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ORTHOGONAL POLYNOMIALS AND QUADRATIC EXTREMAL PROBLEMS

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ABSTRACT. The purpose of this paper is to analyse a class of quadratic extremal problems defined on various Hilbert spaces of analytic functions, thereby generalizing an extremal problem on the Dirichlet space which was solved by S.D. Fisher. Each extremal problem considered here is shown to be connected with a system of orthogonal polynomials. The orthogonal polynomials then determine properties of the extremal function, and provide information about the existence of extremals.

1. Introduction and Definitions

Let Δ be the open unit disk in the complex plane $\mathbb{C}$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers. The Hilbert space $\mathcal{H}(\{\varepsilon_n\})$ is the set of all functions f analytic on Δ for which the norm

$$\|f\|_{\mathcal{H}} = \sqrt{\sum_{n=0}^{\infty} \varepsilon_n |a_n|^2}, \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is finite. The inner product in $\mathcal{H}(\{\varepsilon_n\})$ is given by the formula

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \varepsilon_n a_n \overline{b_n}, \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \ \in \ \mathcal{H}(\{\varepsilon_n\}).$$

Hilbert spaces which arise in this way include the Bergman space ($\varepsilon_n = \frac{1}{n+1}$), the Hardy space H^2 ($\varepsilon_n = 1$), and the Dirichlet space $\mathcal{D}$ (essentially $\varepsilon_n = n+1$).

The number λ and the corresponding extremal function in

$$\lambda = \sup\left\{ Re\frac{1}{\pi}\int_{\Delta} \overline{f(z)} f'(z) dA(z), \quad f \in \mathcal{D},\ \|f\|_{\mathcal{D}} = 1 \right\} \tag{1.1}$$

were shown to exist in 1995 by S.D. Fisher (see [5]), thereby solving a problem posed by B. Korenblum. Fisher calculated the number λ to be the reciprocal of the smallest zero of J_0, where J_n is the Bessel function of index n, and showed that the extremal function, which is unique up to rotation, has Taylor coefficients $a_n = J_n\left(\frac{1}{\lambda}\right)$, $n \geq 1$.

This paper shows that the problem (1.1) is connected with the Lommel orthogonal polynomials, and that all reciprocals of the zeros of J_0 appear as extreme

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values over the unit sphere of some subspace of the Dirichlet space. It also gives an analysis of extremal problems generalizing (1.1), namely

$$\sup\left\{\psi\left(f\right):f\in\mathcal{H},\ \|f\|_{\mathcal{H}}=1\right\},$$

where ψ is a functional of the form

$$\psi(f)=Re\frac{1}{\pi}\int_{\Delta}\overline{f(z)}f'(z)dA(z)+\sum_{n=0}^{\infty}c_n\left|f^{(n)}(0)\right|^2,\quad f\in\mathcal{H}\left(\{\varepsilon_n\}\right),\quad c_n\in\mathbb{R}.$$

The question of the existence of extrema can be highly sensitive to the choice of coefficients c_n. For example, one can consider the functional ψ_c defined by

$$\psi_c(f)=Re\frac{1}{\pi}\int_{\Delta}\overline{f(z)}f'(z)dA(z)+c\left|f(0)\right|^2,\quad f\in\mathcal{H}\left(\{\varepsilon_n\}\right),\quad c\in\mathbb{R} \tag{1.2}$$

(the problem (1.1) is then recovered by the choice $c=0$ and $\varepsilon_n=n+1$). It was shown in [5] that although ψ_0 attains a maximum on the unit sphere of the Dirichlet space, there is no maximum on the Hardy space H^2. Here it is shown that in fact ψ_c attains a maximum in the Hardy space if and only if $c>\frac{1}{2}$.

Necessary conditions can be given for the existence of an extremal. This is accomplished by showing that a maximum value of the functional ψ must be a 'critical value' as defined in the third section. Critical values can then be characterised either as eigenvalues of a Jacobi operator, or as values connected with a system of orthogonal polynomials. The existence of such values is then necessary for the existence of an extremal.

The paper is organised as follows: section 2 describes an equivalent formulation of the extremal problems on a weighted l^2 sequence space. Necessary and sufficient conditions are then given for functionals to be bounded.

In section 3, a variational argument shows that the Taylor coefficients for an extremal must satisfy a three term recurrence relation. The recurrence relation provides a natural way to define critical points and values; local maxima of ψ over the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ are shown to be examples of critical values. The set of critical values is characterised as the set of eigenvalues of a tri-diagonal operator on a weighted l^2 space. Results about the eigenvalues of tri-diagonal operators provide some information about the extrema of ψ. In particular, when the problem is 'compact', the Courant-Hilbert minimax principle applies.

Section 4 introduces the connection with orthogonal polynomials. The three term recurrence relation of section 3 determines uniquely, through Favard's theorem, a system of orthogonal polynomials connected to the functional ψ. This connection is complete in the sense that every system of orthogonal polynomials is determined by a functional of the form ψ over some Hilbert space $\mathcal{H}\left(\{\varepsilon_n\}\right)$. Moreover, through the work [8] of M. Stone on Jacobi matrices, the critical values of ψ can be characterised by the measure μ with respect to which the orthgonal polynomials are orthogonal: the set of critical values of ψ coincides with the set of isolated points in the support of μ. For any given system of orthogonal polynomials, the suport of the measure is generally not a straightforward calculation. However, T.S. Chihara's analysis [1] of the modified Chebyshev polynomials together with the results of section 4 determine precisely the functionals in (1.2) which attain an extremum on the Hardy space.

Finally in section 5, necessary conditions are given for the extremal function to be entire. The problem (1.1) is also reconsidered: The extremal function is shown

to have the alternative form

$$C\sum_{n=1}^{\infty} h_n(\frac{1}{\lambda})z^n$$

where $\{h_n\}$ are the Lommel polynomials of index one. Also, it is shown that every element of the set $\left\{\frac{1}{j} : J_0(j) = 0\right\}$ of reciprocals of zeros of J_0 appears as a extremum for ψ_0 over some subspace of the Dirichlet space.

2. Preliminaries

Let $\{\alpha_n\}$ be a real sequence and $\{\varepsilon_n\}$ a positive sequence. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\{\varepsilon_n\})$, and define

$$\psi(f) = Re\frac{1}{\pi}\int_{\Delta} \overline{f(z)}f'(z)dA(z) + \sum_{n=0}^{\infty} \alpha_n\varepsilon_n|a_n|^2. \tag{2.1}$$

By identifying the complex function f with its sequence of Taylor coefficients $\mathbf{a} = \{a_n\}$, a new functional Φ is defined (in (2.3)) as a function of $\mathbf{a}$ rather than f. This functional is analogous to ψ in the sense it has the same local and global extrema, but has the advantage of being represented in terms of a tri-diagonal operator, and being homogeneous of degree zero. The Taylor coefficients occurring in product pairs inspired the term 'quadratic functional'. Definitions follow:

Let $l^2(\{\varepsilon_n\})$ be the space of all complex sequences $\mathbf{a} = \{a_n\}$ with finite norm

$$\|\mathbf{a}\| = \sqrt{\sum_{n=0}^{\infty} \varepsilon_n|a_n|^2}.$$

Let

$$\mathbf{e}_n = \left(\overbrace{0,0,0,...,0}^{n \text{ zeros}},1,0,0,...\right).$$

On the space $l^2(\{\varepsilon_n\})$, define the tri-diagonal operator A with respect to the basis $\{\mathbf{e}_n\}$ by

$$A = \begin{pmatrix} \alpha_0 & \frac{1}{2\varepsilon_0} & & & & \\ \frac{1}{2\varepsilon_1} & \alpha_1 & \frac{1}{2\varepsilon_1} & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & \frac{1}{2\varepsilon_n} & \alpha_n & \frac{1}{2\varepsilon n} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.2}$$

Note that $\{\mathbf{e}_n\}$ is orthogonal, but is not orthonormal since $\|\mathbf{e}_n\| = \sqrt{\varepsilon_n}$. The operator A is self adjoint.

Proposition 2.1. *The operator A is self adjoint, and is bounded on $l^2(\{\varepsilon_n\})$ if and only if $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Moreover, A is compact if and only if $\{\alpha_n\}$ converges to 0, and $\lim_{n\to\infty} \varepsilon_{n-1}\varepsilon_n = \infty$.*

Let Φ be defined on $l^2(\{\varepsilon_n\})\setminus\{\mathbf{0}\}$ by

$$\Phi(\mathbf{a}) = \frac{\langle A\mathbf{a},\mathbf{a}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle} = \frac{Re\sum_{n=0}^{\infty}\overline{a_n}a_{n+1} + \sum_{n=0}^{\infty}\varepsilon_n\alpha_n|a_n|^2}{\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2}. \tag{2.3}$$

The next proposition shows that the functional Φ remains finite on $l^2(\{\varepsilon_n\})$ precisely when $\{\alpha_n\}$ is bounded and $\{\varepsilon_{n-1}\varepsilon_n\}$ is bounded from zero. The corollary states that this condition also characterises those functionals Φ which are bounded.

Proposition 2.2. *Let $\{\varepsilon_n\}$ be a positive sequence, $\{\alpha_n\}$ be a real sequence and the functional Φ be defined as in* (2.3)*. Then $\Phi(\mathbf{a})$ is finite for every $\mathbf{a}\in l^2(\{\varepsilon_n\})\setminus\{\mathbf{0}\}$ if and only if $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty}\varepsilon_{n-1}\varepsilon_n > 0$.*

Proof. Suppose that $\{|\alpha_n|\}$ is bounded by the constant M and that $\liminf_{n\to\infty}\varepsilon_{n-1}\varepsilon_n > \delta^2$ for some $\delta > 0$. A proof similar to that of theorem 5(i) in [5] shows that for every $\mathbf{a}\in l^2(\{\varepsilon_n\})\setminus\{\mathbf{0}\}$, we have $|\Phi(\mathbf{a})| \le \frac{1}{\delta} + M$. Thus Φ is bounded.

If $\{\alpha_n\}$ is unbounded, then there is a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ for which $n_k - n_{k-1} > 1$ and $\alpha_{n_k}\to\infty$ or $\alpha_{n_k}\to-\infty$. Suppose the former, and let w_{n_k} be a non-negative sequence such that $\sum w_{n_k}^2 < \infty$ but $\sum \alpha_{n_k} w_{n_k}^2 = \infty$. Then the sequence $\{a_n\}$ defined by

$$a_n = \begin{cases} \frac{w_n}{\sqrt{\varepsilon_n}}, & n = n_k \text{ for some } k,\\ 0, & n \ne n_k \text{ for any } k \end{cases}$$

is in $l^2(\{\varepsilon_n\})$ but $\Phi(\{a_n\})$ is not finite. A similar argument can be made when $\{|a_n|\}$ is unbounded from below.

Now suppose that $\{|\alpha_n|\}$ is bounded by the constant M but that $\{\varepsilon_{n-1}\varepsilon_n\}$ is not bounded from zero. Then there is a subsequence $\{\varepsilon_{n_k-1}\varepsilon_{n_k}\}$ of $\{\varepsilon_{n-1}\varepsilon_n\}$ which converges to zero. A further subsequence can be chosen such that $n_k - n_{k-1} > 2$ and $\sum_{k=0}^{\infty}\sqrt{\varepsilon_{n_k-1}\varepsilon_{n_k}} < \infty$. Define the sequence $\{a_n\}$ by

$$a_n = \begin{cases} \sqrt[4]{\frac{\varepsilon_{n+1}}{\varepsilon_n}}, & n = n_k \text{ for some } k,\\ \sqrt[4]{\frac{\varepsilon_{n-1}}{\varepsilon_n}}, & n = n_k + 1 \text{ for some } k,\\ 0, & n \ne n_k,\ n \ne n_{k+1} \text{ for any } k. \end{cases}$$

Then

$$\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2 = \sum_{k=0}^{\infty} 2\sqrt{\varepsilon_{n_k}\varepsilon_{n_k+1}} < \infty,$$

so $\mathbf{a}\in l^2(\{\varepsilon_n\})$, but

$$\begin{aligned}\Phi(\mathbf{a}) &\ge \frac{Re\sum_{k=0}^{\infty}a_{n_k}a_{n_k+1}}{\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2} - M\frac{\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2}{\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2}\\ &\ge \frac{Re\sum_{k=0}^{\infty}1}{\sum_{n=0}^{\infty}\varepsilon_n|a_n|^2} - M \ = \infty.\end{aligned}$$

□

Corollary 2.3. *Let $\{\alpha_n\}$ be a real sequence and $\{\varepsilon_n\}$ be a positive sequence. Then $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty}\varepsilon_{n-1}\varepsilon_n > 0$ if and only if Φ is bounded, in the sense that there is a constant C such that*

$$|\Phi(\mathbf{a})| \le C, \quad \mathbf{a}\in l^2(\{\varepsilon_n\})\setminus\{\mathbf{0}\}.$$

The next proposition states that the extrema of the functional ψ on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ are the same as those of Φ on $l^2(\{\varepsilon_n\}) \setminus \{\mathbf{0}\}$.

Proposition 2.4. *Suppose that $\{\alpha_n\}$ is a bounded real sequence, and $\{\varepsilon_n\}$ is a positive sequence such that $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Let Φ be the functional defined on $l^2(\{\varepsilon_n\}) \setminus \{\mathbf{0}\}$ as in* (2.3)*, and let ψ be the functional defined on $\mathcal{H}(\{\varepsilon_n\})$ by* (2.1)*. The functionals ψ and Φ have the same local and global extrema over the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$. Moreover, the functional ψ attains an extreme value at $g(z) = \sum b_n z^n$ on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ if and only if Φ attains the same (extreme) value at $C\{b_n\}$ where $C = \frac{1}{\|\{b_n\}\|}$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an element of the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$, and let $\delta > 0$. Then ψ and Φ take on the same sets of values on corresponding open sets:

$$\left\{\psi(g) : \|g\|_{\mathcal{H}} = 1,\ \|f-g\|_{\mathcal{H}} < \delta,\ g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}(\{\varepsilon_n\})\right\}$$
$$= \left\{\Phi(\mathbf{b}) : \|\mathbf{a}-\mathbf{b}\| < \delta,\ \mathbf{b} \in l^2(\{\varepsilon_n\}) \setminus \{\mathbf{0}\}\right\}.$$

The proposition is easily verified. □

Combining corollary 2.3 with this proposition gives the following:

Theorem 2.5. *Suppose that $\{\alpha_n\}$ is a bounded real sequence, and $\{\varepsilon_n\}$ is a positive sequence. The functional ψ defined in* (2.1) *is bounded on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ if and only if $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$.*

Example 2.6. Let $\{\alpha_n\}$ be any bounded real sequence.

1. The functional
$$\psi(f) = Re\frac{1}{\pi}\int_{\Delta} \overline{f} f' dA + \sum_{n=0}^{\infty} \alpha_n |a_n|^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is bounded on the unit sphere of the Hardy space.
2. The functional
$$\psi(f) = Re\frac{1}{\pi}\int_{\Delta} \overline{f} f' dA + \sum_{n=1}^{\infty} n\alpha_n |a_n|^2, \quad f(z) = \sum_{n=1}^{\infty} a_n z^n$$
is bounded on the unit sphere of the Dirichlet space.
3. The functional
$$\psi(f) = Re\frac{1}{\pi}\int_{\Delta} \overline{f} f' dA + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} |a_n|^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is unbounded on the unit sphere of the Bergman space.

Proof. The sequence $\{\alpha_n\}$ is bounded, so in each case theorem 2.5 shows that the functional ψ will be bounded if and only if $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. The Hardy space H^2 is $\mathcal{H}(\{\varepsilon_n\})$ with $\varepsilon_n = 1$. The Bergman space is $\mathcal{H}(\{\varepsilon_n\})$ with $\varepsilon_n = \frac{1}{n+1}$. The Dirichlet space $\mathcal{D}$ is defined here, as in [5], to be those f analytic in Δ for which $f(0) = 0$, and the norm $\|f\|_{\mathcal{D}} = \sqrt{\sum_{n=1}^{\infty} n|a_n|^2}$ is finite. The sets

$$\left\{\psi(f) : \|f\|_{\mathcal{D}} = 1,\ f \in \mathcal{D}\right\} \text{ and } \left\{\psi(g) : \|g\|_{\mathcal{H}} = 1,\ g \in \mathcal{H}(\{n+1\})\right\}$$

are identical; ψ is bounded on $\mathcal{H}(\{n+1\})$. □

3. Critical Values and Eigenvalues

Throughout this section, let the functional Φ be defined on $l^2(\{\varepsilon_n\})$ by (2.3), where $\{\varepsilon_n\}$ and $\{c_n\}$ are appropriate sequences. Let A be the corresponding operator defined by (2.2).

Suppose that the number λ is a local extremum of a functional of the form Φ. Then the corresponding point in $l^2(\{\varepsilon_n\})$ at which the value λ is assumed satisfies a three term recurrence relation. Through this recurrence relation, extreme values of Φ are connected with the eigenvalues of the operator A. The existence of extrema then becomes a question of the existence of eigenvalues.

Proposition 3.1. *Let $\{\alpha_n\}$ be a bounded real sequence, and let $\{\varepsilon_n\}$ be a positive sequence such that $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Suppose that λ is a local extremum for Φ which is attained at $\mathbf{a} \in l^2(\{\varepsilon_n\})$. Then the complex numbers a_n are co-linear, and satisfy the recurrence relation*

$$a_{n+1} = 2\varepsilon_n a_n(\lambda - \alpha_n) - a_{n-1}, \qquad n \geq 0, \quad a_{-1} = 0. \tag{3.1}$$

Proof. Suppose that λ is a local maximum of Φ. The proof for a local minimum is similar. For every $\mathbf{y} \in l^2(\{\varepsilon_n\})$ and for every $\delta \in \mathbb{R}$ for which $|\delta|$ is small enough ($|\delta|$ depends on $\mathbf{y}$),

$$\Phi(\mathbf{a} + \delta\mathbf{y}) \leq \lambda.$$

Let $a_{-1} = 0$ and $\mathbf{y} = \mathbf{e}_k$. Then

$$\mathbf{a} + \delta\mathbf{y} = (a_0, a_1, a_2, ..., a_{k-1}, a_k + \delta, a_{k+1}, a_{k+2}, ...).$$

A calculation shows that the inequality $\Phi(\mathbf{a} + \delta\mathbf{y}) \leq \Phi(\mathbf{a})$ implies

$$\delta\left(Re\left(a_{k-1} + a_{k+1} - 2\varepsilon_k a_k(\lambda - \alpha_k)\right)\right) \leq \delta^2(\lambda\varepsilon_k - \alpha_k\varepsilon_k).$$

Thus the sequence $\{\mathbf{a}\}$ satisfies

$$Re\left(a_{k-1} + a_{k+1} - 2\varepsilon_k a_k(\lambda - \alpha_k)\right) = 0.$$

Since Φ is homogeneous of degree zero, $e^{i\alpha}\mathbf{a}$ satisfies the same equality for all real α. Therefore the recurrence relation holds:

$$a_{k+1} = 2\varepsilon_k a_k(\lambda - \alpha_k) - a_{k-1}, \quad k \geq 1, \quad a_{-1} = 0.$$

Note that the coefficients a_n all lie on the line $\left\{z = re^{i\arg(a_0)}\right\}$. □

Define $\mathbf{a} \in l^2(\{\varepsilon_n\})$ to be a **critical point** for Φ if for some $\lambda \in \mathbb{R}$, the recurrence relation (3.1) holds. The value $\Phi(\mathbf{a})$ is defined to be the **critical value** corresponding to the critical point $\mathbf{a}$. This terminology is substantiated by the fact that a local extremum is a critical value, as the following proposition shows.

Proposition 3.2. *Let $\{\alpha_n\}$ be a bounded real sequence, and let $\{\varepsilon_n\}$ be a positive sequence such that $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. If $\mathbf{a} \in l^2(\{\varepsilon_n\})$ is a critical point for Φ, then the number λ in the recurrence relation* (3.1) *is the critical value corresponding to* $\mathbf{a}$.

Proof. Let $\mathbf{a}$ be a critical point for Φ. Then one can calculate directly from the recurrence relation that $\Phi(\mathbf{a}) = \lambda$. □

Corollary 3.3. *Under the hypotheses of the proposition, a local extremum of Φ is a critical value of Φ.*

The next proposition shows that the set of eigenvalues of the operator A is in fact the set of critical values of the functional Φ.

Proposition 3.4. *Let $\{\alpha_n\}$ be a bounded real sequence, and $\{\varepsilon_n\}$ a positive sequence for which $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Then the number λ is a critical value for Φ if and only if λ is an eigenvalue of the operator A. Moreover, $\mathbf{a}$ is a critical point of Φ corresponding to λ if and only if $\mathbf{a}$ is an eigenvector of A corresponding to λ, that is, the recurrence relation holds if and only if*

$$A\mathbf{a} = \lambda\mathbf{a}.$$

Proof. The recurrence relation (3.1) can be written

$$\frac{1}{2\varepsilon_n}a_{n-1} + \alpha_n\varepsilon_n + \frac{1}{2\varepsilon_n}a_{n+1} = \lambda a_n, \quad n \geq 1, \quad a_{-1} = 0.$$

This is precisely $A\mathbf{a} = \lambda\mathbf{a}$. □

With respect to the orthonormal basis $\left\{\frac{\mathbf{e}_n}{\sqrt{\varepsilon_n}}\right\}$, the operator A defined in (2.2) takes the form,

$$J = \begin{pmatrix} \alpha_0 & \frac{1}{2\sqrt{\varepsilon_1\varepsilon_0}} & & & & \\ \frac{1}{2\sqrt{\varepsilon_1\varepsilon_0}} & \alpha_1 & \frac{1}{2\sqrt{\varepsilon_2\varepsilon_1}} & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & \frac{1}{2\sqrt{\varepsilon_n\varepsilon_{n-1}}} & \alpha_n & \frac{1}{2\sqrt{\varepsilon_{n+1}\varepsilon_n}} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} = DAD^{-1}$$

and so has the same eigenvalues. Results from [3] about the eigenvalues of J have corresponding implications for the critical values of Φ, and consequently (proposition 2.4) for the extrema of ψ.

Theorem 3.5. *Let $\{\varepsilon_{n-1}\varepsilon_n\}$ decrease (not necessarily strictly) to a positive limit. The functional ψ_0 attains neither a maximum nor a minimum on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$.*

Proof. The sequence $\{\alpha_n\}$ is identically zero, so J has main diagonal entries equal to zero, and off-diagonal entries increasing to a positive limit. By [3], example 1, page 251, J has no eigenvalues. □

The following corollary was proved directly in [5].

Corollary 3.6. *The functional ψ_0 attains neither a maximum nor a minimum on the unit sphere of the Hardy space H^2.*

The next theorem describes a class of Hilbert spaces on which the extreme values of the functional ψ_0 are bounded from zero.

Theorem 3.7. *Suppose that $\{\varepsilon_{n-1}\varepsilon_n\}$ increases (not necessarily strictly) to a positive limit l. Any extreme values of ψ_0 on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ must lie outside the interval $\left[-\frac{1}{\sqrt{l}}, \frac{1}{\sqrt{l}}\right]$.*

Proof. The off-diagonal elements of J decrease to $\frac{1}{2\sqrt{l}}$, so by [3], p. 252, the operator J has no eigenvalues in the interval $\left[-\frac{1}{\sqrt{l}}, \frac{1}{\sqrt{l}}\right]$. □

When A is a compact operator, the critical values arise from the Courant-Hilbert minimax principle, providing criteria for the existence of extrema. The next proposition and theorem show this for a class of operators slightly broader than compact operators.

Proposition 3.8. *Let $\{\alpha_n\}$ be a real sequence converging to the real number α, and suppose that $\{\varepsilon_{n-1}\varepsilon_n\}$ diverges to ∞. Then Φ has critical values, all of which are assumed over some subspace of $l^2(\{\varepsilon_n\})$.*

Proof. Without loss of generality, it may be assumed that $\alpha = 0$, for suppose that $\alpha_n \to \alpha$. Let $\tilde{\Phi}$ be defined as in (2.3) from the positive sequence $\{\varepsilon_n\}$ and the real sequence $\{\alpha_n - \alpha\}$ which converges to 0. Then for every $\mathbf{a}$ in $l^2(\{\varepsilon_n\})\setminus\{\mathbf{0}\}$,

$$\Phi(\mathbf{a}) = \tilde{\Phi}(\mathbf{a}) + \alpha.$$

Since Φ and $\tilde{\Phi}$ differ only by a constant, they have the same set of critical points.

The operator $A-\alpha I$ is compact, so it has eigenvalues. Moreover, by the Courant-Hilbert minimax principle for compact operators (see for example [6], p. 25), the eigenvalues of $A - \alpha I$ are assumed as extrema over some subspace of $l^2(\{\varepsilon_n\})$. □

The theorem which follows is a consequence of propositions 2.4 and 3.8.

Theorem 3.9. *Let $\{\alpha_n\}$ be a real sequence converging to the real number α, and suppose that $\{\varepsilon_{n-1}\varepsilon_n\}$ diverges to ∞. The functional ψ determined by* (2.1) *achieves an extremum on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$.*

Example 3.10. Let $\{\alpha_n\}$ be a real sequence converging to the real number α. On the unit sphere of the Dirichlet space, the functional

$$\psi(f) = Re\frac{1}{\pi}\int_\Delta \overline{f} f' dA + \sum_{n=1}^{\infty} n\alpha_n |a_n|^2, \quad f(z) = \sum_{n=1}^{\infty} a_n z^n$$

achieves an extreme value. Note that this example encompasses ψ_0.

4. Extreme Values and Orthogonal Polynomials

In this section it is shown that the set of extreme values for a functional Φ can be described in terms of a system of orthogonal polynomials. In fact it will be shown that every system of orthogonal polynomials determines a functional of the form (2.1) (although this functional will not necessarily be bounded).

Suppose for $n > 0$ that P_n is a polynomial of degree n, with a positive leading coefficient. The set of polynomials $\{P_n\}$ is said to be **orthogonal** if there exist a positive measure μ and positive constants K_n such that

$$\int_{\mathbb{R}} P_n P_m d\mu = K_n \delta_{n,m}.$$

In determining if a given family of polynomials is orthogonal, there is no loss of generality in considering monic polynomials. Favard's theorem ([1], p. 21) states that every family of monic orthogonal polynomials $\{p_n\}$ satisfies a three term recurrence relation of the form

(4.1)
$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad n \geq 0,$$

where $\{\alpha_n\}$ is some real sequence ($n \geq 0$) and $\{\beta_n\}$ is some positive sequence ($n \geq 1$). The converse to this theorem is also true: if $\{\alpha_n\}$ is any real sequence, and

TABLE 1. Some systems of orthogonal polynomials, together with the (bounded) sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which generate them (for a more complete listing, see [1], p. 217)).

Orthogonal System	$\{\alpha_n\} \quad n \geq 0$	$\{\beta_n\} \quad n \geq 1$
Chebyshev $\{T_n(x)\}$	0	$\beta_1 = \frac{1}{2}, \quad \beta_n = \frac{1}{4}, \quad n \geq 2$
Chebyshev $\{U_n(x)\}$	0	$\beta_n = \frac{1}{4}$
Lommel $\{h_{n,\nu}(x)\}$	0	$\frac{1}{4(n+\nu)(n+\nu-1)}$
Jacobi $\left\{P_n^{(\alpha,\beta)}(x)\right\}$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
Legendre	0	$\frac{n^2}{(2n-1)(2n+1)}$

$\{\beta_n\}$ is any positive sequence, then the polynomials defined in (4.1) are orthogonal with respect to some positive measure on the real line.

The recurrence relation connects each functional of the form ψ with a system of orthogonal polynomials $\{P_n\}$. The polynomials themselves determine the form of the extremals of ψ, since an extremal's sequence of Taylor coefficients can be written as the sequence of polynomials $\{P_n\}$. The main result of this section will be to prove that the set of critical values of Φ is characterised by the set of isolated points in the support of the measure with respect to which the set $\{P_n\}$ is orthogonal. To obtain the system of orthogonal polynomials for a given functional ψ, write $\lambda = x$, $a_1 = 1$, and $a_n = P_n(x)$ in (3.1). Then

(4.2)
$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad P_{n+1}(x) = 2\varepsilon_n(x-\alpha_n)P_n(x) - P_{n-1}(x), \quad n \geq 0,$$

where $P_n(x)$ is a real polynomial of degree n. The next proposition uses Favard's theorem to show that the polynomials in (4.2) are in fact orthogonal.

Proposition 4.1. *Let $\{\alpha_n\}$ be a real sequence, and $\{\varepsilon_n\}$ be a positive sequence. Let $\{P_n\}$ be defined as in* (4.2)*. Then $\{P_n\}$ is a family of orthogonal polynomials, and the corresponding monic polynomials $\{p_n\}$ satisfy the relation* (4.1)*, where*

$$\beta_n = \frac{1}{4\varepsilon_n\varepsilon_{n-1}}, \quad n \geq 1, \quad \beta_0 = 1.$$

Proof. The leading coefficient of the polynomial P_n is $2^n\varepsilon_{n-1}\varepsilon_{n-2}\varepsilon_{n-3}...\varepsilon_1\varepsilon_0$. Replacing P_n by $2^n\varepsilon_{n-1}\varepsilon_{n-2}\varepsilon_{n-3}...\varepsilon_1\varepsilon_0 p_n$ in the recurrence relation, we obtain

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \frac{1}{4\varepsilon_n\varepsilon_{n-1}}p_{n-1}(x), \quad n \geq 1. \tag{4.3}$$

□

The critical points of Φ now take a specific form in relation to the orthogonal polynomial system.

Proposition 4.2. *Let $\{\alpha_n\}$ be a bounded real sequence, and $\{\varepsilon_n\}$ be a positive sequence for which $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Suppose that $\{P_n\}$ is defined as in* (4.2)*, and that Φ is defined as in* (2.3)*. Then*

1. *the number λ is a critical value for Φ if and only if*
$$\sum_{n=0}^{\infty} \varepsilon_n \left|P_n(\lambda)\right|^2 < \infty$$
2. *if $\mathbf{a}$ is a critical point for Φ corresponding to the critical value λ, then for some constant C,*
$$a_n = CP_n(\lambda).$$

Proof. A critical point corresponding to the critical value λ has the form $\{P_n(\lambda)\}$. The sequence $\{P_n(\lambda)\}$ satisfies the recurrence relation (3.1) for every real λ. Thus λ is a critical value if and only if $\{P_n(\lambda)\} \in l^2(\{\varepsilon_n\})$, that is if and only if $\sum_{n=0}^{\infty} \varepsilon_n \left|P_n(\lambda)\right|^2 < \infty$. □

M. Stone's book [8] on Hilbert spaces describes a relationship between systems of orthogonal polynomials and tri-diagonal operators known as Jacobi matrices. The key result of Stone's to be used here is that these Jacobi matrices are cyclic. Specifically, let the orthogonal polynomials $\{p_n\}$ be defined by (4.1), where $\{\alpha_n\}$ is a bounded sequence and $\{\beta_n\}$ is a bounded positive sequence. Let the operator J be defined with respect to an orthonormal basis $\{\mathbf{j}_n\}_{n=0}^{\infty}$ on Hilbert space by

$$J = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & \sqrt{\beta_n} & \alpha_n & \sqrt{\beta_{n+1}} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4.4}$$

Stone shows that the operator J is cyclic, meaning that the set
$$\{p(J)\mathbf{j}_0 : p \text{ is a polynomial}\}$$
is dense in the Hilbert space (see [8], chapter VII). Through the spectral decomposition of the self adjoint operator J, a measure ν is defined as follows: Let $\sigma(J)$ be the spectrum of J, and let E be the spectral family for which
$$J = \int_{\sigma(J)} t dE(t).$$
That is to say, for every $\mathbf{x}$ and $\mathbf{y}$ in the Hilbert space,
$$\langle J\mathbf{x}, \mathbf{y}\rangle = \int_{\sigma(J)} t dE_{\mathbf{x},\mathbf{y}}(t),$$
where $E_{\mathbf{x},\mathbf{y}}$ is the complex measure such that
$$E_{\mathbf{x},\mathbf{y}}(U) = \langle E(U)\mathbf{x}, \mathbf{y}\rangle$$
for every Borel set U in $\sigma(J)$. The existence of this spectral family is guaranteed by the spectral theorem (see [7], theorem 12.23). Define the positive measure $\nu = E_{\mathbf{j}_0,\mathbf{j}_0}$, where $\mathbf{j}_0$ is the first vector from the orthonormal basis. Then
$$\nu(U) = \langle E(U)\mathbf{j}_0, \mathbf{j}_0\rangle \tag{4.5}$$
for every Borel set U. The following proposition shows that the eigenvalues of J are determined completely by the measure ν.

Proposition 4.3. *Let $\{\alpha_n\}$ be a bounded real sequence and $\{\beta_n\}$ be a positive bounded sequence. Let J be defined by* (4.4) *on the Hilbert space with orthonormal basis $\{\mathbf{j}_n\}$. Suppose that the measure ν is defined by* (4.5)*. Then the set of isolated points in the support of ν is the set of eigenvalues of J.*

Proof. Let λ be isolated in the support of ν. Then

$$\nu(\{\lambda\}) = \langle E(\{\lambda\})\mathbf{j}_0, \mathbf{j}_0\rangle \neq 0$$

so that $E(\{\lambda\}) \neq 0$. Thus λ is an eigenvalue of J (see [7], theorem 12.29 b).

Now suppose that λ is an eigenvalue of J which does not belong to supp ν. Then $J\mathbf{x} = \lambda\mathbf{x}$ for some element $\mathbf{x}$ of the Hilbert space, and $E(\{\lambda\})$ is a self adjoint projection onto the eigenspace corresponding to λ. Therefore

$$0 = \int_{\sigma(J)} \chi_{\{\lambda\}}\, d\nu = \langle E(\{\lambda\})\mathbf{j}_0, \mathbf{j}_0\rangle = \|E(\{\lambda\})\mathbf{j}_0\|^2$$

and since $\mathbf{x}$ is already in the eigenspace corresponding to λ,

$$\langle \mathbf{j}_0, \mathbf{x}\rangle = \langle \mathbf{j}_0, E(\{\lambda\})\mathbf{x}\rangle = \langle E(\{\lambda\})\mathbf{j}_0, \mathbf{x}\rangle = 0.$$

Also,

$$\langle \mathbf{j}_0, \mathbf{x}\rangle = \frac{1}{\bar{\lambda}}\langle \mathbf{j}_0, J\mathbf{x}\rangle = \frac{1}{\bar{\lambda}}\langle J\mathbf{j}_0, \mathbf{x}\rangle = \frac{1}{\bar{\lambda}^n}\langle J^n\mathbf{j}_0, \mathbf{x}\rangle$$

so $\langle J^n\mathbf{j}_0, \mathbf{x}\rangle = 0$ for all n . This contradicts the fact that the operator J is cyclic. Thus the eigenvalues of J are isolated points in the support of ν. □

By choosing the β_n appropriately in proposition 4.3, a relationship is obtained between the critical values of Φ and the measure of orthogonality μ, as described in the following theorem.

Theorem 4.4. *Let $\{\alpha_n\}$ be a bounded real sequence and $\{\varepsilon_n\}$ be a positive sequence for which $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Let Φ be defined as in* (2.3)*, and let the system of orthogonal polynomials $\{P_n\}$ defined as in* (4.2) *be orthogonal with respect to the positive measure μ. Then λ is a critical value for Φ if and only if λ is isolated in the support of the measure μ.*

Proof. Let $\beta_n = \frac{1}{4\varepsilon_n\varepsilon_{n-1}}$, $\mathbf{j}_n = \frac{\mathbf{e}_n}{\sqrt{\varepsilon_n}}$, and J be defined on $l^2(\{\varepsilon_n\})$ as in (4.4). Then J is also the operator (3) on the Hilbert space $l^2(\{\varepsilon_n\})$; the eigenvalues of J are the critical values of Φ. Let ν be the measure defined by (4.5). By proposition 4.3, the eigenvalues of J are the isolated points in the support of ν. It remains to be shown that ν is in fact the measure of orthogonality μ for the polynomials $\{P_n\}$. Stone defines the family of polynomials

$$R_{-1}(x) = 0,\ R_0(x) = 1,$$

$$R_{n+1}(x) = \frac{1}{\sqrt{\beta_{n+1}}}(x-\alpha_n)R_n(x) - \frac{\sqrt{\beta_n}}{\sqrt{\beta_{n+1}}}R_{n-1}(x), \quad n \geq 1,$$

and shows that they are orthogonal with respect to the measure ν (see chapter VII of [8]). A proof by induction shows that

$$P_n = \sqrt{\frac{\varepsilon_0}{\varepsilon_n}}R_n.$$

Thus the polynomials $\{P_n\}$ are orthogonal with respect to both μ and ν. The measure with respect to which the polynomials $\{p_n\}$ are orthogonal is bounded if

and only if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded (see [1], p. 109). By theorem 5.6, p. 67 of [1], the measure of orthogonality is unique up to multiplication by a constant. Thus $\mu = c\nu$ for some constant c, and the critical values of Φ are the isolated points in the support of the measure of orthogonality μ. □

The next proposition shows that every system of polynomials orthogonal with respect to a measure of bounded support is connected to a corresponding functional Φ, in the sense of proposition 4.2 and theorem 4.4.

Proposition 4.5. *Suppose that $\{p_n\}$ is any system of monic polynomials which is orthogonal with respect to a measure μ of bounded support, and that $\{p_n\}$ satisfies the recurrence relation* (4.1)*. Let the functional Φ be defined as in* (2.3)*, where $\{\varepsilon_n\}$ is a sequence satisfying $\beta_n = \frac{1}{4\varepsilon_n\varepsilon_{n-1}}$. If $P_n = 2^n\varepsilon_{n-1}\varepsilon_{n-2}...\varepsilon_1\varepsilon_0 p_n$, then the critical points of Φ are the isolated points in the support of the measure μ. Moreover, the critical points corresponding to a critical value λ are the multiples of $\{P_n(\lambda)\}$.*

Proof. The measure μ is of bounded support if and only if $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$. Thus the functional Φ is bounded, and by proposition 4.2, the critical points are of the form described. □

The next theorem summarises the information about ψ which can be obtained from the measure of orthogonality.

Theorem 4.6. *Let $\{\alpha_n\}$ be a bounded sequence and let $\{\varepsilon_n\}$ be a positive sequence. Let ψ be defined as in* (2.1)*. Suppose that $\{\beta_n\}$ is a sequence satisfying $\beta_n = \frac{1}{4\varepsilon_n\varepsilon_{n-1}}$, $n \geq 1$, and that $\{p_n\}$ is a system of monic polynomials satisfying the recurrence relation* (4.1)*. Let μ be the measure with respect to which $\{p_n\}$ is orthogonal.*

1. *If the support of μ has no isolated points then the functional ψ has no local extrema.*
2. *The set of local extrema of ψ is contained in the set of isolated points of the support of μ.*
3. *The functional ψ is bounded if and only if the measure μ is of bounded support.*

Proof. Proposition 2.4 shows that the set of local extrema of ψ on the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ is equal to the set of local extrema of the functional Φ defined by (2.3). By propositions 3.1, 3.4 and theorem 4.4, the set of local extrema is contained in the set of isolated points in the support of μ. Statements (i) and (ii) then follow. For (iii), note that ψ is bounded if and only if $\{\alpha_n\}$ is bounded and $\liminf_{n\to\infty} \varepsilon_{n-1}\varepsilon_n > 0$ (corollary 2.3), that is, if and only if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded. By [1], p. 109, this is true if and only if the support of μ is bounded. □

It was established in [5], and in example 3.6, that for $c = 0$, the functional ψ_c of (1.2) attains no maximum on the unit sphere of the Hardy space. However the recurrence relation determined by the sequences $\varepsilon_n = 1$, $\alpha_0 = c$, and $\alpha_n = 0$, $n \neq 0$, generate a system of orthogonal polynomials which satisfy a recurrence relation which is almost identical to that of the Chebyshev polynomials of the second kind, but with a modified degree one polynomial, namely $P_1(x) = 2(x - c)$. The corresponding measures of orthogonality have been studied by Chihara. The next theorem shows that the functional ψ_c achieves an extremum if $|c| > \frac{1}{2}$, but fails to do so if $|c| \leq \frac{1}{2}$; this non-existence of an extremum when $|c| < \frac{1}{2}$ is a consequence of theorem 4.6.

Theorem 4.7. *Let c be any real number, and let ψ_c be the functional defined as in (1.2). If $|c| \leq \frac{1}{2}$ then ψ_c attains neither a maximum nor a minimum on the unit sphere of H^2. If $|c| > \frac{1}{2}$, then ψ_c attains a single extremum on the unit sphere of H^2. The extreme value of ψ_c is $c + \frac{1}{4c}$, which is assumed at the rational function*

$$F(z) = C\frac{2c}{2c - z}$$

where C is chosen so that $\|F\|_{H^2} = 1$.

Proof. Let $\varepsilon_n = 1$, and $\alpha_n = 0,\ n \geq 1$, and $\alpha_0 = 0$. Then $\mathcal{H}(\{\varepsilon_n\}) = H^2$. The orthogonal system $\{P_n\}$ is

$$P_0(x) = 1, \quad P_1(x) = 2(x - c), \quad P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \quad n \geq 1.$$

When $c = 0$, the polynomials $\{P_n\}$ are in fact the Chebyshev polynomials. Let the function τ be defined by

$$\tau(x) = \begin{cases} \frac{2}{\pi}\int_{-1}^{x} \frac{\sqrt{1-t^2}}{4c^2+1-4ct}dt, & x \in [0,1], \\ 0, & x < 0 \text{ or } x > 1, \end{cases}$$

and let μ_a be the Stieltjes measure on $[0,1]$ determined there by the non-decreasing function τ of bounded variation. The measure of orthogonality μ is

$$\mu = \begin{cases} \mu_a + \left(1 + \frac{1}{4c^2}\right)\delta_{c+\frac{1}{4a}}, & |c| > \frac{1}{2}, \\ \mu_a, & |c| \leq \frac{1}{2} \end{cases}$$

(see [1], p. 205). This measure μ has a single isolated point in its support, namely $c + \frac{1}{4c}$, if and only if $|c| > \frac{1}{2}$. By theorem 4.6 (1), ψ_c does not attain an extremum if $|c| \leq \frac{1}{2}$. Suppose that $c > \frac{1}{2}$. Then $c + \frac{1}{4c}$ is a critical value of Φ which is attained at say $\mathbf{a} \in l^2(\{\varepsilon_n\})$. Thus $\psi(\sum_{n=0}^{\infty} a_n z^n) = c + \frac{1}{4c}$. The following argument based on the Cauchy-Schwarz inequality shows that $c + \frac{1}{4c}$ is also an upper bound for ψ_c.

Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$, and $\|g\|_{H^2} = 1$. Then $|b_0| \leq 1$ and

$$\begin{aligned} |\psi(g)| &= \left|\frac{1}{\pi}Re\int_{\Delta}\overline{g(z)}g'(z)dA(z) + c|g(0)|^2\right| \\ &= \left|Re\sum_{n=0}^{\infty}\overline{b_n}b_{n+1} + cb_0^2\right| \\ &\leq \sqrt{\sum_{n=0}^{\infty}|b_n|^2}\sqrt{\sum_{n=1}^{\infty}|b_n|^2} + |c||b_0|^2 \\ &= \sqrt{\sum_{n=1}^{\infty}|b_n|^2} + |c||b_0|^2 \\ &= \sqrt{1 - |b_0|^2} + |c||b_0|^2. \end{aligned}$$

The function

$$h(x) = \sqrt{1 - x} + cx$$

is bounded on the interval $[-1, 1]$ by $c + \frac{1}{4c}$. Therefore, $c + \frac{1}{4c}$ is an upper bound for ψ_c on the unit sphere of H^2. A similar argument shows that $c + \frac{1}{4c}$ is a minimum of ψ_c if $c < \frac{1}{2}$.

To determine the function at which this extremum is achieved, let $\lambda = c + \frac{1}{4c}$. From the recurrence relation,

$$b_1 = 2\lambda b_0 - 2cb_0 \quad \text{and} \quad b_{n+1}z^n = 2\lambda b_n z^n - b_{n-1}z^n.$$

Summing over n, the stated form for F is obtained. □

5. Properties of Extremals

Theorem 4.7 provides an example of an extremal which is not entire. The next proposition gives a sufficient condition for an extremal to be entire.

Theorem 5.1. *Let $\{\varepsilon_n\}$ be a sequence which diverges to infinity, and let $\{\alpha_n\}$ converge to α. Let ψ be defined by* (2.1)*, and Φ by* (4.2)*.*

1. *If $\mathbf{a}$ is a critical point for Φ, then there is a number N such that*

$$|a_N| > |a_{N+1}| > |a_{N+2}| > |a_{N+3}| > \cdots.$$

   Moreover, if a maximum is attained at $\mathbf{a}$ and a_0 is positive, then the coefficients a_n are all positive, and

$$a_N > a_{N+1} > a_{N+2} > a_{N+3} > \cdots.$$

2. *The extremals of ψ over the unit sphere of $\mathcal{H}(\{\varepsilon_n\})$ are entire functions.*

Proof. Without loss of generality, suppose that the sequence $\{\alpha_n\}$ converges to 0. This assumption will affect the critical values of Φ, but not the critical points. Suppose that $\mathbf{a}$ is a critical point for Φ. Such a point exists by theorem 3.8. Since ε_n becomes arbitrarily large, and $\alpha_n \to 0$, there is a number N such that $\varepsilon_n(\lambda - \alpha_n) > 1$ if $n > N$. Choose $n > N$ such that $|a_{n+1}| < |a_n|$. This is possible for arbitrarily large N since $\sum_{n=0}^{\infty} \varepsilon_n |a_n|^2 = 1$.

$$\begin{aligned}
4|a_n|^2 &< 4\varepsilon_n^2(\lambda - \alpha_n)^2|a_n|^2 \text{ since } 1 < \varepsilon_n(\lambda - \alpha_n) \\
&= |a_{n-1} + a_{n+1}|^2 \quad \text{(this is the square of the recurrence relation (3.1))} \\
&\le |a_{n-1}|^2 + 2|a_{n-1}||a_{n+1}| + |a_{n+1}|^2 \\
&\le 2(|a_{n-1}|^2 + |a_{n+1}|^2) \\
&< 2(|a_{n-1}|^2 + |a_n|^2) \text{ since } |a_{n+1}| < |a_n|,
\end{aligned}$$

so $2|a_n|^2 < 2|a_{n-1}|^2$ and $\{|a_{N+n}|\}_{n=0}^{\infty}$ is increasing.

Since $|a_{N+n}| < |a_{N+n-1}|$ in the inequality

$$4\varepsilon_{N+n}^2(\lambda - \alpha_{N+n})^2|a_{N+n}|^2 < 2(|a_{N+n-1}|^2 + |a_{N+n}|^2),$$

it follows that

$$\frac{|a_{N+n}|^2}{|a_{N+n-1}|^2} < \frac{1}{\varepsilon_{N+n}^2(\lambda - \alpha_{N+n})^2}$$

holds. The ratio test now shows that F is entire.

Finally, suppose that Φ assumes a maximum at $\mathbf{a} = \{a_n\}$, and that a_0 is positive. Then we can assume that the coefficients are all positive, because $\Phi(\{a_n\}) \le \Phi(\{|a_n|\})$. □

This paper concludes with two theorems about the functional ψ_0 over various spaces equivalent in norm to the Dirichlet space, and shows that the reciprocal of any zero of any Bessel function of positive index appears as a local extremal of ψ_0. The recurrence relation for ψ_0 over the unit sphere of the Dirichlet space is satisfied

by both the Lommel polynomials, and the Bessel functions. The modified Lommel polynomials are defined for a fixed complex number $\nu \neq 0, -1, -2, ...$ by

(5.1)
$$h_{-1,\nu}(x) = 0,\ h_{0,\nu}(x) = 1,\ h_{n+1,\nu}(x) = 2x\,(n+\nu)\,h_{n,\nu}(x) - h_{n-1,\nu}(x),\ n \geq 0,$$

and are orthogonal if and only if $\nu > 0$ (see [4]).

Let J_ν be the Bessel function of index ν and let $\{j_{n,\nu}\}$ be the set of zeros of J_ν ordered so that $j_{0,\nu}$ is the least positive zero, that is,

$$\cdots < j_{-2,\nu} < j_{-1,\nu} < 0 < j_{0,\nu} < j_{1,\nu} \cdots$$

As shown in [5], ψ_0 achieves a maximum of $\frac{1}{j_{0,0}}$ at the function

$$F(z) = C \sum_{n=1}^{\infty} J_n(j_{0,0}) z^n.$$

For a fixed number $\nu > 0$, let $\mathcal{D}_\nu$ be the space $\mathcal{H}(\{n+\nu\})$. Note that all the spaces $\mathcal{D}_\nu$ for $\nu > 0$ are equivalent in norm.

Theorem 5.2. *Let $\nu > 0$, and $\{h_{n,\nu}\}$ be the modified Lommel polynomials of index ν. Then the functional*

$$\psi_0(f) = Re \frac{1}{\pi} \int_\Delta \overline{f(z)} f'(z) dA(z)$$

attains its maximum $\frac{1}{j_{0,\nu-1}}$ on the unit sphere of $\mathcal{D}_\nu$. The maximum is attained at $F(z)$ where

$$F(z) = C \sum_{n=0}^{\infty} h_{n,\nu}(\frac{1}{j_{0,\nu-1}}) z^n = C \sum_{n=0}^{\infty} J_{n+\nu}(j_{0,\nu-1}) z^n. \tag{5.2}$$

The extremal F is unique up to rotation, and is also entire. Further, the reciprocal of every zero of $J_{\nu-1}$, namely $\frac{1}{j_{i,\nu-1}}$ where $i \in \mathbb{Z}$, is an extremum for ψ_0 over the unit sphere of a subspace of $\mathcal{D}_\nu$; this value is attained at the entire function

$$G(z) = C \sum_{n=0}^{\infty} h_{n,\nu}(\frac{1}{j_{i,\nu-1}}) z^n = C \sum_{n=0}^{\infty} J_{n+\nu}(j_{i,\nu-1}) z^n.$$

Proof. Let $\alpha_n = 0$ and $\varepsilon_n = n + \nu$. The functional Φ defined by (2.3) has the same extreme values as ψ_0 (proposition 2.4). Let λ be a critical value for Φ. By proposition 4.2, the corresponding critical point is of the form $C\{P_n(\lambda)\}$ where

$$P_{-1}(x) = 0,\ P_0(x) = 1,\ P_{n+1}(x) = 2x\,(n+\nu)\,P_n(x) - P_{n-1}(x),\ n \geq 0.$$

These polynomials are recognized from (5.1) as the modified Lommel polynomials of index ν. Thus a critical value λ is assumed by Φ at the corresponding critical point $\{h_{n,\nu}(\lambda)\}$. D. Dickinson has shown in [2] that the measure of orthogonality μ_ν for the Lommel polynomials $\{h_{n,\nu}\}$ is given by

$$\mu_\nu = \sum_{n \in \mathbb{Z}} \left(\frac{1}{j_{n,\nu-1}}\right)^2 \delta_{\frac{1}{j_{n,\nu-1}}}.$$

The isolated points in the support of μ_ν are precisely the reciprocals of zeros of the Bessel function $J_{\nu-1}$, which in turn are the critical values of Φ (theorem 4.4); the greatest of the critical values is $\frac{1}{j_{0,\nu-1}}$. Therefore the maximum value for Φ is

$\frac{1}{j_{0,\nu-1}}$, and the corresponding critical point is $C\left\{h_{n,\nu}(\frac{1}{j_{0,\nu-1}})\right\}$. By proposition 2.4, the maximum value for ψ_0 over the unit sphere $\mathcal{D}_\nu$ is also $\frac{1}{j_{0,\nu-1}}$, and is attained at

$$F(z) = C\sum_{n=0}^{\infty} h_{n,\nu}(\frac{1}{j_{0,\nu-1}})z^n.$$

By proposition 3.8, the critical values of Φ all appear as maxima or minima over subspaces of $l^2(\{n+\nu\})$. Thus the reciprocals of all other zeros of $J_{\nu-1}$ appear as extrema for ψ_0 over the unit sphere of the corresponding subspaces of $\mathcal{D}_\nu$. For example, if $\psi_0(F) = \frac{1}{j_{0,\nu-1}}$ then

$$\frac{1}{j_{1,\nu-1}} = \max\left\{\psi_0(g) : g \in \mathcal{D}_\nu, \|g\|_\nu = 1, \langle F, g\rangle = 0\right\}.$$

The extremal over the orthogonal subspace $\{g \in \mathcal{D}_\nu, \|g\|_\nu = 1, \langle F, g\rangle = 0\}$ is then

$$G(z) = C\sum_{n=0}^{\infty} h_{n,\nu}(\frac{1}{j_{1,\nu-1}})z^n.$$

The Lommel polynomials of index ν are related to the Bessel functions of the first kind through their common recurrence relation. For any (complex) w, the Bessel functions satisfy

$$J_{w+1}(z) = \frac{2w}{z}J_w(z) - J_{w-1}(z). \tag{5.3}$$

(see [9]). Let $w = n+\nu$, and $z = j_{i,\nu-1}$ for some fixed $i \in \mathbb{Z}$. Then the sequence $\{J_{n+\nu}(j_{i,\nu-1})\}_{n=-1}^{\infty}$ satisfies

$$J_{n+\nu+1}(j_{i,\nu-1}) = 2\left(\frac{1}{j_{i,\nu-1}}\right)(n+\nu)J_{n+\nu}(j_{i,\nu-1}) - J_{n+\nu-1}(j_{i,\nu-1}),$$

where $J_{\nu-1}(j_{i,\nu-1}) = 0$. By (5.1), the sequence $\left\{h_{n,\nu}(\frac{1}{j_{i,\nu-1}})\right\}_{n=-1}^{\infty}$ also satisfies

$$h_{n+1,\nu}(\frac{1}{j_{i,\nu-1}}) = 2\left(\frac{1}{j_{i,\nu-1}}\right)(n+\nu)\,h_{n,\nu}(\frac{1}{j_{i,\nu-1}}) - h_{n-1,\nu}(\frac{1}{j_{i,\nu-1}}), \quad n \geq 0,$$

where $h_{-1,\nu}(x) = 0$. Thus the two sequences

$$\left\{h_{n,\nu}(\frac{1}{j_{i,\nu-1}})\right\}_{n=-1}^{\infty} \quad \text{and} \quad \{J_{n+\nu}(j_{i,\nu-1})\}_{n=-1}^{\infty}$$

are constant multiples of one another, so that

$$C\sum_{n=0}^{\infty} h_{n,\nu}(\frac{1}{j_{i,\nu-1}})z^n = C'\sum_{n=0}^{\infty} J_{n+\nu}(j_{i,\nu-1})z^n.$$

The sequence $\varepsilon_n = n+\nu$ diverges to ∞, so by theorem 5.1, the extremal functions F and G are entire. □

When the previous example is considered with $\nu = 1$, results from [5] about the maximum value of the functional ψ_0 are reproduced:

Theorem 5.3. *The maximum value of the functional*

$$\psi_0(f) = Re\frac{1}{\pi}\int_\Delta \overline{f(z)}f'(z)dA(z)$$

over the unit sphere of the Dirichlet space is $\frac{1}{j_{0,0}}$. *This value is attained at the entire function*

$$F(z) = C \sum_{n=1}^{\infty} J_n(j_{0,0}) z^n. \tag{5.4}$$

Moreover, the reciprocal of every zero of J_0, *namely* $\frac{1}{j_{i,0}}$, *where* $i \in \mathbb{Z}$, *appears as an extremum for* ψ_0 *over the unit sphere of a subspace of the Dirichlet space; this value* $\frac{1}{j_{i,0}}$ *is attained at the entire function*

$$G(z) = C \sum_{n=1}^{\infty} h_{n,1}(\frac{1}{j_{i,0}}) z^n = C \sum_{n=1}^{\infty} J_n(j_{i,0}) z^n.$$

Proof. Let Φ be determined by $\{\alpha_n\}$ and $\{\varepsilon_n\}$ where $\alpha_n = 0$ and $\varepsilon_n = n+1$. Then by theorem 5.2, $\sup\left\{\psi_0(f) : \|f\|_{D_1} = 1,\ f \in \mathcal{D}_1\right\} = \frac{1}{j_{0,0}}$, and this maximum is attained at $C\sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{D}_1$, where $b_n = J_{n+1}(\frac{1}{j_{0,0}})$. The argument in example 2.5 then shows that $\sup\{\psi_0(f) : \|f\|_D = 1,\ f \in \mathcal{D}\} = \frac{1}{j_{0,0}}$, and that this maximum is attained at $F(z) = zC\sum_{n=0}^{\infty} b_n z^n = C\sum_{n=1}^{\infty} b_{n-1} z^n = C\sum_{n=1}^{\infty} J_n(\frac{1}{j_{0,0}}) z^n$ in $\mathcal{D}$. A similar argument proves the statement about G. □

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