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SURFACES WITH $p_g = q = 3$

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ABSTRACT. We classify minimal complex surfaces of general type with $p_g = q = 3$. More precisely, we show that such a surface is either the symmetric product of a curve of genus 3 or a free $\mathbb{Z}_2-$quotient of the product of a curve of genus 2 and a curve of genus 3. Our main tools are the generic vanishing theorems of Green and Lazarsfeld and the characterization of theta divisors given by Hacon in Corollary 3.4 of *Fourier transforms, generic vanishing theorems and polarizations of abelian varieties.*

1. INTRODUCTION

The purpose of this paper is to show how the generic vanishing theorems of Green and Lazarsfeld can be concretely applied to the birational classification of complex projective (irregular) surfaces. We believe that these techniques can be applied in a variety of contexts and are of independent interest.

Let X be a smooth minimal complex surface of general type. The smallest possible value of the Euler–Poincaré characteristic $\chi(X) = 1 + p_g(X) - q(X)$ is 1, and for $\chi(X) = 1$ one has the bounds $1 \leq K_X^2 \leq 9$. If, in addition, the surface is irregular, i.e., $q(X) > 0$, then one also has $K_X^2 \geq 2p_g(X)$ (cf. [De]), so that $p_g(X) \leq 4$. The limit case $p_g(X) = 4$ corresponds to the product of two curves of genus 2 (cf. [Be2]). Here we consider the case $q(X) = 3$ (cf. Theorem 2.2) and prove that X is either the symmetric product of a curve of genus 3 ($K_X^2 = 6$) or a free $\mathbb{Z}_2-$quotient of the product of a curve of genus 3 and a curve of genus 2 ($K_X^2 = 8$). Both surfaces have already been described in [CCM], where it is also shown that the former is the only example with $K_X^2 = 6$ and the latter is the only example with a pencil of curves of genus 2.

However, our approach is independent of the results of [CCM], and, as far as we know, it is a new one in the classification of surfaces of general type. We are able to identify the surfaces by looking at the locus $V^1(X) := \{P \in \mathrm{Pic}^0(X) | h^1(-P) > 0\}$, whose properties have been described very precisely by several authors (cf. [GL1], [GL2], [Be1], [Si], [EL]). Roughly speaking, if $V^1(X)$ is $0-$dimensional we prove that X is the symmetric product of a curve of genus 3 by using a cohomological characterization of theta divisors, due to the first author (cf. Theorem 2.9). If instead $V^1(X)$ has positive dimension, then we show the existence of a pencil of curves of genus 2 by using the infinitesimal description of $V^1(X)$ (cf. [GL1]) and the Castelnuovo–de Franchis Theorem. As a by-product of the classification, we obtain a description of the moduli space of surfaces of general type with $p_g = q = 3$ (cf.

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Corollary 2.4) and we determine the degree of the bicanonical map (cf. Corollary 2.3).

We remark that it is possible to generalize the techniques of this paper to higher dimensional varieties. For instance, in [HP], we show:

If X is a smooth projective variety of maximal Albanese dimension such that $h^0(\omega_X) = q(X) = \dim(X) + 1$ and $\dim V^1(\omega_X) = 0$, then X is birational to a theta divisor in $\mathrm{Alb}(X)$.

Moreover, if $\dim V^1(\omega_X) > 0$, then by the results of Green and Lazarsfeld there exists a surjective morphism $X \to Y$ onto a variety of maximal Albanese dimension Y with $0 < \dim(Y) < \dim(X)$ (i.e., if $\dim(X) = 2$ there exists an irrational pencil). However, unlike the surface case, for $\dim(X) \geq 3$, the methods become technically more involved and it seems unrealistic to classify all cases in which $\dim V^1(\omega_X) > 0$ or in which X is not of maximal Albanese dimension.

Acknowledgments. Gian Pietro Pirola has informed us that he has also proven our main result Theorem 2.2. His proof ([Pi]) uses different methods, based on the techniques and results of [CCM].

We are indebted to Margarida Mendes Lopes for drawing our attention to the problem of classifying surfaces of general type with $p_g = q = 3$, for several useful conversations on this subject and for pointing out some inaccuracies in an earlier version of this paper.

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Notations and conventions. We work over the complex numbers; all varieties are projective. We use the standard notation of algebraic geometry; we just recall here the notation for the invariants of a surface X: K_X is the *canonical class*, $p_g(X) = h^0(X, K_X)$ the *geometric genus* and $q(X) = h^1(X, \mathcal{O}_X)$ the *irregularity*. An *irrational pencil of genus g* on a surface X is a fibration $p\colon X \to B$ with B a smooth curve of genus $g > 0$.

2. The classification theorem

Throughout the paper we make the following assumption:

Assumption 2.1. *X is a smooth minimal complex projective surface of general type with $p_g(X) = q(X) = 3$.*

We denote by A the Albanese variety of X and by $a\colon X \to A$ the Albanese map. We also assume that we have fixed a Kähler metric on X, so that for every $P \in \mathrm{Pic}^0(X)$ and $i \geq 0$ there is an antilinear isomorphism $H^i(X, -P) \simeq H^0(X, \Omega^i_X(P))$ (cf. [GL1], 2.5).

Here are two examples of surfaces satisfying Assumption 2.1:

Example 1. Let C be a smooth curve of genus 3. Then the symmetric product of C is a surface X with $K_X^2 = 6$ satisfying Assumption 2.1. Let $\mathrm{Pic}^2(C)$ denote the subset of $\mathrm{Pic}(C)$ consisting of the line bundles of degree 2 and let $\beta\colon X \to \mathrm{Pic}^2(C)$ be the map that sends an unordered pair $\{p, q\} \in X$ to the linear equivalence class of $p + q$. The image D of β is a principal polarization by [LB], Corollary 11.2.2. By the Riemann's Singularity Theorem (cf. [LB], Theorem 11.2.5) D is smooth if C is not hyperelliptic, while it has a double point at the canonical class if C is hyperelliptic. By Riemann–Roch on C, β is one-to-one if C is not hyperelliptic, while if C is hyperelliptic it contracts to a singular point of type A_1, the -2 curve of X corresponding to the canonical series. In either case D is the canonical model of X. It follows that A is the Jacobian of C and that, up to the choice of an identification of $\mathrm{Pic}^2(C)$ with $\mathrm{Pic}^0(C)$, β is the Albanese map of X.

Example 2. Let C_1 be a curve of genus 2 with an elliptic involution σ_1 and C_2 a curve of genus 3 with a free involution σ_2. We write $B_1 := C_1/\langle\sigma_1\rangle$ and $B_2 := C_2/\langle\sigma_2\rangle$. The curve B_1 has genus 1 and B_2 has genus 2. We let $\mathbb{Z}_2$ act freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$. The quotient surface $X := (C_1 \times C_2)/\mathbb{Z}_2$ is a surface with $K_X^2 = 8$ satisfying Assumption 2.1. The projections of $C_1 \times C_2$ onto C_1 and C_2 induce fibrations $p_i \colon X \to B_i$, $i = 1, 2$. The singular fibers of p_1 are two double fibers with smooth support, occurring at the branch points of $C_1 \to B_1$, while the fibers of p_2 are all smooth. The Albanese variety of X is isogenous to the product of the Jacobians of B_1 and B_2.

Our aim is to prove the following:

Theorem 2.2. *The possibilities for a smooth minimal surface X with $p_g(X) = q(X) = 3$ are the following:*

i) $K_X^2 = 6$ *and X is the surface of Example 1;*
ii) $K_X^2 = 8$ *and X is the surface of Example 2.*

Before proving Theorem 2.2 we deduce some consequences from it.

Corollary 2.3. *The bicanonical map of a minimal surface of general type X with $p_g(X) = q(X) = 3$ has degree 2.*

Proof. By Theorem 2.2 X is either the surface of Example 1 or the surface of Example 2. In the former case the bicanonical map has degree 2 by Proposition 3.17 of [CCM]. In the latter case, the degree is 2 by Theorem 5.6 of [Xi]. $\square$

Corollary 2.4. *The moduli space $\mathcal{M}$ of surfaces of general type with $p_g = q = 3$ has two irreducible connected components of dimensions respectively 6 and 5.*

Proof. By Theorem 2.2, $\mathcal{M}$ is a disjoint union $\mathcal{M}_6 \cup \mathcal{M}_8$, where $\mathcal{M}_\alpha := \{[X] \in \mathcal{M} | K_X^2 = \alpha\}$. The sets $\mathcal{M}_\alpha$ are open since K^2 is a topological invariant, thus we only have to show that $\mathcal{M}_6$ and $\mathcal{M}_8$ are irreducible of dimension 6 and 5 respectively. The points of $\mathcal{M}_6$ are in one-to-one correspondence with the isomorphism classes of curves of genus 3, thus the result is well known in this case.

If $[X] \in \mathcal{M}_8$, then $p_2 \colon X \to B_2$ is the only irrational pencil of genus 2 of X. This can be seen in several ways, for instance by observing that a base-point free pencil is determined uniquely by the span of its class in $H^{1,1}(X, \mathbb{C})$ and that $H^{1,1}(X, \mathbb{C})$, being of dimension 2, has only two isotropic lines, spanned by the classes of the general fiber of p_1 and p_2. It follows that the double cover $C_1 \times C_2 \to X$ is also determined uniquely, since it is the étale cover of X that kills the monodromy of the pencil $p_2 \colon X \to B_2$. So $[X]$ is determined by the choice, up to isomorphism, of a curve C_1 of genus 2 with an elliptic involution σ_1 and of a curve C_2 of genus 3 with a free involution σ_2. The pair (C_1, σ_1) is determined by the quotient curve $E_1 := C_1/\sigma_1$, by the line bundle L of degree 1 on E_1 such that $\psi_* \mathcal{O}_{C_1} = \mathcal{O}_{E_1} \oplus L^{-1}$, where $\psi \colon C_1 \to E_1$ is the quotient map, and by the branch divisor $D \equiv 2L$ of ψ. So, taking into account the action of the automorphism group of E_1, we see that (C_1, σ_1) depends on 2 parameters. In addition, it is not difficult to write down an irreducible global family containing all the isomorphism classes of double covers of genus 3 of elliptic curves. Thus the isomorphism classes of the pairs (C_1, σ_1) form an irreducible family of dimension 2. Analogously, the isomorphism class of (C_2, σ_2) is determined by the genus 2 curve $B_2 := C_2/\sigma_2$ and by the choice of a $2-$torsion line bundle L of B_2. The pairs (B_2, L) form a $3-$dimensional irreducible family (cf. for instance [LB], Chapter 8, §3). □

We now turn to the proof of Theorem 2.2. Surfaces satisfying Assumption 2.1 are studied in section 3 of [CCM], where it is proven that if $K_X^2 = 6$ then X is the surface of Example 1. We recall some general facts from [CCM].

Proposition 2.5. *Let X be a surface as in Assumption 2.1. Then:*

i) $6 \le K_X^2 \le 9$*;*
ii) *the Albanese image of X is a surface;*
iii) *if X has an irrational pencil of genus $g > 1$, then X is the surface of Example 2.*

Proof. Statement i) follows from the Miyaoka–Yau inequality $K_X^2 \le 9\chi$ and from the inequality $K_X^2 \ge 2p_g$ of [De]. Statements ii) and iii) correspond to Proposition 3.1, i) and Theorem 3.23 of [CCM], respectively. □

The proof of Theorem 2.2 is based on the study of the set $V^1(X) := \{P \in \operatorname{Pic}^0(X) | \ h^1(-P) > 0\}$. The sets $V^i(Y) := \{P \in \operatorname{Pic}^0(Y) | \ h^i(-P) > 0\}$, for Y a variety of any dimension, have been studied by Green and Lazarsfeld ([GL1], [GL2]), by Simpson ([Si]), and in the case of surfaces by Beauville ([Be1]). We recall here only those properties of $V^1(X)$ that we are going to use:

Theorem 2.6. *Let X be a complex surface of maximal Albanese dimension. Then:*

i) *$V^1(X)$ is a proper subvariety of $\operatorname{Pic}^0(X)$;*
ii) *the irreducible components of $V^1(X)$ are translates of abelian subvarieties of $\operatorname{Pic}^0(X)$ by torsion points;*
iii) *$V^1(X)$ has a component of dimension > 1 iff X has an irrational pencil of genus > 1;*
iv) *Let $P \in V^1(X)$ be a point, let $v \in H^1(X, \mathcal{O}_X)$ be a nonzero vector and let $\sigma \in H^0(X, \Omega_X^1)$ be the conjugate of v. If v is in the tangent cone to $V^1(X)$ at P, then the map $H^0(X, \Omega_X^1(P)) \xrightarrow{\wedge\sigma} H^0(X, \omega_X(P))$ is not injective;*
v) *Let $P \in V^1(X)$. Then $\{P\}$ is a component of $V^1(X)$ iff the map*

$$H^0(X, \Omega_X^1(P)) \xrightarrow{\wedge\sigma} H^0(X, \omega_X(P))$$

is injective for all nonzero $\sigma \in H^0(X, \Omega_X^1)$.

Proof. Statement i) is Theorem 1 of [GL1]. The fact that the components of $V^1(X)$ are translates of abelian subvarieties follows from Theorem 0.1 of [GL2] and the fact that the translation is by a torsion point follows from Theorem 4.2 of [Si]. Statement iii) follows from [Be1], Corollary 2.3. Statement iv) follows by combining Theorem 1.6, Lemma 2.3 and Lemma 2.6 of [GL1]. Statement v) is a consequence of Theorem 1.2, (1.2.3) of [EL]. □

If X is the surface of Example 2, then the set $V^1(X)$ has a component of dimension 2 by Theorem 2.6, iii). On the other hand, using adjunction on $A := \operatorname{Pic}^2(C)$, it is easy to show that for the surface of Example 1 one has $V^1(X) = \{\mathcal{O}_X\}$. The results that follow show that the structure of $V^1(X)$ characterizes X.

Proposition 2.7. *If X does not have an irrational pencil of genus > 1, then the dimension of $V^1(X)$ is 0.*

Proof. We remark first of all that, in view of the assumption, we have $\dim V^1(X) \leq 1$ by Theorem 2.6, iii).

Assume now that there is a $1-$dimensional component T of $V^1(X)$. By Theorem 2.6, ii), $T = T_0 + Q$, where T_0 is an abelian subvariety of $\mathrm{Pic}^0(X)$ and Q is a torsion point. Notice that $Q \notin T_0$, since by Proposition 4.1 of [GL1] $\mathcal{O}_X$ is an isolated point of $V^1(X)$. Fix a torsion point $P \in T$, denote by n its order and assume that n is minimal. This is the same as saying that if $kP \in T_0$ then $kP = 0$. Let $v \in H^1(\mathcal{O}_X)$ be a nonzero vector tangent to T and let $\sigma \in H^0(\Omega^1_X)$ be the conjugate of v. The vector v lies in the tangent cone to $V^1(X)$ at P and therefore the map $H^0(\Omega^1_X(P)) \stackrel{\wedge\sigma}{\to} H^0(\omega_X(P))$ is not injective by Theorem 2.6, iv). Denote by τ a nonzero element in the kernel of this map. Let $\pi \colon Y \to X$ be the connected étale cyclic cover associated to the finite subgroup $\langle P \rangle \subset \mathrm{Pic}^0(X)$. The group $\langle P \rangle$ can be naturally identified with the dual group of the Galois group $G \cong \mathbb{Z}_n$ of π, and $\pi^* H^0(\Omega^1_X(P))$ is the eigenspace of $H^0(\Omega^1_Y)$ on which G acts via the character corresponding to P. Since $\pi^*\sigma \wedge \pi^*\tau = 0$, by the Castelnuovo–de Franchis Theorem there exists a fibration $q \colon Y \to B$ onto a curve of genus at least 2, such that $\pi^*\sigma$ and $\pi^*\tau$ are pull-backs from B. The fibers of q are integral curves of $\pi^*\sigma$. Since $\pi^*\sigma$ is $G-$invariant, it follows that G permutes the fibers of q and thus induces a pencil $p \colon X \to E := B/G$. More precisely, we have a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\bar{\pi}} & E \end{array}$$

Let $\bar{\sigma}$ be the $1-$form on B such that $\pi^*\sigma = q^*\bar{\sigma}$. By commutativity of the diagram, $\bar{\sigma}$ is $G-$invariant and thus it is a pull-back from E. This shows that E is not rational. In view of the assumptions, we conclude that E has genus 1. Since σ is a pull-back from E, comparing the tangent spaces at the origin one sees that $T_0 = p^*\mathrm{Pic}^0(E)$.

Now let $\bar{\tau}$ be the $1-$form on B such that $\pi^*\tau = q^*\bar{\tau}$. By commutativity of the diagram, $\bar{\tau}$ is an eigenvector for G with character corresponding to P. Since this character generates the dual group G^*, the action of G on B is effective, i.e. $\bar{\pi} \colon B \to E$ is a $G-$cover.

We claim that $\bar{\pi}$ is totally ramified, namely it does not factor through an étale cover of E. Assume otherwise, and denote by $H \subsetneq G$ the subgroup generated by the elements that do not act freely on B. Set $Y' := Y/H$, $B' := B/H$. Then we have a commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{\pi'} & X \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\bar{\pi}'} & E \end{array}$$

where the maps are the obvious ones. In particular, $\bar{\pi}'$ is étale by construction. If we denote by $d < n$ the order of H, then the cover π' corresponds to the subgroup of $\mathrm{Pic}^0(X)$ generated by dP. On the other hand, π' is obtained from $\bar{\pi}'$ by taking base change with p, so the subgroup of $\mathrm{Pic}^0(X)$ corresponding to π' is actually a subgroup of $p^*\mathrm{Pic}^0(E)$, i.e. $dP \in p^*\mathrm{Pic}^0(E) = T_0$. This contradicts the minimality assumption on n, and we conclude that $\bar\pi$ is totally ramified. The $G-$action on B gives a decomposition $\bar\pi_*\mathcal{O}_B = \mathcal{O}_E \oplus_{\chi\in G^*-\{1\}} L_\chi^{-1}$, where the L_χ are line bundles and L_χ^{-1} is the eigenspace of $\bar\pi_*\mathcal{O}_B$ on which G acts via the character $\chi \in G^*$. Saying that $\bar\pi$ is totally ramified is the same as saying that $\deg L_\chi > 0$ for every $\chi \in G^* - \{1\}$. The standard formulas for abelian covers (cf. [Pa], Proposition 4.1), give $g(B) = g(E) + \sum_{\chi\in G^*-\{1\}} h^0(\omega_E \otimes L_\chi) = 1 + \sum_{\chi\in G^*-\{1\}} \deg L_\chi \geq n$. Denote by f the genus of the general fiber of q. The Corollary on page 344 of [Be2] gives: $\chi(Y) = n\chi(X) = n \geq (f-1)(g(B)-1) \geq (f-1)(n-1)$. Thus we either have $f = 2$ or $n = 2$, $f = 3$. Recall that f is also the genus of the general fiber of p. Then $f = 2$ is impossible: indeed by the Lemma on page 345 of [Be2] the equality $3 = q(X) = g(E) + f$ would imply that X is the product of E with a curve of genus 2, contradicting the fact that X is of general type. So we are left with the case $n = 2$, $f = 3$. Notice that in this case we also have $g(B) = 2$, so that $2 = \chi(Y) = (g(B)-1)(f-1)$. So, again by the Corollary of page 344 of [Be2], the fibration q is isotrivial with smooth fibers, and the only singular fibers of p are two double fibers with smooth support, occurring at the two branch points of $\bar\pi\colon B \to E$. Hence by [Se], Theorem 2.1 and Remark 2.3, there exist a curve C_1 of genus 3, a curve C_2 of genus g_2, and a finite group G acting faithfully both on C_1 and on C_2 such that:

i) C_2/G is isomorphic to E;

ii) the diagonal action on $C_1 \times C_2$ defined by $(x,y) \overset{g}{\mapsto} (gx, gy)$ is free;

iii) X is isomorphic to the quotient $(C_1 \times C_2)/G$, and the fibration p corresponds to the map $(C_1\times C_2)/G \to C_2/G = E$ induced by the second projection $C_1 \times C_2 \to C_2$.

In addition, the following hold:

a) C_1/G has genus 2 (since $q(X) = g(C_2/G) + g(C_1/G) = 1 + g(C_1/G)$ by Proposition 2.2 of [Se]);

b) $2(g_2 - 1) = |G|$ (since $2(g_2-1) = \chi(C_1 \times C_2) = |G|\chi(X) = |G|$).

Applying the Hurwitz formula to the quotient map $C_1 \to C_1/G$ we see that G has order 2. Condition b) now implies $g_2 = 2$, hence X is the surface of Example 2 and p is the pencil p_1. Thus X has also an irrational pencil of genus 2, contrary to the assumptions. □

Proposition 2.8. *If X does not have an irrational pencil of genus > 1, then $V^1(X) = \{\mathcal{O}_X\}$.*

Proof. Let $\mathcal{O}_X \neq P \in V^1(X)$. By Proposition 2.7, $\{P\}$ is a component of $V^1(X)$. It follows from Theorem 2.6, v) that for all $0 \neq \sigma \in H^0(X, \Omega^1_X)$ the map $H^0(X, \Omega^1_X(P)) \overset{\wedge\sigma}{\to} H^0(X, \omega_X(P))$ is injective. Using Hodge theory with twisted coefficients it follows that the map induced by the cup product

$$H^1(-P) \otimes H^1(X, \mathcal{O}_X) \longrightarrow H^2(-P)$$

is nonzero on (nonzero) simple tensors. Therefore, by a result of H. Hopf (see [ACGH] p. 108) $h^2(-P) \geq h^1(-P) + h^1(\mathcal{O}_X) - 1$. Hence

$$1 = \chi(\mathcal{O}_X) = \chi(-P) = h^2(-P) - h^1(-P) \geq 2.$$

This is the required contradiction. □

The last ingredient of the proof of Theorem 2.2 is the following result from [Hac]:

Theorem 2.9. *Let Z be a smooth complex projective variety of dimension n, let A be an abelian variety and let $f\colon Z \to A$ be a generically finite morphism such that $f(Z)$ is a divisor. Assume that:*

i) $h^0(Z, \Omega^i_Z) = \binom{n+1}{i}$ for $0 \le i \le n$;

ii) $h^i(Z, \omega_Z(f^*P)) = 0$ for all $\mathcal{O}_A \neq P \in \operatorname{Pic}^0(A)$ and all $i > 0$.

Then A is principally polarized and $f(Z)$ is a theta divisor.

Proof. Cf. [Hac], Corollary 3.4. □

Proof of Theorem 2.2. By Proposition 2.5, iii), X has an irrational pencil of genus > 1 iff X is the surface of Example 1.

If X has no irrational pencil of genus > 1, then $V^1(X) = \{\mathcal{O}_X\}$ by Proposition 2.8. Thus we can apply Theorem 2.9 to the Albanese map $a\colon X \to A$. It follows that A is principally polarized and $Y := a(X)$ is a theta divisor. It is well known (cf. [LB], Ch. 11, Corollary 8.2,b)) that an abelian threefold with an irreducible principal polarization is the Jacobian of a curve C of genus 3. As already explained in Example 1, the theta divisor is the canonical model of the symmetric product of C. Thus K_Y is an ample Cartier divisor with $K_Y^2 = 6$. To finish the proof, it suffices to show that the map $a\colon X \to Y$ is birational. If we denote by R the divisorial part of the ramification locus of a, the adjunction formula gives $K_X = a^*K_Y + R$. If we denote by d the degree of a, then we have:

$$K_X^2 = a^*K_YK_X + RK_X \ge a^*K_YK_X \ge (a^*K_Y)^2 = 6d$$

where the first inequality follows from the fact that K_X is nef and the second one from the fact that a^*K_Y is nef. Since $K_X^2 \le 9$ by Proposition 2.5, i), it follows that $d = 1$ and X and Y are birational. □

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