ such that

$$\forall\, i \in I, \quad Z_i = \sum_{j\in I} b_{i,j} X_j.$$

As in the proof of Lemma 2.6, observe that for any sequence $\xi = (\xi_i)$ with finitely many non-zero entries we can find a function $f \in \mathcal{B}(G)$ such that $X_i f(e) = \xi_i$ (here we use the independence of the family X). Thus, (3.3) yields

$$\forall\, \xi, \zeta \in \mathbb{R}^{(I)}, \quad \sum_I \xi_i \zeta_i = \sum_i \sum_{n,m} b_{i,n} b_{i,m} \xi_n \zeta_m.$$

That is

$$\forall\, n, m \in I, \quad \sum_i b_{i,n} b_{i,m} = \delta_{n,m} \tag{3.4}$$

where $\delta_{n,m} = 1$ if $n = m$ and 0 otherwise.

Now, write

$$\begin{aligned} \sum_{\ell\in I^k} |Z^\ell f|^2 &= \sum_{\ell\in I^k} \sum_{n,m\in I^k} b_{\ell_1,n_1}\cdots b_{\ell_k,n_k} b_{\ell_1,m_1}\cdots b_{\ell_k,m_k} X^n f X^m f \\ &= \sum_{n\in I^k} |X^n f|^2 \end{aligned}$$

where the last equality uses (3.4) for each $(n_j, m_j) \in I \times I$. □

The following proposition gathers some important properties of the spaces $\mathcal{S}_X^k$.

Proposition 3.6. *Fix a projective family $X = (X_i)_I$.*

(1) *Let μ be a Borel measure of total mass $\|\mu\| = |\mu|(G)$. Then*

$$\forall\, f \in \mathcal{S}_X^k, \;\; S_X^k(\mu * f) \le \|\mu\| S_X^k(f).$$

(2) *A function f is in $\mathcal{S}_L^k$ if and only if, for any $\ell \le k$ and $j \in I^\ell$, the functions $x \mapsto X^j f(x)$, and $x \mapsto |D_x^\ell f|_L$ exists and are continuous on G.*

(3) *Let $\phi_n \in L^1(G)$, $\phi_n \to \delta_e$ as n tends to infinity. Then, for any function $f \in \mathcal{S}_L^k$, the sequence $f_n = \phi_n * f$ converges to f in $\mathcal{S}_L^k$.*

(4) *The spaces $\mathcal{S}_L^k$ are algebras for pointwise multiplication and, for any $f, g \in \mathcal{S}_L^k$,*

$$|D^k(fg)|_X \le 4^k \sup_{n\le k}\{|D^n f|_X\} \sup_{n\le k}\{|D^n g|_X\}.$$

Proof of (1). For any $f \in \mathcal{B}(G)$ and $\ell \in I^m$,

$$X^\ell(\mu * f)(x) = \int_G X^\ell f(y^{-1}x)\, d\mu(y). \tag{3.5}$$

Minkowski's inequality and (3.5) yield

$$|D^m_x(\mu * f)|_X \leq \int_G |D^m_{y^{-1}x} f|_X d|\mu|(y) \leq \|\mu\| \|D^m f|_X\|_\infty$$

for any integers m. The desired conclusion follows. □

Proof of (2) and (3). Assume that $f \in \mathcal{S}^k_X$. Then, for each $m \leq k$ and each $\ell \in I^m$, the function $X^\ell f$ is continuous as the uniform limit of continuous functions. The function

$$|D^m_x f|_X : \ x \mapsto |D^m_x f|_X$$

is also continuous as the uniform limit of continuous functions. Indeed, if $f_n \to f$ in $\mathcal{S}^k_X$ and $f_n \in \mathcal{B}(G)$, $x \mapsto |D^m_x f_n|_X$ is a continuous function since it is, in fact, a finite sum of continuous functions.

Keeping the same notation, assume now that f, $X^\ell f$, $|D^m f|_X$ are continuous functions. Let $\phi_n \in \mathcal{B}(G)$, $\phi_n \to \delta_e$ and set $f_n = \phi_n * f \in \mathcal{B}(G)$. Note that

$$K = \sup_n \int_G |\phi_n| d\nu < +\infty.$$

By a classical argument $X^\ell f_n = \phi_n * [X^\ell f]$ tends to $X^\ell f(x)$, uniformly in G. As $x \mapsto |D^m_x f|_X$ is continuous, Dini's theorem shows that the partial sums $\sum_{\ell \in J} |X^\ell f(x)|^2$ converge uniformly to $|D^m_x f|^2_X$ as the finite set $J \subset I^m$ increases to I^m. Hence, for any $\epsilon > 0$ there exists a finite set J such that

$$\forall\, x \in G, \ \sum_{\ell \in J^c} |X^\ell f(x)|^2 \leq \epsilon.$$

As

$$\sup_{x \in G} \left(\sum_{j \in J^c} |X^\ell f_n(x)|^2 \right)^{1/2} \leq K \sup_{x \in G} \left(\sum_{j \in J^c} |X^\ell f(x)|^2 \right)^{1/2},$$

we obtain

$$|D^m_x (f_n - f)|^2_X \leq \sum_{j \in J} |X^\ell (f_n - f)(x)|^2 + (1 + K)\epsilon.$$

This shows that $S^k_X(f_n - f) \to 0$. Hence, f belongs to $\mathcal{S}^k_X$ as desired. The same line of reasoning proves (3). □

Proof of (4). For $f, g \in \mathcal{B}(G)$, and $i \in I^m$, $m \leq k$, write

$$X^i(fg) = \sum_{\epsilon \in \{0,1\}^m} X^{\epsilon, i} f X^{\epsilon', i} g$$

where ϵ' is the "complement" of ϵ obtained by adding 1 modulo 2 to each coordinate and

$$X^{\epsilon, i} = X^{\epsilon_1}_{i_1} \cdots X^{\epsilon_m}_{i_m}, \ \ X^1_j h = X_i h, \quad X^0_j h = h.$$

Thus, setting $|\epsilon| = \sum_1^m \epsilon_i$,

$$\begin{aligned}\sum_{i\in I^m} |X^i(fg)|^2 &\leq 2^m \sum_{\epsilon\in\{0,1\}^m} \sum_{i\in I^m} |X^{\epsilon,i} f|^2 |X^{\epsilon',i} g|^2 \\ &= 2^m \sum_{\epsilon\in\{0,1\}^m} \left(\sum_{j\in I^{|\epsilon|}} |X^j f|^2\right)\left(\sum_{j\in I^{|\epsilon'|}} |X^j g|^2\right) \\ &\leq 4^m \sup_{n\leq m}\{|D^n f|_X^2\} \sup_{n\leq m}\{|D^n g|_X^2\}.\end{aligned}$$

Now, if $f_n, g_n \in \mathcal{B}(G)$ and $f_n \to f$, $g_n \to g$ in $\mathcal{S}_X^k$, it easily follows from the inequality above that $S_X^k(f_n g_n - fg) \to 0$. This proves (4). □

3.3. **The spaces $\mathcal{T}_L^k$ associated with bi-invariant L.** Now let L be the infinitesimal generator of a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$. The hypothesis that $(\mu_t)_{t>0}$ is central is equivalent to the fact that L is bi-invariant. This section introduces some spaces of smooth functions precisely adapted to L. Let $X = (X_i)_{i\in I}$ be a projective family such that (such a family always exists by Lemma 2.4)

$$L = -\sum_{i\in I} X_i^2.$$

By Proposition 3.5, we can denote the spaces $\mathcal{S}_X^k$ by $\mathcal{S}_L^k$ since they depend only on L. In fact, when L is bi-invariant, one can describe $\mathcal{S}_L^k$ intrinsically as follows. Recall that the iterated gradient Γ_n is defined recursively for $n = 1, 2, 3, \ldots$, by

$$\Gamma_n(f,g) = \frac{1}{2}\left(-L\Gamma_{n-1}(f,g) + \Gamma_{n-1}(f, Lg) + \Gamma_{n-1}(Lf, g)\right)$$

with $\Gamma_0(f,g) = fg$. See [1, 25] and the references therein. Higher iterated gradients, are difficult to compute in general but, since L is bi-invariant, we have

$$\forall\, f,g \in \mathcal{B}(G),\ \ \Gamma_n(f,g) = \sum_{(\ell_1,\ldots,\ell_n)\in I^n} (X_{\ell_1}\cdots X_{\ell_n} f)(X_{\ell_1}\cdots X_{\ell_n} g).$$

In particular $|D^n f|_X^2 = \Gamma_n(f,f)$ and we set

$$|D^n f|_L^2 = |D^n f|_X^2 = \Gamma_n(f,f).$$

Now, for two integers n, m, define $w(n,m) = n + 2m$ and set, for any $f \in \mathcal{B}(G)$,

$$M_L^k(f) = \sup_{x\in G} \sup_{\substack{(n,m)\in\mathbb{N}^2 \\ w(n,m)\leq k}} \left\{|D_x^n L^m f|_L\right\}. \tag{3.6}$$

Definition 3.7. Given a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$ on G with infinitesimal generator $-L$, let $\mathcal{T}_L^k$ be the closure of $\mathcal{B}(G)$ for the norm $M_L^k(f)$. Define $\mathcal{T}_L = \mathcal{T}_L^\infty$ to be the space

$$\mathcal{T}_L^\infty = \bigcap_{k\in\mathbb{N}} \mathcal{T}_L^k$$

equipped with the topology defined by the family of seminorms M_L^k, $k = 0, 1, 2, \ldots$.

Note that, if $X = (X_i)$ is a projective family such that $L = -\sum X_i^2$, we have

$$\mathcal{B}(G) \subset \mathcal{T}_L^k \subset \mathcal{S}_L^k \subset \mathcal{C}_X^k \subset \mathcal{C}(G).$$

Proposition 3.6 has an exact analog concerning the spaces $\mathcal{T}_L^k$. For instance, these spaces are algebras for pointwise multiplication. See [11]. For the purpose of the present paper, we only need to record the following alternative description of $\mathcal{T}_L^k$. The proof is entirely similar to that of Proposition 3.6(2).

Proposition 3.8. *A function f is in $\mathcal{T}_L^k$ if and only if, for any pair of integers (n,m) with $w(n,m) \le k$ and $j \in I^n$, the Bruhat distributions $X^j L^m f$ can be represented by continuous functions and $x \mapsto |D_x^n L^m f|_L$ is continuous on G.*

4. Heat kernel derivative estimates

4.1. **Gaussian estimates for derivatives.** Fix a central symmetric Gaussian semigroup $(\mu_t)_{t>0}$ on G with infinitesimal generator $-L$.

Definition 4.1. Let $\rho : G \times G \to [0, +\infty)$ be a bi-invariant continuous distance function on G and set $\rho(x) = \rho(e, x)$. We say that ρ is adapted to L (equivalently, to $(\mu_t)_{t>0}$) if it has the following property: for any non-negative function $\phi \in \mathcal{B}(G)$ such that $\int \phi d\nu = 1$, $\rho * \phi$ satisfies

$$\Gamma(\rho * \phi, \rho * \phi) = \sum_i |X_i(\rho * \phi)|^2 \le 1.$$

Examples will be given in Section 4.2.

Theorem 4.2. *Let $(\mu_t)_{t>0}$ be a central symmetric Gaussian semigroup on G with infinitesimal generator $-L$. Let $X = (X)_I$ be a projective family such that $-L = \sum_{i\in I} X_i^2$.*

1. *Assume that $(\mu_t)_{t>0}$ satisfies property* (CK)*. Then, for all $t > 0$, the continuous density $x \mapsto \mu_t(x)$ of the measure μ_t belongs to $\mathcal{T}_L$.*
2. *Let ρ be an adapted distance. Assume that $(\mu_t)_{t>0}$ satisfies* (CK)*. Assume further that there is a positive decreasing continuous function $M(t)$ and a constant $B_0 > 0$ such that*

$$\forall\, t \in (0,1),\ \ \forall\, x \in G, \quad \mu_t(x) \le \exp\left(M(t) - \frac{\rho(x)^2}{B_0 t}\right).$$

Then, for any $(k,n) \in \mathbb{N}^2$, there exist positive constants A, B, C, a such that

$$\forall\, t \in (0,1),\ \ \forall\, x \in G, \quad |D_x^k L^n \mu_t|_L \le C t^{-n-k/2} \exp\left(AM(at) - \frac{\rho(x)^2}{Bt}\right).$$

Proof of "$\mu_t \in \mathcal{T}_L$". We start with the following simple lemma.

Lemma 4.3. *For any $f \in \mathcal{B}(G)$,*

$$\int f(L^k f) d\nu = \int |D^k f|_L^2 d\nu.$$

Proof. To see this observe that

$$\begin{aligned}\int f(L^k f) d\nu &= -\int f[\sum_{i\in I} X_i^2 L^{k-1} f] d\nu \\ &= \sum_{i\in I} \int [X_i f][X_i L^{k-1} f] d\nu = \sum_{i\in I} \int [X_i f][L^{k-1} X_i f] d\nu.\end{aligned}$$

The lemma follows by induction. ∎

Remark 4.4. The lemma above depends heavily on the fact that L is central, i.e., commutes with any left-invariant vector field. In general, the correct statement is in terms of iterated gradients. Namely, $\int f(L^k f) d\nu = \int \Gamma_k(f,f) d\nu$. See [25].

Lemma 4.5. *Let $(\mu_t)_{t>0}$ be a central symmetric Gaussian semigroup on G satisfying* (CK)*. Then $\mu_t \in \mathcal{T}_L$ and, for any pair of integers (n,m), we have*

$$|D_x^n L^m \mu_t|_L \le n^{n/2} m^m \left(\frac{4}{t}\right)^{w(n,m)/2} \mu_{t/2}(e).$$

Proof. As $\mathcal{B}(G)$ is dense in the $L^2(G,d\nu)$-domain $\mathcal{D}_2^{2k}$ of L^k, the identity of Lemma 4.3 extends to any function in $\mathcal{D}_2^{2k}$. In particular,

$$\||D^n L^m \mu_t|_L\|_2^2 = \int (L^m\mu_t)(L^{n+m}\mu_t)d\nu = \|L^{m+n/2}\mu_t\|_2^2.$$

Let $H_t = e^{-tL}$ be the semigroup of operators defined at (1.1). Then, for any $f \in L^2(G,\nu)$,

$$2\|L^{1/2}H_t f\|_2^2 = 2\langle LH_t f, H_t f\rangle = -\partial_t \|H_t f\|_2^2$$

is a non-negative decreasing function. As

$$2\int_0^t \langle LH_s f, H_s f\rangle ds = \|f\|_2^2 - \|H_t f\|_2^2$$

it follows that

$$2t\|L^{1/2}H_t f\|_2^2 \le \|f\|_2^2.$$

In other words,

$$\|L^{1/2}H_t\|_{2\to 2} \le \left(\frac{1}{2t}\right)^{1/2}.$$

By the semigroup property, this implies

$$\|L^{k/2}H_t\|_{2\to 2} \le \left(\frac{k}{2t}\right)^{k/2}$$

and

$$\begin{aligned}\|L^{k/2}\mu_t\|_2^2 &= \|L^{k/2}H_t\|_{2\to\infty}^2 \le \|L^{k/2}H_{t/2}\|_{2\to 2}^2 \|H_{t/2}\|_{2\to\infty}^2 \\ &\le \left(\frac{k}{t}\right)^k \mu_t(e).\end{aligned}$$

Finally, for any integer p, q, we obtain

$$\||D^p L^{q/2}\mu_t|_L\|_2^2 = \|L^{(p+q)/2}\mu_t\|_2^2 \le \left(\frac{p+q}{t}\right)^{p+q} \mu_t(e).$$

To prove Lemma 4.5, use the semigroup property once more and write

$$\begin{aligned}|D_x^n L^m \mu_t|_L^2 &= \sum_{\ell\in I^n} |X^\ell \mu_{t/2} * L^m \mu_{t/2}(x)|^2 \\ &= \sum_{\ell\in I^n} \left| \int X^\ell \mu_{t/2}(y^{-1}x) L^m \mu_{t/2}(y) d\nu(y)\right|^2 \\ &\le \sum_{\ell\in I^n} \|X^\ell \mu_{t/2}\|_2^2 \|L^m \mu_{t/2}\|_2^2 \\ &= \||D^n \mu_{t/2}|_L\|_2^2 \|L^m\mu_{t/2}\|_2^2 \le n^n m^{2m} \left(\frac{4}{t}\right)^{n+2m} \mu_{t/2}(e)^2.\end{aligned}$$

To see that $x \mapsto X^\ell L^m f(x)$, $\ell \in I^n$, and $x \mapsto |D^n_x L^m f|_L$ are continuous, observe that both $|X^\ell L^m f(x) - X^\ell L^m f(y)|^2$ and $\big||D^n_x L^m \mu_t|_L - |D^n_y L^m \mu_t|_L\big|^2$ are bounded by

$$\begin{aligned}\sum_{\ell\in I^n} \big|X^\ell L^m \mu_{t/2} * \mu_{t/2}(x) - X^\ell \mu_{t/2} * \mu_{t/2}(y)\big|^2 \\ \leq \sum_{\ell\in I^n} \left(\int |X^\ell L^m \mu_{t/2}(z)||(\mu_{t/2}(xz^{-1}) - \mu_{t/2}(yz^{-1})|\right)^2 d\nu(z) \\ \leq \||D^n L^m \mu_{t/2}|_L\|_2^2 \sup_{z\in G} |\mu_{t/2}(xz^{-1}) - \mu_{t/2}(yz^{-1})|^2.\end{aligned}$$

This, together with finiteness of $\||D^n L^m \mu_{t/2}|_L\|_2$ and uniform continuity of the density μ_t, shows that $x \mapsto X^\ell L^m \mu_t(x)$, $\ell \in I^n$ and $x \to |D^n_x L^m \mu_t|_L$ are continuous. □

Clearly, Lemma 4.5 shows that $\mu_t \in \mathcal{T}_L$. □

Proof of the Gaussian upper-bounds. We start with the following lemma.

Lemma 4.6. *Under the hypothesis of Theorem* 4.2*,we have*

$$\forall\, t \in (0,1),\ \ \forall\, x \in G, \quad |L^k \mu_t(x)| \leq (18)^k k! t^{-k} \exp\left(M(t/2) - \frac{2\rho(x)^2}{3B_0 t}\right).$$

Proof. After observing that $(-L)^k \mu_t = \partial_t^k \mu_t$, this follows from the hypotheses and [17, Theorem 4] by taking $y = e$, $\delta = 1/2$, $\epsilon = 1/9$, $a = b = \exp(M(t))$, $c = \exp(-\rho(x)^2/[B_0 t])$ in that theorem. □

Thus we are left with proving the corresponding bounds for

$$|D^k_x L^n \mu_t|_L, \quad k = 1, 2, \dots, \quad n = 0, 1, \dots.$$

We claim it suffices to prove that, for any $k = 0, 1, \dots$, $n = 0, 1, \dots$, there exist positive constants A, a, B, C (depending on (n,k)) such that

(4.1)
$$\forall\, \alpha > 0,\ \ \forall\, t \in (0,1), \quad \|e^{\alpha\rho}|D^k L^n \mu_t|_L\|_\infty \leq C t^{-n-k/2} \exp(AM(at) + B\alpha^2 t).$$

Indeed, given (4.1), write

$$|D^k_x L^n \mu_t|_L \leq C t^{-n-k/2} \exp(AM(at) + B\alpha^2 t - \alpha\rho(x))$$

and choose $\alpha = \rho(x)/[2Bt]$. This yields

$$|D^k_x L^n \mu_t|_L \leq C t^{-n-k/2} \exp\left(AM(at) - \frac{\rho(x)^2}{4Bt}\right)$$

as desired.

Next, we claim that it suffices to prove that, for any $k = 0, 1, \dots$, $n = 0, 1, \dots$, there exist positive constants A, a, B, C such that

$$\text{(4.2)}\quad \forall\, \alpha > 0,\ \ \forall\, t \in (0,1), \quad \|e^{\alpha\rho}|D^k L^n \mu_t|_L\|_2 \leq C t^{-n-k/2} \exp(AM(at) + B\alpha^2 t).$$

Indeed, assuming that (4.2) holds and using the triangle inequality $\rho(x) \leq \rho(y^{-1}x) + \rho(y)$, we have

$$\begin{aligned} & e^{2\alpha\rho(x)}|D_x^k L^n \mu_t|_L^2 \\ \leq & \sum_{\ell \in I^k} \left| \int \left[e^{\alpha\rho(y^{-1}x)} X^\ell L^n \mu_{t/2}(y^{-1}x) \right] \left[e^{\alpha\rho(y)} \mu_{t/2}(y) \right] d\nu(y) \right|^2 \\ \leq & \|e^{\alpha\rho}|D^k L^n \mu_{t/2}|_L\|_2^2 \|e^{\alpha\rho}\mu_{t/2}\|_2^2. \end{aligned}$$

By the postulated Gaussian upper-bound on μ_t and the elementary inequality

$$\forall\, \alpha, b, t, \rho > 0, \quad \alpha\rho - \frac{\rho^2}{bt} \leq \frac{\alpha^2 bt}{4} \tag{4.3}$$

we have

$$\|e^{\alpha\rho}\mu_{t/2}\|_2^2 \leq \exp\left(2M(t/2) + \frac{\alpha^2 B_0 t}{4}\right).$$

Thus (4.2) implies (4.1) as claimed.

In order to prove (4.2) we will proceed by induction on k. By Lemma 4.6 and (4.3), the upper-bound (4.2) is satisfied for $k = 0$. Assume it is satisfied for some integer k. Fix a non-negative function $\phi \in \mathcal{B}(G)$ and let $\varrho = \phi * \rho$. Then write

$$\begin{aligned} \int e^{2\alpha\varrho}|D^{k+1}L^n\mu_t|_L^2 d\nu &= \sum_{\ell \in I^{k+1}} \int e^{2\alpha\varrho}|X^\ell L^n \mu_t|^2 d\nu \\ &= \sum_{i \in I} \sum_{\ell \in I^k} \int e^{2\alpha\varrho}|X_i X^\ell L^n \mu_t|^2 d\nu \\ &= -\sum_{i \in I} \sum_{\ell \in I^k} \int \big([X_i e^{2\alpha\varrho}][X^\ell L^n \mu_t][X_i X^\ell L^n \mu_t] \\ &\qquad\qquad + e^{2\alpha\varrho}[X^\ell L^n \mu_t][X_i^2 X^\ell L^n \mu_t]\big)\, d\nu \\ &= -2\alpha \sum_{\ell \in I^k} \int e^{2\alpha\varrho}[\sum_i (X_i\varrho)(X_i X^\ell L^n \mu_t)][X^\ell L^n \mu_t] d\nu \\ &\quad + \sum_{\ell \in I^k} \int e^{2\alpha\varrho}[X^\ell L^n \mu_t][L X^\ell L^n \mu_t] d\nu \\ &\leq 2\alpha \sum_{\ell \in I^k} \int e^{2\alpha\varrho} \left(\sum_i |X_i\varrho||X_i X^\ell L^n \mu_t| \right) |X^\ell L^n \mu_t| d\nu \\ &\quad + \sum_{\ell \in I^k} \int e^{2\alpha\varrho}|X^\ell L^n \mu_t||X^\ell L^{n+1} \mu_t| d\nu \\ &= 2\alpha E_1 + E_2. \end{aligned} \tag{4.4}$$

Note that to obtain the formula which gives E_2 we have used the fact that L commutes with any left-invariant vector field. Next, recall that our hypothesis on

ρ implies that $\sum_i |X_i\varrho|^2 \leq 1$ and write

$$\begin{aligned}
E_1 &= \sum_{\ell\in I^k} \int e^{2\alpha\varrho} \left(\sum_i |X_i\varrho||X_iX^\ell L^n\mu_t| \right) |X^\ell L^n\mu_t| d\nu \\
&\leq \int e^{2\alpha\varrho} \sum_{\ell\in I^k} \left(\sum_i |X_iX^\ell L^n\mu_t|^2 \right)^{1/2} |X^\ell L^n\mu_t| d\nu \\
&\leq \int e^{2\alpha\varrho} \left(\sum_{\ell\in I^k} \sum_i |X_iX^\ell L^n\mu_t|^2 \right)^{1/2} \left(\sum_{\ell\in I^k} |X^\ell L^n\mu_t|^2 \right)^{1/2} d\nu \\
&\leq \left(\int \sum_{\ell\in I^{k+1}} |X^\ell L^n\mu_t|^2 d\nu \right)^{1/2} \left(\int e^{4\alpha\varrho} \sum_{\ell\in I^k} |X^\ell L^n\mu_t|^2 d\nu \right)^{1/2} \\
&\leq \||D^{k+1}L^n\mu_t|_L\|_2 \|e^{2\alpha\varrho}|D^kL^n\mu_t|_L\|_2. \qquad (4.5)
\end{aligned}$$

To bound E_2, write

$$E_2 \leq \|e^{\alpha\varrho}|D^kL^n\mu_t|_L\|_2 \|e^{\alpha\varrho}|D^kL^{n+1}\mu_t|_L\|_2. \qquad (4.6)$$

By (4.4), (4.5), (4.6), the induction hypothesis and Lemma 4.5, we obtain

$$\begin{aligned}
\|e^{\alpha\varrho}|D^{k+1}L^n\mu_t|_L\|_2^2 &\leq 2\alpha\||D^{k+1}L^n\mu_t|_L\|_2 \|e^{2\alpha\varrho}|D^kL^n\mu_t|_L\|_2 \\
&\quad + \|e^{\alpha\varrho}|D^kL^n\mu_t|_L\|_2 \|e^{\alpha\varrho}|D^kL^{n+1}\mu_t|_L\|_2 \\
&\leq C(1+2\alpha t^{1/2}) t^{-2n-k-1} \exp\left(AM(at) + B\alpha^2 t\right) \\
&\leq C' t^{-2n-k-1} \exp\left(AM(at) + B'\alpha^2 t\right).
\end{aligned}$$

This finishes the inductive proof of (4.2) and the proof of Theorem 4.2. □

4.2. **Examples of Gaussian estimates.** Let us fix a symmetric central Gaussian semigroup $(\mu_t)_{t>0}$. Given a $\lambda > 0$, we say that $(\mu_t)_{t>0}$ satisfies (CKλ) if property (CK) holds and the continuous density $\mu_t(x)$ satisfies

$$\kappa = \sup_{0<t<1} \{t^\lambda \log \mu_t(e)\} < +\infty. \qquad (4.7)$$

In order to apply Theorem 4.2, we need to have some basic Gaussian estimates for the density $\mu_t(x)$ of our Gaussian semigroup in terms of some bi-invariant distance adapted to L. The next definition provides adapted distance candidates.

Definition 4.7. Given a symmetric Gaussian semigroup $(\mu_t)_{t>0}$ on G with infinitesimal generator $-L$, we set

$$d(x,y) = d_L(x,y) = \sup\{|f(x) - f(y)| : f \in \mathcal{B}(G),\ \ \Gamma(f,f) \leq 1\}$$

and

$$\delta(x,y) = \delta_L(x,y) = \sup\{|f(x) - f(y)| : f \in \mathcal{B}(G),\ \ \Gamma(f,f) \leq 1,\ \ |Lf| \leq 1\}.$$

These quasi-distances are called, respectively the intrinsic distance and the relaxed distance associated with L.

The distances d and δ are not necessarily adapted because it may happen that they are not continuous. However, if d (resp. δ) is continuous, then it is not hard to show that it is adapted. Indeed, if d (resp. δ) is continuous, then it satisfies $\Gamma(d,d) \leq 1$ (resp. $\Gamma(\delta,\delta) \leq 1$) almost everywhere.

Gaussian estimates involving either the intrinsic distance d or the relaxed distance δ introduced in Definition 4.7 have been obtained in [6] under various hypotheses. We now recall these results which are crucial in the sequel.

The following result is taken from [6, 7].

Theorem 4.8. *Let $(\mu_t)_{t>0}$ be a symmetric Gaussian semigroup satisfying property* (CK) *and let $x \mapsto \mu_t(x)$ be its continuous density.*

(1) *Assume that $(\mu_t)_{t>0}$ satisfies* (CK$*$). *Then the relaxed distance δ is a continuous distance function which defines the topology of G and*

$$\forall\, t \in (0,1),\ \forall\, x \in G, \quad \mu_t(x) \leq \exp\left(M(t) - \frac{\delta(x)^2}{Ct}\right)$$

where M satisfies $\lim_{t\to 0} tM(t) = 0$.

(2) *Fix $\lambda \in (0,1)$. Assume that $(\mu_t)_{t>0}$ satisfies* (CKλ). *Then the intrinsic distance d is a continuous distance function which defines the topology of G and*

$$\forall\, t \in (0,1),\ \forall\, x \in G, \quad \mu_t(x) \leq \exp\left(\frac{A}{t^\lambda} - \frac{d(x)^2}{Ct}\right).$$

Let us comment that, in Theorem 4.8(1), the intrinsic distance d might well be equal to $+\infty$ almost everywhere in which case no Gaussian estimate involving the intrinsic distance can possibly hold. Thus the relaxed distance plays a crucial role in this case.

Applying Theorem 4.2 and the above result we obtain the following corollary.

Corollary 4.9. *Let d, δ denote the intrinsic and relaxed distances, respectively.*

(1) *Assume that $(\mu_t)_{t>0}$ satisfies* (CK$*$). *Then, for each fixed k and n there exists $C = C(k,n)$ such that*

$$\forall\, t \in (0,1),\ \forall\, x \in G, \quad |D_x^k L^n \mu_t|_L \leq \exp\left(M(t) - \frac{\delta(x)^2}{Ct}\right)$$

where M satisfies $\lim_{t\to 0} tM(t) = 0$.

(2) *Fix $\lambda \in (0,1)$. Assume that $(\mu_t)_{t>0}$ satisfies* (CKλ) *and let κ be as in* (4.7). *Then, for each fixed k and n there exists $A = A(\kappa, \lambda, k, n)$ and $C = C(k,n)$ such that,*

$$\forall\, t \in (0,1),\ \forall\, x \in G, \quad |D_x^k L^n \mu_t|_L \leq \exp\left(\frac{A}{t^\lambda} - \frac{d(x)^2}{Ct}\right).$$

In terms of potential theory, the importance of condition (CK$*$) and of the Gaussian bound stated in Theorem 4.8(1) is that it implies that

$$\lim_{t\to 0} \sup_{x\in K} \mu_t(x) = 0 \tag{4.8}$$

for any compact K which does not contain e. See [4, 5, 6] where we call this property (CK#). The next corollary gives a bound on the Green function $g = \int_0^\infty e^{-t}\mu_t dt$. As defined, g is a measure. In [8], it is proved that g is absolutely continuous w.r.t. Haar measure and admits a continuous density on $G \setminus \{e\}$ if and only if property (CK$*$) holds true. In this case, we denote by $x \mapsto g(x)$ the continuous density of g on $G \setminus \{e\}$. The following result easily follows from the bounds of Corollary 4.9. The proof is omitted.

Corollary 4.10. *Let $(\mu_t)_{t>0}$ be a symmetric central Gaussian semigroup.*

1. *Assume that $(\mu_t)_{t>0}$ satisfies* (CK$*$). *Then, for any function $\phi \in \mathcal{B}(G)$ with support in $G \setminus \{e\}$, $\phi\, g$ belongs to $\mathcal{T}_L$. In particular, for any integers n, m and $j \in I^m$, the Bruhat distribution $X^j L^n g$ can be represented in $G \setminus \{e\}$ by a continuous function.*
2. *Fix $\lambda \in (0,1)$. Assume that $(\mu_t)_{t>0}$ satisfies* (CKλ). *Then, for any fixed integers n, m, there exists a constant C such that*

$$\forall\, x \in G \setminus \{e\}, \quad \log\left(|D_x^m L^n g|_L\right) \le C d(x)^{-\frac{2\lambda}{1-\lambda}}.$$

In [11], in order to study hypoellipticity questions, we will use the following result which is in the same spirit as (4.8). For each $\Omega \subset G$ and $f \in \mathcal{T}_L$, set

$$M_L^k(\Omega, f) = \sup_{x \in \Omega} \sup_{\substack{(n,m) \in \mathbb{N} \\ w(n,m) \le k}} \left\{|D_x^n L^m f|_L\right\}.$$

Corollary 4.11. *Assume that $(\mu_t)_{t>0}$ is a symmetric central Gausssian semigroup satisfying condition* (CK$*$). *Then, for any compact set K with $e \notin K$, any integer k, any $\sigma > 0$, there exists a constant C (which depends on $(\mu_t)_{t>0}$, K, k and σ) such that*

$$\sup_{t \in (0,1)} \left\{t^{-\sigma} M_L^k(K, \mu_t)\right\} \le C.$$

Proof. Under condition (CK$*$) the relaxed distance is continuous and defines the topology of G. As K is compact and does not contain e it follows that $\inf_K \delta(x) > 0$. The desired result thus follows from Corollary 4.9. □

Example 4.12. Let $G = \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{R} = \mathbb{R}^\infty$ and $\mathbf{Z} = \mathbb{Z}^\infty$, as in the example of Section 2.3. Write $\mathbf{T} = (\mathbb{R}/2\pi\mathbb{Z})^\infty$ as an infinite product of circles and consider the projective basis $Y = (Y_i)_1^\infty$ where $Y_i = \partial_i$ is identified with partial differentiation in the i-th coordinate. For any sequence $a = (a_i)$, $a_i > 0$, let $(\mu_t^a)_{t>0}$ the symmetric Gaussian semigroup with generator $-L = \sum a_i \partial_i^2$. Set $N_a(s) = \#\{i : a_i \le s\}$. Then, $(\mu_t^a)_{t>0}$ satisfies (CK$*$) if and only if $N_a(s) = o(s)$ as s tends to infinity. For any fixed $\lambda > 0$, $(\mu_t^a)_{t>0}$ satisfies (CKλ) if and only if $N_a(s) = O(s^\lambda)$ as s tends to infinity. See [2, 3, 6]. Thus, assuming that $N_a(s) = O(s^\lambda)$ at infinity, for some $\lambda \in (0,1)$, Corollary 4.9 gives the following bound on the first order spatial derivatives:

$$\forall\, t \in (0,1), \;\; \forall\, x \in \mathbf{T}, \quad \sum_i a_i |\partial_i \mu_t^a(x)|^2 \le \exp\left(At^{-\lambda} - \frac{d(x)^2}{Ct}\right).$$

Example 4.13. Keeping the notation of Example 4.12, consider an increasing sequence $b = (b_i)$ of positive numbers. Set

$$A = \begin{pmatrix} b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & \cdot & \cdot \\ b_1 & b_2 & b_2 & b_2 & b_2 & b_2 & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_3 & b_3 & b_3 & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_4 & b_4 & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_5 & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

This generalizes the matrix A_1 of the example of Section 2.3 which corresponds to $b_i = i$ for all i. Thus, $A = (a_{i,j})$ with $a_{i,j} = \min\{b_i, b_j\}$. Set $X_i = \sum_{j\geq i} Y_i$. Then the family $(X_i)_1^\infty$ is a projective basis of the Lie algebra $\mathbf{R}$ of $\mathbf{T}$. Moreover, as observed in Section 2.3, $\mathbf{Z}_X = \mathbf{Z}$ so that $\mathbf{T}$ is in fact the direct product of the circles given by the one parameter subgroup generated by the X_i's. Let $(\mu_t)_{t>0}$ be the symmetric Gaussian semigroup with infinitesimal generator $-L = \sum_{i,j} a_{i,j} Y_i Y_j$. In the basis X, we have $-L = \sum_i u_i X_i^2$ where $u_1 = b_1$ and $u_i = b_i - b_{i-1}$ for $i \geq 2$. Thus $(\mu_t)_{t>0}$ satisfies (CK$*$) if and only if $\#\{u_i \leq s\} = o(s)$ at infinity. For instance, this is the case if $b_i = i^2 \log(1+i)$. $(\mu_t)_{t>0}$ satisfies (CKλ) if and only if $\#\{u_i \leq s\} = O(s^\lambda)$ at infinity, e.g., if $b_i = i^{1+1/\lambda}$. Corollary 4.9 gives Gaussian upper-bounds for derivatives in the directions of the basis X but, since $\partial_i = Y_i = X_i - X_{i+1} \in \mathcal{H}_L$, one easily deduces Gaussian bounds in the directions of the basis Y. For instance, assuming that $b_i = i^{1+1/\lambda}$ for some fixed $\lambda \in (0,1)$, we obtain

$$\forall\, i,\ \forall\, t \in (0,1),\ \forall\, x \in \mathbf{T}, \quad i^{1/\lambda} |\partial_i \mu_t(x)| \leq \exp\left(At^{-\lambda} - \frac{d(x)^2}{Ct} \right).$$

Example 4.14. A compact connected group is semisimple if it is equal to its commutator subgroup. See [21]. For any semisimple group G, there exists a family (Σ_k) of compact connected simple Lie groups, and a closed central subgroup H of $\Sigma = \prod \Sigma_k$ such that $G = \Sigma/H$. Since we assume that G is metrizable, the family (Σ_k) is countable. The center of Σ, being a product of finite groups, is totally disconnected. Thus, so is H. It follows that Σ and G have the same Lie algebra (see [8]). The infinitesimal generator $-L$ of any given symmetric central Gaussian semigroup $(\mu_t)_{t>0}$ on G has the form $-L = \sum a_k \Delta_k$ where $a_k \geq 0$ and Δ_k is the Laplace-Beltrami operator of the canonical Killing metric on Σ_k (i.e., the Casimir operator). Let also $|\nabla_k f|_k$ denote the length of the gradient in the Killing metric on Σ_k. In what follows we assume that L is not degenerate, i.e., $a_k > 0$ for all k. Set

$$N(s) = \sum_{a_k \leq s} n_k$$

where n_k is the topological dimension of Σ_k. Then $(\mu_t)_{t>0}$ satisfies (CK$*$) if and only if $N(s) = o(s)$ as s tends to infinity. It satisfies (CKλ) if and only if $N(s) = O(s^\lambda)$. See [4, 8]. Assuming that $N(s) = O(s^\lambda)$ for some $\lambda \in (0,1)$, we obtain that

$$\sum_k a_k |\nabla_k \mu_t(x)|_k^2 \leq \exp\left(At^{-\lambda} - \frac{d(x)^2}{Ct} \right)$$

and similar estimates for higher derivatives.

References

1. Bakry D. *Transformation de Riesz pour les semigroupes symétriques.* Séminaire de Probabilités XIX, Lect. Notes in Math. 1123, Springer, 1985, 130-174. MR **89h:**42023
2. Bendikov A. *Potential Theory on Infinite Dimensional Abelian Groups.* Walter De Gruyter & Co., Berlin-New York, 1995. MR **98i:**31007
3. Bendikov A. *Symmetric stable semigroups on the infinite dimensional torus.* Expo. Math., 13, 1995, 39-79. MR **96h:**60123
4. Bendikov A. and Saloff-Coste L. *Elliptic diffusions on infinite products.* J. reine angew. Math. 493, 1997, 171-220. MR **99k:**58173
5. Bendikov A. and Saloff-Coste L. *Potential theory on infinite products and groups.* Potential Analysis, 11, 325-358, 1999. MR **2001a:**31012

6. Bendikov A. and Saloff-Coste L. *On and off-diagonal heat kernel behaviors on certain infinite dimensional local Dirichlet spaces.* American J. Math. 122, 2000, 1205-1263. CMP 2001:05
7. Bendikov A. and Saloff-Coste L. *Gaussian semigroups of measures on locally compact locally connected groups*, Preprint 1999.
8. Bendikov A. and Saloff-Coste L. *Central Gaussian semigroups of measures with continuous density*, J. Funct. Anal. to appear.
9. Bendikov A. and Saloff-Coste L. *On the absolute continuity of Gaussian measures on locally compact groups*, J. Theoret. Prob. to appear.
10. Bendikov A. and Saloff-Coste L. *On the hypoellipticity of sub-Laplacians on infinite dimensional compact groups.* Forum Math. to appear.
11. Bendikov A. and Saloff-Coste L. *Hypoellipticity for bi-invariant Laplacians on infinite dimensional compact groups.* Preprint 2001.
12. Berg Ch. *Potential theory on the infinite dimensional torus.* Invent. Math. 32, 1976, 49-100. MR **53:**5915
13. Born E. *Projective Lie Algebra Bases of a Locally Compact Group and Uniform Differentiability.* Math. Zeit. 200, 1989, 279-292. MR **90f:**22008
14. Born E. *An Explicit Lévy-Hinčin Formula for Convolution Semigroups on Locally Compact Groups.* J. Theor. Prob. 2, 1989, 325-342. MR **91a:**60023
15. Bruhat F. *Distributions sur un groupe localement compact et application à l'étude des représentations des groupes p-adiques.* Bull. Soc. Math. France, 89, 1961, 43-75. MR **25:**4354
16. Davies E.B. *Heat kernels and spectral theory.* 1989, Cambridge University Press. MR **90e:**35123
17. Davies E.B. *Non-Gaussian aspects of heat kernel behaviour.* J. London Mat. Soc. 55, 1997, 105-125. MR **97i:**58169
18. Fukushima M., Oshima Y. and Takeda, M. *Dirichlet forms and symmetric Markov processes,* W. De Gruyter, 1994. MR **96f:**60126
19. Glushkov V.N. *The structure of locally compact groups and Hilbert's Fifth Problem.* AMS Translations 15, 1960, 55-94. MR **22:**5690
20. Heyer H. *Probability Measures on Locally Compact Groups.* Ergeb. der Math. und ihrer Grenzgeb. 94, Springer, Berlin-Heildelberg-New York, 1977. MR **58:**18648
21. Hofmann K. and Morris S. *The structure of compact groups.* W. de Gruyter, 1998. MR **99k:**22001
22. Hörmander L. *Hypoelliptic second order differential equations.* Acta Math. 119, 1967, 147-171. MR **36:**5526
23. Hunt G. A. *Semi-groups of measures on Lie groups.* Trans. A.M.S., 81, 1956, 264-293. MR **18:**54a
24. Kusuoka S. and Stroock D. *Applications of the Malliavin calculus, Part II.* J. Fac. Sci., The University of Tokyo, Sec IA, 32, 1985, 1-76. MR **86k:**60100b
25. Ledoux M. *L'algèbre des gradients itérés d'un générateur Markovien — Développements de moyennes et entropies.* Ann. scient. Éc. Norm. Sup. 4ème série, 28, 1995, 435-460. MR **96f:**60130
26. Saloff-Coste L. *Analyse sur les groupes de Lie nilpotents.* C. R. Acad. Sci. Paris, 302, Série I, 1986, 499-502. MR **88j:**22008
27. Siebert E. *Absolute continuity, singularity and supports of Gaussian semigroups on a Lie group.* Monatshefte für Math. 93, 1982, 239-253. MR **83m:**60018
28. Sturm K-T. *On the geometry defined by Dirichlet forms.* In Seminar on Stochastic Processes, Random Fields and Applications, Ascona, E. Bolthausen et al., Eds. Birkhäuser, Progress in Probability, Vol 36, 1995, 231-242. MR **96j:**31009
29. Varopoulos N., Saloff-Coste L. and Coulhon T., *Analysis and geometry on groups.* Cambridge University Press, 1992. MR **95f:**43008

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