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RANDOM POINTS ON THE BOUNDARY OF SMOOTH CONVEX BODIES

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Abstract. The convex hull of n independent random points chosen on the boundary of a convex body $K \subset \mathbb{R}^d$ according to a given density function is a random polytope. The expectation of its i–th intrinsic volume for $i = 1, \ldots, d$ is investigated. In the case that the boundary of K is sufficiently smooth, asymptotic expansions for these expected intrinsic volumes as $n \to \infty$ are derived.

1. Introduction and statement of results

Approximation of d–dimensional convex bodies by random polytopes has been considered frequently in stochastic geometry. Most of the investigations concentrate on random polytopes with vertices chosen in the interior of a given convex body. However, the approximation is improved if the vertices of the random polytope lie on the boundary of the convex body. Thus in this paper we investigate random polytopes with vertices chosen according to a density function concentrated on the boundary of the given convex body. To determine how well the random polytope approximates the convex body, the expected value of the difference of the i–th intrinsic volume V_i of the convex body and the random polytope is used, $i = 1, \ldots, d$. Here, for instance, V_d is the volume, $2V_{d-1}$ the surface area, and V_1 is a multiple of the mean width. Since – except for special cases (cf. [6], [12]) – explicit formulae seem to be out of reach, we are interested in asymptotic results as the number of vertices of the random polytope tends to infinity.

Let $\mathcal{K}^k_+$ denote the set of convex bodies, i.e., compact convex sets, in $\mathbb{R}^d$, with boundary of differentiability class $\mathcal{C}^k$ and everywhere positive Gaussian curvature. Fix $K \in \mathcal{K}^k_+$, and choose points $P_1, \ldots, P_n$ randomly, independently, and according to some density function d_K on the boundary of K. We call the convex hull $[P_1, \ldots, P_n]$ a random polytope.

For any convex body K the expected value $\mathbb{E}_n(V_i)$ of the i–th intrinsic volume of $[P_1, \ldots, P_n]$ tends to $V_i(K)$ as n tends to infinity, and the shape of the boundary of K determines the asymptotic behaviour of $V_i(K) - \mathbb{E}_n(V_i)$. If $K \in \mathcal{K}^2$, the boundary of K is determined by the principal curvatures $k_1(x), \ldots, k_{d-1}(x)$ of K at x. To state our results we need the following notions: denote by $H_j(x)$ the j-th normalized elementary symmetric function of the principal curvatures of K at x;

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thus $H_0(x) = 1$ and

$$H_j(x) = \binom{d-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le d-1} k_{i_1}(x) \cdots k_{i_j}(x) \,. \tag{1}$$

In particular, $H_{d-1}(x)$ is the Gaussian curvature and $H_1(x)$ the mean curvature of K at x. In Theorem 1 we prove that for $K \in \mathcal{K}^2_+$ the difference $V_i(K) - \mathbb{E}_n(V_i)$ is of order $n^{-2/(d-1)}$ for $i = 1, \dots, d$, thus solving a problem of Schneider (Problem 9.1.4 in [23]).

Theorem 1. *Let $K \in \mathcal{K}^2_+$. Denote by $\mathbb{E}_n(V_i)$, $i = 1, \dots, d$, the expected i–th intrinsic volume of the convex hull of n random points on ∂K chosen independently and according to a continuous, positive density $d_K(x)$. Then*

$$V_i(K) - \mathbb{E}_n(V_i) = c_2^{(i,d)} \int\limits_{\partial K} d_K(x)^{-\frac{2}{d-1}} \, H_{d-1}(x)^{\frac{1}{d-1}} \, H_{d-i}(x) \, dx \;\; n^{-\frac{2}{d-1}} + o\left(n^{-\frac{2}{d-1}}\right) \tag{2}$$

as $n \to \infty$. The constant $c_2^{(i,d)}$ only depends on i and on the dimension d.

If K is sufficiently smooth even more can be proved: we extend (2) to an asymptotic expansion for $V_i(K) - \mathbb{E}_n(V_i)$ as $n \to \infty$.

Theorem 2. *Let $K \in \mathcal{K}^{k+1}_+$, $k \ge 2$. Choose n random points on ∂K independently and according to a positive density $d_K \in \mathcal{C}^{k-1}$ with uniformly bounded derivatives up to order $k-1$. Then*

$$\begin{aligned} V_i(K) - \mathbb{E}_n(V_i) = {} & c_2^{(i,d)}(K)\, n^{-\frac{2}{d-1}} + c_3^{(i,d)}(K)\, n^{-\frac{3}{d-1}} \\ & + \cdots + c_{k-2}^{(i,d)}(K)\, n^{-\frac{k-2}{d-1}} + O\left(n^{-\frac{k-1}{d-1}}\right) \end{aligned}$$

as $n \to \infty$. If d is even, then $c_{2m-1}^{(i,d)}(K) = 0$ for $m \le \frac{d}{2}$, and if d is odd, then $c_{2m-1}^{(i,d)}(K) = 0$ for all m.

In principle the coefficients $c_m^{(i,d)}(K)$ in this expansion can be given explicitly:

$$c_m^{(i,d)}(K) = \int_{\partial K} f_m^{(i,d)}(x) \, dx$$

with suitable functions $f_m^{(i,d)}(x)$. For each $x \in \partial K$ these functions $f_m^{(i,d)}(x)$ depend on expectations of weighted volumes of random simplices in the indicatrix of K at x. The weight functions are determined by the radial function of K. But, for $m \ge 3$, there is no simple representation for the coefficients $c_m^{(i,d)}(K)$ similar to the representation of the coefficient $c_2^{(i,d)}(K)$ given in Theorem 1.

For convex bodies of class $\mathcal{K}^\infty_+$ Theorem 2 yields asymptotic series expansions for $\mathbb{E}_n(V_i)$ as $n \to \infty$.

For $i = 1$ and $i = d$ asymptotic formulae for $V_i(K) - \mathbb{E}_n(V_i)$ as in Theorem 1 are already known: for $K \in \mathcal{K}^3_+$ Buchta, Müller and Tichy [6] (for points chosen uniformly from ∂K) and Müller [13] (for arbitrary densities $d_K(x)$) determined the asymptotic behaviour of $V_1(K) - \mathbb{E}_n(V_1)$:

$$V_1(K) - \mathbb{E}_n(V_1) = c_2^{(1,d)} \int\limits_{\partial K} d_K(x)^{-\frac{2}{d-1}} \, H_{d-1}(x)^{\frac{d}{d-1}} \, dx \; n^{-\frac{2}{d-1}} + o\left(n^{-\frac{2}{d-1}}\right) \tag{3}$$

as $n \to \infty$, where $c_2^{(1,d)}$ is a given constant depending on d. This result was extended to convex bodies $K \in \mathcal{K}_+^2$ by Gruber [8], who in the same paper also proved the case $i = 1$ of Theorem 2. Müller remarked that by Hölder's inequality the integral occurring in (3) attains its minimum for the density

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{\frac{d}{d+1}}}{\int\limits_{\partial K} H_{d-1}(x)^{\frac{d}{d+1}}\,dx}.$$

In the planar case Schneider [21] showed that, with probability 1,

$$\lim_{n\to\infty} n^2 \left(V_1(K) - V_1([P_1,\ldots,P_n])\right) = \frac{1}{4}\int\limits_{\partial K} d_K(x)^{-2} H_1(x)^2\,dx\,,$$

$$\lim_{n\to\infty} n^2 \left(V_2(K) - V_2([P_1,\ldots,P_n])\right) = \frac{1}{2}\int\limits_{\partial K} d_K(x)^{-2} H_1(x)\,dx$$

for $K \in \mathcal{K}_+^3$. Using Hölder's inequality, Schneider determined those densities $d_K^{\min}(x)$ which minimize the integrals. In the special case that K is the d–dimensional unit ball and $d_K = 1$ the asymptotic behaviour of $V_i(B^d) - \mathbb{E}_n(V_i)$ is already known for all $i = 1,\ldots,d$: Müller [14] for $i = d-1, d$, and Affentranger [1] for general i.

Recently Schütt and Werner [25] proved that

$$V_d(K) - \mathbb{E}_n(V_d) = c_2^{(d,d)} \int\limits_{\partial K} d_K^{-\frac{2}{d-1}}\, H_{d-1}(x)^{\frac{1}{d-1}}\,dx\; n^{-\frac{2}{d-1}} + o\left(n^{-\frac{2}{d-1}}\right) \tag{4}$$

as $n \to \infty$ for arbitrary convex bodies $K \in \mathcal{K}$ only fulfilling some weak regularity conditions on the boundary of K. Here $c_2^{(d,d)}$ is a given constant depending only on the dimension d, and $H_{d-1}(x)$ denotes the generalized Gaussian curvature of K at x. Using Hölder's inequality, they deduce that the integral attains its minimum for

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{\frac{1}{d+1}}}{\int\limits_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}}\,dx}.$$

We determine those densities which minimize $V_i(K) - \mathbb{E}_n(V_i)$ for $i \in \{1,\ldots,d\}$ as n tends to infinity. Choose n random points on the boundary of K independently and according to the density function

$$d_K^{\min}(x) = \frac{H_{d-1}(x)^{\frac{1}{d+1}} H_{d-i}(x)^{\frac{d-1}{d+1}}}{\int\limits_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}} H_{d-i}(x)^{\frac{d-1}{d+1}}\,dx}.$$

Then by Theorem 1

$$V_i(K) - \mathbb{E}_n(V_i) = c_2^{(i,d)} \left(\int\limits_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}}\, H_{d-i}(x)^{\frac{d-1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}} n^{-\frac{2}{d-1}} + o\left(n^{-\frac{2}{d-1}}\right)$$

as $n \to \infty$. On the other hand, for an arbitrary density function $d_K(x)$ with $\int_{\partial K} d_K(x)\,dx = 1$, it follows from Hölder's inequality that

$$\left(\int\limits_{\partial K} d_K(x)^{-\frac{2}{d-1}} \, H_{d-1}(x)^{\frac{1}{d-1}} \, H_{d-i}(x) \, dx \right)^{\frac{d-1}{d+1}} \geq \int\limits_{\partial K} H_{d-1}(x)^{\frac{1}{d+1}} \, H_{d-i}(x)^{\frac{d-1}{d+1}}(x) \, dx \,.$$

This implies that the right hand side of (2) is minimized if the density function $d_K(x)$ is equal to $d_K^{\min}(x)$.

Surveys on random polytopes and related questions are due to Affentranger [1], Buchta [5], Schneider [21], [23], and Weil and Wieacker [26] (section "Random points in a convex body"). In a recent survey article by Gruber [9] approximation of convex bodies by random polytopes is compared to approximation by 'best approximating' polytopes. For results corresponding to Theorems 1 and 2 for random points chosen in the interior of a convex body, see Bárány [3], Bárány and Buchta [4], Schütt [24], and also [15].

2. Formulae of Blaschke-Petkantschin type

The use of the Blaschke-Petkantschin formula (cf., e.g., Santaló [19], II.12.3)

$$\begin{aligned} &\int\limits_K \cdots \int\limits_K \cdots \; dP_1 \cdots dP_d \\ (5) \qquad &= \int\limits_{H \in \mathcal{H}(d,d-1)} \int\limits_{K \cap H} \cdots \int\limits_{K \cap H} \cdots (d-1)! \, V_{d-1}([P_1, \dots, P_d]) \, dP_1 \cdots dP_d \, dH \end{aligned}$$

is standard in papers dealing with the convex hull of random points in a given convex body K. It relates the d–dimensional volume elements dP_j of the points $P_j \in K$ to the $(d-1)$–dimensional volume elements dP_j of points $P_j \in K \cap H$, where H is a random hyperplane in $\mathbb{R}^d$. (Throughout this paper we denote by dP the j–dimensional volume element corresponding to the j–dimensional Hausdorff measure on a given space. The space itself, and thus its dimension, and thus the precise meaning of dP, is determined by the range of integration.) The differential dH corresponds to the suitably normalized rigid motion invariant Haar measure on the Grassmannian $\mathcal{H}(d, d-1)$ of $(d-1)$–dimensional planes. The hyperplane can be parameterized by its unit normal vector $u \in S^{d-1}$ and its distance h to the origin. Denoting by du the element of surface area on S^{d-1}, we have $dH = du\,dh$.

In this paper we need an analogous formula for points distributed on the boundary of a given convex body. A theorem which involves such a formula as a special case has been proved by Zähle [28]. Using our notation, it reads as follows:

$$\int\limits_{\partial K} \cdots \int\limits_{\partial K} \cdots dP_1 \cdots dP_d =$$

$$\int\limits_{H \in \mathcal{H}(d,d-1)} \int\limits_{\partial K \cap H} \cdots \int\limits_{\partial K \cap H} \cdots (d-1)! \, V_{d-1}([P_1, \dots, P_d]) \prod_{j=1}^{d} l_H(P_j) \, dP_1 \cdots dP_d \, dH$$

with suitable functions $l_H(P_j)$. Such a formula cannot hold if the boundary of K contains a flat part, because with positive probability the affine hull of the random points $P_1, \dots, P_d$ would coincide with the affine hull of the flat part on the boundary of K. Thus the formula proved by Zähle contains an additional factor vanishing in this case. We avoid such a situation by assuming that $K \in \mathcal{K}_+^2$.

Lemma 1 (Zähle [28])**.** *Let $K \in \mathcal{K}_+^2$ and let $g(P_1, \dots, P_d)$ be a continuous function. Denote by $H(h,u)$ the hyperplane with unit normal vector $u \in S^{d-1}$ and distance h to the origin. Then*

$$\int\limits_{\partial K} \cdots \int\limits_{\partial K} g(P_1, \dots, P_d)\, dP_1 \cdots dP_d$$

$$= \int\limits_{u \in S^{d-1}} \int\limits_{h=0}^{\infty} \int\limits_{\partial K \cap H(h,u)} \cdots \int\limits_{\partial K \cap H(h,u)} g(P_1, \dots, P_d)\,(d-1)!\, V_{d-1}([P_1, \dots, P_d]) \times \prod_{j=1}^{d} l_{H(h,u)}(P_j)\, dP_1 \cdots dP_d\, dh\, du$$

with

$$l_{H(h,u)}(P_j) = \|\mathrm{proj}_{H(h,u)} n_K(P_j)\|^{-1},$$

where $n_K(P)$ denotes the outer unit normal vector of K at P and proj_H *the orthogonal projection onto the hyperplane H.*

Remark. A direct and elementary proof of Lemma 1 can be given along the following lines. Define

$$\mathcal{I}(g) = \int\limits_{\partial K} \cdots \int\limits_{\partial K} g(P_1, \dots, P_d) \prod_{j=1}^{d} \|\mathrm{proj}_H n_K(P_j)\|\, dP_1 \cdots dP_d$$

where $H = H(P_1, \dots, P_d)$ denotes the hyperplane which is the affine hull of $P_1, \dots, P_d$ and dP denotes the $(d-1)$–dimensional element of surface area. For a point $P \notin K$ we denote by $\bar{P}$ the point in K nearest to P, i.e., the unique point with $|P - \bar{P}| = \min_{Q \in K} |P - Q|$. Steiner's formula implies that

$$\mathcal{I}(g) = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon}\right)^d \int\limits_{(K+\varepsilon B)\setminus K} \cdots \int\limits_{(K+\varepsilon B)\setminus K} g(\bar{P}_1, \dots, \bar{P}_d) \times \prod_{j=1}^{d} \|\mathrm{proj}_H n_K(\bar{P}_j)\|\, dP_1 \cdots dP_d,$$

where – as indicated by the range of integration – dP now denotes the d–dimensional volume element. We apply the Blaschke-Petkantschin formula (5) and obtain

$$\mathcal{I}(g) = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon}\right)^d \int\limits_{u \in S^{d-1}} \int\limits_{h=0}^{\infty} \int\limits_{((K+\varepsilon B)\setminus K)\cap H(h,u)} \cdots \int\limits_{((K+\varepsilon B)\setminus K)\cap H(h,u)} g(\bar{P}_1, \dots, \bar{P}_d) \times \prod_{j=1}^{d} \|\mathrm{proj}_{H(h,u)} n_K(\bar{P}_j)\|\ (d-1)!\, V_{d-1}([P_1, \dots, P_d])\, dP_1 \cdots dP_d\, dh\, du,$$

where dP denotes the $(d-1)$–dimensional volume element in $H(h,u)$.

Each point $P_j \in ((K+\varepsilon B)\backslash K)\cap H(h,u)$ is determined by the point in $K\cap H(h,u)$ nearest to P_j and its distance $t(P_j)$ to $\partial K \cap H(h,u)$. We denote by $\lambda_{H(h,u)}(P_j)$ the length of the intersection of $(K+\varepsilon B)\backslash K$ with the line in $H(h,u)$ containing P_j and orthogonal to $\partial K \cap H(h,u)$. Then $0 \le t = t(P_j) \le \lambda_{H(h,u)}(P_j)$, and using a generalization of Steiner's formula for $(K+\varepsilon B)\cap H(h,u)$ (cf. Sangwine–Yager [18]), the continuity of the integrand implies

$$\mathcal{I}(g) = \lim_{\varepsilon\to 0} \int\limits_{u\in S^{d-1}} \int\limits_{h=0}^{\infty} \int\limits_{\partial K\cap H(h,u)} \cdots \int\limits_{\partial K\cap H(h,u)} g(P_1,\dots,P_d)\,(d-1)!\,V_{d-1}([P_1,\dots,P_d]) \times \prod_{j=1}^{d}\left(\varepsilon^{-1}\,\|\mathrm{proj}_{H(h,u)} n_K(P_j)\| \int\limits_0^{\lambda_{H(h,u)}(P_j)} dt\right) dP_1\cdots dP_d\,dh\,du\,.$$

Since

$$\lambda_{H(h,u)}(P_j) \le \varepsilon \|\mathrm{proj}_{H(h,u)} n_K(P_j)\|^{-1}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\lambda_{H(h,u)}(P_j) = \|\mathrm{proj}_{H(h,u)} n_K(P_j)\|^{-1}$$

this proves

$$\mathcal{I}(g) = \int\limits_{u\in S^{d-1}} \int\limits_{h=0}^{\infty} \int\limits_{\partial K\cap H(h,u)} \cdots \int\limits_{\partial K\cap H(h,u)} g(P_1,\dots,P_d)\,(d-1)! \times V_{d-1}([P_1,\dots,P_d])\,dP_1\cdots dP_d\,dh\,du\,.$$

□

In the proof of Theorem 1 and Theorem 2 for the case $i \le d-1$ we need an analogous formula for integrals of the type

$$\int\limits_{\partial K} \cdots \int\limits_{\partial K} g(\mathrm{proj}_G P_1,\dots,\mathrm{proj}_G P_i)\,dP_1\cdots dP_i,$$

where G is an arbitrary i–dimensional linear subspace of $\mathbb{R}^d$. Such a result can be deduced using the original Blaschke-Petkantschin formula. Again we assume the convex body to be of differentiability class $\mathcal{K}^2_+$, and thus at every boundary point of K the outer unit normal vector is uniquely determined.

We calculate each integral

$$\int\limits_{\partial K} \cdots\ dP_k$$

by first integrating over all points P_k with common point P_k^G of projection onto the linear subspace G and then integrating over $P_k^G \in \mathrm{proj}_G K$. This gives

$$\int\limits_{\mathrm{proj}_G K} \int\limits_{\substack{P_k\in\partial K\\ \mathrm{proj}_G P_k = P_k^G}} \cdots \frac{1}{\|\mathrm{proj}_{G_\perp} n_K(P_k)\|}\,dP_k\,dP_k^G,$$

where $G_\perp$ is the orthogonal complement of G. We obtain

$$\begin{aligned}&\int_{\partial K}\cdots\int_{\partial K} g(\mathrm{proj}_G P_1,\dots,\mathrm{proj}_G P_i)\,dP_1\cdots dP_i\\&=\int_{\mathrm{proj}_G K}\cdots\int_{\mathrm{proj}_G K}\ \int_{\substack{P_i\in\partial K\\ \mathrm{proj}_G P_i=P_i^G}}\cdots\int_{\substack{P_1\in\partial K\\ \mathrm{proj}_G P_1=P_1^G}} g(P_1^G,\dots,P_i^G)\\&\quad\times\prod_{j=1}^{i}\frac{1}{\|\mathrm{proj}_{G_\perp} n_K(P_j)\|}\,dP_1\cdots dP_i\,dP_1^G\cdots dP_i^G\,.\end{aligned}$$

The affine hull of the points $P_1^G,\dots,P_i^G$ is an $(i-1)$–dimensional plane H^G in G. Thus we use the Blaschke-Petkantschin formula in G, which yields

$$\begin{aligned}&\int_{\mathrm{proj}_G K}\cdots\int_{\mathrm{proj}_G K}\cdots\, dP_1^G\cdots dP_i^G\\&=(i-1)!\int_{S^{d-1}\cap G}\int_0^\infty\int_{\mathrm{proj}_G K\cap H^G(h,u)}\cdots\int_{\mathrm{proj}_G K\cap H^G(h,u)}\\&\qquad\cdots\ V_{i-1}([P_1^G,\dots,P_i^G])dP_1^G\cdots dP_i^G\,dh\,du,\end{aligned}\tag{6}$$

where $H^G(h,u)$ is the $(i-1)$–dimensional affine subspace of G with unit normal vector u and distance h to the origin, and the points $P_1,\dots,P_i$ are chosen in $H^G(h,u)$. Hence we have the following integral:

$$\begin{aligned}&(i-1)!\int_{S^{d-1}\cap G}\int_0^\infty\int_{\mathrm{proj}_G K\cap H^G(h,u)}\cdots\int_{\mathrm{proj}_G K\cap H^G(h,u)}\\&\times\int_{\substack{P_i\in\partial K\\ \mathrm{proj}_G P_i=P_i^G}}\cdots\int_{\substack{P_1\in\partial K\\ \mathrm{proj}_G P_1=P_1^G}} g(P_1^G,\dots,P_i^G)V_{i-1}([P_1^G,\dots,P_i^G])\\&\times\prod_{j=1}^{i}\frac{1}{\|\mathrm{proj}_{G_\perp} n_K(P_j)\|}\,dP_1\cdots dP_i\,dP_1^G\cdots dP_i^G\,dh\,du\,.\end{aligned}$$

To write the integrations with respect to $P_k^G\in\mathrm{proj}_G K\cap H^G(h,u)$ and $P_k\in\partial K$, $\mathrm{proj}_G P_k = P_k^G$ as a single integration with respect to $P_k\in\partial K$ we define the hyperplane

$$H(h,u) := H^G(h,u) + G_\perp$$

and obtain

$$\int\limits_{\mathrm{proj}_G K\cap H^G(h,u)} \int\limits_{\substack{P_k\in\partial K\\ \mathrm{proj}_G P_k = P_k^G}} \cdots\, dP_k\, dP_k^G$$

$$= \int\limits_{\partial K\cap H(h,u)} \cdots \;\|\mathrm{proj}_{G_\perp} n_{K\cap H(h,u)}(P_k)\|\, dP_k\,.$$

Here $n_{K\cap H}(P)$ denotes the outer unit normal vector of the $(d-1)$–dimensional convex body $K\cap H$ at P in the hyperplane H.

We combine our results:

$$\int\limits_{\partial K}\cdots\int\limits_{\partial K} g(\mathrm{proj}_G P_1,\ldots,\mathrm{proj}_G P_i)\, dP_1\cdots dP_i$$

$$= (i-1)! \int\limits_{S^{d-1}\cap G}\int\limits_0^\infty \int\limits_{\partial K\cap H(h,u)}\cdots\int\limits_{\partial K\cap H(h,u)} g(\mathrm{proj}_G P_1,\ldots,\mathrm{proj}_G P_i)$$

$$\times V_{i-1}(\mathrm{proj}_G[P_1,\ldots,P_i])\prod_{j=1}^i \frac{\|\mathrm{proj}_{G_\perp} n_{K\cap H(h,u)}(P_j)\|}{\|\mathrm{proj}_{G_\perp} n_K(P_j)\|}\, dP_1\cdots dP_i\, dh\, du\,.$$

Now $\|\mathrm{proj}_{G_\perp} n_K(P_j)\| = \|\mathrm{proj}_{H(h,u)} n_K(P_j)\|\;\|\mathrm{proj}_{G_\perp} n_{K\cap H(h,u)}(P_j)\|$, since $u\in S^{d-1}\cap G$ and thus $G_\perp\subset H(h,u)$. Setting, as before,

$$l_{H(h,u)}(P_j) = \|\mathrm{proj}_{H(h,u)} n(P_j)\|^{-1},$$

we obtain the following analogon to Lemma 1:

Let $K\in\mathcal{K}^2_+$ and let G be an i–dimensional linear subspace, $i\le d-1$. Denote by $H(h,u)$ the hyperplane with unit normal vector $u\in S^{d-1}\cap G$ and distance h to the origin. Then

$$\int\limits_{\partial K}\cdots\int\limits_{\partial K} g(\mathrm{proj}_G P_1,\ldots,\mathrm{proj}_G P_i)\, dP_1\cdots dP_i$$

$$= (i-1)! \int\limits_{S^{d-1}\cap G}\int\limits_0^\infty \int\limits_{\partial K\cap H(h,u)}\cdots\int\limits_{\partial K\cap H(h,u)} g(\mathrm{proj}_G P_1,\ldots,\mathrm{proj}_G P_i) \tag{7}$$

$$\times V_{i-1}(\mathrm{proj}_G[P_1,\ldots,P_i])\prod_{j=1}^i l_{H(h,u)}(P_j)\, dP_1\cdots dP_i\, dh\, du\,.$$

3. Proof of Theorem 2: the case $i = d$

We split the proof of Theorem 2 into the case $i = d$ and the case $i < d$, for the following reason: Most of the main ideas of the proof can be made clear in the case $i = d$ without the additional notation required in the case $i < d$.

So we elaborate in detail the case $i = d$ in this section. The necessary additional considerations for the cases $i = 1,\ldots,d-1$ are carried out in Section 4. The proof is closely related to the proof of Theorem 1 in [15] concerning random polytopes with vertices in the interior of a given convex body.

The proof is divided into three parts. In 3.1 and 3.2 we present two tools and a modification of Crofton's boundary theorem needed in the subsequent sections. In 3.3–3.6 we prove the existence of an asymptotic expansion

$$\mathbb{E}_n(V_d) = c_0^{(d,d)}(K) + c_1^{(d,d)}(K)\, n^{-\frac{1}{d-1}} + c_2^{(d,d)}(K)\, n^{-\frac{2}{d-1}} + \cdots + c_{k-2}^{(d,d)}(K)\, n^{-\frac{k-2}{d+1}} + O\left(n^{-\frac{k-1}{d+1}}\right),$$

and in Section 3.7 we show the properties of the coefficients $c_i^{(d,d)}(K)$ stated in Theorem 2.

3.1. **Tools.** The first tool is a precise description of the local behaviour of the boundary of a smooth convex body. It is a straightforward generalization of a result of Schneider [20] concerning convex bodies of class $\mathcal{K}_+^3$ to convex bodies of class K_+^k.

Lemma 2. *Let $K \in \mathcal{K}_+^{k+1}$, $k \geq 2$, be given. Then there are constants α, $\beta > 0$, only depending on K, such that the following holds for every boundary point $x \in \partial K$. Identify the support plane of K at x with $\mathbb{R}^{d-1}$, and x with the origin. Then the α-neighborhood of x in ∂K can be represented by a convex function $f(y)$ of differentiability class $\mathcal{C}^{k+1}$, $y \in \mathbb{R}^{d-1}$. Furthermore, the absolute values of the partial derivatives of $f(y)$ up to order $k+1$ are uniformly bounded by β.*

The next lemma concerns the Taylor expansions of inverse functions. It is a refinement of well known results on the inversion of analytic functions (cf., e.g., Henrici [10], § 1.7. and § 1.9.), due to Gruber [8].

Lemma 3. *Let*

$$z = z(w,t) = b_m(w)t^m + \cdots + b_k(w)t^k + O\left(t^{k+1}\right)$$

for $0 \leq t \leq \alpha$, $2 \leq m \leq k$, be a strictly increasing function in t for each fixed w in a given set. Assume that $b_m(\cdot)$ is bounded between positive constants, that $b_{m+1}(\cdot), \ldots, b_k(\cdot)$ are bounded, and that the constant in $O(\cdot)$ may be chosen independent of w. Then there are coefficients $c_1(\cdot), \ldots, c_{k-m+1}(\cdot)$, and a constant $\gamma > 0$ independent of w, such that for each fixed w the inverse function $t(w,\cdot)$ of $z(w,\cdot)$ has the representation

$$t = t(w,z) = c_1(w)z^{\frac{1}{m}} + \cdots + c_{k-m+1}(w)z^{\frac{k-m+1}{m}} + O\left(z^{\frac{k-m+2}{m}}\right) \tag{8}$$

for $0 \leq z \leq \gamma$. The coefficients $c_1(\cdot), \ldots, c_{k-m+1}(\cdot)$ can be determined explicitly in terms of $b_m(\cdot)$, $\ldots$, $b_k(\cdot)$; in particular,

$$c_1(\cdot) = \frac{1}{b_m(\cdot)^{\frac{1}{m}}}, \quad c_2(\cdot) = -\frac{b_{m+1}(\cdot)}{mb_m(\cdot)^{\frac{m+2}{m}}},$$
$$c_3(\cdot) = -\frac{b_{m+2}(\cdot)}{mb_m(\cdot)^{\frac{m+3}{m}}} + \frac{(m+3)b_{m+1}(\cdot)^2}{2m^2 b_m(\cdot)^{\frac{2m+3}{m}}},$$
$$c_4(\cdot) = -\frac{b_{m+3}(\cdot)}{mb_m(\cdot)^{\frac{m+4}{m}}} + \frac{(m+4)b_{m+1}(\cdot)b_{m+2}(\cdot)}{m^2 b_m(\cdot)^{\frac{2m+4}{m}}} - \frac{(m+2)(m+4)b_{m+1}(\cdot)^3}{3m^3 b_m(\cdot)^{\frac{3m+4}{m}}}.$$

The coefficients are bounded, and if $b_m(\cdot), \ldots, b_k(\cdot)$ are continuous, so are $c_1(\cdot), \ldots, c_{k-m+1}(\cdot)$, and the constant in $O(\cdot)$ may be chosen independent of w.

Remark. It is easy to check the following additional property of the coefficients $c_i(\cdot)$:

If $b_m(w)$, $b_{m+2}(w)$, $b_{m+4}(w), \dots$ are even functions, and $b_{m+1}(w)$, $b_{m+3}(w)$, $b_{m+5}(w), \dots$ are odd functions of w, then $c_1(w)$, $c_3(w)$, $c_5(w), \dots$ are even functions, and $c_2(w)$, $c_4(w)$, $c_6(w), \dots$ are odd functions of w. Further if $b_{m+1}(w)$, $b_{m+3}(w)$, $b_{m+5}(w), \dots$ vanish, then also $c_2(w)$, $c_4(w)$, $c_6(w), \dots$ vanish.

3.2. **Random Simplices in a Family of Convex Bodies.** Consider a family $K(s)$ of convex bodies with the origin in their interiors which converge (with respect to the usual Hausdorff metric) for $s \to 0$ to a convex body $K(0)$. We call such a family *k–smooth* if the radial function $r(u;s)$ of $K(s)$ satisfies

$$r(u;s) = r_0(u) + r_1(u)s + r_2(u)s^2 + \cdots + r_k(u)s^k + O\left(s^{k+1}\right) \tag{9}$$

and the outer unit normal $n(u,s)$ of $K(s)$ at the point $r(u,s)$ satisfies

$$n(u;s) = n_0(u) + n_1(u)s + n_2(u)s^2 + \cdots + n_k(u)s^k + O\left(s^{k+1}\right) \tag{10}$$

uniformly for all $u \in S^{d-1}$. I.e., there exists a constant β such that for all $u \in S^{d-1}$ and $m \in \{0, \dots, k\}$ the coefficients $r_m(u)$ and the coordinates of $n_m(u)$ are bounded by β, and

$$|r(u;s) - \sum_{m=0}^{k} r_m(u)s^m| \leq \beta s^{k+1} \tag{11}$$

and

$$\|n(u;s) - \sum_{m=0}^{k} n_m(u)s^m\| \leq \beta s^{k+1} \,. \tag{12}$$

Clearly $r_0(u) = r(u;0)$ and $n(u;0) = n_0(u)$. Let the same conditions hold for some weight function $g(P;s) = g(r(u;s)u;s)$, i.e., assume that

$$g(r(u;s)u;s) = g_0(u) + g_1(u)s + g_2(u)s^2 + \cdots + g_k(u)s^k + O\left(s^{k+1}\right), \tag{13}$$

where for all $u \in S^{d-1}$ and $m \in \{0, \dots, k\}$ the coefficients $g_m(u)$ are bounded by β and

$$|g(r(u;s)u;s) - \sum_{m=0}^{k} g_m(u)s^m| \leq \beta s^{k+1} \,. \tag{14}$$

In this section we investigate for such functions $g(P;s)$ the following integral:

$$\mathcal{I}_{K(s)}(V_d^2 \,\Pi\, g) = \int\limits_{\partial K(s)} \cdots \int\limits_{\partial K(s)} V_d([P_1, \dots, P_{d+1}])^2 \prod_{j=1}^{d+1} g(P_j;s)\; dP_1 \cdots dP_{d+1} \,.$$

Note that $\mathcal{I}_{K(s)}(V_d^2 \,\Pi\, g)$ divided by $V_{d-1}(K(s))^{d+1}$ is the mean value of the random variable $V_d([P_1, \dots, P_{d+1}])^2 \prod g(P_j)$, i.e., the second moment of the volume of a random simplex whose vertices are chosen independently and according to the weight function g on $\partial K(s)$.

Lemma 4. *Let $K(s)$ be a k–*smooth *family of convex bodies and let g satisfy (13) and (14). Then $\mathcal{I}_{K(s)}(V_d^2 \,\Pi\, g)$ satisfies*

$$\mathcal{I}_{K(s)}(V_d^2 \,\Pi\, g) = \mathcal{I}_0 + \mathcal{I}_1 s + \cdots + \mathcal{I}_k s^k + O\left(s^{k+1}\right) \,. \tag{15}$$

Clearly $\mathcal{I}_0 = \mathcal{I}_{K(0)}(V_d^2\, g)$*. Further, there exists a constant,* γ *only depending on* β*, such that* $|\mathcal{I}_m| \le \gamma$*,* $m = 0, \dots, k$*, and such that*

$$|\mathcal{I}_{K(s)}(V_d^2\, \Pi\, g) - \sum_{m=0}^{k} \mathcal{I}_m s^m| \le \gamma s^{k+1}\,. \tag{16}$$

Proof of Lemma 4. First we rewrite the integral using polar coordinates:

$$\mathcal{I}_{K(s)}(V_d^2\, \Pi\, g) = \int\limits_{S^{d-1}} \cdots \int\limits_{S^{d-1}} V_d([r(u_1;s)u_1, \dots, r(u_{d+1};s)u_{d+1}])^2 \tag{17}$$
$$\prod_{k=1}^{d+1} \left(r(u_k;s)^{d-1}\, g(r(u_k;s)u_k)\ \frac{1}{u_k \cdot n(r(u_k;s)u_k)} \right) du_1\ \cdots\ du_{d+1}\,,$$

where $u_k \in S^{d-1}$ and du_k denotes surface area measure on S^{d-1}. In this integral only the integrand depends on the variable s.

The convex hull of the random points $P_1, \dots, P_{d+1}$ is a simplex whose volume $V_d([P_1, \dots, P_{d+1}])$ is given by the absolute value of the determinant

$$\frac{1}{d!}\ \det \begin{pmatrix} 1 & x_1^1 & \cdots & x_1^d \\ 1 & x_2^1 & \cdots & x_2^d \\ 1 & : & : & : \\ 1 & x_{d+1}^1 & \cdots & x_{d+1}^d \end{pmatrix}, \tag{18}$$

where $P_k = (x_k^1, \dots, x_k^d)$. The value of the determinant can be positive or negative according to the orientation of the ordered $(d+1)$–tuple of points $P_1, \dots, P_{d+1}$. Observe that this determinant is a polynomial $Q_d(\cdot)$ in the variables x_k^j, and thus $V_d([P_1, \dots, P_{d+1}])^2 = Q_d(\cdot)^2$ is a polynomial in the variables x_k^j and hence is an analytic function with bounded derivatives. Since the origin is contained in the interior of all convex bodies $K(s)$, the factor $u_k \cdot n(r(u_k;s)u_k)$ is bounded from below by a positive constant and thus $(u_k \cdot n(r(u_k;s)u_k))^{-1}$ has a Taylor expansion. Therefore the existence of a Taylor formula is obvious.

By (18) the coefficients of the polynomial $Q_d(\cdot)^2$ are bounded by $d!^{-2}$. By (9) – (14) the coefficients of the Taylor expansion as well as the error term of $r(\cdot)$, $g(\cdot)$ and $u_k \cdot n(\cdot)$ are uniformly bounded. Thus the same holds for the Taylor expansion of $V_d(\cdot) r(\cdot)^{d-1}\, g(\cdot)\ (u_k \cdot n(\cdot))^{-1}$ with respect to s, where the coefficients and the error term are bounded by constants only depending on β. Integrating the uniformly bounded coefficients of the Taylor expansion just proves

$$\mathcal{I}_{K(s)}(V_d^2\, \Pi\, g) = \mathcal{I}_0 + \mathcal{I}_1 s + \cdots + \mathcal{I}_k s^k + O\left(s^{k+1}\right), \tag{19}$$

and thus the statement of our lemma. □

In Section 3.5 we deal with radial functions $r(u;s)$, outer unit normal vectors $n(u;s)$, and weight functions $g(P;s)$ having the further property that the coefficients $r_{2m}(u), g_{2m}(u)$ and the coordinates of $n_{2m+1}(u)$ are even functions, and the coefficients $r_{2m+1}(u), g_{2m+1}(u)$ and the coordinates of $n_{2m}(u)$ are odd functions of $u \in S^{d-1}$. Consider the expansion of

$$V_d([r(u_1;s)u_1, \dots, r(u_{d+1};s)u_{d+1}])^2$$
$$\times \prod_{k=1}^{d+1} \left(r(u_k, s)^{d-1}\, g(r(u_k;s)u_k)\ \frac{1}{u_k \cdot n(r(u_k;s)u_k)} \right)$$

in a Taylor series using (9)–(14). Our assumption implies that the coefficient of s^{2m+1}, $m = 0, \dots, \lfloor \frac{k-1}{2} \rfloor$, is an odd function on S^{d-1}, and thus the integration over S^{d-1} yields

$$\mathcal{I}_1 = \mathcal{I}_3 = \mathcal{I}_5 = \cdots = 0\,. \tag{20}$$

3.3. Assume without loss of generality that the origin is contained in the interior of the convex body K. In a first step we transform the integral

$$\mathbb{E}_n(V_d) = \int\limits_{\partial K} \cdots \int\limits_{\partial K} V_d([P_1, \dots, P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n$$

into a form which is useful for further investigations, using ideas of Rényi and Sulanke [16], [17], and Wieacker [27]. With probability 1 the random polytope $[P_1, \dots, P_n]$ is simple, and thus the facets are $(d-1)$-dimensional simplices. Denote by $F^j = [P_{j_1}, \dots, P_{j_d}]$, $j = 1, \dots, k$, the facets of the random polytope and by H_+^j the halfspace which satisfies $F^j \subset \partial H_+^j$ and which contains the origin. Without loss of generality assume that for $j = 1, \dots, k_+$ the halfspace H_+^j contains the random polytope $[P_1, \dots, P_n]$.

The volume of the random polytope is given by

$$V_d([P_1, \dots, P_n]) = \sum_{j=1}^{k_+} V_d([0, P_{j_1}, \dots, P_{j_d}]) - \sum_{j=k_+ +1}^{k} V_d([0, P_{j_1}, \dots, P_{j_d}])\,.$$

Define

$$S_+^j = \int\limits_{\partial K \cap H_+^j} d_K(P)\, dP\,, \tag{21}$$

which is the weighted surface area of $\partial K \cap H_+^j$. The $(d-1)$–dimensional simplex defined by the points $P_{j_1}, \dots, P_{j_d}$ is a facet of $[P_1, \dots, P_n]$ if the remaining $n-d$ points lie either all in H_+^j (i.e., $j \le k_+$) or all in $\mathbb{R} \backslash H_+^j$ (i.e., $j > k_+$). The first event occurs with probability $(S_+^j)^{n-d}$, the second event with probability $(1 - S_+^j)^{n-d}$. (Recall that $d_K(x)$ is a density on ∂K, and so $\int_{\partial K} d_K(P)\, dP = 1$.) The argument is the same for each selection of d points, and thus

(22)
$$\int\limits_{\partial K} \cdots \int\limits_{\partial K} V_d([P_1, \dots, P_n]) \prod_{j=1}^{n} d_K(P_j)\; dP_1 \cdots dP_n$$
$$= \binom{n}{d} \int\limits_{\partial K} \cdots \int\limits_{\partial K} V_d([0, P_1, \dots, P_d])(S_+^{n-d} - (1 - S_+)^{n-d}) \prod_{j=1}^{d} d_K(P_j)\, dP_1 \cdots dP_d,$$

where H_+ is the halfspace satisfying $[P_1, \dots, P_d] \subset \partial H_+$ and which contains the origin, and S_+ is the weighted surface area of $\partial K \cap H_+$.

The halfspace H_+ and thus the hyperplane $H := \partial H_+$ can be parameterized by the outer unit normal vector $u \in S^{d-1}$ of H_+, and the distance h of H to the origin, $H_+ = H_+(h, u)$ and $H = H(h, u)$. Also define the weighted surface area $S_+(h, u)$ of $\partial K \cap H_+(h, u)$ analogously to (21). Observe that the integrand vanishes for h

larger than the support function $h_K(u)$ of K in direction u. Using Lemma 1, we obtain

$$= \binom{n}{d} \frac{(d-1)!}{d} \int_{S^{d-1}} \int_0^{h_K(u)} h(S_+(h,u)^{n-d} - (1-S_+(h,u))^{n-d})$$
$$\times \int_{\partial K \cap H(h,u)} \cdots \int_{\partial K \cap H(h,u)} V_{d-1}([P_1,\dots,P_d])^2$$
$$\times \prod_{j=1}^{d} \left(d_K(P_j) l_{H(h,u)}(P_j) \right) \, dP_1 \cdots dP_d \, dh \, du$$

$$= \binom{n}{d} \frac{(d-1)!}{d} \int_{S^{d-1}} \int_0^{h_K(u)} h(S_+(h,u)^{n-d} - (1-S_+(h,u))^{n-d})$$
$$\times \mathcal{I}_{\partial K \cap H(h,u)}(V_{d-1}^2 \Pi \, d_K \, l_{H(h,u)}) \, dh \, du,$$

where $\mathcal{I}_{\partial K \cap H(h,u)}(V_{d-1}^2 \Pi \, d_K \, l_{H(h,u)})$ is, up to a factor $V_{d-1}(\partial K \cap H(h,u))^{-d}$, the mean value of the random variable $V_{d-1}([P_1,\dots,P_d])^2 \prod d_K(P_j) \, l_{H(h,u)}(P_j)$, where the points $P_1,\dots,P_d$ are chosen at random in $\partial K \cap H(h,u)$. (Recall that dP denotes the j–dimensional volume element corresponding to the j–dimensional Hausdorff measure on a given space, which is determined by the range of integration.)

Since $d_K > 0$, the following holds uniformly for $u \in S^{d-1}$: for given $\varepsilon > 0$ sufficiently small there exists a constant $0 < \eta < 1$ such that $1 - S_+(h,u) < \eta$ for all $h \in [0, h_K(u)]$, and $S_+(h,u) < \eta$ for $0 \le h \le h_K(u) - \varepsilon$. Hence

$$\mathbb{E}_n(V_d) = \binom{n}{d} \frac{(d-1)!}{d} \int_{S^{d-1}} \int_{h_K(u)-\varepsilon}^{h_K(u)} h \, S_+(h,u)^{n-d} \times \mathcal{I}_{\partial K \cap H(h,u)}(V_{d-1}^2 \Pi \, d_K \, l_{H(h,u)}) \, dh \, du + O(\eta^n) \,. \tag{23}$$

Now let $u \in S^{d-1}$ be fixed. In Section 3.4 we investigate the local behaviour of ∂K at the boundary x with outer unit normal vector u. In Section 3.5 we show that $h \, \mathcal{I}_{\partial K \cap H(h,u)}(V_{d-1}^2 \Pi \, d_K \, l_{H(h,u)})$ can be expanded in a suitable Taylor series, which in Section 3.6 leads to an asymptotic series expansion of the integral

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} h \, S_+(h,u)^{n-d} \mathcal{I}_{\partial K \cap H(h,u)}(V_{d-1}^2 \Pi \, d_K \, l_{H(h,u)}) \, dh \qquad \text{as } n \to \infty \tag{24}$$

for given u. Taking the remaining integrations into account, we obtain our theorem.

3.4. Fix u and let x be the point on ∂K with outer unit normal vector u. For brevity we write S_+ and H instead of $S_+(h,u)$ and $H(h,u)$, respectively. In this section we give local representations of K, $K \cap H$, l_H, and the outer unit normal vectors in a neighborhood of x, and thus for h sufficiently small, using cylinder coordinates. Let $\mathbb{R}^d = (\mathbb{R}^+ \times S^{d-2}) \times \mathbb{R}$ and thus denote by (rv, z) a point in $\mathbb{R}^d$, $v \in S^{d-2}$, $r \in \mathbb{R}^+$, $z \in \mathbb{R}$. Identify the support plane of ∂K at x with the plane $z = 0$ and x with the origin so that K is contained in the halfspace $z \ge 0$. Note that if z is the distance from a hyperplane H with normal vector u to the support plane of ∂K at x, then $h = h_K(u) - z$. Since $K \in \mathcal{K}_+^{k+1}$, by Lemma 2 there is a neighborhood of x in ∂K such that ∂K can be represented by a convex function $f(rv)$ which in polar coordinates reads as

$$z = f(rv) = b_2(v)r^2 + b_3(v)r^3 + \cdots + b_k(v)r^k + O\left(r^{k+1}\right) . \tag{25}$$

The coefficients are bounded by a constant independent of x and v, and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. By choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ the coefficient $b_2(v)$ can be written as

$$b_2(v) = \frac{1}{2}\,(k_1(x)\,(v\cdot e_1)^2 + \cdots + k_{d-1}(x)\,(v\cdot e_{d-1})^2), \tag{26}$$

and since for all boundary points x of K the principal curvatures $k_i(x)$ are bounded from below and above by positive constants, the same holds for $b_2(v)$. Since (25) is a Taylor series, the coefficients $b_2(v), b_4(v), b_6(v), \ldots$ are even functions, and $b_3(v), b_5(v), b_7(v), \ldots$ are odd functions of $v \in S^{d-2}$.

Inverting this series using Lemma 3 gives

$$r = c_1(v)z^{\frac{1}{2}} + \cdots + c_{k-1}(v)z^{\frac{k-1}{2}} + O\left(z^{\frac{k}{2}}\right), \tag{27}$$

which is the radial function of $K \cap H$, where H is the hyperplane with distance z to the parallel hyperplane $z = 0$. Again the coefficients as well as the constant in $O(\cdot)$ are uniformly bounded, independent of x and v. Further, the coefficients $c_1(v), c_3(v), \ldots$ are even functions, and the coefficients $c_2(v), c_4(v), \ldots$ are odd functions of $v \in S^{d-2}$.

On the other hand, the Taylor expansion of $f(y)$, $y \in \mathbb{R}^{d-1}$, implies a Taylor expansion of $f(y)_i$, $i = 1, \ldots, d-1$, where $f(y)_i$ is the i-th partial derivative of $f(y)$. In cylinder coordinates this Taylor expansion reads as

$$f(rv)_i = \bar{d}_{i,1}(v)r + \bar{d}_{i,2}(v)r^2 + \cdots + \bar{d}_{i,k-1}(v)r^{k-1} + O\left(r^k\right) . \tag{28}$$

The coefficients are bounded by a constant independent of x and v, and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $\bar{d}_{i,1}(v), \bar{d}_{i,3}(v), \ldots$ are odd functions, and $\bar{d}_{i,2}(v), \bar{d}_{i,4}(v), \ldots$ are even functions of $v \in S^{d-2}$. Hence

$$\begin{aligned}(\mathrm{grad} f(rv))^2 &= \bar{d}_2(v)r^2 + \bar{d}_3(v)r^3 + \cdots + \bar{d}_k(v)r^k + O\left(r^{k+1}\right)\\ &= d_2(v)z + d_3(v)z^{\frac{3}{2}} + \cdots + d_k(v)z^{\frac{k}{2}} + O\left(z^{\frac{k+1}{2}}\right),\end{aligned} \tag{29}$$

where the coefficients are bounded by a constant independent of x and v and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. Again, by choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$ the coefficient $d_2(v)$ can be written as

$$d_2(v) = \frac{k_1(x)^2\,(v\cdot e_1)^2 + \cdots + k_{d-1}(x)^2\,(v\cdot e_{d-1})^2}{\frac{1}{2}\,(k_1(x)\,(v\cdot e_1)^2 + \cdots + k_{d-1}(x)\,(v\cdot e_{d-1})^2)}, \tag{30}$$

and since for all boundary points of K the principal curvatures k_i are bounded from below and above by positive constants, the same holds for $d_2(v)$. The coefficients $d_2(v), d_4(v), \ldots$ are even functions, and $d_3(v), d_5(v), \ldots$ are odd functions of $v \in S^{d-2}$.

Therefore the element of surface area $\sqrt{1+(\mathrm{grad}f)^2}$ has a Taylor expansion up to order $O\left(z^{\frac{k+1}{2}}\right)$:

$$\sqrt{1+(\mathrm{grad}f(rv))^2} = 1+e_2(v)z+e_3(v)z^{\frac{3}{2}}+\cdots+e_k(v)z^{\frac{k}{2}}+O\left(z^{\frac{k+1}{2}}\right) . \tag{31}$$

All coefficients are bounded by a constant independent of x and v, and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $e_2(v), e_4(v), \dots$ are even functions, and $e_3(v), e_5(v), \dots$ are odd functions of $v \in S^{d-2}$.

As $K \in \mathcal{K}^3_+$, for every boundary point there is a unique outer unit normal vector. Thus by (29) and (31)

$$\begin{aligned} l_H(rv) &= (\|\mathrm{proj}_H n(rv)\|)^{-1} \\ &= \frac{\sqrt{1+(\mathrm{grad}f(rv))^2}}{\sqrt{(\mathrm{grad}f(rv))^2}} \\ &= z^{-\frac{1}{2}}\left(l_0(v)+l_1(v)z^{\frac{1}{2}}+\cdots+l_{k-2}(v)z^{\frac{k-2}{2}}+O\left(z^{\frac{k-1}{2}}\right)\right) . \end{aligned} \tag{32}$$

By (30), all coefficients are bounded by a constant independent of x and v, and are continuous in v for fixed x. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $l_0(v), l_2(v), \dots$ are even functions, and $l_1(v), l_3(v), \dots$ are odd functions of $v \in S^{d-2}$.

3.5. Now we prove that the function $h\,\mathcal{I}_{\partial K\cap H}(V^2_{d-1}\Pi\, d_K\, l_H)$, which is part of the integrand in (24), has a Taylor expansion in $z^{\frac{1}{2}}$. Recall that z is the distance from H to the support plane of ∂K at x, and thus $h = h_K(u) - z$.

By (27) $r\,z^{-\frac{1}{2}}$ is the radial function of the convex body $z^{-\frac{1}{2}}(K\cap H)$, which tends by definition – up to a factor $2^{-\frac{1}{2}}$ – to the indicatrix of K at x, which is a $(d-1)$–dimensional ellipsoid. Set

$$s = z^{\frac{1}{2}} .$$

We define a family of convex bodies by $K(s) = K(z^{\frac{1}{2}}) := z^{-\frac{1}{2}}(K\cap H)$. Then

$$r(v;s) = c_1(v)+c_2(v)s+\cdots+c_{k-1}(v)s^{k-2}+O\left(s^{k-1}\right)$$

is the radial function of $K(s)$. By (28) and (29) the outer unit normal vector of $K(s)$ at the boundary point $r(v;s)v$ has a Taylor expansion in s:

$$\begin{aligned} n(v;s) &= \frac{\mathrm{grad}f(r(v;s)v)}{\sqrt{(\mathrm{grad}f(r(v;s)v))^2}} \\ &= n_0(v)+n_1(v)s+n_2(v)s^2+\cdots+n_{k-2}(v)s^{k-2}+O\left(s^{k-1}\right) . \end{aligned} \tag{33}$$

All coefficients are bounded by a constant independent of x and v. The constant in $O(\cdot)$ can be chosen independent of x and v. Thus $K(s)$ is a $(k-2)$–smooth family of convex bodies. The coefficients $c_1(v), c_3(v), \dots$ and the coordinates of $n_{2m+1}(v)$ are even functions, and the coefficients $c_2(v), c_4(v), \dots$ and the coordinates of $n_{2m}(v)$ are odd functions of $v \in S^{d-1}$.

Let $g = z^{\frac{1}{2}} d_K l_H$. As $d_K \in \mathcal{C}^{k-1}$, we obtain a Taylor expansion for d_K in terms of $r(v;z)$ where the first term equals $d_K(x)$, which implies the existence of functions

$d_{K,m}(v)$ with

$$d_K(r(v,z)v) = d_K(x) + d_{K,1}(v)z^{\frac{1}{2}} + \cdots + d_{K,k-2}(v)z^{\frac{k-2}{2}} + O\left(z^{\frac{k-1}{2}}\right) . \tag{34}$$

All coefficients are bounded by a constant independent of x and v. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $d_{K,2}(v), d_{K,4}(v), \dots$ are even functions, and the coefficients $d_{K,1}(v), d_{K,3}(v), \dots$ are odd functions of $v \in S^{d-1}$. By (32) there is a Taylor expansion for $z^{\frac{1}{2}} l_H$. Hence there is a Taylor expansion for g in terms of the variable s:

$$g(r(v;s)v;s) = g_0(v) + g_1(v)s + g_2(v)s^2 + \cdots + g_{k-2}(v)s^{k-2} + O\left(s^{k-1}\right) .$$

All coefficients are bounded by a constant independent of x and v. The constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $g_0(v), g_2(v), \dots$ are even functions, and the coefficients $g_1(v), g_3(v), \dots$ are odd functions of $v \in S^{d-1}$.

Thus Lemma 4 yields

$$\begin{aligned} \mathcal{I}_{\partial K \cap H}(V_{d-1}^2 \Pi\, d_K\, l_H) &= z^{\frac{d^2-d-2}{2}}\ \mathcal{I}_{K(s)}(V_{d-1}^2\, \Pi\, g) \\ &= z^{\frac{d^2-d-2}{2}} \left(\mathcal{I}_0 + \mathcal{I}_1 s + \cdots + \mathcal{I}_{k-2}s^{k-2} + O\left(s^{k-1}\right)\right) \\ &= z^{\frac{d^2-d-2}{2}} \left(\mathcal{I}_0 + \mathcal{I}_1 z^{\frac{1}{2}} + \cdots + \mathcal{I}_{k-2} z^{\frac{k-2}{2}} + O\left(z^{\frac{k-1}{2}}\right)\right) . \end{aligned} \tag{35}$$

The coefficients and the constant in $O(\cdot)$ are bounded independent of x. By the remarks at the end of Section 3.2, $\mathcal{I}_1 = \mathcal{I}_3 = \cdots = 0$.

Collecting terms, we obtain

$$(h_K(u) - z)\mathcal{I}_{\partial K \cap H}(V_{d-1}^2 \Pi\, d_K\, l_H) = z^{\frac{d^2-d-2}{2}} \left(P^{\lfloor \frac{k-2}{2} \rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right), \tag{36}$$

where $P^{\lfloor \frac{k-2}{2} \rfloor}(z)$ is a polynomial in the variable z of degree $\lfloor \frac{k-2}{2} \rfloor$. The coefficients of $P^{\lfloor \frac{k-2}{2} \rfloor}(\cdot)$ and the constant in $O(\cdot)$ are bounded independent of x.

3.6. In the last step we expand the integral (24),

$$\int\limits_{h_K(u)-\varepsilon}^{h_K(u)} h\, S_+^{n-d} \mathcal{I}_{\partial K \cap H}(V_{d-1}^2 \Pi\, d_K\, l_H)\, dh = \int\limits_0^{\varepsilon} S_+^{n-d}\, z^{\frac{d^2-d-2}{2}} \left(P^{\lfloor \frac{k-2}{2} \rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right) dz$$

in an asymptotic series in powers of $n^{-\frac{1}{d-1}}$. First we prove that this can be written as

$$\int\limits_0^{1-S_+(\varepsilon)} (1-s)^{n-d} s^{d-1} \left(P^{\lfloor \frac{k-2}{2} \rfloor}(s^{\frac{2}{d-1}}) + O\left(s^{\frac{k-1}{d-1}}\right)\right) ds \,. \tag{37}$$

To this end, define the function $s_+(z)$ by

$$s_+(z) = \int\limits_{S^{d-2}} d_K(r(v,z)v) \sqrt{1 + (\mathrm{grad} f(r(v,z)v))^2}\ r(v,z)^{d-2}\ \frac{\partial r(v,z)}{\partial z}\ dv \,.$$

By (34), (31), and (27) the integrand can be evaluated in a Taylor series in $z^{\frac{1}{2}}$:

$$s_1(v)z^{\frac{d-3}{2}} + s_2(v)z^{\frac{d-2}{2}} + \cdots + s_{k-1}(v)z^{\frac{d+k-5}{2}} + O\left(z^{\frac{d+k-4}{2}}\right).$$

All coefficients are bounded by a constant independent of x and v, and the constant in $O(\cdot)$ can be chosen independent of x and v. The coefficients $s_2(v), s_4(v), \dots$ are odd functions of $v \in S^{d-1}$. Thus

(38)

$$\begin{aligned} s_+(z) &= \int\limits_{S^{d-2}} s_1(v)z^{\frac{d-3}{2}} + s_2(v)z^{\frac{d-2}{2}} + \cdots + s_{k-1}(v)z^{\frac{d+k-5}{2}} + O\left(z^{\frac{d+k-4}{2}}\right) dv \\ &= s_1 z^{\frac{d-3}{2}} + s_2 z^{\frac{d-2}{2}} + \cdots + s_{k-1} z^{\frac{d+k-5}{2}} + O\left(z^{\frac{d+k-4}{2}}\right) \end{aligned}$$

and

$$\frac{1}{s_+(z)} = z^{-\frac{d-3}{2}} \left(\bar{s}_1 + \bar{s}_2 z^{\frac{1}{2}} + \cdots + \bar{s}_{k-1} z^{\frac{k-2}{2}} + O\left(z^{\frac{k-1}{2}}\right)\right), \tag{39}$$

where the coefficients $s_2, s_4, \dots$ and $\bar{s}_2, \bar{s}_4, \dots$ vanish. Since s_1 is bounded from below by a positive constant, all coefficients are bounded by a constant independent of x. The constant in $O(\cdot)$ can be chosen independent of x.

By the definition of the weighted surface area $S_+ = S_+(z)$ we have

$$1 - S_+(z_0) = \int d_K(P)\, dP,$$

where the integral is extended over all points on the boundary of K with $z(P) \leq z_0$. We rewrite this integral using cylinder coordinates:

(40)

$$\begin{aligned} 1 - S_+(z_0) &= \int\limits_{f(y)\leq z_0} d_K(y)\sqrt{1+(\mathrm{grad} f(y))^2}\, dy \\ &= \int\limits_{S^{d-2}} \int\limits_{r\leq r(v,z_0)} d_K(rv)\sqrt{1+(\mathrm{grad} f(rv))^2}\; r^{d-2}\, dr\, dv \\ &= \int\limits_{S^{d-2}} \int\limits_{z\leq z_0} d_K(r(v,z)v)\sqrt{1+(\mathrm{grad} f(r(v,z)v))^2}\; r(v,z)^{d-2}\; \frac{\partial r(v,z)}{\partial z}\, dz\, dv \\ &= \int\limits_{z\leq z_0} s_+(z)\, dz, \end{aligned}$$

which means that $d(1 - S_+(z))/dz = s_+(z)$.

We introduce the new variable $s := 1 - S_+$, which is a function of z. To describe the relation between these variables we use (38) and obtain

$$s = 1 - S_+(z) = \frac{2s_1}{d-1}\, z^{\frac{d-1}{2}} + \frac{2s_2}{d}\, z^{\frac{d}{2}} + \cdots + \frac{2s_{k-1}}{d+k-3}\, z^{\frac{d+k-3}{2}} + O\left(z^{\frac{d+k-2}{2}}\right).$$

Inverting this series in the variable $z^{\frac{1}{2}}$ using Lemma 3 gives

$$z^{\frac{1}{2}} = g_1 s^{\frac{1}{d-1}} + g_2 s^{\frac{2}{d-1}} + \cdots + g_{k-1} s^{\frac{k-1}{d-1}} + O\left(s^{\frac{k}{d+1}}\right), \tag{41}$$

where the coefficients $g_2, g_4, \dots$ vanish. All coefficients are bounded by a constant independent of x, and the constant in $O(\cdot)$ can be chosen independent of x.

Now substituting $s = 1 - S_+(z)$ yields

$$\int_0^{\varepsilon} S_+^{n-d}\, z^{\frac{d^2-d-2}{2}} \left(P^{\lfloor\frac{k-2}{2}\rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right) dz$$
$$= \int_0^{1-S_+(\varepsilon)} (1-s)^{n-d}\, z^{\frac{d^2-d-2}{2}} \left(P^{\lfloor\frac{k-2}{2}\rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right) \frac{1}{s_+(z)}\, ds\,.$$

Using (41), we define the polynomial $\bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(\cdot)$ by

$$z^{\frac{d^2-d-2}{2}} \left(P^{\lfloor\frac{k-2}{2}\rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right) \frac{1}{s_+(z)} = s^{d-1}\left(\bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(s^{\frac{2}{d-1}}) + O\left(s^{\frac{k-1}{d-1}}\right)\right),$$

which proves (37). The coefficients of $\bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(\cdot)$ and the constant in $O(\cdot)$ are bounded independent of x.

Finally, the integral

$$\int_0^{1-S_+(\varepsilon)} (1-s)^{n-d} s^{d-1}\left(\bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(s^{\frac{2}{d-1}}) + O\left(s^{\frac{k-1}{d-1}}\right)\right) ds$$

can be evaluated by the substitution $e^{-t} := 1 - s$. Observe that – taking the remarks before (23) into account – $S_+(\varepsilon) < \eta < 1$, independent of x. Consider the integral of a single term $s^{\frac{m}{d-1}}$. Up to an error term which decreases exponentially in n, this is the following Laplace transform:

$$\begin{aligned}\int_0^{1-S_+(\varepsilon)} (1-s)^{n-1}\, s^{\frac{m}{d-1}}\, ds &= \int_0^{-\ln(S_+(\varepsilon))} e^{-tn}(1-e^{-t})^{\frac{m}{d-1}}\, dt \\ &= \mathcal{L}\left\{(1-e^{-t})^{\frac{m}{d-1}}\right\}(n) + O\left(e^{-n}\right) \\ &= \mathcal{L}\left\{t^{\frac{m}{d-1}}\left(1 - \frac{m}{2(d-1)}t + \cdots\right)\right\}(n) + O\left(e^{-n}\right)\,.\end{aligned}$$

Using an Abelian theorem, cf., e.g., Doetsch [7], chap. 3, § 1, we obtain

$$= \Gamma\left(\frac{m}{d-1}+1\right) n^{-\frac{m}{d-1}-1} - \frac{m}{2(d-1)}\Gamma\left(\frac{m}{d-1}+2\right) n^{-\frac{m}{d-1}-2} + \cdots + O\left(e^{-n}\right)\,.$$

In particular,

$$\mathcal{L}\left\{\, O\left(t^{\frac{m}{d-1}+j+1}\right)\right\} = O\left(n^{-\frac{m}{d-1}-j-2}\right) \quad \text{as } n \to \infty\,.$$

Therefore terminating the Taylor expansion of $(1-e^{-t})^{\frac{m}{d-1}}$ after the term of order $t^{\frac{m}{d-1}+j}$, and taking into account the error term $O\left(t^{\frac{m}{d-1}+j+1}\right)$ of the same order as the first term omitted, results in an expansion of the Laplace transform up to order $n^{-\frac{m}{d-1}-j-1}$ with an error term of order $O\left(n^{-\frac{m}{d-1}-j-2}\right)$. Choose j as the smallest integer such that $\frac{m}{d-1}+j+2>\frac{k-1}{d-1}+d$. Then expanding $(1-s)^{-d+1}$ in (37) gives

$$\int_0^{s(\varepsilon)} (1-s)^{n-d} s^{d-1} \left(\bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(s^{\frac{2}{d-1}}) + O\left(s^{\frac{k-1}{d-1}}\right)\right) ds$$
$$= h_0 n^{-d} + h_1 n^{-\frac{1}{d-1}-d} + \cdots + h_{k-2}, n^{-\frac{k-2}{d-1}-d} + O\left(n^{-\frac{k-1}{d-1}-d}\right)$$

where the coefficients $h_m = h_m(u)$ and the constant in $O(\cdot)$ are bounded independent of x and thus independent of the outer unit normal vector u of K at x. Taking into account that

$$\binom{n}{d} = \frac{n^d}{d!}\left(1 - \frac{(d+1)(d+2)}{2n} + \cdots\right), \tag{42}$$

the integration in (23) concerning $u \in S^{d-1}$ yields

$$\mathbb{E}_n(V_d) = c_0^{(d,d)}(K) + c_1^{(d,d)}(K) n^{-\frac{1}{d-1}} + c_2^{(d,d)}(K) n^{-\frac{2}{d-1}}$$
$$+ \cdots + c_{k-2}^{(d,d)}(K) n^{-\frac{k-2}{d+1}} + O\left(n^{-\frac{k-1}{d+1}}\right),$$

which proves the case $i = d$ of Theorem 2.

3.7. The following facts concerning the coefficients $c_m^{(d,d)}(K)$ are easily checked:

$\mathbb{E}_n(V_d)$ tends to $V_d(K)$ as $n \to \infty$, and thus $c_0^{(d,d)}(K) = V_d(K)$.

If $d-1$ is even, then $(1-s)^{-d+1}\, s^{d-1} \bar{P}^{\lfloor\frac{k-2}{2}\rfloor}(s^{\frac{2}{d-1}})$ has a Taylor series in powers $\frac{2}{d-1}$ of s, which yields that for odd d

$$c_1^{(d,d)}(K) = c_3^{(d,d)}(K) = c_5^{(d,d)}(K) = \cdots = 0\,.$$

Let $d-1$ be odd. The integral concerning the constant $\bar{c}_0$ in the polynomial $\bar{P}(\cdot)$ gives

$$\binom{n}{d} \int_0^{1-S_+(\varepsilon)} (1-s)^{n-d} s^{d-1} \bar{c}_0\, ds = \frac{\bar{c}_0}{d} + O\left(e^{-n}\right)\,.$$

The Taylor expansion of $(1-s)^{-d+1}$ implies a series in $s^{\frac{1}{d-1}}$ of $(1-s)^{-d+1}(\bar{P}(\cdot)-\bar{c}_0)$ where the coefficients of $s^0, s^{\frac{1}{d-1}}, s^{\frac{3}{d-1}}, \ldots, s^{\frac{d-1}{d-1}}$ vanish. This yields that for d even

$$c_1^{(d,d)}(K) = c_3^{(d,d)}(K) = \cdots = c_{d-1}^{(d,d)}(K) = 0\,.$$

□

4. Proof of Theorem 2: the case $i < d$

We prove the case $i < d$ of Theorem 2 using Kubota's integral recursion. The main part of the proof is already contained in Section 3. We only point out the differences and work out the required additional considerations. The proof is presented in the same format as the proof of the case $i = d$.

4.1. See 3.1.

4.2. It is easy to obtain the more general statement concerning projections of an i–dimensional simplex onto a (fixed) i–dimensional subspace $G \subset \mathbb{R}^d$, $i = 1, \dots, d$. So denote by proj_G the orthogonal projection of $\mathbb{R}^d$ onto G. The following lemma concerns the integral

$$\mathcal{I}^G_{K(s)}(V_i^2 \,\Pi\, g) = \int\limits_{\partial K(s)} \cdots \int\limits_{\partial K(s)} V_i(\mathrm{proj}_G[P_1, \dots, P_{i+1}])^2 \prod_{j=1}^{i+1} g(P_j; s)\ dP_1 \cdots dP_{i+1}.$$

Note that the case $i = d$ is just Lemma 4.

Lemma 5. *Let $K(s)$ be a k–*smooth *family of convex bodies and let g satisfy (13) and (14). Then $\mathcal{I}^G_{K(s)}(V_i^2 \,\Pi\, g)$ satisfies*

$$\mathcal{I}^G_{K(s)}(V_i^2 \,\Pi\, g) = \mathcal{I}^G_0 + \mathcal{I}^G_1 s + \cdots + \mathcal{I}^G_k s^k + O\left(s^{k+1}\right). \tag{43}$$

Clearly $\mathcal{I}^G_0 = \mathcal{I}^G_{K(0)}(V_i^2\, g)$. The coefficients $\mathcal{I}^G_m$ are continuous functions of G. Furthermore, there exists a constant γ, only depending on β, such that $|\mathcal{I}^G_m| \le \gamma$, $m = 0, \dots, k$, and such that

$$|\mathcal{I}^G_{K(s)}(V_i^2 \,\Pi\, g) - \sum_{m=0}^{k} \mathcal{I}^G_m s^m| \le \gamma s^{k+1}. \tag{44}$$

Proof of Lemma 5. The proof is the same as the proof of Lemma 4. It relies heavily on the fact that $V_i(\mathrm{proj}_G[P_1, \dots, P_{i+1}])^2$ is a polynomial and hence an analytic function with coefficients depending continuously on G. □

4.3. Assume that the origin is contained in the interior of the convex body K. Using Kubota's integral recursion for quermassintegrals, we obtain

(45)

$$\begin{aligned}\mathbb{E}_n(V_i) &= \int\limits_{\partial K} \cdots \int\limits_{\partial K} V_i([P_1, \dots, P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n \\ &= \frac{\binom{d}{i}\kappa_d}{\kappa_{d-i}\kappa_i} \int\limits_{SO(d)} \int\limits_{\partial K} \cdots \int\limits_{\partial K} V_i(\mathrm{proj}_{\rho G}[P_1, \dots, P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n\ d\rho,\end{aligned}$$

where G is a fixed i–dimensional linear subspace of $\mathbb{R}^d$, $i = 1, \dots, d-1$, ρG is the image of G under the rotation ρ, and $\mathrm{proj}_{\rho G}$ denotes orthogonal projection onto ρG. $d\rho$ corresponds to the normalized Haar measure on the group $SO(d)$ of proper rotations of $\mathbb{R}^d$, cf. Schneider [22].

Thus for ρG fixed we are led to consider the expected i–dimensional volume of the convex hull of random points in a convex body $\mathrm{proj}_{\rho G} K$. We rewrite this integral, using again the ideas of Rényi and Sulanke [16]. Let H_+ be the halfspace whose boundary hyperplane is the affine hull of $[P_1, \dots, P_i] + (\rho G)_\perp$ and which contains the origin. $(\rho G)_\perp$ is the orthogonal complement of ρG. Denoting by S_+ the weighted surface area of $\partial K \cap H_+$ (see (21)) and using the same arguments which led to (22), we obtain

$$\int_{\partial K} \cdots \int_{\partial K} V_i(\mathrm{proj}_{\rho G}[P_1, \ldots, P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n$$
$$= \binom{n}{i} \int_{\partial K} \cdots \int_{\partial K} V_i(\mathrm{proj}_{\rho G}[0, P_1, \ldots, P_i])(S_+^{n-i} - (1 - S_+)^{n-i})$$
$$\times \prod_{j=1}^{i} d_K(P_j)\; dP_1 \cdots dP_i\,.$$

The analogon (7) to Lemma 1 implies that

$$\mathbb{E}_n(V_i) = \frac{\binom{n}{i}\binom{d}{i}\,(i-1)!\,\kappa_d}{i\kappa_{d-i}\kappa_i} \int_{SO(d)} \int_{S^{d-1}\cap\rho G} \int_0^{h_K(u)}$$
$$h\,(S_+(h,u)^{n-i} - (1 - S_+(h,u))^{n-i})\,\mathcal{I}_{\partial K\cap H(h,u)}(V_{i-1}^2 \Pi\, d_K\, l_{H(h,u)})\; dh\; du\; d\rho,$$

where

$$\mathcal{I}_{\partial K\cap H(h,u)}(V_{i-1}^2 \Pi\, d_K\, l_{H(h,u)}) = \int_{\partial K\cap H(h,u)} \cdots \int_{\partial K\cap H(h,u)}$$
$$\times V_{i-1}(\mathrm{proj}_{\rho G}[P_1, \ldots, P_i])^2 \prod_{j=1}^{i} d_K(P_j) l_{H(h,u)}(P_j)\, dP_1 \cdots dP_i$$

and $H(h,u)$ and $S_+(h,u)$ again depend on the normal vector u and the distance h from H to the origin. In the case $i = 1$, V_{i-1} is the Euler characteristic and thus equals one.

Therefore the asymptotic behaviour of $\mathbb{E}_n(V_i)$ is – up to an error term of order $O(\eta^n)$ – determined by

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} h\, S_+(h,u)^{n-i}\,\mathcal{I}_{\partial K\cap H(h,u)}(V_{i-1}^2 \Pi\, d_K\, l_{H(h,u)})\; dh$$

as $n \to \infty$.

4.4. See 3.4

4.5. Since $K(s) := z^{-\frac{1}{2}}(K \cap H)$ is a $(k-2)$–*smooth* family of convex bodies, Lemma 5 proves that

$$\mathcal{I}_{\partial K\cap H}(V_{i-1}^2 \Pi\, d_K\, l_H) = z^{\frac{di-i-2}{2}} \left(\mathcal{I}_0 + \mathcal{I}_1 z^{\frac{1}{2}} + \cdots + \mathcal{I}_{k-2} z^{\frac{k-2}{2}} + O\left(z^{\frac{k-1}{2}}\right)\right), \tag{46}$$

which implies

$$(h_K(u) - z)\mathcal{I}_{\partial K\cap H}(V_{i-1}^2 \Pi\, d_K\, l_H) = z^{\frac{di-i-2}{2}} \left(P^{\lfloor\frac{k-2}{2}\rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right), \tag{47}$$

where $P^{\lfloor\frac{k-2}{2}\rfloor}(z)$ is a polynomial in the variable z of degree $\lfloor\frac{k-2}{2}\rfloor$. The coefficients of $P^{\lfloor\frac{k-2}{2}\rfloor}(\cdot)$ and the constant in $O(\cdot)$ are bounded independent of x and ρ.

4.6. In order to expand the integral

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} h\, S_+^{n-i} \mathcal{I}_{\partial K\cap H}(V_{i-1}^2 \Pi\, d_K\, l_H)\ dh$$
$$= \int_0^{\varepsilon} S_+^{n-i}\, z^{\frac{di-i-2}{2}}\ \left(P^{\lfloor \frac{k-2}{2} \rfloor}(z) + O\left(z^{\frac{k-1}{2}}\right)\right)\ dz,$$

we use the same substitutions as in Section 3.6 and obtain

$$\int_0^{1-S_+(\varepsilon)} (1-s)^{n-i}\ s^{i-1} \left(\bar{P}^{\lfloor \frac{k-2}{2} \rfloor}(s^{\frac{2}{d-1}}) + O\left(s^{\frac{k-1}{d-1}}\right)\right)\ ds$$
$$= h_0 n^{-i} + h_1 n^{-\frac{1}{d-1}-i} + \cdots + h_{k-2} n^{-\frac{k-2}{d-1}-i} + O\left(n^{-\frac{k-1}{d-1}-i}\right),$$

and thus

$$\mathbb{E}_n(V_i) = c_0^{(i,d)}(K) + c_1^{(i,d)}(K) n^{-\frac{1}{d-1}} + c_2^{(i,d)}(K) n^{-\frac{2}{d-1}}$$
$$+ \cdots + c_{k-2}^{(i,d)}(K) n^{-\frac{k-2}{d+1}} + O\left(n^{-\frac{k-1}{d+1}}\right)\ .$$

4.7. The following properties of the coefficients $c_m^{(i,d)}(K)$ are easily checked:

$\mathbb{E}_n(V_i)$ tends to $V_i(K)$ as $n \to \infty$, and thus $c_0^{(i,d)}(K) = V_i(K)$.

If $d-1$ is even, then $(1-s)^{-i+1}\, s^{i-1} \bar{P}^{\lfloor \frac{k-2}{2} \rfloor}(s^{\frac{2}{d-1}})$ has a Taylor series in powers $\frac{2}{d-1}$ of s, which yields that for odd d

$$c_1^{(i,d)}(K) = c_3^{(i,d)}(K) = c_5^{(i,d)}(K) = \cdots = 0\ .$$

Let $d-1$ be odd. The integral concerning the constant $\bar{c}_0$ in the polynomial $\bar{P}(\cdot)$ gives

$$\begin{pmatrix} n \\ i \end{pmatrix} \int_0^{1-V_+(\varepsilon)} (1-s)^{n-i} s^{i-1} \bar{c}_0\, ds = \frac{\bar{c}_0}{i} + O\left(e^{-n}\right)\ .$$

The Taylor expansion of $(1-s)^{-i+1}$ implies a series in $s^{\frac{1}{d-1}}$ of $(1-s)^{-i+1}(\bar{P}(\cdot) - \bar{c}_0)$ where the coefficients of $s^0, s^{\frac{1}{d-1}}, s^{\frac{3}{d-1}}, \ldots, s^{\frac{d-1}{d-1}}$ vanish. This yields that for d even

$$c_1^{(i,d)}(K) = c_3^{(i,d)}(K) = \cdots = c_{d-1}^{(i,d)}(K) = 0\ .$$

□

5. Proof of Theorem 1

The proof of Theorem 1 uses a slightly different approach than that given in the proofs of Theorem 2. Nevertheless we present the proof in an order similar to that of the proofs of Theorem 2.

5.1. **Tools.** Again we need a precise description of the local behaviour of the boundary of a convex body $K \in \mathcal{K}_+^2$. The essential point in Lemma 2 was the existence of a convex function f with uniformly bounded derivatives representing ∂K in a neighborhood of an arbitrary boundary point x of K. This implied a Taylor expansion of f with a uniformly bounded error term (cf. Section 3.1 and 3.4).

In the following lemma we prove an analogous result for convex bodies of differentiability class $\mathcal{K}_+^2$. In principle, such a result is well known and has already been used before, but we think that an explicit statement of this result will be helpful.

So fix $K \in \mathcal{K}_+^2$. At every boundary point x of K there is a paraboloid $Q_2^{(x)}$, given by a quadratic form $b_2^{(x,x)}$, osculating ∂K at x. $Q_2^{(x)}$ and $b_2^{(x,x)}$ can be defined in the following way. Identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. Then there is a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y = (y^1, \ldots, y^{d-1}) \in \mathbb{R}^{d-1}$, representing ∂K in a neighborhood of x, i.e., $(y, f^{(x)}(y)) \in \partial K$. Denote by $f_{ij}^{(x)}(0)$ the second partial derivatives of $f^{(x)}$ at the origin. Then

$$b_2^{(x,x)}(y) := \frac{1}{2} \sum_{i,j} f_{ij}^{(x)}(0) y^i y^j$$

and

$$Q_2^{(x)} := \{(y,z) \,|\, z \geq b_2^{(x,x)}(y)\}\,.$$

We prove that at each boundary point x of K the deviation of $\partial Q_2^{(x)}$ from ∂K is uniformly bounded in a suitable neighborhood of x.

Lemma 6. *Let $K \in \mathcal{K}_+^2$ be given. Choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$, only depending on δ and K, such that for each boundary point x of K the following holds. Identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the origin. The λ–neighborhood U^λ of x in ∂K defined by $\mathrm{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda B^{d-1}$ can be represented by a convex function $f^{(x)}(y) \in \mathcal{C}^2$, $y \in \lambda B^{d-1}$. Furthermore*

$$(1+\delta)^{-1} b_2^{(x,x)}(y) \leq f^{(x)}(y) \leq (1+\delta)\, b_2^{(x,x)}(y) \quad \textit{for } y \in \lambda B^{d-1}\ , \tag{48}$$

$$\sqrt{1 + (\mathrm{grad} f^{(x)}(y))^2} \leq (1+\delta) \quad \textit{for } y \in \lambda B^{d-1}\ , \tag{49}$$

$$(1+\delta)^{-1} d_K(x) \leq d_K(p) \leq (1+\delta)\, d_K(x) \quad \textit{for } p \in U^\lambda\,, \tag{50}$$

and

$$(1+\delta)^{-1} 2 b_2^{(x,x)}(y) \leq (y,0)\cdot n_K(y) \leq (1+\delta)\, 2 b_2^{(x,x)}(y) \quad \textit{for } y \in \lambda B^{d-1}, \tag{51}$$

where $n_K(y)$ is the outer unit normal vector of K at the boundary point $(y, f^{(x)}(y))$.

Proof of Lemma 6. Let $p \in \partial K$. Identify the support plane $H(p)$ of K at p with $\mathbb{R}^{d-1}$. Since $K \in \mathcal{K}_+^2$, there are a function $f^{(p)}$ and an open neighborhood $U(p) \subset \partial K$ of p such that the projection of $U(p)$ onto $\mathbb{R}^{d-1}$ is an open $(d-1)$–dimensional ball with center in p, and $U(p)$ is represented in the form

$$U(p) = \Big\{ (y, f^{(p)}(y)) \mid y \in \mathrm{proj}_{\mathbb{R}^{d-1}} U(p) \Big\}\,,$$

where $f^{(p)}$ is a convex function of differentiability class $\mathcal{C}^2$.

For each $y \in \mathrm{proj}_{\mathbb{R}^{d-1}} U(p)$ we use the second term of the Taylor expansion of $f^{(p)}$ at y to define a quadratic form $b_2^{(p,y)}(v)$ by

$$b_2^{(p,y)}(v) = \frac{1}{2} \sum_{i,j} f_{ij}^{(p)}(y)\, v^i v^j \tag{52}$$

with $v = (v^1, \ldots, v^{d-1}) \in \mathbb{R}^{d-1}$. Since $K \in \mathcal{K}_+^2$, the quadratic forms $b_2^{(p,y)}$ are positive and continuous in p and x. Choosing $U(p)$ sufficiently small, we get

$$(1+\bar\delta)^{-1}\, b_2^{(p,p)}(v) \le b_2^{(p,y)}(v) \le (1+\bar\delta)\, b_2^{(p,p)}(v) \qquad \text{for } y \in \mathrm{proj}_{\mathbb{R}^{d-1}} U(p)\,,\ v \in \mathbb{R}^{d-1}$$

with $\bar\delta > 0$. Since d_K is a continuous function,

$$(1+\bar\delta)^{-1}\, d_K(p) \le d_K(x) \le (1+\bar\delta)\, d_K(p) \qquad \text{for } x \in U(p)\,.$$

As $f_i(0) = 0$ for $i = 1, \ldots, d-1$, we further can assume that

$$\sqrt{1 + \sum_i f_i^{(p)}(y)^2} \le (1+\bar\delta) \qquad \text{for } y \in \mathrm{proj}_{\mathbb{R}^{d-1}} U(p)\,.$$

Consider an additional point $x \in U(p)$. We choose a Cartesian coordinate system in $\mathbb{R}^{d-1}$ such that the intersection of $\mathbb{R}^{d-1}$ with the support plane $H(x)$ to K at x is parallel to the first $d-2$ coordinate axes. Analogously we choose in $H(x)$ a coordinate system such that the origin coincides with x and the intersection with $\mathbb{R}^{d-1}$ is parallel to the first $d-2$ coordinate axes. Thus, if $y = \mathrm{proj}_{\mathbb{R}^{d-1}} x$, then $f_i(y) = 0$ for $i = 1, \ldots, d-2$, and we can assume that $f_{d-1}(y) > 0$. With respect to these corresponding coordinate systems the continuity of $b_2^{(p,p)}$ implies

$$(1+\bar\delta)^{-1}\, b_2^{(p,p)}(v) \le b_2^{(x,x)}(v) \le (1+\bar\delta)\, b_2^{(p,p)}(v) \qquad \text{for } x \in U(p)\,,\ v \in \mathbb{R}^{d-1}\,.$$

So let $\bar\delta > 0$ be arbitrary, but fixed. The union of the sets $U(p) = U^{\bar\delta}(p)$ for all $p \in \partial K$ covers the boundary of K, and as ∂K is compact there exist points p_l, $l = 1, \ldots, m$, say, and corresponding open neighborhoods $U^{\bar\delta}(p_l)$ such that the union of the sets $U^{\bar\delta}(p_l)$ covers ∂K, and, further, the projection of $U^{\bar\delta}(p_l)$ onto the support plane at p_l is an open $(d-1)$–dimensional ball with center in p_l,

$$(1+\bar\delta)^{-1}\, b_2^{(p_l,p_l)}(v) \le b_2^{(p_l,y)}(v) \le (1+\bar\delta)\, b_2^{(p_l,p_l)}(v)\,, \tag{53}$$

$$(1+\bar\delta)^{-1}\, b_2^{(p_l,p_l)}(v) \le b_2^{(x,x)}(v) \le (1+\bar\delta)\, b_2^{(p_l,p_l)}(v)\,, \tag{54}$$

$$\sqrt{1 + \sum_i f_i^{(p_l)}(y)^2} \le (1+\bar\delta)\,, \tag{55}$$

$$(1+\bar\delta)^{-1}\, d_K(p_l) \le d_K(x) \le (1+\bar\delta)\, d_K(p_l)\,, \tag{56}$$

for $x \in U^{\bar\delta}(p_l)$, $y \in \mathrm{proj}_{\mathbb{R}^{d-1}} U^{\bar\delta}(p_l)$, and $v \in \mathbb{R}^{d-1}$.

As $K \in \mathcal{K}_+^2$, there is a constant $\lambda = \lambda(\bar\delta) > 0$ such that for each $x \in \partial K$ there exists a suitable $l \in \{1, \ldots, m\}$ with $x \in U^{\bar\delta}(p_l)$, and, further, the projection of $U^{\bar\delta}(p_l)$ onto the support plane $H(x)$ of ∂K at x contains a $(d-1)$–dimensional ball with radius λ and center x, i.e.,

$$B(x, \lambda) \cap H(x) \subset \mathrm{proj}_{H(x)} U^{\bar\delta}(p_l)\,. \tag{57}$$

Now we prove that the conditions (53)–(57) imply (48)–(50) at each boundary point of K. So fix $x \in \partial K$ and thus the point p_l with $x \in U^{\delta}(p_l)$. Again we identify the support plane of ∂K at p_l with $\mathbb{R}^{d-1}$ and choose corresponding coordinate systems in $\mathbb{R}^{d-1}$ and $H(x)$. To distinguish between corresponding coordinates in $H(x)$ and $\mathbb{R}^{d-1}$ we add a $\tilde{}$ to coordinates in $H(x)$, for example, $x = (\tilde{0}, \ldots, \tilde{0})$. For abbreviation put $y := \mathrm{proj}_{\mathbb{R}^{d-1}} x$, i.e., $x = (y, f^{(p_l)}(y))$. Define $z = z(y+t)$, $t \in \mathbb{R}^{d-1}$, as the distance from the point $(y+t, f^{(p_l)}(y+t))$ on ∂K to the support plane $H(x)$ of ∂K at x. Taylor's theorem implies

$$\begin{aligned} z = & \frac{f^{(p_l)}(y+t) - f^{(p_l)}(y) - \sum\limits_i f_i^{(p_l)}(y) t^i}{\sqrt{1 + \sum\limits_i f_i^{(p_l)}(y)^2}} \\ = & \frac{\frac{1}{2} \sum\limits_{i,j} f_{ij}^{(p_l)}(y + \theta t) t^i t^j}{\sqrt{1 + \sum\limits_i f_i^{(p_l)}(y)^2}} \end{aligned}$$

with $\theta \in [0,1]$, and $t = (t^1, \ldots, t^{d-1})$. Combining this with (53) and (55), we have

$$(1+\bar{\delta})^{-2}\, b_2^{(p_l,p_l)}(t) \le z \le (1+\bar{\delta})\, b_2^{(p_l,p_l)}(t) \tag{58}$$

for $y + t \in \mathrm{proj}_{\mathbb{R}^{d-1}} U^{\bar{\delta}}(p_l)$.

To each point $y+t$ there corresponds a point $t_\pi \in H(x)$ with

$$t_\pi := \mathrm{proj}_{H(x)}(y+t, f^{(p_l)}(y+t)) .$$

Consider z as a function of t_π. Then $f^{(x)}$ defined by $z = f^{(x)}(t_\pi)$ represents the boundary of K for $t_\pi \in B(x,\lambda) \cap H(x)$. To prove (48) we replace $b_2^{(p_l,p_l)}(t)$ in (58) first by $b_2^{(p_l,p_l)}(t_\pi)$ and then by $b_2^{(x,x)}(t_\pi)$. Observe that

$$\begin{aligned} \tilde{t}_\pi^j &= t^j \quad \text{for } d = 1, \ldots, d-2, \\ \tilde{t}_\pi^{d-1} &= t^{d-1} \sqrt{1 + f_{d-1}^{(p_l)}(y)^2} + z f_{d-1}^{(p_l)}(y) \end{aligned}$$

with $t_\pi = (\tilde{t}_\pi^1, \ldots, \tilde{t}_\pi^{d-1})$. By (55) and (58) this implies

$$\begin{aligned} \|t - t_\pi\| &= t^{d-1} \left(\sqrt{1 + f_{d-1}^{(p_l)}(y)^2} - 1 \right) + z f_{d-1}^{(p_l)}(y) \\ &\le \sqrt{\bar{\delta}}\, (t^{d-1} + 4 b_2^{(p_l,p_l)}(t)) . \end{aligned}$$

Since $K \in \mathcal{K}_+^2$, the coefficients of the quadratic forms $b_2^{(p,p)}$ are bounded for arbitrary $p \in \partial K$. Therefore there exist constants $c_1, c_2, c_3 > 0$ independent of p such that for all $t_1, t_2 \in \mathbb{R}^{d-1}$ and for all p on the boundary of K, $b_2^{(p,p)}(t_1)$ is bounded from below by $c_1 \|t_1\|^2$ and from above by $c_2 \|t_1\|^2$, and the absolute value of $b_2^{(p,p)}(t_1) - b_2^{(p,p)}(t_2)$ is bounded from above by $c_3 \|t_1 - t_2\| \max\{\|t_1\|, \|t_2\|\}$. Here $\|t\|$ denotes the Euclidean norm with respect to the coordinate system with origin in p_l.

Hence

$$\|t - t_\pi\| \le \sqrt{\bar{\delta}}\, (\|t\| + 4 b_2^{(p_l,p_l)}(t))$$

implies

$$|b_2^{(p_l,p_l)}(t_\pi) - b_2^{(p_l,p_l)}(t)| \le c_2 \sqrt{\bar{\delta}}\ \left((1+\bar{\delta}) c_1^{-1} + 8 b_2^{(p_l,p_l)}(t) \right)$$

for $\|t\|, \|t_\pi\| \le 1$, or, equivalently,

$$(1+\alpha\sqrt{\bar\delta})^{-1} b_2^{(p_l,p_l)}(t_\pi) \le b_2^{(p_l,p_l)}(t) \le (1+\alpha\sqrt{\bar\delta}) b_2^{(p_l,p_l)}(t_\pi) \tag{59}$$

for $\bar\delta$ small and with a suitable $\alpha > 0$ depending only on K. Combining this with (58) yields

$$(1+\alpha\sqrt{\bar\delta})^{-1}(1+\bar\delta)^{-3} b_2^{(x,x)}(t_\pi) \le f^{(x)}(t_\pi) \le (1+\alpha\sqrt{\bar\delta})(1+\bar\delta)^2 b_2^{(x,x)}(t_\pi) \tag{60}$$

for $t_\pi \in B(x,\lambda) \cap H(x)$.

It remains to prove (51). Define

$$t_p := (t^1, \dots, t^{d-1}, f_{d-1}^{(p_l)}(y) t^{d-1})$$

which is chosen so that t_p is parallel to the support plane $H(x)$ and its projection onto $\mathbb{R}^{d-1}$ equals t. The outer unit normal vector $n_K(y+t)$ clearly is given by $(f_1^{(p_l)}(y+t), \dots, f_{d-1}^{(p_l)}(y+t), -1)$ divided by its length. Since t_p is parallel to $H(x)$ it is orthogonal to $n_K(y)$. So Taylor's theorem yields

$$\begin{aligned} t_p \cdot n_K(y+t) = &\ \frac{\sum_i f_i^{(p_l)}(y+t) t^i - f_{d-1}^{(p_l)}(y) t^{d-1}}{\sqrt{1+\sum_i f_i^{(p_l)}(y+t)^2}} \\ = &\ \frac{\sum_{i,j} f_{ij}^{(p_l)}(y+\theta t) t^i t^j}{\sqrt{1+\sum_i f_i^{(p_l)}(y+t)^2}} \end{aligned}$$

with $\theta \in [0,1]$. Together with (53) and (55), this shows that

$$(1+\bar\delta)^{-2}\, 2b_2^{(p_l,p_l)}(t) \le t_p \cdot n_K(y+t) = (x+t_p)\cdot n_K(y+t) \le (1+\bar\delta)\, 2b_2^{(p_l,p_l)}(t)\,.$$

Changing the coordinate system, the vector $x + t_p$ in coordinates with respect to $H(x)$ has the form

$$\begin{aligned} \tilde t_p^j, &= t^j \quad \text{for } d = 1, \dots, d-2, \\ \tilde t_p^{d-1} &= t^{d-1}\sqrt{1+f_{d-1}^{(p_l)}(y)^2}, \\ \tilde t_p^d &= 0, \end{aligned}$$

since $x + t_p \in H(x)$. Thus to replace $(x+t_p)\cdot n_K(y+t)$ by $(t_\pi, \tilde 0)\cdot n_K(y+t)$ we investigate

$$|((x+t_p) - (t_\pi, \tilde 0))\cdot n_K(y+t)| \le \|(x+t_p) - (t_\pi,\tilde 0)\| = z f_{d-1}(y),$$

which by (55) and (58) gives

$$\begin{aligned} (1+\bar\delta)^{-2}(1+2\sqrt{\bar\delta})^{-1}\, 2b_2^{(p_l,p_l)}(t) &\le (t_\pi, \tilde 0)\cdot n_K(y+t) \\ &\le (1+\bar\delta)(1+2\sqrt{\bar\delta})\, 2b_2^{(p_l,p_l)}(t) \end{aligned}$$

for $\bar\delta$ small. We already know how to replace $b_2^{(p_l,p_l)}(t)$ by $b_2^{(x,x)}(t_\pi)$: using (54) and (59), we get

$$\begin{aligned} (1+\bar\alpha\sqrt{\bar\delta})^{-2}(1+\delta)^{-3}\, 2b_2^{(x,x)}(t_\pi) &\le (t_\pi, \tilde 0)\cdot n_K(y+t) \\ &\le (1+\bar\alpha\sqrt{\bar\delta})^2(1+\delta)^2\, 2b_2^{(x,x)}(t_\pi) \end{aligned} \tag{61}$$

where $\bar\alpha$ denotes $\max\{\alpha, 2\}$.

This completes the proof of Lemma 6: for given δ we choose $\bar{\delta}$ sufficiently small such that (60), (55), (56), and (61) imply (48), (49), (50), and (51). $\square$

5.2. Assume that the origin is contained in the interior of the convex body K. We investigate

$$V_i(K) - \mathbb{E}_n(V_i) = \int\limits_{\partial K} \cdots \int\limits_{\partial K} (V_i(K) - V_i([P_1, \ldots, P_n])) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n,$$

which by Kubota's integral recursion (cf. Section 4.3) equals

(62)
$$\frac{\binom{d}{i}\kappa_d}{\kappa_{d-i}\kappa_i} \int\limits_{SO(d)} \int\limits_{\partial K} \cdots \int\limits_{\partial K} V_i(\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1, \ldots, P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n \; d\rho\, .$$

Choose $\varepsilon > 0$ sufficiently small. Ensure in particular that $h_K(u) - \varepsilon > 0$ for all $u \in S^{d-1}$. Since the probability that $h_K(u) - h_{[P_1,\ldots,P_n]}(u) \geq \varepsilon$ tends to zero of order $O\left(e^{-n}\right)$ (62) can be written as

$$\frac{\binom{d}{i}\kappa_d}{\kappa_{d-i}\kappa_i} \int\limits_{SO(d)} \underbrace{\int\limits_{\partial K} \cdots \int\limits_{\partial K}}_{h_K(u) - h_{[P_1,\ldots,P_n]}(u) \leq \varepsilon} V_i(\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1, \ldots, P_n])$$
$$\times \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n \; d\rho + O\left(e^{-n}\right),$$

where the integral is extended over all choices of random points such that $h_K(u) - h_{[P_1,\ldots,P_n]}(u) \leq \varepsilon$ for all $u \in S^{d-1}$.

Let ρG be fixed. The projection of $[P_1, \ldots, P_n]$ onto ρG is a polytope with, say, k facets F^j and outer unit normal vectors $u^j \in S^{d-1} \cap \rho G$. With probability 1 the random polytope is simple, and thus the facets are $(i-1)$–dimensional simplices. We need the following notion:

$$F_c^j := \{P \in \mathrm{proj}_{\rho G} K \mid [0, P] \cap F^j \neq \emptyset\}.$$

F_c^j is that part of $\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1, \ldots, P_n]$ which is behind the facet F^j looked at from the origin.

Clearly the union of the sets F_c^j, $j = 1, \ldots, k$, equals $\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1, \ldots, P_n]$, whence

$$V_i(\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1, \ldots, P_n]) = \sum_{j=1}^{k} V_i(F_c^j)\, .$$

Denote by H_+^j the halfspace whose boundary hyperplane is the affine hull of $F^j + (\rho G)_\perp$ and which contains the origin (and thus $[P_1, \ldots, P_n]$). The simplex $F^j = \mathrm{proj}_{\rho G}[P_{j_1}, \ldots, P_{j_i}]$ is a facet of the random polytope $\mathrm{proj}_{\rho G}[P_1, \ldots, P_n]$ if the remaining $(n-i)$ points are contained in H_+^j. This event occurs with probability $(S_+^j)^{n-i}$, where S_+^j is the weighted surface area of $\partial K \cap H_+^j$ (see (21)). The argument is the same for each selection of i points, and since the points are distributed independently we obtain

$$\underbrace{\int_{\partial K} \cdots \int_{\partial K}}_{h_K(u)-h_{[P_1,\dots,P_n]}(u)\le\varepsilon} V_i(\mathrm{proj}_{\rho G} K \backslash \mathrm{proj}_{\rho G}[P_1,\dots,P_n]) \prod_{j=1}^{n} d_K(P_j)\, dP_1 \cdots dP_n \; d\rho$$

$$= \binom{n}{i} \underbrace{\int_{\partial K} \cdots \int_{\partial K}}_{h_K(u)-h_{[P_1,\dots,P_n]}(u)\le\varepsilon} S_+^{n-i}\, V_i(F_c) \prod_{j=1}^{i} d_K(P_j)\; dP_1 \cdots dP_i$$

as $n \to \infty$, where $F = \mathrm{proj}_{\rho G}[P_1,\dots,P_i]$, and thus H_+, S_+, and F_c depend on the positions of the points $P_1,\dots,P_i$. u denotes the outer unit normal vector of F in ρG.

We rewrite this integral using Lemma 1 in the case $i = d$ and (7) for $i \le d-1$. Again define $H_+(h,u)$, $H(h,u)$, and $S_+(h,u)$ as in Section 3.3:

$$= \binom{n}{i} (i-1)! \int_{S^{d-1}\cap\rho G} \int_{h_K(u)-\varepsilon}^{h_K(u)} \int_{\partial K\cap H(h,u)} \cdots \int_{\partial K\cap H(h,u)}$$
$$\times\, S_+(h,u)^{n-i}\, V_{i-1}(F)\, V_i(F_c) \prod_{j=1}^{i} d_K(P_j)\, l_{H(h,u)}(P_j)\; dP_1 \cdots dP_i\, dh\, du,$$

where $H(h,u)$ is the boundary hyperplane of $H_+(h,u)$. Therefore we obtain

(63)
$$V_i(K) - \mathbb{E}_n(V_i) = \frac{\binom{n}{i}\binom{d}{i}(i-1)!\,\kappa_d}{i\kappa_{d-i}\kappa_i}$$
$$\times \int_{SO(d)} \int_{S^{d-1}\cap\rho G} \int_{h_K(u)-\varepsilon}^{h_K(u)} S_+(h,u)^{n-i}\, \mathcal{I}_{\partial K\cap H(h,u)}(h,u,\rho) dh\; du\; d\rho + O\left(e^{-n}\right)$$

with

$$\mathcal{I}_{\partial K\cap H(h,u)}(h,u,\rho)$$
$$= \int_{\partial K\cap H(h,u)} \cdots \int_{\partial K\cap H(h,u)} V_{i-1}(F)\, V_i(F_c) \prod_{j=1}^{i} d_K(P_j)\, l_{H(h,u)}(P_j)\; dP_1 \cdots dP_i$$

as $n \to \infty$. In the case $i = 1$, V_{i-1} is the Euler characteristic and thus equals one.

Therefore the asymptotic behaviour of $V_i(K) - \mathbb{E}_n(V_i)$ is – up to an error term of order $O\left(e^{-n}\right)$ – determined by

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} S_+(h,u)^{n-i}\, \mathcal{I}_{\partial K\cap H(h,u)}(h,u,\rho)\; dh$$

as $n \to \infty$ for given ρ and u.

5.3. Fix ρ and u, and let x be the point on ∂K with outer unit normal vector u. For brevity we write S_+ and H instead of $S_+(h,u)$ and $H(h,u)$, respectively. In this section we give local representations of K, $K \cap H$, l_H and the outer unit normal vectors in a neighborhood of the point x using cylinder coordinates. Let $\mathbb{R}^d = (\mathbb{R}^+ \times S^{d-2}) \times \mathbb{R}$, and denote by (rv, z) a point in $\mathbb{R}^d$, $v \in S^{d-2}$, $r \in \mathbb{R}^+$, $z \in \mathbb{R}$. Identify the support plane of ∂K at x with the plane $z = 0$ and x with the origin, so that K is contained in the halfspace $z \geq 0$. Recall that $h = h_K(u) - z$. Since $K \in \mathcal{K}^2_+$, by Lemma 6 there is a λ–neighborhood of x in ∂K such that ∂K can be represented by a convex function $f^{(x)}(rv)$ which satisfies (48)–(51). For abbreviation write $b_2(\cdot)$ and $f(\cdot)$ instead of $b_2^{(x,x)}(\cdot)$ and $f^{(x)}(\cdot)$. Put $z = f(rv)$. In polar coordinates (48) reads as

$$(1+\delta)^{-1}\, b_2(v)\, r^2 \leq z \leq (1+\delta)\, b_2(v)\, r^2, \tag{64}$$

which implies that (compare this to (27))

$$(1+\delta)^{-\frac{1}{2}}\, b_2(v)^{-\frac{1}{2}}\, z^{\frac{1}{2}} \leq r \leq (1+\delta)^{\frac{1}{2}}\, b_2(v)^{-\frac{1}{2}}\, z^{\frac{1}{2}} \tag{65}$$

for $r \leq \lambda$.

In the following section we need a similar formula for the expression

$$l_H(rv)\ \frac{1}{v \cdot n_{K\cap H}(rv)}\,.$$

Fix rv, and let H be the hyperplane containing $(rv, f(rv))$ and parallel to $\mathbb{R}^{d-1}$. The intersection of K with H is a $(d-1)$–dimensional convex body whose outer unit normal vector $n_{K\cap H}$ in H at $(rv, f(rv))$ is determined by the projection of the outer unit normal vector n_K of K at the same point:

$$\begin{aligned} l_H(rv)\ \frac{1}{v \cdot n_{K\cap H}(rv)} &= \frac{1}{\|\mathrm{proj}_{\mathbb{R}^{d-1}} n_K(rv)\|}\ \frac{\|\mathrm{proj}_{\mathbb{R}^{d-1}} n_K(rv)\|}{v \cdot \mathrm{proj}_{\mathbb{R}^{d-1}} n_K(rv)} \\ &= \frac{1}{v \cdot n_K(rv)}. \end{aligned}$$

Hence, using (51) combined with (65), we obtain

(66)

$$(1+\delta)^{-\frac{3}{2}}\, 2^{-1} b_2(v)^{-\frac{1}{2}}\, z^{-\frac{1}{2}} \leq l_H(rv)\ \frac{1}{v \cdot n_{K\cap H}(rv)} \leq (1+\delta)^{\frac{3}{2}}\, 2^{-1} b_2(v)^{-\frac{1}{2}}\, z^{-\frac{1}{2}}$$

for $r \leq \lambda$.

5.4. Now we investigate the function $\mathcal{I}_{\partial K\cap H}(h, u, \rho G)$, which is part of the integrand in (63), in the case when u is the outer unit normal vector of x and h is sufficiently small. Ignoring the density $d_K(x)$, which by (50) is locally nearly constant, we compare this to the same quantity where we replace the boundary of the convex body K by the osculating paraboloid $Q_2^{(x)}$. Note that z is the distance from H to the support plane of ∂K at x, and thus $z = h_K(u) - h$.

As a first step we replace the random points chosen on $\partial K \cap H$ which determine the facet F by random points chosen on the intersection of H with the paraboloid $Q_2^{(x)}$. Let the facet F be the convex hull of the projection of the points $P_1, \dots, P_i$ with $P_j = r(v_j)v_j$, $j = 1, \dots, i$, where $r(v_j)$ is the radial function of $K \cap H$ and thus given in (65). To determine $V_{i-1}(F)$ it is essential that V_{i-1} is a linear function in the variables $r(v_j)$. E.g. for $j=1$ we have

$$V_{i-1}(F) = V_{i-2}(\mathrm{proj}_{\rho G}[r(v_2)v_2, \dots, r(v_i)v_i])\, \alpha\, |r(v_1) - r_0|$$

with suitably chosen $\alpha \le 1$ and r_0, depending on the points $r(v_j)v_j$, $j = 2, \dots, i$, and on v_1. If $r(v_1)$ is contained in the interval $[(1+\delta)^{-1} r_2(v_1), (1+\delta)\, r_2(v_1)]$, it follows immediately that

$$|r_2(v_1) - r_0| - \delta r_2(v_1) \le |r(v_1) - r_0| \le |r_2(v_1) - r_0| + \delta r_2(v_1),$$

which implies

$$\begin{aligned} &|V_{i-1}(F) - V_{i-1}(\mathrm{proj}_{\rho G}[r_2(v_1)v_1, r(v_2)v_2, \dots, r(v_i)v_i])| \\ &\qquad \le \delta\, V_{i-2}(\mathrm{proj}_{\rho G}[r(v_2)v_2, \dots, r(v_i)v_i])\, r_2(v_1) \end{aligned} \tag{67}$$

since $\alpha \le 1$. Let $r_2(v)$ be $b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}}$. Then by (65)

$$r(v) \in [(1+\delta)^{-1} r_2(v_1), (1+\delta)\, r_2(v_1)].$$

In particular (65) proves $r = O\left(z^{\frac{1}{2}}\right)$, where the constant in $O(\cdot)$ can be chosen independent of x, ρ, and δ. Hence the right hand side of (67) is of order $\delta\, O\left(z^{\frac{i-1}{2}}\right)$. This holding true also for $j = 2, \dots, i$ proves that

$$|V_{i-1}(F) - V_{i-1}(F_2)| \le \delta\, O\left(z^{\frac{i-1}{2}}\right) \tag{68}$$

as $n \to \infty$, where the constant in $O(\cdot)$ can be chosen independent of x, ρ, and δ, and

$$F_2 := \mathrm{proj}_{\rho G}[r_2(v_1)v_1, \dots, r_2(v_i)v_i]\,.$$

Next we compare F_c to that part of $\mathrm{proj}_{\rho G} K$ which is below the facet F in direction orthogonal to $\mathbb{R}^{d-1}$, i.e., which is contained in the convex hull of F and the projection of F onto $\mathbb{R}^{d-1}$,

$$F_\Pi := \mathrm{proj}_{\rho G} K \cap [F, \mathrm{proj}_{\mathbb{R}^{d-1}} F].$$

We show that the difference between $V_i(F_c)$ and $V_i(F_\Pi)$ is of small order. Choose δ_0 sufficiently small so that due to (64) $z < \frac{\varepsilon}{2}$ for all v, x, and $\delta \le \delta_0$, where ε has already been chosen after (62). This ensures that the angle between the line from the origin to an arbitrary point in $H \cap K$ and the line orthogonal to H is bounded away from $\frac{\pi}{2}$. In particular, this condition and (49) show that for any point $p \in F$ the angle between the line from the origin through p and the line through p orthogonal to $\mathbb{R}^{d-1}$ is bounded. Hence there exists a constant α, independent of F, x, δ, and ρ, such that

$$|V_i(F_c) - V_i(F_\Pi)| \le \alpha z^2 V_{i-2}(\partial F),$$

and thus

$$V_{i-1}(F) V_i(F_c) = V_{i-1}(F) V_i(F_\Pi) + V_{i-1}(F) V_{i-2}(\partial F) O\left(z^2\right)$$

as $n \to \infty$. The differentiability class $\mathcal{K}^2_+$ of K implies the existence of a constant R such that K slides freely inside a ball of radius R. Hence

$$\begin{aligned} V_{i-1}(F) V_{i-2}(\partial F)\, z^2 &\le V_{i-1}(K \cap H) V_{i-2}(\partial K \cap H)\, z^2 \\ &\le (i-1)\kappa_{i-1}^2 (R^2 - (R-z)^2)^{\frac{2i-3}{2}}\, z^2 = O\left(z^{\frac{2i+1}{2}}\right) \end{aligned}$$

as $n \to \infty$, where the constant in $O(\cdot)$ can be chosen independent of x, ρ, and δ, and κ_{i-1} is the volume of the $(i-1)$–dimensional unit ball.

Now we replace F_Π by $\mathrm{proj}_{\rho G}[r_2(v_1)v_1,\dots,r_2(v_i)v_i]_\Pi$, which is analogously defined as the part of $\mathrm{proj}_{\rho G}K$ below $\mathrm{proj}_{\rho G}[r_2(v_1)v_1,\dots,r_2(v_i)v_i] = \mathrm{proj}_{\rho G}F_2$. Both sets are located between the hyperplane H and $\mathbb{R}^{d-1}$, which are at a distance z from each other. This implies

$$|V_i(F_\Pi) - V_i(\mathrm{proj}_{\rho G}[r_2(v_1)v_1,\dots,r_2(v_i)v_i]_\Pi)| \le z\,|V_{i-1}(F) - V_{i-1}(F_2)|\,.$$

Hence it is an easy consequence of (68) that

$$|V_i(F_\Pi) - V_i(\mathrm{proj}_{\rho G}[r_2(v_1)v_1,\dots,r_2(v_i)v_i]_\Pi)| \le \delta\, O\left(z^{\frac{i+1}{2}}\right) \tag{69}$$

as $n \to \infty$, where the constant in $O(\cdot)$ can be chosen independent of x, ρ, and δ.

In the definition of $\mathrm{proj}_{\rho G}[r_2(v_1)v_1,\dots,r_2(v_i)v_i]_\Pi$ we want to replace the boundary of K by the boundary of Q_2. So denote by $F_{2\Pi}$ that part of $\mathrm{proj}_{\rho G}Q_2$ which is contained in the convex hull of F_2 and the projection of F_2 onto $\mathbb{R}^{d-1}$. The definition of F_Π, (69) and (64) imply that $V_i(F_\Pi)$ is bounded from below by

$$V_i(F_{2\Pi}) - \delta\, z\, V_{i-1}(F_2) - \delta O\left(z^{\frac{i+1}{2}}\right) = V_i(F_{2\Pi}) - \delta O\left(z^{\frac{i+1}{2}}\right)$$

and from above by

$$V_i(F_{2\Pi}) + \delta\, 2z\, V_{i-1}(F_2) + \delta O\left(z^{\frac{i+1}{2}}\right) = V_i(F_{2\Pi}) + \delta O\left(z^{\frac{i+1}{2}}\right)$$

as $n \to \infty$. Again, the differentiability class $\mathcal{K}_+^2$ of K implies the existence of a constant R such that K slides freely inside a ball of radius R. Hence $V_{i-1}(F)$ is of order $O\left(z^{\frac{i-1}{2}}\right)$, and $V_i(F_\Pi)$ is of order $O\left(z^{\frac{i+1}{2}}\right)$. Combining these results yields

$$\mathcal{I}_{\partial K\cap H}(h,u,\rho G) = \int\limits_{\partial K\cap H}\cdots\int\limits_{\partial K\cap H} \left(V_{i-1}(F_2)V_i(F_{2\Pi}) + \delta O\left(z^i\right) + O\left(z^{\frac{2i+1}{2}}\right)\right)$$
$$\times \prod_{j=1}^{i} d_K(P_j)\, l_H(P_j)\; dP_1\cdots dP_i$$

as $n \to \infty$, where the constant in $O(\cdot)$ can be chosen independent of x, ρ, and δ. To evaluate the integral concerning $\delta O(\cdot) + O(\cdot)$, note that d_K is bounded, and the integrations concerning $l_H(P_j)\, dP_j$ result by (66) in terms of order

$$O\left(z^{-\frac{1}{2}}\right) V_{d-2}(\partial K\cap H),$$

that is,

$$\int\limits_{\partial K\cap H(h,u)}\cdots\int\limits_{\partial K\cap H(h,u)} \left(\delta O\left(z^i\right) + O\left(z^{\frac{2i+1}{2}}\right)\right)\prod_{j=1}^{i} d_K(P_j)\, l_H(P_j)\; dP_1\cdots dP_i$$
$$= \left(\delta O\left(z^i\right) + O\left(z^{\frac{2i+1}{2}}\right)\right) O\left(z^{-\frac{i}{2}}\right) V_{d-2}(\partial K\cap H)^i$$
$$= \delta O\left(z^{\frac{(d-1)i}{2}}\right) + O\left(z^{\frac{(d-1)i+1}{2}}\right)$$

as $n \to \infty$, where the constants in $O(\cdot)$ can be chosen independent of x, ρ, and δ.

It remains to investigate

$$\int\limits_{\partial K\cap H}\cdots\int\limits_{\partial K\cap H} V_{i-1}(F_2)V_i(F_{2\Pi})\prod_{j=1}^{i} d_K(P_j)\, l_H(P_j)\; dP_1\cdots dP_i\,.$$

We rewrite this as an integral over $S^{d-2}\subset H$,

$$=\int\limits_{S^{d-2}}\cdots\int\limits_{S^{d-2}} V_{i-1}(F_2)V_i(F_{2\Pi})\prod_{j=1}^{i} d_K(r(v_j)v_j)\,\frac{l_H(r(v_j)v_j)\,r(v_j)^{d-2}}{v\cdot n_{K\cap H}(r(v_j)v_j)}\,dv_1\cdots dv_i\,.$$

Together with (66), (50), and (65) this yields

$$\begin{aligned}(1+\delta)^{-3i}2^{-i}d_K(x)^i\mathcal{I}_2(x)z^{-i}+\delta O\left(z^{\frac{(d-1)i}{2}}\right)+O\left(z^{\frac{(d-1)i+1}{2}}\right)\le\\ \le \mathbb{E}_{\partial K\cap H}(h,u,\rho G)\le\\ \le(1+\delta)^{3i}2^{-i}d_K(x)^i\mathcal{I}_2(x)z^{-i}+\delta O\left(z^{\frac{(d-1)i}{2}}\right)+O\left(z^{\frac{(d-1)i+1}{2}}\right)\end{aligned}$$

as $n\to\infty$, where

$$\mathcal{I}_2(x)=\int\limits_{S^{d-2}}\cdots\int\limits_{S^{d-2}} V_{i-1}(F_2)V_i(F_{2\Pi})\prod_{j=1}^{i} r_2(v_j)^{d-1}\; dv_1\cdots dv_i$$

and

$$r_2(v)=b_2(v)^{-\frac12}z^{\frac12}\,.$$

In the case $i=d$ the result of Schütt and Werner [25] indicates that $\mathcal{I}_2(x)$ is invariant under volume-preserving affinities acting in the hyperplane H. This implies that $\mathcal{I}_2(x)$ only depends on the Gaussian curvature $H_{d-1}(x)$ of K at x, and leads to (4). We prove the analogous property for general $i\in\{1,\dots,d\}$.

Define the ellipsoid E_2 as the $(d-1)$–dimensional convex body with radial function $b_2(v)^{-\frac12}$. E_2 is the intersection of Q_2 with the hyperplane $z=1$. Since $V_{i-1}(\cdot)$ and $V_i(\cdot)$ are homogeneous, $\mathcal{I}_2(x)$ can be rewritten as an integral where the random points are chosen in the interior of E_2 according to the uniform distribution,

$$\begin{aligned}\mathcal{I}_2(x)&=z^{\frac{(d+1)i}{2}}\int\limits_{S^{d-2}}\cdots\int\limits_{S^{d-2}}\int\limits_{0}^{b_2(u_i)^{-\frac12}}\cdots\int\limits_{0}^{b_2(u_1)^{-\frac12}}\\ &\quad\times V_{i-1}(\mathrm{proj}_{\rho G}[b_2(u_1)^{-\frac12}u_1,\dots,b_2(u_i)^{-\frac12}u_i])\\ &\quad\times V_i(\mathrm{proj}_{\rho G}[b_2(u_1)^{-\frac12}u_1,\dots,b_2(u_i)^{-\frac12}u_i]_\Pi)\\ &\quad\times\prod_{k=1}^{i}((d-1)\,t_k^{d-2}))\,dt_1\ \cdots\ dt_i\,du_1\ \cdots\ du_i\\ &=z^{\frac{(d+1)i}{2}}(d-1)^i\int\limits_{E_2}\cdots\int\limits_{E_2}V_{i-1}(\mathrm{proj}_{\rho G}[\bar P_1,\dots,\bar P_i])\\ &\quad\times V_i(\mathrm{proj}_{\rho G}[\bar P_1,\dots,\bar P_i]_\Pi)\,dP_1\ \cdots\ dP_i,\end{aligned}$$

where $\bar P_k$ is the projection of the point P_k onto the boundary of E_2, i.e.,

$$\bar P_k=\frac{P_k}{\|P_k\|}\;r_E\!\left(\frac{P_k}{\|P_k\|}\right)\,.$$

Clearly, the volume-elements dP_k as well as $V_{i-1}(\cdot)V_i(\cdot)$ are homogeneous and invariant with respect to volume preserving affinities acting in the affine subspace $\rho G \cap \{z = 1\}$ or in the affine subspace $\rho G_\perp \cap \{z = 1\}$. Observe that the volume of E_2 equals $2^{\frac{d-1}{2}} H_{d-1}(x)^{-\frac{1}{2}} \kappa_{d-1}$ and the volume of $\mathrm{proj}_{\rho G} E_2$ equals $2^{\frac{i-1}{2}} {}_{\rho G}H_{d-1}(x)^{-\frac{1}{2}} \kappa_{i-1}$, where $H_{d-1}(x)$, resp. ${}_{\rho G}H_{d-1}(x)$, is the Gaussian curvature of K, resp. $\mathrm{proj}_{\rho G} K$, at x and κ_j denotes the volume of the j–dimensional unit ball. Transforming the ellipsoid E_2 into the unit ball B^{d-1} using a suitable affinity, and rewriting the integral again as an integral over S^{d-2}, we finally obtain

$$\begin{aligned}
\mathcal{I}_2(x) &= {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{-\frac{i}{2}}\, z^{\frac{(d+1)i}{2}} \\
&\quad \times \int\limits_{S^{d-2}} \cdots \int\limits_{S^{d-2}} V_{i-1}(\mathrm{proj}_{\rho G}[P_1, \dots, P_i])\, V_i(\mathrm{proj}_{\rho G}[P_1, \dots, P_i]_\Pi)\, dP_1 \,\cdots\, dP_i \\
&= {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{-\frac{i}{2}}\, z^{\frac{(d+1)i}{2}}\, \mathbb{E}(i, d),
\end{aligned}$$

where $\mathbb{E}(i, d)$ is a suitable constant, depending only on i and d, since the integral is invariant under rotations and hence is independent of ρ.

Collecting terms, we obtain

$$\begin{aligned}
&(1+\delta)^{-3i}\mathbb{E}(i,d)2^{-i} d_K(x)^i\, {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{-\frac{i}{2}}\, z^{\frac{(d-1)i}{2}} \\
&\quad + \delta O\left(z^{\frac{(d-1)i}{2}}\right) + O\left(z^{\frac{(d-1)i+1}{2}}\right) \le \\
&\qquad\qquad \le \mathcal{I}_{\partial K \cap H}(h, u, \rho G) \le \\
&\quad \le (1+\delta)^{3i}\mathbb{E}(i,d)2^{-i} d_K(x)^i\, {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{-\frac{i}{2}} z^{\frac{(d-1)i}{2}} \\
&\quad + \delta O\left(z^{\frac{(d-1)i}{2}}\right) + O\left(z^{\frac{(d-1)i+1}{2}}\right)
\end{aligned}$$

as $n \to \infty$ with a suitable constant $\mathbb{E}(i, d)$, and where the constants in $O(\cdot)$ can be chosen independent of x, ρ, and δ.

5.5. In order to expand the integral

$$\int\limits_{h_K(u)-\varepsilon}^{h_K(u)} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h, u, \rho G)\, dh = \int\limits_0^{\varepsilon} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h_K(u) - z, u, \rho G)\, dz \tag{70}$$

we use similar substitutions as in Section 3.6. By (49), (50), and (65) it follows from

$$\begin{aligned}
1 - S_+ &= \int\limits_{f(y) \le z} d_K(y) \sqrt{1 + (\mathrm{grad} f(y))^2}\, dy \\
&= \int\limits_{S^{d-2}} \int\limits_{r \le r(v,z)} d_K(rv) \sqrt{1 + (\mathrm{grad} f(rv))^2}\, r^{d-2}\, dr\, dv
\end{aligned}$$

that

$$s^- \le 1 - S_+ \le s^+ \tag{71}$$

with

$$s^- = (1+\delta)^{-d} 2^{\frac{d-1}{2}} d_K(x) H_{d-1}(x)^{-\frac{1}{2}} \kappa_{d-1} z^{\frac{d-1}{2}}$$

and

$$s^+ = (1+\delta)^{d+1} 2^{\frac{d-1}{2}} d_K(x) H_{d-1}(x)^{-\frac{1}{2}} \kappa_{d-1} z^{\frac{d-1}{2}} .$$

Since the coefficients of the quadratic forms $b_2(\cdot)$ are bounded from below by a positive constant, it follows from (64) that for all $z \le \eta_z$ the radial function r of $K \cap H$ is bounded by λ, i.e., (65) holds for all $z \le \eta_z$ with a suitably chosen constant $\eta_z > 0$ depending only on λ. This also implies that (71) holds for all $s^-, s^+ \le \eta_s$ with a suitable constant $\eta_s > 0$ depending only on λ and thus on δ.

So we obtain the following lower bound for (70):

$$\begin{aligned}
&\int_0^{\varepsilon} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h_K(u) - z, u, \rho G)\ dz \\
&\quad = \int_0^{\eta_z} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h_K(u) - z, u, \rho G)\ dz\ + O\left(e^{-n}\right) \\
&\quad \ge \int_0^{\eta_z} (1-s^+)^{n-i} \Big((1+\delta)^{-3i} \mathbb{E}(i,d) 2^{-i} d_K(x)^i {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{-\frac{i}{2}}\, z^{\frac{(d-1)i}{2}} \\
&\quad\quad + \delta O\left(z^{\frac{(d-1)i}{2}}\right) + O\left(z^{\frac{(d-1)i+1}{2}}\right)\Big)\ dz\ + O\left(e^{-n}\right)
\end{aligned}$$

as $n \to \infty$.

Introducing s^+ as a new variable and then substituting $1 - s^+ = e^{-t}$ as in Section 3.6 leads to Laplace transforms, and using an Abelian theorem we obtain

(72)
$$\begin{aligned}
&\int_0^{\varepsilon} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h_K(u) - z, u, \rho G)\ dz \\
&\quad \ge (1+\delta)^{-(d+4)i - \frac{2(d+1)}{d-1}}\, c(i,d)\, d_K(x)^{-\frac{2}{d-1}}\, {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{\frac{1}{d-1}}\, n^{-i-\frac{2}{d-1}} \\
&\quad\quad + O\left(n^{-i-1-\frac{2}{d-1}}\right) + \delta O\left(n^{-i-\frac{2}{d-1}}\right) + O\left(n^{-i-\frac{3}{d-1}}\right)
\end{aligned}$$

as $n \to \infty$ with

$$c(i,d) = \mathbb{E}(i,d) 2^{-\frac{(d+1)i}{2}} \frac{1}{d-1} \kappa_{d-1}^{-i-\frac{2}{d-1}} \Gamma(i + \frac{2}{d-1})$$

and where the constants in $O(\cdot)$ can be chosen independent of x, ρ, and δ. Analogously, using $S_+ \le 1 - s^-$, we obtain

(73)
$$\begin{aligned}
&\int_0^{\varepsilon} S_+^{n-i} \mathcal{I}_{\partial K \cap H}(h_K(u) - z, u, \rho G)\ dz \\
&\quad \le (1+\delta)^{(d+3)i + \frac{2d}{d-1}}\, c(i,d)\, d_K(x)^{-\frac{2}{d-1}}\, {}_{\rho G}H_{d-1}(x)^{-1}\, H_{d-1}(x)^{\frac{1}{d-1}}\, n^{-i-\frac{2}{d-1}} \\
&\quad\quad + O\left(n^{-i-1-\frac{2}{d-1}}\right) + \delta O\left(n^{-i-\frac{2}{d-1}}\right) + O\left(n^{-i-\frac{3}{d-1}}\right)
\end{aligned}$$

as $n \to \infty$.

Concerning the remaining two integrations, we introduce the j-th normalized elementary symmetric function $H_j(x(u))$ of the principal curvatures of K and the

j-th normalized elementary symmetric function ${}_{\rho G}H_j(x(u))$ of the principal curvatures of $\mathrm{proj}_{\rho G}K$ at the boundary point x with outer unit normal vector u which are defined in (1). Since the constants in the above result are independent of ρ and u, only ${}_{\rho G}H_{i-1}(x(u))$ – depending on ρ and u – and $d_K(x)$ and $H_{d-1}(x(u))$ – depending on u only – are of interest for the remaining integrations. Changing the order of integration gives

$$\int\limits_{SO(d)} \int\limits_{S^{d-1}\cap\rho G} d_K(x(u))^{-\frac{2}{d-1}}\, H_{d-1}(x(u))^{\frac{1}{d-1}}\, {}_{\rho G}H_{i-1}(x(u))^{-1}\, du\, d\rho$$
$$= \int\limits_{S^{d-1}} d_K(x(u))^{-\frac{2}{d-1}}\, H_{d-1}(x(u))^{\frac{1}{d-1}} \int\limits_{SO(d)|_u} {}_{\rho G}H_{i-1}(x(u))^{-1}\, d\rho|_u\, du,$$

where $SO(d)|_u$ denotes the set of all rotations ρ such that $u \in \rho G$, and $d\rho|_u$ corresponds to the normalized Haar measure on $SO(d)|_u$. Using a local version of an integral geometric projection formula for surface area measures which is contained in Hug [11], Lemma 4.3, this equals

$$\int\limits_{S^{d-1}} d_K(x(u))^{-\frac{2}{d-1}}\, H_{d-1}(x(u))^{\frac{1}{d-1}}\, \frac{H_{d-i}(x(u))}{H_{d-1}(x(u))}\; du$$
$$(74)\qquad = \int\limits_{\partial K} d_K(x(u))^{-\frac{2}{d-1}}\, H_{d-1}(x)^{\frac{1}{d-1}}\, H_{d-i}(x)\, dx\,.$$

We summarize our results: by (63), and by (42), by (72), (73), and (74) there exists a constant $c_2^{(i,d)}$, only depending on i and on d, such that

$$(1+\delta)^{-(d+4)i-\frac{2(d+1)}{d-1}}\, c_2^{(i,d)} \int\limits_{\partial K} d_K(x)^{-\frac{2}{d-1}}\, H_{d-1}(x)^{\frac{1}{d-1}}\, H_{d-i}(x)\, dx\;\; n^{-\frac{2}{d-1}}$$
$$+\,\delta O\left(n^{-\frac{2}{d-1}}\right) + O\left(n^{-\frac{3}{d-1}}\right) \le$$
$$\le V_i(K) - \mathbb{E}_n(V_i) \le$$
$$\le (1+\delta)^{(d+3)i+\frac{2d}{d-1}}\, c_2^{(i,d)} \int\limits_{\partial K} d_K(x)^{-\frac{2}{d-1}}\, H_{d-1}(x)^{\frac{1}{d-1}}\, H_{d-i}(x)\, dx\;\; n^{-\frac{2}{d-1}}$$
$$+\,\delta O\left(n^{-\frac{2}{d-1}}\right) + O\left(n^{-\frac{3}{d-1}}\right)$$

as $n \to \infty$. This, holding for each $\delta > 0$, proves Theorem 1. □

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