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A PIERI-TYPE FORMULA FOR ISOTROPIC FLAG MANIFOLDS

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Abstract. We give the formula for multiplying a Schubert class on an odd orthogonal or symplectic flag manifold by a special Schubert class pulled back from the Grassmannian of maximal isotropic subspaces. This is also the formula for multiplying a type B (respectively, type C) Schubert polynomial by the Schur P-polynomial p_m (respectively, the Schur Q-polynomial q_m). Geometric constructions and intermediate results allow us to ultimately deduce this formula from formulas for the classical flag manifold. These intermediate results are concerned with the Bruhat order of the infinite Coxeter group $\mathcal{B}_\infty$, identities of the structure constants for the Schubert basis of cohomology, and intersections of Schubert varieties. We show that most of these identities follow from the Pieri-type formula, and our analysis leads to a new partial order on the Coxeter group $\mathcal{B}_\infty$ and formulas for many of these structure constants.

Introduction

The cohomology of a flag manifold G/B has an integral basis of Schubert classes $\mathfrak{S}_w$ indexed by elements w of the Weyl group of G. Consequently, there are integral structure constants $c^w_{u\,v}$ [17, p. 103] defined by the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \;=\; \sum_w c^w_{u\,v}\, \mathfrak{S}_w\,.$$

The constant $c^w_{u\,v}$ is non-negative as it is the number of points in a suitable triple intersection of Schubert varieties. There exist algorithms for computing these numbers $c^w_{u\,v}$: The algebraic structure of these rings is known [9] with respect to a monomial basis, and there are methods (Schubert polynomials) for expressing the $\mathfrak{S}_w$ in terms of this basis [6, 7, 11, 14, 16, 22, 31]. These algorithms do not show the non-negativity of the $c^w_{u\,v}$. When $\mathfrak{S}_v$ is a hypersurface Schubert class, the $c^w_{u\,v}$ are either 0, 1, or 2, by Chevalley's formula [10], which determines the ring structure of the cohomology of G/B with respect to the Schubert basis. It remains an open problem to give a closed or bijective formula for the rest of these constants. The $c^w_{u\,v}$ are expected to count certain chains in the Bruhat order of the Weyl group (see [3] and the references therein).

Of particular interest are Pieri-type formulas which describe the constants $c^w_{u\,v}$ when $\mathfrak{S}_v$ is a special Schubert class pulled back from a Grassmannian projection (G/P, P maximal parabolic), as these determine the ring structure with respect to the Schubert basis for the cohomology of G/P when P is any parabolic subgroup. Pieri-type formulas for Grassmannians of classical groups are known. When $G =$

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$SL_n\mathbb{C}$, this is the classical Pieri formula, and for other groups G these formulas are due to Boe and Hiller [8] and to Pragacz and Ratajski [28, 29, 30]. A goal of this paper is to begin extending these results to all parabolic subgroups P.

When G is $SL_n\mathbb{C}$, a Pieri-type formula for multiplication by a special Schubert class was described [22] in terms of the Weyl group element wu^{-1}. A formula in terms of chains in the Bruhat order was conjectured [1], given geometric [32], algebraic [25] and combinatorial [20] proofs. Our main results are the analogous formulas when G is $Sp_{2n}\mathbb{C}$ or $SO_{2n+1}\mathbb{C}$ and $\mathfrak{S}_v$ is a special Schubert class pulled back from a Grassmannian of maximal isotropic subspaces. These are common generalizations of the Pieri-type formulas for $SL_n\mathbb{C}$, Chevalley's formula, and the formula of Boe and Hiller.

Our proof uses results on the Bruhat order, identities of these structure constants, a decomposition of intersections of Schubert varieties, and formulas in the cohomology of the $SL_n\mathbb{C}$-flag manifold to explicitly determine a triple intersection of Schubert varieties. This shows the coefficients in the Pieri-type formula are the intersection number of a linear space with a collection of quadrics and hence are either 0 or a power of 2. Some intermediate results, including a fundamental identity and some additional identities of the structure constants, are deduced from constructions on $SL_n\mathbb{C}$-flag manifolds [3]. This analysis leads to other results, including a new partial order on the infinite Coxeter group $\mathcal{B}_\infty$. We show how the Pieri-type formula implies our fundamental identity, and use the identities to express many structure constants in terms of the Littlewood-Richardson coefficients for the multiplication of Schur P- (or Q-) functions [34].

1. Statement of results

Schubert classes in the cohomology of the flag manifolds $SO_{2n+1}\mathbb{C}/B$ and $Sp_{2n}\mathbb{C}/B$ form integral bases indexed by elements of the Weyl group $\mathcal{B}_n$. We represent $\mathcal{B}_n$ as the group of permutations w of $\{-n,\ldots,-2,-1,1,\ldots,n\}$ satisfying $w(-a) = -w(a)$ for $1 \leq a \leq n$. Let $\mathfrak{B}_w$ denote the Schubert class indexed by $w \in \mathcal{B}_n$ in $H^*(SO_{2n+1}\mathbb{C}/B)$ and $\mathfrak{C}_w$ in $H^*(Sp_{2n}\mathbb{C}/B)$. The degree of these classes is $2 \cdot \ell(w)$, where the *length* $\ell(w)$ of w is

$$\#\{0 < i < j \leq n \mid w(i) > w(j)\} \quad + \quad \sum_{i>0>w(i)} |w(i)|\,.$$

For an integer i, let $\bar{\imath}$ denote $-i$. For each $1 \leq m \leq n$, define $v_m \in \mathcal{B}_n$ by

$$\overline{m} = v_m(1) < 0 < v_m(2) < \cdots < v_m(n)\,.$$

This indexes a (maximal isotropic) special Schubert class in either cohomology ring, written as $p_m := \mathfrak{B}_{v_m}$ and $q_m := \mathfrak{C}_{v_m}$. We first state the Pieri-type formula for the products $\mathfrak{B}_w \cdot p_m$ and $\mathfrak{C}_w \cdot q_m$ in terms of chains in the Bruhat order on $\mathcal{B}_n$. For this, we need some definitions.

Definition 1.1. The 0-*Bruhat order* $\leq_0$ on $\mathcal{B}_n$ is defined recursively as follows: $u \lessdot_0 w$ is a cover in the 0-Bruhat order if and only if

(1) $\ell(u) + 1 = \ell(w)$, and
(2) $u^{-1}w$ is a reflection of the form $(\bar{\imath},\, i)$ or $(\bar{\imath},\, j)(\bar{\jmath},\, i)$ for some $0 < i < j \leq n$.

With these definitions, we may state Chevalley's formula [10] for multiplication by p_1 and q_1.

$$
\begin{aligned}
\mathfrak{B}_u \cdot p_1 &= \sum_{u \lessdot_0 w} \mathfrak{B}_w, \\
\mathfrak{C}_u \cdot q_1 &= \sum_{u \lessdot_0 w} \chi(u^{-1}w)\mathfrak{C}_w\,,
\end{aligned}
\tag{1.1}
$$

where $\chi(u^{-1}w)$ is the number of transpositions in the reflection $u^{-1}w$ (1 or 2).

We enrich the 0-Bruhat order in two complementary ways. Write the two types of covers in the 0-Bruhat order as $u \lessdot_0 (\overline{\beta},\, \beta)u$ and $u \lessdot_0 (\overline{\beta},\, \overline{\alpha})(\alpha,\, \beta)u$ where $0 < \alpha < \beta \leq n$. (Observe that these reflections act on the left, on the values of u, while those in Definition 1.1(2) act on the right, on the positions of u.) The *labeled* 0-*Bruhat réseau* is a labeled directed multigraph with vertex set $\mathcal{B}_n$ and labeled edges between covers in the 0-Bruhat order given by the following rule: If $u \lessdot_0 (\overline{\beta},\, \beta)u$, then a single edge is drawn with label β. If $u \lessdot_0 (\overline{\beta},\, \overline{\alpha})(\alpha,\, \beta)u$, then two edges are drawn with respective labels $\overline{\alpha}$ and β. Thus if $u \lessdot_0 w$, then $\chi(u^{-1}w)$ counts the edges from u to w in this 0-Bruhat réseau. The *labeled* 0-*Bruhat order* is obtained from this réseau by removing edges with negative integer labels. With this definition, the coefficient of $\mathfrak{B}_w$ given by the Chevalley formula (1.1) is the number of chains from u to w in the 0-Bruhat order, and the coefficient of $\mathfrak{C}_w$ similarly counts chains in the 0-Bruhat réseau.

Given a (saturated) chain γ in either of these structures, let $\mathrm{end}(\gamma)$ denote the endpoint of γ. A *peak* in a chain γ is an index $i \in \{2, \dots, m-1\}$ with $a_{i-1} < a_i > a_{i+1}$, where $a_1, a_2, \dots, a_m$ is the sequence of edge labels in γ. A *descent* is an index $i < m$ with $a_i > a_{i+1}$ and an *ascent* is an index $i < m$ with $a_i < a_{i+1}$.

Theorem A (Pieri-type formula)**.** *Let $u \in \mathcal{B}_n$ and $m > 0$.*

(1) (Odd-orthogonal Pieri-type formula) *We have*

$$\mathfrak{B}_u \cdot p_m \;=\; \sum \mathfrak{B}_{\mathrm{end}(\gamma)}\,,$$

the sum over all chains γ in the labeled 0-Bruhat order of $\mathcal{B}_n$ which begin at u, have length m, and have no peaks.

(2) (Symplectic Pieri-type formula) *There are two equivalent formulae*

$$\mathfrak{C}_u \cdot q_m \;=\; \sum \mathfrak{C}_{\mathrm{end}(\gamma)}\,,$$

either (a) the sum over all chains γ in the labeled 0-Bruhat réseau of $\mathcal{B}_n$ which begin at u, have length m, and have no descents, or (b) the same sum, except with no ascents.

Theorem A shows that the coefficient $b^w_{u\,m}$ of $\mathfrak{B}_u$ in $\mathfrak{B}_u \cdot p_m$ counts certain chains in the Bruhat order (likewise for the coefficient $c^w_{u\,m}$). It generalizes both Chevalley's formula (1.1) and the Pieri-type formula for $SL_n\mathbb{C}/B$, which is expressed in [1, 32] as a sum of certain labeled chains in the Bruhat order on the symmetric group $\mathcal{S}_n$ with no ascents/no descents. The duality of these two formulas, one in terms of peaks for an order, and the other in terms of descents/ascents for an enriched structure on that order has connections to Stembridge's theory of enriched P-partitions [35], where peak and descent sets play a complementary role. These relations are explored in [2], which extends the theory developed in [4] to the ordered structures of this manuscript.

$b^w_{u\,3}$	**0**	**1**	**1**	**1**	**1**
$c^w_{u\,3}$	**0**	**1**	**1**	**1**	**2**
w	$\overline{3}\overline{2}\overline{1}4$	$\overline{4}2\overline{1}3$	$\overline{3}1\overline{2}4$	$\overline{1}2\overline{3}4$	$12\overline{4}3$

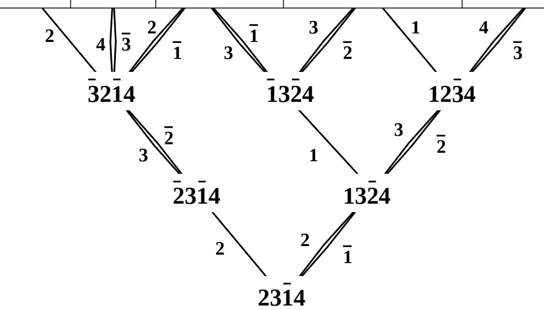


FIGURE 1. Chains above $23\overline{1}4$.

Example 1.2. Represent permutations $w \in \mathcal{B}_4$ by their values $w(1)w(2)w(3)w(4)$. Consider the products $\mathfrak{B}_{23\overline{1}4} \cdot p_3$ and $\mathfrak{C}_{23\overline{1}4} \cdot q_3$. Figure 1 shows the part of the 0-Bruhat réseau of height 3 above $23\overline{1}4$ in $\mathcal{B}_4$. (Erase edges with negative labels to obtain its analog in the 0-Bruhat order.) The entries in the first row of the table count the peakless chains in the 0-Bruhat order, so by Theorem A(1), we have

$$\mathfrak{B}_{23\overline{1}4} \cdot p_3 \;=\; \mathfrak{B}_{\overline{4}2\overline{1}3} + \mathfrak{B}_{\overline{3}1\overline{2}4} + \mathfrak{B}_{\overline{1}2\overline{3}4} + \mathfrak{B}_{12\overline{4}3}\,.$$

The entries in its second row count the chains with no descents in the 0-Bruhat réseau. Thus, by Theorem A(2), we have

$$\mathfrak{C}_{3\overline{1}42} \cdot p_3 \;=\; \mathfrak{C}_{\overline{4}2\overline{1}3} + \mathfrak{C}_{\overline{3}1\overline{2}4} + \mathfrak{C}_{\overline{1}2\overline{3}4} + 2\mathfrak{C}_{12\overline{4}3}\,.$$

Note also that the numbers of chains with no descents and chains with no ascents are equal.

If λ is a *strict partition* (decreasing integral sequence $n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$), then λ determines a unique *Grassmannian permutation* $v(\lambda) \in \mathcal{B}_n$ where $v(i) = \overline{\lambda_i}$ for $i \leq k$ and $0 < v(k+1) < \cdots < v(n)$. If $k = 1$ and $\lambda_1 = m$, then $v_m = v(\lambda)$. The Schubert classes $P_\lambda := \mathfrak{B}_{v(\lambda)}$ and $Q_\lambda := \mathfrak{C}_{v(\lambda)}$ are pullbacks of Schubert classes from the Grassmannians of maximal isotropic subspaces $SO_{2n+1}\mathbb{C}/P_0$ and $Sp_{2n}\mathbb{C}/P_0$, where P_0 is the maximal parabolic associated to the simple root of exceptional length. Formulas for products of these P- and Q-classes are known [34] as these classes are specializations of Schur P- and Q-functions [18, 27].

Our proof of Theorem A uses identities among the structure constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$ defined by the following formulas

$$\mathfrak{B}_u \cdot P_\lambda \;=\; \sum_w b^w_{u\,\lambda}\, \mathfrak{B}_w \qquad \text{and} \qquad \mathfrak{C}_u \cdot Q_\lambda \;=\; \sum_w c^w_{u\,\lambda}\, \mathfrak{C}_w\,.$$

If $u, w, v(\lambda) \in \mathcal{B}_n$, then these constants do not depend upon n.

Iterating Chevalley's formula (1.1) shows that if either of $b^w_{u\,\lambda}$ or $c^w_{u\,\lambda}$ is non-zero, then $u \leq_0 w$ and $\ell(w) - \ell(u)$ equals $|\lambda|$, the sum of the parts of λ. In fact the constant $b^w_{u\,\lambda}$ determines and is determined by the constant $c^w_{u\,\lambda}$: Let $s(w)$ count the sign changes ($\{i \mid i > 0 > w(i)\}$) in w. Then the map $\mathfrak{C}_w \mapsto 2^{s(w)}\mathfrak{B}_w$ embeds

$H^*(Sp_{2n}\mathbb{C}/B)$ into $H^*(SO_{2n+1}\mathbb{C}/B)$. Thus it suffices to work in $H^*(Sp_{2n}\mathbb{C}/B)$. This is fortunate, as a key geometric result, Theorem 3.5, holds only for $Sp_{2n}\mathbb{C}/B$.

Let f_u^w count the saturated chains in the interval $[u,w]_0$ and g_u^w count the saturated chains in the réseau $[u,w]_0$. Iterating Chevalley's formula (1.1) with $u=e$, the identity permutation, we obtain the following expressions

$$p_1^m \;=\; \sum_{|\lambda|=m} f_e^{v(\lambda)}\, P_\lambda \qquad \text{and} \qquad q_1^m \;=\; \sum_{|\lambda|=m} g_e^{v(\lambda)}\, Q_\lambda\,.$$

Multiplying the first expression by $\mathfrak{B}_u$ and collecting the coefficients of $\mathfrak{B}_w$ in the resulting expansion (likewise for the second expression) gives the following proposition.

Proposition 1.3. *Let $u,w\in\mathcal{B}_n$. Then*

$$f_u^w \;=\; \sum_{|\lambda|=\ell(w)-\ell(u)} f_e^{v(\lambda)}\, b_{u\,\lambda}^w \qquad \textit{and} \qquad g_u^w \;=\; \sum_{|\lambda|=\ell(w)-\ell(u)} g_e^{v(\lambda)}\, c_{u\,\lambda}^w\,.$$

Theorem A and Proposition 1.3 show a close connection between chains in the 0-Bruhat order/réseau and the structure constants $b_{u\,\lambda}^w$ and $c_{u\,\lambda}^w$. This justifies an elucidation of the basic properties of the 0-Bruhat order and réseau, which we do in Sections 2 and 6. These structures have a remarkable property and there are related fundamental identities among the structure constants.

Theorem B. *Suppose $u<_0 w$ and $x<_0 z$ in $\mathcal{B}_n$ with $wu^{-1}=zx^{-1}$. Then*

(1) *The map $v\mapsto vu^{-1}x$ induces an isomorphism of labeled intervals in both the 0-Bruhat order and the 0-Bruhat réseau $[u,w]_0 \xrightarrow{\sim} [x,z]_0$.*
(2) *For any strict partition λ,*

$$b_{u\,\lambda}^w \;=\; b_{x\,\lambda}^z \qquad \textit{and} \qquad c_{u\,\lambda}^w \;=\; c_{x\,\lambda}^z\,.$$

We prove a strengthening of Theorem B(1) in Section 2.2 using combinatorial methods. Theorem B(2) is a consequence of a geometric result (Theorem 3.3) proven in Section 4. Both parts of Theorem B are key to our proof of the Pieri-type formula. By Theorem 8.1(3), the Pieri-type formula and Theorem B(1) together imply Theorem B(2).

Let $\zeta\in\mathcal{B}_n$. By Theorem B(1), we may define $\eta\preceq\zeta$ if there is a $u\in\mathcal{B}_n$ with $u\leq_0 \eta u\leq_0 \zeta u$ and also define $\mathcal{L}(\zeta):=\ell(\zeta u)-\ell(u)$ whenever $u\leq_0\zeta u$. Then $(\mathcal{B}_n,\prec)$ is a graded partial order with rank function $\mathcal{L}(\cdot)$, which we call the *Lagrangian order*. We transfer the labeling from the 0-Bruhat order to obtain the labeled Lagrangian order. In the same fashion, we transfer the labeling and multiple edges of the 0-Bruhat réseau to $(\mathcal{B}_n,\prec)$, obtaining the (labeled) *Lagrangian réseau*. By the identity of Theorem B(2), we may define $b_\lambda^\zeta := b_{u\,\lambda}^{\zeta u}$ and $c_\lambda^\zeta := c_{u\,\lambda}^{\zeta u}$ for any $u\in\mathcal{B}_n$ with $u\leq_0\zeta u$ and $|\lambda|=\mathcal{L}(\zeta)$.

These coefficients satisfy one obvious identity, $c_\lambda^\zeta = c_\lambda^{\zeta^{-1}}$, as $c_{u\,v}^w = c_{\omega_0 w,\,v}^{\omega_0 u}$ and $\omega_0\zeta\omega_0=\zeta$, where $\omega_0\in\mathcal{B}_n$ is the longest element. They also satisfy two non-obvious identities, which we now describe. Let $\rho\in\mathcal{B}_n$ be the permutation defined by $\rho(i)=i-1-n$ for $1\leq i\leq n$. Then ρ is the element with largest rank in $(\mathcal{B}_n,\prec)$. Let $\gamma\in\mathcal{B}_n$ be defined by $\gamma(1)=2,\gamma(2)=3,\ldots,\gamma(n)=1$.

Theorem C. *For any $\zeta\in\mathcal{B}_n$,*

(1) *$\mathcal{L}(\zeta)\;=\;\mathcal{L}(\rho\zeta\rho)$ and for any strict partition λ, we have $b_\lambda^\zeta \;=\; b_\lambda^{\rho\zeta\rho}$ and $c_\lambda^\zeta \;=\; c_\lambda^{\rho\zeta\rho}$.*

(2) *If $a\cdot\zeta(a)>0$ for all a, then $\mathcal{L}(\zeta)=\mathcal{L}(\gamma\zeta\gamma^{-1})$ and for any strict partition λ, we have $b^{\zeta}_{\lambda} = b^{\gamma\zeta\gamma^{-1}}_{\lambda}$ and $c^{\zeta}_{\lambda} = c^{\gamma\zeta\gamma^{-1}}_{\lambda}$.*

This is a consequence of a geometric result, Theorem 3.4, proven in Section 5. These identities are the analogs of the cyclic shift identity (Theorem H) of [3], which was generalized by Postnikov [26] to the quantum cohomology of the SL_n flag manifold.

Our last major result is a reformulation of Theorem A in terms of the permutation wu^{-1}. For $\zeta\in\mathcal{B}_n$, let $\mathrm{supp}(\zeta):=\{a>0 \mid \zeta(a)\neq a\}$, the *support* of ζ. A permutation $\zeta\in\mathcal{B}_n$ is *reducible* if it has a non-trivial factorization $\zeta=\eta\cdot\xi$ with $\mathcal{L}(\zeta)=\mathcal{L}(\eta)+\mathcal{L}(\xi)$ where η and ξ have disjoint supports ($\eta\cdot\xi=\xi\cdot\eta$). We say that such a product $\eta\cdot\xi$ is *disjoint*. Permutations $w\in\mathcal{B}_n$ have unique factorizations into disjoint irreducible permutations.

For a permutation $\zeta\in\mathcal{B}_n$, define $\delta(\zeta)=1$ if $a\cdot\zeta(a)>0$ for all a, and $\delta(\zeta)=0$ otherwise. In Section 6, we show that if ζ is irreducible, then $\mathcal{L}(\zeta)\geq \#\mathrm{supp}(\zeta)-\delta(\zeta)$. If we have equality, then, as a permutation in $\mathcal{S}_{\pm[n]}$, ζ has $1+\delta(\zeta)$ cycles.

Definition 1.4. If every irreducible factor η of ζ satisfies $\mathcal{L}(\eta)=\#\mathrm{supp}(\eta)-\delta(\eta)$, then we say that ζ is *minimal*. That is, its length is minimal given its support.

If $\zeta\in\mathcal{B}_n$ is minimal, then set

$$\begin{aligned}\theta(\zeta) &:= 2^{\#\{\text{irreducible factors of }\zeta\}-1},\\ \chi(\zeta) &:= 2^{\#\{\text{irreducible factors }\eta\text{ of }\zeta\text{ with }\delta(\eta)\,=\,1\}}.\end{aligned}$$

If ζ is not minimal, then set $\theta(\zeta)=\chi(\zeta)=0$.

We state the Pieri-type formula in terms of the permutation wu^{-1}.

Theorem D. *Let $u,w\in\mathcal{B}_n$ and $m\leq n$. Then*

$$\mathfrak{B}_u\cdot p_m \;=\; \sum\theta(wu^{-1})\mathfrak{B}_w \qquad \textit{and} \qquad \mathfrak{C}_u\cdot q_m \;=\; \sum\chi(wu^{-1})\mathfrak{C}_w\,,$$

the sum over all $w\in\mathcal{B}_n$ with $u\leq_0 w$ and $\ell(w)-\ell(u)=m$.

This is similar to the form of the Pieri-type formula for $SL_n\mathbb{C}/B$ in [22], which is in terms of the cycle structure of the permutation wu^{-1}. Our proof for $Sp_{2n}\mathbb{C}/B$ shows these multiplicities arise from the intersection of a linear subspace of $\mathbb{P}^{2n-1}$ with a collection of quadrics, one for each irreducible factor η of wu^{-1} with $\delta(\eta)=1$, similar to the proof of the Pieri-type formula for Grassmannians of maximal isotropic subspaces in [33]. The two formulas in Theorem D are shown to be equivalent in Section 6.4, and we show the equivalence of the two formulations (Theorems A and D) of the Pieri-type formula in Sections 6.3 and 6.4.

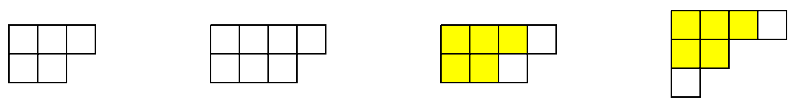
Example 1.5. We return to Example 1.2. For $u=23\overline{1}4$ and w equal to each of $\overline{3}\,\overline{2}\,\overline{1}4$, $\overline{4}2\overline{1}3$, $\overline{3}1\overline{2}4$, $\overline{1}2\overline{3}4$, $12\overline{4}3$ in turn, wu^{-1} is the permutation in $\mathcal{S}_{\pm[4]}$:

$(2,\overline{3})(3,\overline{2})$, $(2,\overline{4},\overline{3},\overline{2},4,3)$, $(1,2,\overline{3},\overline{1},\overline{2},3)$, $(1,3,2,\overline{1},\overline{3},\overline{2})$, and $(1,4,3,2)(\overline{1},\overline{4},\overline{3},\overline{2})$.

These are all irreducible as permutations in $\mathcal{B}_4$. The first is not minimal, while the rest are. Of those, the first three have $\delta=0$, and the last has $\delta=1$. Thus the values of θ are $0,1,1,1,1$ and of χ are $0,1,1,1,2$, which, together with Example 1.2, shows the two forms of the Pieri-type formula agree on this example.

Theorem D generalizes the Pieri-type formula for Grassmannians of maximal isotropic subspaces [8] (see also [17, p. 31]). We describe this. A strict partition λ

can be represented by its Ferrers diagram as a left justified array of boxes with λ_i boxes in the ith row. When $\mu \subset \lambda$, we can consider the skew partition $\lambda/\mu = \lambda - \mu$. Here, the unshaded boxes represent 32, 43, 43/32, and 421/32:



Let $|\lambda/\mu|$ count the boxes in λ/μ. We call λ/μ a *horizontal strip* if there is at most one box in each column. Two boxes in λ/μ are connected if they share a vertex. Let $k(\lambda/\mu)$ count the connected components of λ/μ and let $m(\lambda/\mu)$ count those components not containing a box in the first column. The Pieri-type formulas for these Grassmannians are:

$$P_\mu \cdot p_m = \sum_{\substack{\lambda : |\lambda/\mu| = m \\ \lambda\mu \text{ a horizontal strip}}} 2^{k(\lambda/\mu)-1} P_\lambda,$$

$$Q_\mu \cdot q_m = \sum_{\substack{\lambda : |\lambda/\mu| = m \\ \lambda\mu \text{ a horizontal strip}}} 2^{m(\lambda/\mu)} Q_\lambda.$$

The connection of these formulas to Theorem D is a consequence of the following two facts. The permutation $v(\lambda)v(\mu)^{-1}$ is minimal if and only if λ/μ is a horizontal strip. Connected components of λ/μ correspond to disjoint irreducible factors of $v(\lambda)v(\mu)^{-1}$ and a component has $\delta = 1$ if it does not contain a box in the first column. In the figure above, 43/32 has a single connected component with $\delta = 1$, while 421/32 has two components, one with $\delta = 0$ and the other with $\delta = 1$.

This paper is organized as follows: Section 2 contains basic combinatorial definitions and properties of the Bruhat order on $\mathcal{B}_\infty$ analogous to those of $\mathcal{S}_\infty$ established in [3, 5], and also a strengthened version of Theorem B(1). Section 3 contains our basic geometric definitions and also the geometric Theorems 3.3 and 3.4, which imply Theorems B(2) and C, respectively. Section 4 is devoted to the proof of Theorem 3.3. In Section 5, we prove Theorem 3.4 and derive some useful geometric lemmas. The next two sections form the heart of this paper. In Section 6, we establish further combinatorial properties of the Lagrangian order and réseau, and prove the equivalence of Theorems A and D. These combinatorial results are used in Section 7 to establish the Pieri-type formula, which we prove by first reducing to the case when ζ is irreducible and minimal, and then treating two further subcases separately. Finally, in Section 8, we apply these results to show how the Pieri-type formula implies the identity of Theorem B(2), compute many of the constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$, and deduce some combinatorial consequences for the Bruhat order.

The proof of the Pieri-type formula occupies most of Sections 2 through 7. Those results not needed for the proof are closely related to the other results in those sections. For example, in Section 5 only Lemma 5.9 is used in the proof of the Pieri-type formula—but this lemma requires some other results in Section 5. Figure 2 is a schematic of some main ingredients in the proof of the Pieri-type formula. Here, a solid arrow $\longrightarrow$ indicates a direct implication, while a broken arrow $--\!\!\rightarrow$ indicates that one result is used in the proof of another. Note the centrality of Theorem 3.3 and Sections 6 and 7.

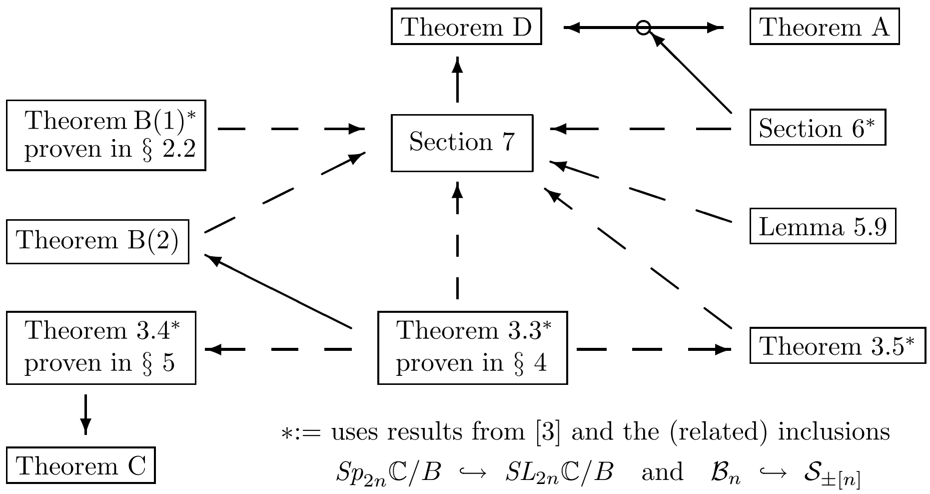


Figure 2. Schematic of Proof of Pieri-type formula.

2. Orders on $\mathcal{B}_\infty$

2.1. **Basic definitions.** We begin with some combinatorial definitions, and then in Section 2.2 derive the basic properties of the 0-Bruhat order on $\mathcal{B}_\infty$ analogous to properties of the k-Bruhat order on $\mathcal{S}_\infty$. We list the various orders with their respective notation and place of definition in Table 1. The entries marked with $(*)$

Table 1. Different orders

Order	notation	rank function	defined in
Bruhat order type B	$<$	ℓ	Sec. 2.1
0-Bruhat order type B	$<_0$	$(*)$	Sec. 2.1
k-Bruhat order type B	$<_k$	$(*)$	Sec. 2.1
Lagrangian order (type B)	$\prec$	$\mathcal{L}$	Def. 2.9
Bruhat order type A	$\lhd$	ℓ_A	Sec. 2.2
k-Bruhat order type A	$\lhd_k$	$(**)$	Sec. 2.2
Grassmannian order (type A)	$\prec\!\!\lhd$	$\mathcal{L}_A$	Def. 2.10

are not ranked posets, but every interval $[x, y]$ is ranked by $\ell(z) - \ell(x)$ for $z \in [x, y]$. Similarly for $(**)$, except that the ranking now is $\ell_A(z) - \ell_A(x)$.

Let $\#S$ be the cardinality of a finite set S. For an integer j, its absolute value is $|j|$ and let $\bar{\jmath} := -j$. Likewise, for a set P of integers, define $\overline{P} := \{\bar{\jmath} \mid j \in P\}$ and $\pm P := P \cup \overline{P}$. Set $[n] := \{1, \ldots, n\}$ and let $\mathcal{S}_{\pm[n]}$ be the group of permutations of $\pm[n]$. Let e be the identity permutation in $\mathcal{S}_{\pm[n]}$ and ω_0 the longest element in $\mathcal{S}_{\pm[n]}$: $\omega_0(i) = \bar{\imath}$. Then $\mathcal{B}_n$ is the subgroup of $\mathcal{S}_{\pm[n]}$ consisting of those w for which $\omega_0 w \omega_0 = w$. We have $\omega_0 \in \mathcal{B}_n$. We also have $\mathcal{B}_n \subset \mathcal{S}_{[\overline{n}, n]}$, the symmetric group on $[\overline{n}, n] := \pm[n] \cup \{0\}$. We refer to elements of these groups as permutations. Permutations $w \in \mathcal{B}_n$ are often represented by their values $w(1)\, w(2)\, \ldots\, w(n)$. For example, $2\,4\,\overline{3}\,\overline{1} \in \mathcal{B}_4$. The *length* $\ell(w)$ of $w \in \mathcal{B}_n$ is [17, p. 66]

$$\ell(w) \quad = \quad \#\{0 < i < j \mid w(i) > w(j)\} \; + \sum_{i > 0 > w(i)} |w(i)|\,.$$

Thus $\ell(2\,4\,\overline{3}\,\overline{1}) = 4 + 4 = 8$. Note that ω_0 is the longest element in $\mathcal{B}_n$.

The inclusion $\pm[n] \hookrightarrow \pm[n{+}1]$ induces inclusions $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ and $\mathcal{S}_{\pm[n]} \hookrightarrow \mathcal{S}_{\pm[n+1]}$. Define $\mathcal{B}_\infty := \bigcup_n \mathcal{B}_n$ and $\mathcal{S}_{\pm\infty} := \bigcup_n \mathcal{S}_{\pm[n]}$. In this representation, $\mathcal{B}_\infty$ has three types of reflections, which are, as elements of $\mathcal{S}_{\pm\infty}$:

$$\begin{array}{lcl} t_{i\,j} & := & (\overline{\jmath},\, \overline{\imath})(i,\, j) \\ t_j & := & (\overline{\jmath},\, j) \\ t_{\overline{\imath}\,j} & := & (\overline{\jmath},\, i)(\overline{\imath},\, j) \end{array} \qquad \text{for } 0 < i < j\,.$$

These reflections act on positions on the right and on values on the left. The *Bruhat order* on $\mathcal{B}_\infty$ is defined by its covers: $u \lessdot w$ if $\ell(u) + 1 = \ell(w)$ and $u^{-1}w$ is a reflection. For each $k = 0, 1, \ldots$, define the *k-Bruhat order* (on $\mathcal{B}_n$ or $\mathcal{B}_\infty$) by its covers: Set $u \lessdot_k w$ if $u \lessdot w$ and

$$u^{-1}w \quad \text{is one of} \quad \left\{ \begin{array}{lll} t_{i\,j} & \text{with } i \leq k < j, & \text{or} \\ t_j & \text{with } k < j, & \text{or} \\ t_{\overline{\imath}\,j} & \text{with } k < j\,. & \end{array} \right.$$

For example, Figure 3 shows all covers $w \in \mathcal{B}_4$ of $u = 2\,4\,\overline{3}\,\overline{1}$, the reflection $u^{-1}w$, and for which k this is a cover in the k-Bruhat order.

An important class of permutations are the *Grassmannian permutations*, those $v \in \mathcal{B}_n$ for which $v(1) < v(2) < \cdots < v(n)$. Such a permutation is determined by its initial negative values. If $v(k) < 0 < v(k+1)$, define $\lambda(v)$ to be the decreasing sequence $\overline{v(1)} > \overline{v(2)} > \cdots > \overline{v(k)}$. Note that $\ell(v) = \overline{v(1)} + \cdots + \overline{v(k)} =: |\lambda(v)|$. Likewise, given a decreasing sequence μ of positive integers (a *strict partition*) with $n \geq \mu_1$, let $v(\mu)$ be the Grassmannian permutation with $\lambda(v(\mu)) = \mu$. We write $\mu \subset \lambda$ for strict partitions μ, λ if $\mu_i \leq \lambda_i$ for all i. For example, $v = \overline{4}\,\overline{1}\,2\,3 \in \mathcal{B}_4$ is Grassmannian with $\lambda(v) = 4\,1$. Since v has no inversions, $\ell(v) = 4 + 1 = |4\,1|$.

2.2. **The 0-Bruhat order.** While these orders are analogous to the k-Bruhat orders on $\mathcal{S}_\infty$ [3, 5, 23, 32], only the 0-Bruhat order on B_∞ enjoys most properties of the k-Bruhat orders on $\mathcal{S}_\infty$. This is because the 0-Bruhat order is an induced suborder of the 0-Bruhat order on $\mathcal{S}_{\pm\infty}$.

The *length* $\ell_{\scriptscriptstyle A}(w)$ of a permutation $w \in \mathcal{S}_{\pm\infty}$ counts the inversions of w:

$$\ell_{\scriptscriptstyle A}(w) \;:=\; \#\{i < j \mid w(i) > w(j)\}\,.$$

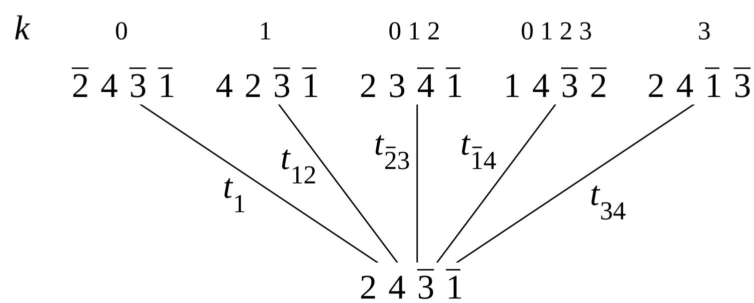


FIGURE 3. Covers of $2\,4\,\overline{3}\,\overline{1}$.

The Bruhat order ($\lessdot$) on $\mathcal{S}_{\pm\infty}$ is defined by its covers: $u \lessdot w$ if and only if wu^{-1} is a transposition and $\ell_A(w) = \ell_A(u) + 1$. Alternatively, $u \lessdot w$ if and only if $(a,\, b) = u^{-1}w$ is a transposition such that $a < b$, $\quad u(a) < u(b)$ and for all $a < i < b$, we have $u(i) < u(a)$ or $u(i) > u(b)$. If $k \in \mathbb{Z}$, this is a cover (written $\lessdot_k$) in the k-Bruhat order ($\lessdot_k$) on $\mathcal{S}_{\pm\infty}$ (or $\mathcal{S}_{\pm[n]}$) if $u^{-1}w = (a,\, b)$ with $a \leq k < b$. The k-Bruhat order has a non-recursive characterization, needed below:

Proposition 2.1 ([3], Theorem A)**.** *Let $u, w \in \mathcal{S}_{\pm\infty}$ and $k \in \mathbb{Z}$. Then $u \lessdot_k w$ if and only if*

(1) *$a \leq k < b$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
(2) *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leq k < b$.*

For the remainder of this paper, we will be concerned with the case $k = 0$.

Theorem 2.2. *The 0-Bruhat order on $\mathcal{B}_\infty$ is the order induced from the 0-Bruhat order on $\mathcal{S}_{\pm\infty}$ by the inclusion $\mathcal{B}_\infty \hookrightarrow \mathcal{S}_{\pm\infty}$.*

Proof. For $u, w \in \mathcal{B}_\infty$, it is straightforward to verify

$$u \lessdot_0 ut_j \iff u \lessdot_0 u(\overline{\jmath},\, j)$$

and

$$u \lessdot_0 ut_{\overline{\imath}\, j} \iff u \lessdot_0 u(\overline{\jmath},\, i) \lessdot_0 u(\overline{\jmath},\, i)(\overline{\imath},\, j)\,.$$

Thus $u <_0 w \Rightarrow u \lhd_0 w$, and so $<_0$ is a suborder of $\lhd_0$.

To show this suborder is induced, we suppose that $u \lhd_0 w$ with $u, w \in \mathcal{B}_\infty$, and argue by induction on $\ell_A(w) - \ell_A(u)$. Suppose $u \lessdot_0 v \lhd_0 w$. If $v = u(\overline{\jmath},\, j) = ut_j$, then $v \in \mathcal{B}_\infty$ and we are done by induction.

Suppose now that $v = u(\overline{\jmath},\, i) \notin \mathcal{B}_\infty$. Let $x \mapsto \overline{x}$ be the involution of $(\mathcal{S}_{\pm\infty}, \lhd_0)$ which fixes $(\mathcal{B}_\infty, <)$ ($\overline{x} := \omega_0 x \omega_0$ when $x, \omega_0 \in \mathcal{S}_{\pm[n]}$). Then $\overline{v} = u(\overline{\imath},\, j)$ also satisfies $u \lessdot_0 \overline{v} \lhd_0 w$. Proposition 2.1(1) show that either $0 < u(i)$ and $0 < u(j)$, or else $u(j) \cdot u(i) < 0$. In the first case, Proposition 2.1(2) forces $u(i) > u(j)$ if and only if $i < j$. We thus have both $u \lessdot_0 u(\overline{\jmath},\, j) \lhd_0 w$ and $u \lessdot_0 u(\overline{\imath},\, i) \lhd_0 w$, and $ut_i,\ ut_j \in \mathcal{B}_\infty$. In the second case $v \lessdot_0 v(\overline{\imath},\, j) \lhd_0 w$ and $v(\overline{\imath},\, j) = ut_{\overline{\imath},\, j} \in \mathcal{B}_\infty$, which completes the proof. $\square$

Remark 2.3. We extract the following useful fact from this proof. If $u(\overline{\jmath},\, i) \lessdot_0 u$, then either $ut_{ij} \lessdot_0 u$ or else both $ut_i \lessdot_0 u$ and $ut_j \lessdot_0 u$.

Example 2.4. Figure 4 illustrates Theorem 2.2. There, the elements of $\mathcal{B}_3$ are boxed.

This relation between the two partial orders $(\mathcal{B}_\infty, <_0)$ and $(\mathcal{S}_{\pm\infty}, \lhd_0)$ makes many properties of $(\mathcal{B}_\infty, <_0)$ easy corollaries of Theorem 2.2 and the analogous results for $(\mathcal{S}_{\pm\infty}, \lhd_0)$ (established in [3, 5]). We discuss these properties in the remainder of this section, leaving some proofs to the reader.

From Proposition 2.1, we deduce the following non-recursive characterization of the 0-Bruhat order on $\mathcal{B}_\infty$.

Proposition 2.5. *Let $u, w \in \mathcal{B}_\infty$. Then $u \leq_0 w$ if and only if*

(1) *$0 < i \Longrightarrow u(i) \geq w(i)$, and*
(2) *$0 < i < j$ and $u(i) < u(j) \Longrightarrow w(i) < w(j)$.*

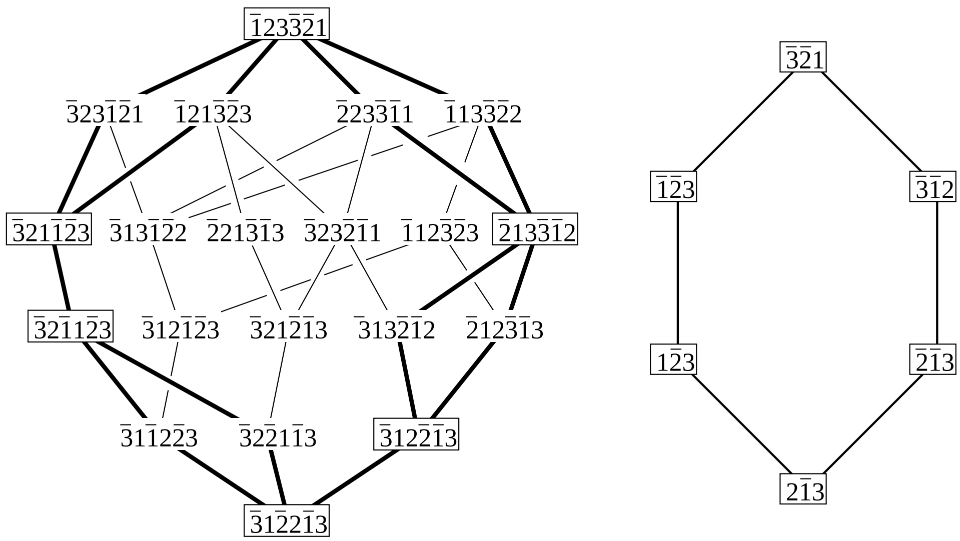


FIGURE 4. The intervals $[\overline{3}\,1\,\overline{2}\,2\,\overline{1}\,3,\ \overline{1}\,2\,3\,\overline{3}\,\overline{2}\,1]_{\lhd_0}$ and $[2\,\overline{1}\,3,\ \overline{3}\,\overline{2}\,1]_0$.

For $P \subset \{1, 2, \dots\} = \mathbb{N}$, let $\#P \in \mathbb{N} \cup \{\infty\}$ be the cardinality of P. For $P = \{p_1, p_2, \dots\}$ where $p_1 < p_2 < \cdots$, the map $i \mapsto p_i$ induces compatible inclusions

$$\begin{tikzcd}
\mathcal{B}_{\#P} \arrow[r, hook, "\varepsilon_P"] \arrow[d, hook] & \mathcal{B}_\infty \arrow[d, hook] \\
\mathcal{S}_{\pm\#P} \arrow[r, hook, "\varepsilon_P"] & \mathcal{S}_{\pm\infty}
\end{tikzcd}$$

Shape-equivalence is the equivalence relation on $\mathcal{B}_\infty$ induced by $u \sim \varepsilon_P(u)$ for $P \subset \mathbb{N}$ and $u \in \mathcal{B}_{\#P}$. The *support* of a permutation $\zeta \in \mathcal{B}_\infty$ is $\{i > 0 \mid \zeta(i) \neq i\}$. Note that the supports of shape-equivalent permutations have the same cardinality. Furthermore, shape-equivalent permutations with the same support are equal. Let $[u, w]_0 := \{v \mid u \leq_0 v \leq_0 w\}$ denote the interval in the 0-Bruhat order between u and w, a finite graded poset. A corollary of Theorem 2.2 and Theorem E(i) of [3] is the following fundamental result about the 0-Bruhat order on $\mathcal{B}_\infty$, which is a strengthening of Theorem B(1).

Theorem B(1)$'$. *Suppose $u, w, x, z \in \mathcal{B}_\infty$ with wu^{-1} shape-equivalent to zx^{-1}. Then $[u, w]_0 \simeq [x, z]_0$. If $\varepsilon_P(wu^{-1}) = zx^{-1}$, then this isomorphism is given by*

$$[u, w]_0 \ni v \ \longmapsto\ \varepsilon_P(vu^{-1})x \in [x, z]_0 .$$

Definition-Example 2.6. We illustrate Theorem B(1)$'$. First, we introduce some notation for permutations in $\mathcal{B}_\infty$. If $\zeta \in \mathcal{B}_\infty$, then as a permutation in $\mathcal{S}_{\pm\infty}$, each of its cycles has one of the following two forms:

$$(a,\, b,\, \dots,\, c) \qquad \text{or} \qquad (a,\, b,\, \dots,\, c,\, \overline{a},\, \overline{b},\, \dots,\, \overline{c})$$

with $|a|, |b|, \dots, |c|$ distinct. Furthermore, every cycle $\eta = (a,\, b,\, \dots,\, c)$ of the first type is paired with another, $\overline{\eta} := (\overline{a},\, \overline{b},\, \dots,\, \overline{c})$, also of the first type. This motivates a 'cycle notation' for permutations $\zeta \in \mathcal{B}_\infty$. Write $\langle a,\, b,\, \dots,\, c\rangle$ for the product

$(a,\,b,\,\ldots\,,\,c)\cdot(\overline{a},\,\overline{b},\,\ldots\,,\,\overline{c})$ and $\langle a,\,b,\,\ldots\,,\,c\,]$ for cycles $(a,\,b,\,\ldots\,,\,c,\,\overline{a},\,\overline{b},\,\ldots\,,\,\overline{c})$ of the second type. Call either of these *cycles* in $\mathcal{B}_\infty$. We will often omit writing the commas.

Consider the isomorphic intervals in the 0-Bruhat orders on $\mathcal{B}_3$ and $\mathcal{B}_4$:

$$\begin{tikzcd}
& \overline{3}\overline{2}1 & \\
\overline{1}\overline{2}3 \arrow[ur, dash] & & \overline{3}\overline{1}2 \arrow[ul, dash] \\
1\overline{2}3 \arrow[u, dash] & & \overline{2}\overline{1}3 \arrow[u, dash] \\
& 2\overline{1}3 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{3}1\overline{2} & \\
\overline{1}3\overline{2} \arrow[ur, dash] & & \overline{3}2\overline{1} \arrow[ul, dash] \\
13\overline{2} \arrow[u, dash] & & \overline{2}3\overline{1} \arrow[u, dash] \\
& 23\overline{1} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{4}12\overline{3} & \\
\overline{1}42\overline{3} \arrow[ur, dash] & & \overline{4}32\overline{1} \arrow[ul, dash] \\
142\overline{3} \arrow[u, dash] & & \overline{3}42\overline{1} \arrow[u, dash] \\
& 342\overline{1} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

If we let u, x, v be the respective bottom elements, and w, z, y the respective top elements, then, $wu^{-1} = zx^{-1} = \langle 12\overline{3}\,]$ and $yv^{-1} = \langle 13\overline{4}\,]$, and these two permutations are shape-equivalent.

This characterization of intervals in the 0-Bruhat order is not shared by the k-Bruhat order on $\mathcal{B}_\infty$, for any $k > 0$. The following example illustrates this for $k = 1$.

Example 2.7. Consider the following two intervals in the 1-Bruhat order on $\mathcal{B}_4$:

$$\begin{tikzcd}
& 3\ \overline{4}\ 1\ 2 & \\
4\ \overline{3}\ 1\ 2 \arrow[ur, dash] & & 2\ \overline{4}\ 1\ 3 \arrow[ul, dash] \\
4\ \overline{2}\ 1\ 3 \arrow[u, dash] & & 2\ \overline{3}\ 1\ 4 \arrow[u, dash] \\
& 3\ \overline{2}\ 1\ 4 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1\ \overline{4}\ 2\ 3 \\
1\ \overline{2}\ 4\ 3 \arrow[u, dash]
\end{tikzcd}$$

If we let u, x be the bottom elements and w, z the top elements, then $wu^{-1} = zx^{-1} = \langle 24\rangle$ but the two intervals are not isomorphic.

Remark 2.8. For any $\zeta \in \mathcal{B}_\infty$, there is a $u \in \mathcal{B}_\infty$ with $u \leq_0 \zeta u$. Suppose we write $\{a \in \pm\mathbb{N} \mid a > \zeta(a)\} = \{a_1, \ldots, a_m\}$ where $\zeta(a_1) < \zeta(a_2) < \cdots < \zeta(a_m)$. Further let $\{a_{m+1} < a_{m+2} < \cdots\} = \mathbb{N} - \{|a_1|, |a_2|, \ldots, |a_m|\}$. If we define $u(j) = a_j$, then Proposition 2.5 implies that $u \leq_0 \zeta u$. Note that if $[m] = \{a > 0 \mid \zeta(a) \neq a\}$, then ζu is a Grassmannian permutation. In this last case we have

$$\#\{l > 0 \mid u(l) < 0\} \quad = \quad \#\{0 > a > \zeta(a)\},$$

as $\zeta(\overline{a}) = \overline{\zeta(a)}$.

This can be seen in the leftmost interval of Definition-Example 2.6. If $\zeta = \langle 12\overline{3}\,]$, then this construction gives $u = 2\overline{1}3$ and $\zeta u = \overline{3}\,\overline{2}1$, which is Grassmannian.

By Theorem B(1)$'$, we may define a new partial order on $\mathcal{B}_\infty$, which we call the *Lagrangian order:* For $\eta, \zeta \in \mathcal{B}_\infty$, set $\eta \preceq \zeta$ if there is a $u \in \mathcal{B}_\infty$ with $u \leq_0 \eta u \leq_0 \zeta u$. By Remark 2.8, the Lagrangian order has the unique minimal element e. This order is graded by the rank, $\mathcal{L}(\zeta)$, where $\mathcal{L}(\zeta) := \ell(\zeta u) - \ell(u)$ whenever $u \leq_0 \zeta u$. These notions have definitions independent of $\leq_0$.

Definition-Theorem 2.9. Let $\eta, \zeta \in \mathcal{B}_\infty$.

(1) Then $\eta \preceq \zeta$ if and only if
  (i) $a \in \pm\mathbb{N}$ with $a > \eta(a) \Longrightarrow \eta(a) \geq \zeta(a)$, and
  (ii) $a, b \in \pm\mathbb{N}$ with $a < b$, $a > \zeta(a), b > \zeta(b)$, and $\zeta(a) < \zeta(b) \Longrightarrow \eta(a) < \eta(b)$.

(2)
$$\mathcal{L}(\zeta) = \sum_{a, 0 > \zeta(a)} |\zeta(a)| - \#\{(a,b) \mid 0 < a < b, a = \zeta(a) > \zeta(b)\}$$
$$- \#\{(a,b) \mid a < b, a > \zeta(a), b > \zeta(b), \zeta(a) > \zeta(b)\} - \sum_{0 > a > \zeta(a)} |a|.$$

Proof. Let u be the permutation with $u \leq_0 \zeta u$ constructed from ζ in Remark 2.8, using Proposition 2.5 and Theorem B(1)$'$. If $u \leq_0 \eta u \leq_0 \zeta u$, then η satisfies the conditions in (1), and conversely.

For (2), consider the difference $\ell(\zeta u) - \ell(u)$. The length of ζu is the first sum, plus the number of inversions of the form $0 < i \leq m < j$ with $\zeta u(i) > \zeta u(j) = u(j)$. (Here, m is as constructed in Remark 2.8.) In the construction of u, each of these is also an inversion in u involving positions $0 < i \leq n < j$, and so are canceled in the difference. The second term counts the remaining inversions of this type in u, the third term counts the inversions with $0 < i < j \leq n$ in u, and the fourth term is $\sum_{i > 0 > u(i)} |u(i)|$. $\square$

We illustrate this result. Consider the rightmost interval of Definition-Example 2.6. It has rank 3, so the permutation $\langle 13\overline{4}]$ has length 3. For this permutation, the four terms in statement 2 are, respectively $7 = 4 + 3$, $2 = \#\{(2,3), (2,4)\}$, $1 = \#\{(\overline{1}, 3)\}$, and 1. Thus $\mathcal{L}\langle 13\overline{4}] = 7 - 2 - 1 - 1 = 3$, as we have already observed.

The Lagrangian order is the $\mathcal{B}_\infty$-counterpart of the Grassmannian Bruhat order $\lessdot$ on $\mathcal{S}_\infty$ [3, 5]. This is defined as follows: Let $\eta, \zeta \in \mathcal{S}_\infty$. Then $\eta \lessdot \zeta$ if and only if there is a $u \in \mathcal{S}_\infty$ with $u \lhd_0 \eta u \lhd_0 \zeta u$. The Grassmannian Bruhat order is ranked with $\mathcal{L}_A(\eta) := \ell_A(\eta u) - \ell_A(u)$ whenever $u \lhd_0 \eta u$. Equivalently, Definition-Theorem 2.9 is the counterpart of the following definition of the Grassmannian order. For $\zeta \in \mathcal{S}_\infty$, let $up(\zeta) = \{j \mid \zeta^{-1}(j) < j\}$ and let $dw(\zeta) = \{j \mid \zeta^{-1}(j) > j\}$.

Definition 2.10 ([3] Sect. 3.2)**.** Let $\eta, \zeta \in \mathcal{S}_\infty$.

(1) Then $\eta \trianglelefteq \zeta$ if and only if
  (1) $a \leq \eta(a) \leq \zeta(a)$ for $a \in \zeta^{-1}\big(up(\zeta)\big)$,
  (2) $a \geq \eta(a) \geq \zeta(a)$ for $a \in \zeta^{-1}\big(dw(\zeta)\big)$,
  (3) $\big(\eta(a) < \eta(b) \Longrightarrow \zeta(a) < \zeta(b)\big)$ for $a < b \in \zeta^{-1}(up(\zeta))$ or $\zeta^{-1}(dw(\zeta))$.

(2) $\mathcal{L}_A(\zeta)$ is given by
$$\#\{(i,j) \in up(\zeta) \times dw(\zeta) \mid i > j\} - \#\{(\zeta(i), \zeta(j)) \in up(\zeta) \times dw(\zeta) \mid i > j\}$$
$$-\#\{(\zeta(i), \zeta(j)) \in up(\zeta)^{\times 2} \mid i < j \text{ and } \zeta(i) > \zeta(j)\}$$
$$-\#\{(\zeta(i), \zeta(j)) \in dw(\zeta)^{\times 2} \mid i < j \text{ and } \zeta(i) > \zeta(j)\}.$$

Let $s(\zeta)$ count the sign changes $\{a > 0 \mid 0 > \zeta(a)\}$ in ζ. We have the following relation between these two orders.

Corollary 2.11.

(1) *$(\mathcal{B}_\infty, \prec)$ is an induced suborder of $(\mathcal{S}_{\pm\infty}, \lessdot\,)$.*
(2) *For $\zeta \in \mathcal{B}_\infty (\subset \mathcal{S}_{\pm\infty})$, we have $\mathcal{L}(\zeta) = \big(\mathcal{L}_A(\zeta) + s(\zeta)\big)/2$.*

Proof. The first statement is a consequence of Theorem 2.2. For the second statement, consider any maximal chain in $[e,\zeta]_\prec$ (in $\mathcal{B}_\infty$). By Theorem 2.2, this gives a maximal chain in $[e,\zeta]_{\lessdot}$ (in $\mathcal{S}_{\pm\infty}$), where covers of the form $\eta \prec t_{ab}\,\eta$ are replaced by $\eta \lessdot (a,b)\eta \lessdot (a,b)(\overline{a},\overline{b})\eta$. Thus $\mathcal{L}_A(\zeta) = \mathcal{L}(\zeta) + \tau$, where τ counts the covers in that chain if the form $\eta \prec t_{ab}\,\eta$. Since only covers of the form $\eta \prec t_b\eta$ contribute to $s(\zeta)$, we have $\mathcal{L}_A(\zeta) = 2\mathcal{L}(\zeta) - s(\zeta)$. □

We remark on a notational convention: we use Latin letters u, v, w, x, y, z or one-line notation for permutations when using the Bruhat orders on $\mathcal{B}_\infty$ or $\mathcal{S}_\infty$, and Greek letters η, ζ, ξ or cycle notation when using the Lagrangian or Grassmannian Bruhat orders.

Let $\eta, \zeta \in \mathcal{B}_\infty$. If $\zeta \cdot \eta = \eta \cdot \zeta$ with $\mathcal{L}(\eta \cdot \zeta) = \mathcal{L}(\eta) + \mathcal{L}(\zeta)$, and neither of ζ or η is the identity, then $\eta \cdot \zeta$ is the *disjoint product* of η and ζ. (In general $\mathcal{L}(\eta \cdot \zeta) \geq \mathcal{L}(\eta) + \mathcal{L}(\zeta)$.) If a permutation cannot be factored in this way, it is *irreducible*. Permutations $\zeta \in \mathcal{B}_\infty$ factor uniquely into disjoint irreducible permutations. This is most easily described in terms of non-crossing partitions [21]: (A non-crossing partition of $\pm\mathbb{N}$ is a set partition such that if $a < c < b < d$ with a, b in a part π and c, d in a part π', then $\pi = \pi'$, as otherwise the parts π, π' are crossing.)

First, consider ζ as an element of $\mathcal{S}_{\pm\infty}$. Let Π be the finest non-crossing partition of $\pm\mathbb{N}$ which is refined by the partition given by the cycles of ζ. For each non-singleton part π of Π, let ζ_π be the product of the cycles of ζ which partition π. (These ζ_π are the disjoint irreducible factors of ζ, as an element of $\mathcal{S}_{\pm\infty}$.) Since $\zeta \in \mathcal{B}_\infty$, for each such part π of Π, either $\pi = \overline{\pi}$ or else one of $\pi, \overline{\pi}$ consists solely of positive integers. In the first case, ζ_π is an irreducible factor of ζ (as an element of $\mathcal{B}_\infty$), and in the second, $\zeta_\pi\zeta_{\overline{\pi}}$ is an irreducible factor of ζ.

For example, $\langle 3\,] \cdot \langle 12\rangle$ is the disjoint product of $\langle 3\,]$ and $\langle 12\rangle$, while $\langle 2\,]\langle 1\overline{3}\rangle$ is irreducible.

The main result concerning this disjointness is the following straightforward consequence of Theorem 2.2 and Theorem G(i) of [3]:

Proposition 2.12. *Suppose $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of $\zeta \in \mathcal{B}_\infty$ into disjoint irreducible permutations. Then the map $(\eta_1, \ldots, \eta_s) \mapsto \eta_1 \cdots \eta_s$ induces an isomorphism*

$$[e,\zeta_1]_\prec \times \cdots \times [e,\zeta_s]_\prec \;\xrightarrow{\;\sim\;}\; [e,\zeta]_\prec\,.$$

We summarize some properties of $(\mathcal{B}_\infty, \prec)$, which follow from previous arguments and Theorem 3.2.3 of [3].

Proposition 2.13.

(1) *$(\mathcal{B}_\infty, \prec)$ is a graded poset with minimal element e and rank function $\mathcal{L}(\cdot)$.*
(2) *The map $\lambda \mapsto v(\lambda)$ exhibits the lattice of strict partitions as an induced suborder of $(\mathcal{B}_\infty, \prec)$.*
(3) *If $u \leq_0 \zeta u$, then $\eta \mapsto \eta u$ induces an isomorphism $[e,\zeta]_\prec \xrightarrow{\sim} [u,\zeta u]_0$.*
(4) *If $\eta \preceq \zeta$, then $\xi \mapsto \xi\eta^{-1}$ induces an isomorphism $[\eta,\zeta]_\prec \xrightarrow{\sim} [e,\zeta\eta^{-1}]_\prec$.*
(5) *For every infinite set $P \subset \mathbb{N}$, the map $\varepsilon_P : \mathcal{B}_\infty \to \mathcal{B}_\infty$ is an injection of graded posets. Thus if $\eta, \zeta \in \mathcal{B}_\infty$ are shape-equivalent, then $[e,\zeta]_\prec \simeq [e,\eta]_\prec$.*
(6) *The map $\eta \mapsto \eta\zeta^{-1}$ induces an order-reversing isomorphism between $[e,\zeta]_\prec$ and $[e,\zeta^{-1}]_\prec$.*

Example 2.14. Figure 5 shows the Lagrangian order on $\mathcal{B}_3$. The thickened lines are between skew Grassmannian permutations $v(\lambda)v(\mu)^{-1}$ for $\mu \subset \lambda$.

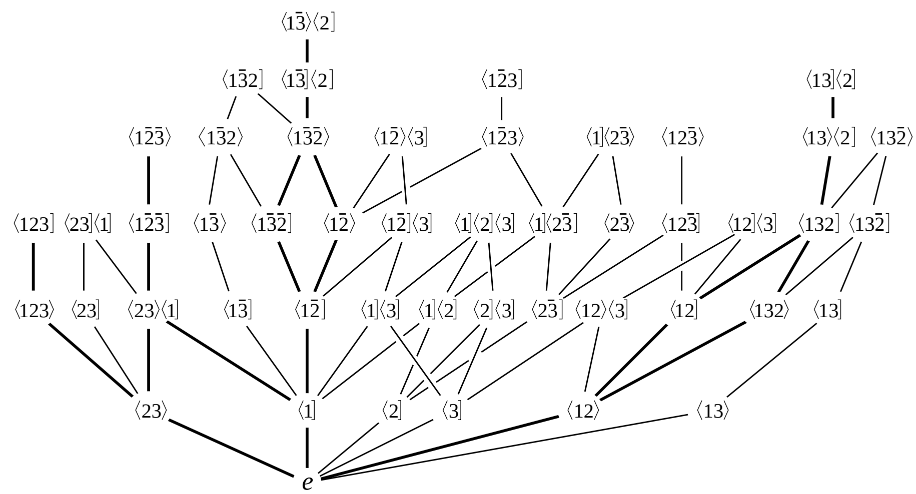


FIGURE 5. The Lagrangian order on $\mathcal{B}_3$.

A chain in either $[u, \zeta u]_0$ or $[e, \zeta]_\prec$ is a particular factorization of ζ into transpositions t_b and $t_{a\,b}$. We give an algorithm for finding a chain in $[e, \zeta]_\prec$, which is the analog of the algorithm given in Remark 3.1.2 of [3]. For this, set $t_{\overline{b}\,b} = t_{b\,\overline{b}} = t_b$.

Algorithm 2.15.
`input:` A permutation $\zeta \in \mathcal{B}_\infty$.
`output:` Permutations $\zeta, \zeta_1, \ldots, \zeta_m = e$ such that

$$e \prec \zeta_{m-1} \prec \cdots \prec \zeta_1 \prec \zeta$$

is a saturated chain in the Lagrangian order.

Output ζ. While $\zeta \neq e$, do

1 Choose $b \in \mathbb{N}$ maximal subject to $b > \zeta(b)$.
2 Choose a minimal subject to $a \leq \zeta(b) < \zeta(a)$.
3 $\zeta := \zeta t_{a\,b}$, output ζ.

Before every execution of 3, $\zeta\, t_{a\,b} \prec \zeta$. Moreover, this algorithm terminates in $\mathcal{L}(\zeta)$ iterations and the reverse of the sequence produced is a chain in $[e, \zeta]_\prec$.

3. Isotropic flag manifolds and maximal Grassmannians

We begin with some basic definitions concerning the flag manifolds G/B for the symplectic and orthogonal groups, then in Section 3.1 we develop properties of their Schubert varieties analogous to those for the $SL_n\mathbb{C}$-flag manifold. We give the statements of geometric results which imply Theorem B(2) and Theorem C, and in Section 3.2 prove an important product decomposition property of certain intersections of Schubert varieties.

Let V denote either $\mathbb{C}^{2n+1}$ equipped with a non-degenerate symmetric bilinear form or $\mathbb{C}^{2n}$ equipped with a non-degenerate alternating bilinear form. In the first case, V is an *odd orthogonal vector space*, and in the second, a *symplectic vector space*. A linear subspace K of V is *isotropic* if the restriction of the form to K is identically zero. Isotropic subspaces have dimension at most n. An *isotropic flag* in V is a sequence $E_\bullet$ of isotropic subspaces:

$$E_\bullet \ :\ E_{\overline{n}} \subset E_{\overline{n-1}} \subset \cdots \subset E_{\overline{1}},$$

where $\dim E_{\overline{\imath}} = n+1-i$. Let $K^\perp$ be the annihilator of a subset K of V. Given an isotropic flag $E_\bullet$ in V, we obtain a canonical complete flag in V (also written $E_\bullet$) by defining $E_i := E^\perp_{\overline{i+1}}$ for $i = 1,\ldots,n$, and in the odd orthogonal case, $E_0 := E^\perp_{\overline{1}}$. Henceforth, flags will always be complete, although we may only specify the subspaces $E_{\overline{n}},\ldots,E_{\overline{1}}$. Indexing flags by elements of $\pm[n]$ or $[\overline{n},n]$ corresponds to our concrete realization of $\mathcal{B}_n$ as permutations.

The group G of linear transformations of V which preserve the given form acts transitively on the set of isotropic flags in V. Since the stabilizer of an isotropic flag is a Borel subgroup B of G, this exhibits the set of isotropic flags as the homogeneous space G/B. Here, G is either $SO_{2n+1}\mathbb{C}$ (odd orthogonal) or $Sp_{2n}\mathbb{C}$ (symplectic). Similarly, G acts transitively on the set of maximal isotropic subspaces of V, exhibiting it as the homogeneous space G/P_0. Here P_0 is the stabilizer of a maximal isotropic subspace, a maximal parabolic subgroup associated to the simple root of exceptional length. Let $\pi : G/B \twoheadrightarrow G/P_0$ be the projection map.

The rational cohomology rings [9] of G/B for both the symplectic and odd-orthogonal flag manifolds are isomorphic to

$$\mathbb{Q}[x_1,\ldots,x_n]/\langle e_i(x_1^2,\ldots,x_n^2), i=1,\ldots,n\rangle\,,$$

where $e_i(a_1,\ldots,a_n)$ is the ith elementary symmetric polynomial in $a_1,\ldots,a_n$. However, their integral cohomology rings differ [13]:

$$\begin{aligned}
H^*(Sp_{2n}\mathbb{C}/B) &\simeq \mathbb{Z}[x_1,\ldots,x_n]/\left\langle e_i(x_1^2,\ldots,x_n^2)\right\rangle.\\
H^*(So_{2n+1}\mathbb{C}/B) &\simeq \mathbb{Z}[x_1,\ldots,x_n,c_1,\ldots,c_n]/I,\\
I &= \left\langle e_i(x_1^2,\ldots,x_n^2),\, 2c_i - e_i(x_1,\ldots,x_n),\, c_{2i} = (-1)^i c_i^2\right\rangle.
\end{aligned}$$

These rings have another description in terms of Schubert classes, given below.

3.1. **Schubert varieties.** Since an isotropic flag $E_\bullet \in G/B$ is also a complete flag in V, we have a canonical embedding $G/B \hookrightarrow \mathbb{F}\ell(V)$, the manifold of complete flags in V. Similarly, there is an embedding $G/P_0 \hookrightarrow \mathbf{G}_n(V)$, the Grassmannian of n-dimensional subspaces of V. We use these maps to understand some structures of G/B.

Given $w \in \mathcal{B}_n$ and an isotropic (complete) flag $E_\bullet \in G/B$, the *Schubert variety* $Y_wE_\bullet$ (or $Y_w(E_\bullet)$) of G/B is the collection of all flags $F_\bullet \in G/B$ satisfying

$$\dim E_i \cap F_{\overline{\jmath}} \quad\geq\quad \#\{n \geq l \geq j \mid w(l) \leq i\}\,, \tag{3.1}$$

for each $j \in [n]$ and $-n \leq i \leq n$ ($i \neq 0$ in the symplectic case) [17, p. 66]. This has codimension $\ell(w)$ in G/B. Also, $Y_wE_\bullet \subset Y_uE_\bullet$ if and only if $u \leq w$ in the Bruhat order. The Schubert cell $Y_u^\circ E_\bullet$ is the set of flags $F_\bullet$ for which equality holds in (3.1). These are the flags in $Y_uE_\bullet$ which are not in any sub-Schubert variety ($Y_wE_\bullet$ with $u < w$).

If now $E_\bullet \in \mathbb{F}\ell(V)$ and $w \in \mathcal{S}_{[\overline{n},n]}$ ($\mathcal{S}_{\pm[n]}$ in the symplectic case), then the Schubert variety $X_wE_\bullet$ of $\mathbb{F}\ell(V)$ is the collection of flags $F_\bullet \in \mathbb{F}\ell(V)$ satisfying (3.1) for all $\overline{n} \leq i,j \leq n$ ($i,j \neq 0$ in the symplectic case). Furthermore, if $w \in \mathcal{B}_n$ and $E_\bullet \in G/B$, then

$$Y_wE_\bullet \;=\; G/B \cap X_wE_\bullet\,.$$

The Schubert cells constitute a cellular decomposition of G/B. Thus *Schubert classes*, the cohomology classes Poincaré dual to the fundamental cycles of Schubert varieties, form $\mathbb{Z}$-bases for these cohomology rings. Write $\mathfrak{B}_w$ for the class $[Y_w E_\bullet]$ in $H^*(SO_{2n+1}\mathbb{C}/B)$ Poincaré dual to the fundamental cycle of $Y_w E_\bullet$ of $SO_{2n+1}\mathbb{C}/B$ and $\mathfrak{C}_w$ for the corresponding class in $H^*(Sp_{2n}\mathbb{C}/B)$. Since these are bases, there are integral structure constants $b^w_{u\,v}$ and $c^w_{u\,v}$ for $u, w, v \in \mathcal{B}_n$ defined by the identities

$$\mathfrak{B}_u \cdot \mathfrak{B}_v \;=\; \sum_w b^w_{u\,v}\, \mathfrak{B}_w \qquad \text{and} \qquad \mathfrak{C}_u \cdot \mathfrak{C}_v \;=\; \sum_w c^w_{u\,v}\, \mathfrak{C}_w\,.$$

Let $s(w)$ count the number of sign changes in the permutation w. Then the isomorphism of rational (coefficients in $\mathbb{Q}$) cohomology rings is induced by the map [7]

$$\mathfrak{C}_w \;\longmapsto\; 2^{s(w)}\, \mathfrak{B}_w\,.$$

Thus

$$2^{s(u)+s(v)}\, b^w_{u\,v} \;=\; 2^{s(w)}\, c^w_{u\,v}\,. \tag{3.2}$$

Hence, it suffices to establish identities and formulas for $Sp_{2n}\mathbb{C}/B$. We do this, because a crucial geometric result (Theorem 3.5) does not hold for $SO_{2n+1}\mathbb{C}/B$. Moreover, since $H^*(Sp_{2n}\mathbb{C}/P) \hookrightarrow H^*(Sp_{2n}\mathbb{C}/B)$, it suffices to work in the ring $H^*(Sp_{2n}\mathbb{C}/B)$ to establish formulas valid in all $H^*(Sp_{2n}\mathbb{C}/P)$.

Two flags $E_\bullet, E'_\bullet$ are *opposite* if $\dim(E_i \cap E'_{\overline{\imath}}) = 1$ for all i. In what follows, Y_u and Y'_v will always denote Schubert varieties defined by fixed, but arbitrary opposite isotropic flags. A consequence of Kleiman's theorem on the transversality of a general translate [19], results in [12], and some combinatorics, is the following proposition.

Proposition 3.1. *Let $u, w \in \mathcal{B}_n$. Then $Y_u \cap Y'_{\omega_0 w} \neq 0$ if and only if $u \leq w$ in the Bruhat order. If $u \leq w$, then $Y_u, Y'_{\omega_0 w}$ meet generically transversally, and the intersection cycle is irreducible of dimension $\ell(w) - \ell(u)$.*

The top-dimensional component of $H^*(G/B)$ is generated by the class of a point $[\mathrm{pt}] = \mathfrak{B}_{\omega_0}$ (or $\mathfrak{C}_{\omega_0}$). The map $\deg : H^*(G/B) \to \mathbb{Z}$ selects the coefficient of $[\mathrm{pt}]$ in a cohomology class. The *intersection pairing* on $H^*(G/B)$ is the composition

$$\beta, \gamma \in H^*(G/B) \;\longmapsto\; \deg(\beta \cdot \gamma)\,.$$

By Proposition 3.1, the product $[Y_u] \cdot [Y_v]$ is the cohomology class $[Y_u \cap Y'_v]$. In particular, when $v = \omega_0 u$, these intersections are single reduced points, so that $[Y_u]$ and $[Y_{\omega_0 u}]$ are dual under the intersection pairing. Thus

$$c^w_{u\,v} \;=\; \deg(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w} \cdot \mathfrak{C}_v)\,,$$

which is also the number of points in the intersection

$$Y_u \cap Y'_{\omega_0 w} \cap Y''_v\,,$$

where Y''_v is defined by a flag $E''_\bullet$ opposite to both $E_\bullet$ and $E'_\bullet$ (which define Y_u and $Y'_{\omega_0 w}$).

We derive a useful description of flags in the intersection of the Schubert cells $Y^\circ_u \cap Y'^\circ_{\omega_0 w}$ when $u \leq_0 w$. For $S \subset V$, let $\langle S \rangle$ be the linear span of S.

Lemma 3.2. *Suppose that $u \leq_0 w$ and $E_\bullet, E'_\bullet$ are opposite isotropic flags in V. Then there are algebraic functions $g_{\overline{\jmath}} : Y^\circ_u E_\bullet \cap Y^\circ_{\omega_0 w} E'_\bullet \to V$ for $1 \leq j \leq n$ such that for each flag $F_\bullet \in Y^\circ_u E_\bullet \cap Y^\circ_{\omega_0 w} E'_\bullet$ and each $1 \leq j \leq n$,*

(1) $F_{\overline{\jmath}} = \langle g_{\overline{n}}(F_\bullet), \ldots, g_{\overline{\jmath}}(F_\bullet)\rangle$, *and*
(2) $g_{\overline{\jmath}}(F_\bullet) \in E_{u(j)} \cap E'_{\overline{w(j)}}$.

Proof. The representation of Schubert cells via parameterized matrices [17, p. 67] gives V-valued functions $f_{\overline{\jmath}}$ defined on the Schubert cell $Y_u^\circ E_\bullet$ such that if $F_\bullet$ is a flag in that cell, then $F_{\overline{\jmath}} = \langle f_{\overline{n}}(F_\bullet), \ldots, f_{\overline{\jmath}}(F_\bullet)\rangle$, and $f_{\overline{\jmath}} \in E_{u(j)}$.

We construct the functions $g_{\overline{\jmath}}$ inductively. First, set $g_{\overline{n}}(F_\bullet) := f_{\overline{n}}(F_\bullet)$ for $F_\bullet \in Y_u^\circ E_\bullet \cap Y^\circ_{\omega_0 w} E'_\bullet$. Since $F_{\overline{n}} \subset E_{u(n)} \cap E'_{\overline{w(n)}}$, conditions (1) and (2) are satisfied for $g_{\overline{n}}$. Suppose we have constructed $g_{\overline{\imath}}$ for $n \geq i > j$. Let $g_{\overline{\jmath}}(F_\bullet)$ be the intersection of $E'_{\overline{w(j)}}$ with the affine space

$$W_j \ := \ f_{\overline{\jmath}}(F_\bullet) \ + \ \langle g_{\overline{\imath}}(F_\bullet) \mid i > j \text{ and } w(i) < w(j)\rangle \,.$$

There is a unique point of intersection: Since $F_\bullet \in Y^\circ_{\omega_0 w} E'_\bullet$,

$$\dim E'_{\overline{w(j)}} \cap F_{\overline{\jmath}} \ = \ \#\{i \mid i \geq j \text{ and } w(i) \geq w(j)\} \,.$$

Since $u \leq_0 w$, if $i > j$ and $w(i) < w(j)$, then necessarily $u(i) < u(j)$, by Proposition 2.5(2). Hence $W_j \subset E_{u(j)}$ and so $g_{\overline{\jmath}}(F_\bullet) \in E_{u(j)} \cap E'_{\overline{w(j)}}$. $\square$

Schubert varieties Υ_λ of G/P_0 are indexed by strict partitions λ, which are decreasing sequences $n \geq \lambda_1 > \cdots > \lambda_k$ of positive integers. The projection map $\pi : G/B \twoheadrightarrow G/P_0$ maps Schubert varieties to Schubert varieties, with $\pi Y_u = \Upsilon_\lambda$, where λ consists of the positive numbers among $\{\overline{u(1)}, \overline{u(2)}, \ldots, \overline{u(n)}\}$ arranged in decreasing order. For a strict partition λ, let $v(\lambda)$ be the Grassmannian permutation whose (initial) negative values are $\overline{\lambda_1} < \overline{\lambda_2} < \cdots < \overline{\lambda_k}$. Let λ^c be the decreasing sequence obtained from the integers in $[n]$ which do not appear in λ. Then an easy argument shows $Y_{v(\lambda)} = \pi^{-1}\Upsilon_\lambda$ and

$$\pi \ : \ Y_{\omega_0 v(\lambda)} \ \longrightarrow \ \Upsilon_{\lambda^c}$$

is an isomorphism of their Schubert cells, and hence is generically one-to-one.

We will let $O(V)$ denote $SO_{2n+1}\mathbb{C}/P_0$ and $Lag(V)$ denote $Sp_{2n}\mathbb{C}/P_0$, and call $Lag(V)$ the *Lagrangian Grassmannian.* Set $P_\lambda := [\Upsilon_\lambda]$ in $H^*(O(V))$ (equivalently $P_\lambda := [Y_{v(\lambda)}]$ in $H^*(SO_{2n+1}\mathbb{C}/B)$) and let Q_λ be the corresponding class in the symplectic case. For $1 \leq m \leq n$, these definitions imply that the *special Schubert variety*, $\Upsilon_{(m)}$, is the collection of all maximal isotropic subspaces which meet a fixed $(n+1-m)$-dimensional isotropic subspace. Let p_m (respectively q_m) denote the class $P_{(m)}$ in either $H^*(SO_{2n+1}\mathbb{C}/B)$ or $H^*(O(V))$ (respectively the class $Q_{(m)}$ in either $H^*(Sp_{2n}\mathbb{C}/B)$ or $H^*(Lag(V))$).

We are particularly interested in the constants $b^w_{u\,\lambda} := b^w_{u\,v(\lambda)}$ and $c^w_{u\,\lambda} := c^w_{u\,v(\lambda)}$ which give the structure of the cohomology of G/B as a module over the cohomology of G/P_0. Using the intersection pairing and the projection formula (see [15, 8.1.7]), we have

$$\begin{aligned} c^w_{u\,\lambda} &= \deg(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w} \cdot \pi^*(Q_\lambda)) \\ &= \deg(\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w}) \cdot Q_\lambda)\,, \end{aligned}$$

and a similar formula for $b^w_{u\,\lambda}$. Our main technique will be to find formulas for $\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w})$ by studying the effect of the map π on the cycle $Y_u \cap Y'_{\omega_0 w}$.

To that end, define $\mathcal{Y}^w_u := \pi(Y_u \cap Y'_{\omega_0 w})$. These cycles $\mathcal{Y}^w_u$ are, like Schubert varieties, defined only up to translation by the group G. In the theorems below, write $\mathcal{Y}_u^w = \mathcal{Y}_x^z$ to mean that the cycles may be carried onto each other by an element of G. (We will be more explicit in their proofs.)

Section 4 is devoted to proving the following result concerning these cycles.

Theorem 3.3. *Let $u, w \in \mathcal{B}_n$ with $u \leq_0 w$. Then*

(1) *The map $\pi : Y_u \cap Y'_{\omega_0 w} \to \mathcal{Y}_u^w$ has degree 1.*
(2) *If we have $x, z \in \mathcal{B}_n$ with $x \leq_0 z$ and wu^{-1} shape-equivalent to zx^{-1}, then $\mathcal{Y}_u^w = \mathcal{Y}_x^z$.*

By Theorem 3.3(1),

$$\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w}) \;=\; \pi_*[Y_u \cap Y'_{w_0 u}] \;=\; [\mathcal{Y}_u^w]\,.$$

Combining this with Theorem 3.3(2) and the projection formula, we deduce the following strengthened version of Theorem B(2):

Theorem B(2)$'$. *If $u, w, x, z \in \mathcal{B}_n$ with $u \leq_0 w$, $x \leq_0 z$, and wu^{-1} shape-equivalent to zx^{-1}, then for any strict partition λ,*

$$b_{u\,\lambda}^w \;=\; b_{x\,\lambda}^z \qquad \textit{and} \qquad c_{u\,\lambda}^w \;=\; c_{x\,\lambda}^z\,.$$

As a consequence of Theorem 3.3(2), define $\mathcal{Y}_\zeta := \mathcal{Y}_u^{\zeta u}$ for any $\zeta, u \in \mathcal{B}_n$ with $u \leq_0 \zeta u$. These cycles satisfy more identities which we establish in Section 5. Define $\rho \in \mathcal{B}_n$ by $\rho(i) = i-1-n$ for $1 \leq i \leq n$ and $\gamma \in \mathcal{B}_n$ by $\gamma(i) = i+1$ for $1 \leq i < n$ and $\gamma(n) = 1$. Then ρ is a reflection ($\rho^2 = 1$), and γ is the n-cycle $\langle 12\ldots n\rangle$.

Theorem 3.4.

(1) *For any $\zeta \in \mathcal{B}_n$, we have $\mathcal{Y}_\zeta \;=\; \mathcal{Y}_{\rho\zeta\rho}$.*
(2) *For any $\zeta \in \mathcal{B}_n$ with $a \cdot \zeta(a) > 0$ for every a, we have $\mathcal{Y}_\zeta \;=\; \mathcal{Y}_{\gamma\zeta\gamma^{-1}}$.*

By Proposition 3.1, $\mathcal{L}(\zeta) = \dim \mathcal{Y}_\zeta$, so Theorem C follows from Theorem 3.4. We prove Theorem 3.4(1) in Section 5.1 and Theorem 3.4(2) in Section 5.2.

3.2. **Product decomposition.** We establish another geometric result concerning these cycles $\mathcal{Y}_\zeta$. Suppose that W is a $2m$-dimensional symplectic vector space and consider the map

$$\Xi \;:\; \mathit{Lag}(V) \times \mathit{Lag}(W) \;\longrightarrow\; \mathit{Lag}(V \oplus W)$$

defined by

$$\Xi \;:\; (H, K) \;\longmapsto\; H \oplus K\,,$$

where $H \subset W$ and $K \subset V$ are maximal isotropic (Lagrangian) subspaces.

Theorem 3.5. *Let $\eta, \zeta \in \mathcal{B}_{n+m}$ with $\eta \cdot \zeta$ a disjoint product and $\#\mathrm{supp}(\eta) \leq m$, $\#\mathrm{supp}(\zeta) \leq n$. Then for any $\eta' \in \mathcal{B}_m$ and $\zeta' \in \mathcal{B}_n$ with $\eta \sim \eta'$ and $\zeta \sim \zeta'$ ($\sim$ is shape-equivalence), there is an element g of $Sp_{2m+2n}\mathbb{C}$ such that*

$$\Xi(\mathcal{Y}_{\zeta'} \times \mathcal{Y}_{\eta'}) \;=\; g(\mathcal{Y}_{\zeta\cdot\eta})\,.$$

Proof. This is a consequence of Lemma 5.2.1 of [3], the analogous fact for the classical flag manifold and Grassmannian. Restricting that result to the symplectic flag manifold and Lagrangian Grassmannian proves the theorem. □

Remark. This does not hold for the odd orthogonal case. In fact, even the map Ξ cannot be defined: If V, W are odd-orthogonal spaces, then $V \oplus W$ is an even-dimensional space.

4. The Fundamental identity of structure constants

We establish Theorem 3.3 which implies Theorem B(2). As in Section 2, many results and methods are similar to those of [3] for analogous results about $SL_n\mathbb{C}/B$. Our discussions are therefore brief. The results here hold for both $SO_{2n+1}\mathbb{C}/B$ and $Sp_{2n}\mathbb{C}/B$, with nearly identical proofs. We only provide justification for $Sp_{2n}\mathbb{C}/B$.

Let $H_2 = \langle h, \overline{h}\rangle \simeq \mathbb{C}^2$ be a symplectic vector space of dimension 2. Then the orthogonal direct sum $V \oplus H_2$ is a symplectic vector space of dimension $2n+2$. For each $1 \leq p \leq n+1$, define embeddings $\psi_p, \psi_{\overline{p}} \;:\; Sp_{2n}\mathbb{C}/B \;\hookrightarrow\; Sp_{2n+2}\mathbb{C}/B$, the space of isotropic flags in $V \oplus H_2$, by

$$\left(\psi_p E_\bullet\right)_j \;=\; \begin{cases} E_{j+1}, & j \leq \overline{p}\,(<0),\\ \langle E_j, h\rangle, & \overline{p} < j < 0. \end{cases}$$

Define $\psi_{\overline{p}}$ by replacing h with $\overline{h}$ in the definition above. We compute the effect of these maps on cohomology by determining the image of a Schubert variety under ψ_p.

First, define two families of maps between $\mathcal{B}_n$ and $\mathcal{B}_{n+1}$. For every $1 \leq p \leq n+1$ and $q \in \pm[n+1]$, define the injection $\varepsilon_{p,q} : \mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ by:

$$\varepsilon_{p,q}(w)(j) \quad=\quad \begin{cases} w(j), & j<p \text{ and } |w(j)|<|q|,\\ w(j)-1, & j<p \text{ and } w(j)\leq -|q|,\\ w(j)+1, & j<p \text{ and } w(j)\geq |q|,\\ q, & j=p,\\ w(j-1), & j>p \text{ and } |w(j)|<|q|,\\ w(j-1)-1, & j>p \text{ and } w(j)\leq -|q|,\\ w(j-1)+1, & j>p \text{ and } w(j)\geq |q|. \end{cases}$$

Let $/_p : \mathcal{B}_{n+1} \to \mathcal{B}_n$ be the left inverse of $\varepsilon_{p,q}$, defined by $\varepsilon_{p,w(p)}(w/_p) = w$. If we represent permutations as permutation matrices, then the effect of $/_p$ on $w \in \mathcal{B}_{n+1}$ is to erase the pth and $\overline{p}$th columns and the $w(p)$th and $\overline{w(p)}$th rows. The effect of $\varepsilon_{p,q}$ on w is to expand its permutation matrix with new pth, $\overline{p}$th columns and qth, $\overline{q}$th rows filled with zeroes, except for ones at positions (q,p) and $(\overline{q},\overline{p})$. For example:

$$\begin{array}{rcl} \varepsilon_{3,\overline{2}}(\overline{2}\,3\,4\,1) &=& \overline{3}\,4\,\overline{2}\,5\,1\\ \varepsilon_{3,2}(\overline{2}\,3\,4\,1) &=& \overline{3}\,4\,2\,5\,1 \end{array} \qquad \text{and} \qquad 4\,\overline{1}\,5\,\overline{2}\,3/_4 \;=\; 3\,\overline{1}\,4\,2\,.$$

These definitions imply the following proposition (*cf.* [32, Lemma 12]).

Proposition 4.1. *Let $w \in \mathcal{B}_n$, $1 \leq p, |q| \leq n+1$, and $E_\bullet$ any isotropic flag. Then*

$$\psi_p\, Y_w E_\bullet \;\subset\; Y_{\varepsilon_{p,q}(w)}(\psi_{\overline{q}} E_\bullet)\,.$$

Recall that e is the identity permutation and Y_e is the flag manifold G/B.

Corollary 4.2. *Let $E_\bullet, E'_\bullet$ be opposite isotropic flags. Then for any $q \in \pm[n+1]$, $\psi_q E_\bullet, \psi_{\overline{q}} E'_\bullet$ are opposite flags, and for any $1 \leq p \leq n+1$, we have*

$$\begin{aligned} \psi_p\, Y_w E_\bullet \quad&=\quad Y_{\varepsilon_{p,\overline{n+1}}(w)}(\psi_{n+1} E_\bullet) \cap Y_{\varepsilon_{p,n+1}(e)}(\psi_{\overline{n+1}} E'_\bullet)\\ &=\quad Y_{\varepsilon_{p,n+1}(w)}(\psi_{\overline{n+1}} E_\bullet) \cap Y_{\varepsilon_{p,\overline{n+1}}(e)}(\psi_{n+1} E'_\bullet)\,. \end{aligned}$$

Proof. It is straightforward to check that the flags are opposite. Moreover, by Proposition 4.1, $\psi_p\, Y_w E_\bullet$ is a subset of either intersection, as $Y_e = Sp_{2n}\mathbb{C}/B$. Since

$$\ell(\varepsilon_{p,\overline{n+1}}(w)) \;=\; \ell(w) + n + p\,, \qquad \ell(\varepsilon_{p,n+1}(w)) \;=\; \ell(w) + n + 1 - p\,,$$

and $\dim Sp_{2n}\mathbb{C}/B = n^2$, Proposition 3.1 implies that all three cycles are irreducible with the same dimension, proving their equality. □

Corollary 4.3. *For any $w \in \mathcal{B}_n$ and $1 \leq p \leq n$, we have*

$$(\psi_p)_*\mathfrak{C}_w \;=\; \mathfrak{C}_{\varepsilon_{p,\overline{n+1}}(w)} \cdot \mathfrak{C}_{\varepsilon_{p,n+1}(e)} \;=\; \mathfrak{C}_{\varepsilon_{p,n+1}(w)} \cdot \mathfrak{C}_{\varepsilon_{p,\overline{n+1}}(e)}\,.$$

Lemma 4.4. *Suppose $u <_0 w$ in $\mathcal{B}_{n+1}$ and $u(p) = w(p) = q$ for some $1 \leq p \leq n+1$. Then*

(1) *$u/_p <_0 w/_p$ and $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$.*
(2) *For any opposite isotropic flags $E_\bullet, E'_\bullet$ in V,*

$$\psi_p\left(Y_{u/_p}E_\bullet \cap Y_{\omega_0 w/_p}E'_\bullet\right) \;=\; Y_u\psi_{\overline{q}}E_\bullet \cap Y_{\omega_0 w}\psi_q E'_\bullet\,.$$

Proof. Since $u <_0 w$ and $u(p) = w(p)$, Proposition 2.5 implies that $u/_p <_0 w/_p$. Moreover, wu^{-1} is shape-equivalent to $w/_p(u/_p)^{-1}$, so the first statement follows from Theorem B(1)$'$.

For (2), Proposition 4.1 gives the inclusion $\subset$. By Corollary 4.2, $\psi_{\overline{q}}E_\bullet$ and $\psi_q E'_\bullet$ are opposite flags. Thus, by Proposition 3.1 and (1), both sides are irreducible and have the same dimension, proving their equality. □

Theorem 4.5. *Suppose $u <_0 w$ in $\mathcal{B}_{n+1}$ and $u(p) = w(p) = q$ for some $1 \leq p \leq n+1$. Then for any strict partition λ, we have*

$$b^w_{u\,\lambda} \;=\; b^{w/_p}_{u/_p,\,\lambda} \qquad \textit{and} \qquad c^w_{u\,\lambda} \;=\; c^{w/_p}_{u/_p,\,\lambda}\,.$$

Proof. We first study the map $\Psi : \mathit{Lag}(V) \hookrightarrow \mathit{Lag}(V + H_2)$, defined by $K \mapsto \langle K, h\rangle$. If $E_\bullet, E'_\bullet$ are opposite isotropic flags in V, the analog of Corollary 4.2 is

$$\Psi(\Upsilon_\lambda E_\bullet) \;=\; \Upsilon_\lambda(\psi_{n+1}E_\bullet) \cap \Upsilon_{(n+1)}(\psi_{\overline{n+1}}E'_\bullet)\,,$$

where $(n{+}1)$ is a decreasing sequence of length 1. We leave this to the reader. As this intersection is generically transverse, $\Psi_*Q_\lambda = Q_\lambda \cdot q_{n+1}$. Expressing Ψ^*Q_λ in the Schubert basis, we have $\Psi^*Q_\lambda = \sum_\mu d^\mu_\lambda Q_\mu$, where

$$\begin{aligned} d^\mu_\lambda \;&:=\; \deg((\Psi^*Q_\lambda)\cdot Q_{\mu^c}) \\ &=\; \deg(Q_\lambda \cdot \Psi_*(Q_{\mu^c})) \\ &=\; \deg(Q_\lambda \cdot q_{n+1} \cdot Q_{\mu^c}) \quad = \quad \delta^\mu_\lambda\,, \end{aligned}$$

the Kronecker delta, by the Pieri-type formula for isotropic Grassmannians [8]. (The product $Q_\lambda \cdot q_{n+1}$ is zero if $\lambda_1 = n+1$ or is $Q_{(n+1,\lambda)}$, if $\lambda_1 < n+1$). In the second case, the Poincaré dual class to $Q_{(n+1,\lambda)}$ is Q_{λ^c}.) Thus

$$\Psi^*Q_\lambda, \;=\; \begin{cases} Q_\lambda, & \lambda_1 < n+1, \\ 0, & \lambda_1 = n+1. \end{cases}$$

Consider the commutative diagram, whose first row is Lemma 4.4(2).

$$\begin{tikzcd} Y_{u/_p} \cap Y'_{\omega_0 w/_p} \arrow[r, "\psi_p"] \arrow[d, "\pi"'] & Y_u \cap Y'_{\omega_0 w} \arrow[d, "\pi"] \\ \mathcal{Y}^{w/_p}_{u/_p} \;=\; \pi(Y_{u/_p} \cap Y'_{\omega_0 w/_p}) \arrow[r, "\Psi"] & \pi(Y_u \cap Y'_{\omega_0 w}) \;=\; \mathcal{Y}^w_u \end{tikzcd} \tag{4.1}$$

Thus $[\mathcal{Y}^w_u] = \Psi_*[\mathcal{Y}^{w/_p}_{u/_p}]$ and the maps π have the same degree δ as the horizontal maps are isomorphisms.

Let λ be a strict partition. Then $c^w_{u\,\lambda} = \delta\cdot\deg(Q_\lambda\cdot[\mathcal{Y}^w_u])$ which is

$$\begin{aligned}\delta\cdot\deg(Q_\lambda\cdot\Psi_*[\mathcal{Y}^{w/_p}_{u/_p}]) &= \delta\cdot\deg(\Psi^*(Q_\lambda)\cdot[\mathcal{Y}^{w/_p}_{u/_p}])\\ &= \delta\cdot\deg(Q_\lambda\cdot[\mathcal{Y}^{w/_p}_{u/_p}]) = c^{w/_p}_{u/_p,\,\lambda}\,. \qquad\square\end{aligned}$$

Lemma 4.6. *Suppose $u <_0 w$ and $x <_0 z$ in $\mathcal{B}_n$ with $wu^{-1} = zx^{-1}$ and $u(i)\neq w(i)$ for all $i\in[n]$. Then, if Y_u, Y_x, (respectively $Y'_{\omega_0 w}, Y'_{\omega_0 z}$) are defined with the same flags, there is commutative diagram*

$$\begin{tikzcd} Y_u\cap Y'_{\omega_0 w} \arrow[r, dashed, "f"] \arrow[d, "\pi"'] & Y_x\cap Y'_{\omega_0 z} \arrow[d, "\pi"] \\ \pi(Y_u\cap Y'_{\omega_0 w}) \arrow[r, phantom, "="] & \pi(Y_x\cap Y'_{\omega_0 z}) \end{tikzcd}$$

where f is an isomorpism between Zariski open subsets of $Y_u\cap Y'_{\omega_0 w}$ and $Y_x\cap Y'_{\omega_0 z}$, the maps π have degree 1, and the equality is set-theoretic.

Proof. Let $g_{\overline{\jmath}}$ for $1\leq j\leq n$ be the functions of Lemma 3.2, defined for all $F_\bullet\in Y^\circ_u\cap {Y'}^\circ_{\omega_0 w}$. For such $F_\bullet$, let $f(F_\bullet)$ be the flag whose $\overline{\jmath}$th subspace is

$$\langle g_{u^{-1}x(\overline{n})}(F_\bullet),\ldots,g_{u^{-1}x(\overline{\jmath})}(F_\bullet)\rangle\,.$$

Then the reader is invited to check that f is an isomorphism between the intersections of Schubert cells, which are Zariski dense in the intersections of Schubert varieties, and the diagram is commutative. Since we may assume w is a Grassmannian permutation, and in this case, the map $\pi : Y_{\omega_0 w}\to\pi(Y_{\omega_0 w})$ has degree 1, it follows that the maps π (which have the same degree) have degree 1. $\square$

Remark 4.7. The hypothesis $u(i)\neq w(i)$ (which implies $x(i)\neq z(i)$) in Lemma 4.6 is due to the equality in the diagram. Suppose $n=1$ and $V=\langle e_{\overline{1}}, e_1\rangle$ with $w=u=1$ and $x=z=\overline{1}$. Then both intersections of Schubert varieties $Y_u\cap Y'_{\omega_0 w}$ and $Y_x\cap Y'_{\omega_0 z}$ are single points (lines in V), with one equal to $\langle e_{\overline{1}}\rangle$ and the other to $\langle e_1\rangle$. This restrictive hypothesis $u(i)\neq w(i)$ could be removed, but at the expense of weakening the statement of Lemma 4.6, and greatly complicating its proof.

Proof of Theorem 3.3. For statement (1), if the support of wu^{-1} is not $[n]$, then there is a number $p\in[n]$ with $u(p)=w(p)$. Since the maps π of diagram (4.1) have the same degree, this degree must be 1, by Lemma 4.6 and an induction on the number of fixed points $\{p\mid u(p)=w(p)\}$ of wu^{-1}.

For (2), since the map Ψ of diagram (4.1) is one-to-one, we may use ideas from the previous paragraph to reduce to the case when the supports of the permutations wu^{-1} and zx^{-1} are both $[n]$. Since these permutations are shape-equivalent with the same support, they are equal and thus the hypotheses of Lemma 4.6 hold. But then $\pi_*[Y_u\cap Y'_{\omega_0 w}] = \pi_*[Y_x\cap Y'_{\omega_0 z}]$, hence $\pi_*(\mathfrak{C}_u\cdot\mathfrak{C}_{\omega_0 w}) = \pi_*(\mathfrak{C}_x\cdot\mathfrak{C}_{\omega_0 z})$, showing $c^w_{u\,\lambda} = c^z_{x\,\lambda}$. $\square$

5. Further identities of structure constants

We prove Theorem 3.4, the geometric counterpart of Theorem C.

5.1. **Reflection identities.** Define the permutation $\rho \in \mathcal{B}_n$ by $\rho(i) = i-1-n$. Then ρ is a reflection. For example, in $\mathcal{B}_4$ we have $\rho = \langle 1\overline{4}\rangle\langle 2\overline{3}\rangle$, in the cycle notation of Definition-Example 2.6. In general, $\rho = \langle 1\,\overline{n}\,\rangle\langle 2\,\overline{n-1}\,\rangle \cdots \langle \lfloor \frac{n+1}{2} \rfloor\, \overline{\lfloor \frac{n+2}{2} \rfloor}\rangle$. We give an example of ζ and $\rho\zeta\rho$ in Example 8.5 below. Here, we prove Theorem 3.4(1), namely that $\mathcal{Y}_\zeta = \mathcal{Y}_{\rho\zeta\rho}$. By Theorem B(2)$'$, we may assume that $\mathrm{supp}(\zeta) = [n]$, and this case is shown in Lemma 5.1 below.

Let $E_\bullet, E'_\bullet$ be opposite isotropic flags in V. Define a flag $\widetilde{E}_\bullet$ by:

$$\widetilde{E}_\bullet \ :\ E_1 \cap E'_{\overline{1}} \ \subset\ \cdots\ \subset\ E_n \cap E'_{\overline{1}} \ \subset\ (E_{\overline{n}} + E'_{\overline{1}}) \ \subset\ \cdots\ \subset\ (E_{\overline{2}} + E'_{\overline{1}}) \ \subset\ V\,.$$

Define $\widetilde{E}'_\bullet$ the same way, but with the roles of $E_\bullet$ and $E'_\bullet$ reversed. This gives opposite flags $\widetilde{E}_\bullet, \widetilde{E}'_\bullet$, and since $(A\cap B)^\perp = (A^\perp + B^\perp)$, they are isotropic.

Lemma 5.1. *Suppose $u, w, x, z \in \mathcal{B}_n$ with $u \leq_0 w$, $x \leq_0 z$, $\rho w u^{-1}\rho = z x^{-1}$, and $u(j) \neq w(j)$ for $1 \leq j \leq n$. Then, for any opposite isotropic flags $E_\bullet, E'_\bullet$ in V, there is a commutative diagram*

$$\begin{tikzcd}
Y_u E_\bullet \cap Y_{\omega_0 w} E'_\bullet \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y_x \widetilde{E}_\bullet \cap Y_{\omega_0 z} \widetilde{E}'_\bullet \arrow[dl, "\pi"] \\
\pi(Y_u E_\bullet \cap Y_{\omega_0 w} E'_\bullet) & = & \pi(Y_x \widetilde{E}_\bullet \cap Y_{\omega_0 z} \widetilde{E}'_\bullet)
\end{tikzcd}$$

with f an isomorpism between Zariski open subsets of the intersections $Y_u E_\bullet \cap Y_{\omega_0 w} E'_\bullet$ and $Y_x \widetilde{E}_\bullet \cap Y_{\omega_0 z} \widetilde{E}'_\bullet$.

Let $G_\bullet, G'_\bullet$ be opposite (not necessarily isotropic) flags in V. Define $G_\bullet^+$ to be

$$G_\bullet^+ \ :\ G_{\overline{n-1}} \cap G'_{n-1} \ \subset\ G_{\overline{n-2}} \cap G'_{n-1} \ \subset\ \cdots\ \subset\ G'_{n-1} \ \subset\ V\,.$$

Define $G'^{+}_\bullet$ to be

$$G'^{+}_\bullet \ :\ G_{\overline{n}} \ \subset\ (G_{\overline{n}} + G'_{\overline{n}}) \ \subset\ \cdots\ \subset\ (G_{\overline{n}} + G'_{n-1}) \ \subset\ V\,.$$

For $\zeta \in \mathcal{S}_{\pm[n]}$, let ζ^+ be the conjugation of ζ by the cycle $(\overline{n}, \ldots, \overline{1}, 1, \ldots, n)$. In Section 5.3 of [3], the following proposition is proven:

Proposition 5.2. *Let $u, w, x, z \in \mathcal{S}_{\pm[n]}$ with $u \lhd_0 w$, $x \lhd_0 z$, $(u^{-1}w)^+ = x^{-1}z$, and w is a Grassmannian permutation with descent 0, ($w(\overline{n}) < \cdots < w(\overline{1})$ and $w(1) < \cdots < w(n)$). If $\pi : \mathbb{F}\ell(V) \twoheadrightarrow \mathbf{G}_n(V)$ is the projection, then there is a commutative diagram*

$$\begin{tikzcd}
X_u G_\bullet \cap X_{\omega_0 w} G'_\bullet \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & X_x G_\bullet^+ \cap X_{\omega_0 z} G'^{+}_\bullet \arrow[dl, "\pi"] \\
\pi(X_u G_\bullet \cap X_{\omega_0 w} G'_\bullet) & = & \pi(X_x G_\bullet^+ \cap X_{\omega_0 z} G'^{+}_\bullet)
\end{tikzcd}$$

with f an isomorphism of Zariski open subsets of the intersections $X_u G_\bullet \cap X_{\omega_0 w} G'_\bullet$ and $X_x G_\bullet^+ \cap X_{\omega_0 z} G'^{+}_\bullet$.

Proof of Lemma 5.1. By Lemma 4.6, we may assume w is a Grassmannian permutation. Observe that $(\widetilde{E}_\bullet, \widetilde{E}'_\bullet)$ is the result of n applications of the map $(E_\bullet, E'_\bullet) \mapsto$

$(E_\bullet{}^+, E'_\bullet{}^+)$. Similarly, $\rho = (\overline{n}, \ldots, \overline{2}, \overline{1}, 1, \ldots, n)^n$. Thus, iterating Proposition 5.2 n times gives the commutative diagram in $\mathbb{F}\ell(V)$ and $\mathbf{G}_n V$:

$$\begin{tikzcd}
X_u E_\bullet \cap X_{\omega_0 w} E'_\bullet \arrow[rr, dashed, "f"] \arrow[d, "\pi"'] & & X_x \widetilde{E}_\bullet \cap X_{\omega_0 z} \widetilde{E}'_\bullet \arrow[d, "\pi"] \\
\pi(X_u E_\bullet \cap X_{\omega_0 w} E'_\bullet) & = & \pi(X_x \widetilde{E}_\bullet \cap X_{\omega_0 z} \widetilde{E}'_\bullet)
\end{tikzcd}$$

Restricting this to the subset of isotropic flags gives the diagram of the lemma. $\square$

These same arguments prove the analog of Lemma 5.1 for $SO_{2n+1}\mathbb{C}/B$.

5.2. **Cyclic identities.** Until now, we have deduced identities in $H^*(Sp_{2n}\mathbb{C}/B)$ by restricting constructions involving Schubert subvarieties of $\mathbb{F}\ell(V)$ to those in $Sp_{2n}\mathbb{C}/B$ via the embedding $Sp_{2n}\mathbb{C}/B \hookrightarrow \mathbb{F}\ell(V)$. This is the geometric counterpart of the embedding $\mathcal{B}_n \hookrightarrow \mathcal{S}_{\pm[n]}$ studied in Section 2. Here, we explore the geometric counterpart of the map $\iota : \mathcal{S}_n \hookrightarrow \mathcal{B}_n$, where a permutation $w \in \mathcal{S}_n$ is extended to act on $-[n]$ by $w(-i) = -w(i)$. We first develop the necessary combinatorial preliminaries. A consequence of Theorem 2.2 is the following lemma.

Lemma 5.3. *The map ι is an embedding of Bruhat orders $(\mathcal{S}_n, \lhd) \hookrightarrow (\mathcal{B}_n, \leq)$ and it respects the length functions in each order. Furthermore, $\iota(\mathcal{S}_n)$ consists of those permutations $\zeta \in \mathcal{B}_n$ with $a \cdot \zeta(a) > 0$ for all numbers a, that is, those ζ with $\delta(\zeta) = 1$.*

Define $\epsilon_k : \mathcal{S}_n \to \mathcal{B}_n$ by

$$(\epsilon_k w)(j) \;=\; \begin{cases} w(j+k), & 1 \leq j \leq n-k, \\ \overline{w(n+1-j)}, & n-k < j \leq n. \end{cases} \tag{5.1}$$

Note that $\epsilon_0 w = \iota(w)$.

Lemma 5.4. *Let $u, w \in \mathcal{S}_n$ with $u \lhd_k w$. Then ϵ_k induces an isomorphism of graded posets $[u,w]_{\lhd_k} \xrightarrow{\sim} [\epsilon_k u, \epsilon_k w]_0$ and $\iota(wu^{-1}) = \epsilon_k w (\epsilon_k u)^{-1}$.*

Proof. A consequence of the definitions is that, for $u, w \in \mathcal{S}_n$,

$$u \lhd_k w \iff \epsilon_k u <_0 \epsilon_k w\,,$$

and $\iota(wu^{-1}) = \epsilon_k w(\epsilon_k u)^{-1}$. The lemma follows from these observations. $\square$

Corollary 5.5. *The map $\iota : (\mathcal{S}_\infty, \prec\!\!\lhd\,) \hookrightarrow (\mathcal{B}_\infty, \prec)$ is an embedding of ranked orders.*

We now introduce the geometric counterpart of the map ι. Let L, L^* be complementary Lagrangian subspaces in V. The pairing $(x,y) \in L \oplus L^* \mapsto \beta(x,y)$, where β is the alternating form, identifies them as linear duals. Given a subspace H of L, let $H^\perp \subset L^*$ denote its annihilator in L^*. Then $H + H^\perp$ is a Lagrangian subspace of V.

Let $\mathbb{F}\ell(L)$ be the space of complete flags $F_\bullet := F_1 \subset F_2 \subset \cdots \subset F_n = L$ in L. Note that here $\dim F_i = i$. For each $k = 0, 1, \ldots, n$, define an injective map

$$\varphi_k \;:\; \mathbb{F}\ell(L) \;\longrightarrow\; Sp_{2n}\mathbb{C}/B$$

by

$$(\varphi_k F_\bullet)_{\overline{j}} \;=\; \begin{cases} F_{n+1-j}, & j \geq n-k+1, \\ F_k + F_{k+j-1}^{\perp}, & j \leq n-k+1. \end{cases} \tag{5.2}$$

Then $(\varphi_k F_\bullet)_{\overline{1}} = (F_k + F_k^{\perp})$ is Lagrangian, showing that $\varphi_k F_\bullet$ is an isotropic flag.

For $w \in \mathcal{S}_n$ the Schubert variety $X_w E_\bullet$ of $\mathbb{F}\ell(L)$ consists of those flags $F_\bullet \in \mathbb{F}\ell(L)$ satisfying

$$\dim E_a \cap F_b \;\geq\; \#\{b \geq l \mid w(l) + a \geq n+1\}\,. \tag{5.3}$$

We determine the image of Schubert varieties of $\mathbb{F}\ell(L)$ under these maps φ_k.

Let $w^\vee$ be defined by $w^\vee(j) = n+1-w(j)$. Then $X_w E_\bullet$ and $X_{w^\vee} E'_\bullet$ are dual under the intersection pairing on $\mathbb{F}\ell(L)$, where $E_\bullet, E'_\bullet$ are opposite flags.

Lemma 5.6. *With these definitions, $\varphi_k X_w E_\bullet$ is a subset of either Schubert variety*

$$Y_{\epsilon_k w}\varphi_n E_\bullet \quad \textit{or} \quad Y_{\omega_0 \epsilon_k w^\vee}\varphi_0 E_\bullet\,.$$

Proof. Let $F_\bullet \in X_w E_\bullet$. We show $\varphi_k F_\bullet \in Y_{\epsilon_k w}\varphi_n E_\bullet$, that is, for each $-n \leq i \leq n$ $(i \neq 0)$ and $1 \leq j \leq n$,

$$\dim(\varphi_n E_\bullet)_i \cap (\varphi_k F_\bullet)_{\overline{j}} \;\geq\; \#\{n \geq l \geq j \mid i \geq \epsilon_k w(l)\}\,. \tag{5.4}$$

Suppose that $j > n-k+1$. Then $(\varphi_k F_\bullet)_{\overline{j}} = F_{n+1-j} \subset L = (\varphi_n E_\bullet)_{\overline{1}}$. If $n \geq l \geq j$, then $(\epsilon_k w)(l) = \overline{w(n{+}1{-}l)} < 0$. Thus if $i > 0$, (5.4) holds as both sides equal $n+1-j$. Suppose $i < 0$. Then $(\varphi_n E_\bullet)_i = E_{n+1-\overline{\imath}}$ and so the left side of (5.4) is

$$\begin{aligned} \dim E_{n+1-\overline{\imath}} \cap F_{n+1-j} \quad &\geq \quad \#\{m \leq n+1-j \mid w(m) + n + 1 - \overline{\imath} \geq n+1\} \\ &= \quad \#\{n \geq l \geq j \mid i \geq \overline{w(n{+}1{-}l)} = \epsilon_k w(l)\}\,. \end{aligned}$$

Now suppose that $j \leq n-k+1$. Then $(\varphi_k F_\bullet)_{\overline{j}} = F_k + F_{k+j-1}^{\perp}$. Thus the left side of (5.4) is

$$\dim(\varphi_n E_\bullet)_i \cap F_k \;+\; \dim(\varphi_n E_\bullet)_i \cap F_{k+j-1}^{\perp}\,. \tag{5.5}$$

If $i < 0$, then $(\varphi_n E_\bullet)_i \subset L$ and only the first term of (5.5) contributes. By the previous paragraph, this is

$$\dim(\varphi_n E_\bullet)_i \cap F_k \;\geq\; \#\{n \geq l \geq k \mid i \geq \epsilon_k w(l)\}\,.$$

If $k \geq l$, then $\epsilon_k w(l) > 0 > i$, showing this equals the right side of (5.4).

If now $i > 0$, then $(\varphi_n E_\bullet)_i = L + E_{n-i}^{\perp}$. Thus (5.5) is

$$k + \dim E_{n-i}^{\perp} \cap F_{k+j-1}^{\perp} \;=\; k + n - \dim(E_{n-1} + F_{k+j-1})\,.$$

But this is

$$k + n - \dim E_{n-1} - \dim F_{k+j-1} + \dim E_{n-1} \cap F_{k+j-1}\,,$$

which is at least

$$i - j + 1 + \#\{k+j-1 \geq m \geq 1 \mid w(m) + n - 1 \geq n+1\}\,.$$

This equals

$$n - j + 1 - \#\{n \geq m \geq k+j \mid w(m) \geq i+1\}$$

Which is

$$k + \#\{n-k \geq l \geq j \mid w(l+k) \leq i\} \;=\; k + \#\{n-k \geq l \geq j \mid \epsilon_k w(l) \leq i\}\,.$$

This last quantity equals the right side of (5.4) since $l > n-k$ implies $\epsilon_k w(l) < 0 < i$.

Similar arguments show $\varphi_k F_\bullet \in Y_{\omega_0\epsilon_k w^\vee}(\varphi_0 E_\bullet)$. $\square$

Corollary 5.7. *Let $u, w \in \mathcal{S}_n$ with $u \lhd_k w$ and $E_\bullet, E'_\bullet \in \mathbb{F}\ell(L)$ be opposite flags. Then $\varphi_n E_\bullet, \varphi_0 E'_\bullet$ are opposite isotropic flags, and*

$$\varphi_k\left(X_u E_\bullet \cap X_{w^\vee} E'_\bullet\right) \;=\; Y_{\epsilon_k u}(\varphi_n E_\bullet) \cap Y_{\omega_0\epsilon_k w}(\varphi_0 E'_\bullet)\,.$$

Proof. Lemma 5.6 gives the inclusion $\subset$ and $\varphi_n E_\bullet$ and $\varphi_0 E'_\bullet$ are opposite flags. By Lemma 5.4, both sides have the same dimension, proving equality. $\square$

For each $k = 1, 2, \ldots, n$, let $\pi_k : \mathbb{F}\ell(L) \to \mathbf{G}_k(L)$ be the projection induced by $E_\bullet \mapsto E_k$. As in Lemma 4.6, if $u \lhd_k w$ in $\mathcal{S}_n$ and $E_\bullet, E'_\bullet$ are opposite flags in $\mathbb{F}\ell(L)$, then the intersection $X_u E_\bullet \cap X_{w^\vee} E'_\bullet$ is mapped birationally onto its image $\pi_k(X_u E_\bullet \cap X_{w^\vee} E'_\bullet)$ in $\mathbf{G}_k(L)$. Furthermore, the image cycle depends only upon $\eta := wu^{-1}$. Denote it by $\mathcal{X}_\eta$.

Define $\Phi_k : \mathbf{G}_k(L) \to \mathit{Lag}(V)$ by $H \mapsto (H + H^\perp)$. Then we have the commutative diagram

$$\tag{5.6}\begin{tikzcd} \mathbb{F}\ell(L) \arrow[r, "\varphi_k"] \arrow[d, "\pi_k"'] & Sp_{2n}\mathbb{C}/B \arrow[d, "\pi"] \\ \mathbf{G}_k(L) \arrow[r, "\Phi_k"] & \mathit{Lag}(V) \end{tikzcd}$$

Thus $\Phi_k \circ \pi_k = \pi \circ \varphi_k$ and so we have the following corollary.

Corollary 5.8. *For any $\eta \in \mathcal{S}_n$, $\mathcal{Y}_{\iota(\eta)} = \Phi_k(\mathcal{X}_\eta)$, where $k = \#\{a \mid a < \eta(a)\}$.*

Recall that γ is the cycle $\iota(1, 2, \ldots, n) = \langle 12 \ldots n\rangle \in \mathcal{B}_n$.

Proof of Theorem 3.4(2). If $\zeta \in \mathcal{B}_n$ with $a \cdot \zeta(a) > 0$ for all a, then $\zeta = \iota(\eta)$ for some $\eta \in \mathcal{S}_n$. Let $k = \#\{a \mid a < \eta(a)\}$. Then by Corollary 5.8 and Proposition 5.2,

$$\mathcal{Y}_{\gamma\zeta\gamma^{-1}} \;=\; \Phi_k(\mathcal{X}_{\gamma\eta\gamma^{-1}}) \;=\; \Phi_k(\mathcal{X}_\eta) \;=\; \mathcal{Y}_\zeta\,. \quad \square$$

We deduce a corollary of Theorem 3.5 and Corollary 5.8, which is needed in Section 7 to establish the Pieri-type formula. For $\zeta \in \mathcal{B}_n$, recall that $\delta(\zeta) = 1$ if ζ is in the image of $\iota : \mathcal{S}_n \hookrightarrow \mathcal{B}_n$ and $\delta(\zeta) = 0$ otherwise. We emphasize that this lemma holds only in the symplectic case.

Lemma 5.9. *Let $\zeta \in \mathcal{B}_n$ and $\zeta = \zeta_1 \cdots \zeta_s$ be the factorization of ζ into disjoint irreducible permutations. For each factor ζ_i with $\delta(\zeta_i) = 1$ there is a quadratic form q_i on V which vanishes on every $K \in \mathcal{Y}_\zeta$. These forms are linearly independent and together they define a reduced complete intersection of codimension r and degree 2^r, where r counts the ζ_i with $\delta(\zeta_i) = 1$.*

Proof. Suppose first that ζ is irreducible with $\delta(\zeta) = 1$. Let L, L^* be complementary Lagrangian subspaces of V. Then the decomposition $V = L \oplus L^*$ allows us to define a quadratic form q on V by

$$q(x) \;=\; \beta(x^+, x^-)\,,$$

where x^+, x^- are the projections of x to the summands L, L^* of V. By Corollary 5.8, every $K \in \mathcal{Y}_\zeta$ has the form $H + H^\perp$ for some $H \subset L$. Since $H^\perp \subset L^*$ annihilates H in the bilinear form associated to q on V, and L, L^* are isotropic for q, we see that $q|_K \equiv 0$.

Now let $\zeta = \zeta_1 \cdots \zeta_s$ be the factorization of ζ into disjoint irreducible permutations. Suppose $\#\mathrm{supp}(\zeta_i) \le n_i$ with $n_1 + \cdots + n_s = n$ and let $\zeta_i' \in \mathcal{B}_{n_i}$ be shape-equivalent to ζ_i for each $i = 1, \ldots, s$. For each i, let V_i be a $2n_i$-dimensional symplectic vector space, and identify V with $V_1 \oplus \cdots \oplus V_s$. By Theorem 3.5, the map

$$(K_1, \ldots, K_s) \longmapsto K_1 + \cdots + K_s$$

gives an isomorphism

$$\mathcal{Y}_{\zeta_1'} \times \cdots \times \mathcal{Y}_{\zeta_s'} \xrightarrow{\sim} \mathcal{Y}_\zeta .$$

If $\delta(\zeta_i) = 1$, so that $\delta(\zeta_i') = 1$, let q_i' be the quadratic form constructed on V_i in the previous paragraph. For each such i, let q_i be the pullback of q_i' to V under the projection $V \twoheadrightarrow V_i$. Since q_i' annihilates each $K_i \in \mathcal{Y}_{\zeta_i'}$, q_i annihilates each $K \in \mathcal{Y}_\zeta$.

Lastly, as each q_i is non-zero only on the summands V_i, the forms q_i are linearly independent. In fact, their monomials are disjoint, and so they define a reduced scheme that is a complete intersection. If r counts the number of ζ_i with $\delta(\zeta_i) = 1$, then this complete intersection has codimension r and degree 2^r. $\square$

6. Minimal permutations and labeled réseaux

Our proof of the Pieri-type formula in Section 7 requires a deeper study of the Lagrangian order $\mathcal{B}_\infty$. Particularly important are permutations whose lengths are minimal given their cycle structure, which we begin studying in Section 6.1. We also relate the two formulations (Theorems A and D) of the Pieri-type formula. This requires a study of labeled chains in both the Lagrangian order and the Lagrangian réseau. We initiate this by studying chains in intervals in the Grassmannian Bruhat order on $\mathcal{S}_\infty$ in Section 6.2, and apply this to chains in the Lagrangian order in Section 6.3 and the Lagrangian réseau in Section 6.4. Throughout, we provide examples to illustrate these results.

We recall some notation developed in Section 2. There are two types of reflections in $\mathcal{B}_\infty$, $t_b = \langle b]$ and $t_{a\,b} = \langle a\,b\rangle$ where $0 < a < |b|$ by convention. As elements in $\mathcal{S}_{\pm\infty}$, these are $(\overline{b}, b)$ and $(\overline{b}, \overline{a})(a, b)$, respectively. For $\zeta \in \mathcal{B}_\infty$, let $s(\zeta)$ count the *sign changes* in ζ, those $a > 0$ with $0 > \zeta(a)$. Let $\iota : \mathcal{S}_\infty \hookrightarrow \mathcal{B}_\infty$ be the map where $\iota(\eta)(a)$ is $\eta(a)$ if $a > 0$ and $\overline{\eta(\overline{a})}$ if $a < 0$. We have that $\delta(\zeta) = 1$ if ζ is in the image of ι and $\delta(\zeta) = 0$ otherwise. The support of a permutation ζ is $\mathrm{supp}(\zeta) := \{a > 0 \mid a \ne \zeta(a)\}$.

Let $\prec$ be the order relation in the Lagrangian order and $\mathcal{L}(\cdot)$ be its length function. Let $\lessdot$ be the order relation in the Grassmannian Bruhat order on both $\mathcal{S}_{\pm\infty}$ and $\mathcal{S}_\infty$, and let $\mathcal{L}_{_A}(\cdot)$ be its length function. As defined in Definition-Example 2.6, a permutation $\zeta \in \mathcal{B}_\infty$ is a *cycle* either if ζ is a single cycle as an element of $\mathcal{S}_{\pm\infty}$, or if ζ is the product of two complementary cycles η and $\overline{\eta}$, where $\eta \in \mathcal{S}_{\pm\infty}$ is a cycle whose numbers have distinct absolute values.

Two permutations $\zeta, \eta \in \mathcal{B}_\infty$ are *disjoint* if they have disjoint supports and if $\mathcal{L}(\eta \cdot \zeta) = \mathcal{L}(\eta) + \mathcal{L}(\zeta)$. This occurs if we do not have $a > b > \zeta(a) > \eta(b)$ (or $a > b > \eta(a) > \zeta(b)$), by the characterization of disjointness in terms of non-crossing partitions. A permutation is *irreducible* if it has no non-trivial factorization into disjoint permutations. Given a permutation $\zeta \in \mathcal{B}_\infty$, it has a unique factorization into irreducible, pairwise disjoint permutations. A *minimal cycle* is a cycle $\zeta \in \mathcal{B}_\infty$ for which $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$. By Lemma 6.1 below, this is a cycle of minimal length, given its form.

For example, consider Figure 5 in Section 2. The permutation $\langle 1\,]\langle 2\overline{3}\,\rangle$ is a disjoint product of $\langle 1\,] = t_1$ and $\langle 2\overline{3}\,\rangle$. The permutation $\langle 2\overline{3}\,\rangle$ is not a minimal cycle, but $\langle 2\overline{3}\,]$ and $\langle 2\,3\rangle$ are: In $\mathcal{S}_{\pm 3}$, $\langle 2\overline{3}\,\rangle = (2,\overline{3})(\overline{2},3)$ while $\langle 2\overline{3}\,] = (2,\overline{3},\overline{2},3)$ and $\langle 23\rangle = (\overline{3},\overline{2})(2,3)$. The permutation $\langle 13\,]\langle 2\,]$ is irreducible as $\mathcal{L}(\langle 13\,]\langle 2\,]) = 5 \neq 2+1 = \mathcal{L}(\langle 13\,]) + \mathcal{L}(\langle 2\,])$.

6.1. **Minimal permutations.** We study the relationship between the length of a permutation in $\mathcal{B}_\infty$ and the support of its disjoint irreducible factors. Those with minimal length play a special role in the Pieri-type formula, and we call these *minimal permutations*. An irreducible minimal permutation is a cycle as introduced in Definition-Example 2.6. In Lemma 6.5 we establish the main technical result about minimal cycles.

Lemma 6.1. *Let $\zeta \in \mathcal{B}_\infty$ be a cycle. Then $\mathcal{L}(\zeta) \geq \#\mathrm{supp}(\zeta) - \delta(\zeta)$. If $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$, then $s(\zeta) + \delta(\zeta) = 1$.*

Proof. A saturated chain in $[e,\zeta]_{\lessdot}$ (in $\mathcal{S}_{\pm\infty}$) gives a factorization of ζ into transpositions. If ζ consists of two cycles in $\mathcal{S}_{\pm\infty}$, then $\mathcal{L}_A(\zeta) \;\geq\; 2(\#\mathrm{supp}(\zeta) - 1)$ and by Corollary 2.11(2),

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - 1 + \lceil s(\zeta)/2 \rceil \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta)\,,$$

with equality only if $s(\zeta) = 0$, that is, only if $\delta(\zeta) = 1$.

Similarly, if ζ is a single cycle in $\mathcal{S}_{\pm\infty}$, then $\mathcal{L}_A(\zeta) \;\geq\; 2\#\mathrm{supp}(\zeta) - 1$. Since $\delta(\zeta) = 0$ and $s(\zeta) \geq 1$, Corollary 2.11(2) gives

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta)\,,$$

with equality when $s(\zeta) = 1$. □

Consider again Figure 5 in Section 2. We have the following minimal cycles

$$\begin{aligned} \mathcal{L}(\langle 123\rangle) &= 2 = 3-1 = \#\mathrm{supp}(\langle 123\rangle) - \delta(\langle 123\rangle), \\ \mathcal{L}(\langle 1\overline{3}\,]) &= 2 = 2-0 = \#\mathrm{supp}(\langle 1\overline{3}\,]) - \delta(\langle 1\overline{3}\,])\,. \end{aligned}$$

Not all cycles are minimal. For example $\mathcal{L}(\langle 1\overline{2}3\,]) = 5 > 3 - 0 = \#\mathrm{supp}(\langle 1\overline{2}3\,]) - \delta(\langle 1\overline{2}3\,])$ and so $\langle 1\overline{2}3\,]$ is a cycle which is *not* minimal. On the other hand, all permutations $\zeta \in \mathcal{B}_3$ with $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$ are cycles. More generally,

Corollary 6.2. *$\zeta \in \mathcal{B}_\infty$ is a minimal cycle if and only if it is irreducible and $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$.*

Proof. The forward implication is clear. For the converse, recall that if $\eta, \xi \in \mathcal{B}_\infty$ have disjoint supports, then $\mathcal{L}(\eta\cdot\xi) \geq \mathcal{L}(\eta) + \mathcal{L}(\xi)$, with equality only when $\eta\cdot\xi$ is a disjoint product. Thus, by Lemma 6.1,

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta)\,,$$

with equality only when $s(\zeta) + \delta(\zeta) = 1$, which implies that ζ is a cycle. □

In view of Definition 1.4 we have that a permutation $\zeta \in \mathcal{B}_\infty$ is *minimal* if each of its disjoint irreducible factors are minimal cycles. For example the permutation $\langle 13\,]\langle 2\,]$ above is irreducible but not minimal as it is not a cycle, and the permutation $\langle 1\,]\langle 2\overline{3}\,]$ is minimal since both irreducible factors $\langle 1\,]$ and $\langle 2\overline{3}\,]$ are minimal cycles.

Corollary 6.3. *If $\eta, \zeta \in \mathcal{B}_\infty$ with $\eta \prec \zeta$ and ζ is minimal, then so is η.*

Proof. By Proposition 2.12, we may assume ζ is irreducible. Then ζ is a single cycle and the result follows by induction on $\mathcal{L}(\zeta)$, similar to the proof of Lemma 6.1. $\square$

If $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of $\zeta \in \mathcal{B}_\infty$ into disjoint irreducible permutations, then we have $\mathcal{L}(\zeta) = \sum_i \mathcal{L}(\zeta_i)$ and also $\#\mathrm{supp}(\zeta) = \sum_i \#\mathrm{supp}(\zeta_i)$. We deduce the following important inequality concerning minimal permutations.

Corollary 6.4. *Let $\zeta \in \mathcal{B}_\infty$. If $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of ζ into disjoint irreducible permutations, then*

$$\mathcal{L}(\zeta) \ \geq \ \#\mathrm{supp}(\zeta) - \sum_i \delta(\zeta_i)\,,$$

with equality only if ζ is minimal.

We establish our main technical result concerning minimal cycles.

Lemma 6.5. *If $\zeta \in \mathcal{B}_\infty$ is a minimal cycle with $\delta(\zeta) = 0$, then there is a unique $a \in \mathrm{supp}(\zeta)$ with $a > 0 > \zeta(a)$. Let α be $\overline{\zeta(a)}$. Furthermore,*

$$\begin{array}{lll} 1 \prec t_a \preceq \zeta & & \text{if } a \leq \alpha, \\ 1 \preceq t_\alpha\zeta \prec \zeta & & \text{if } a \geq \alpha\,. \end{array}$$

For example, let $\zeta = \langle 132\overline{4}]$. We have $\delta(\zeta) = 0$. Since $\zeta(2) = \overline{4}$ and $2 \leq 4$, we have $t_2 \prec \zeta$. (See Figure 6 in Section 6.3 for the Hasse diagram of the interval $[e, \zeta]_\prec$.)

Proof. Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle with $\delta(\zeta) = 0$. Then $s(\zeta) = 1$, so there is a unique $a > 0$ with $0 > \zeta(a)$. Set $\alpha = \overline{\zeta(a)}$. We prove the lemma when $a > \alpha$: If $a = \alpha$, then $\zeta = t_a$, as ζ is irreducible and if $a < \alpha$, then replacing ζ by ζ^{-1}, reduces to the case $a > \alpha$, by Proposition 2.13 (6).

Suppose $a > \alpha$. We claim that if $b > a$, then $\zeta(b) > \alpha$. The lemma follows from this claim. Indeed, then condition (ii) of Definition 2.9(1) is satisfied, and hence $t_\alpha\zeta \prec \zeta$. Since $a > \alpha$, we have $\mathrm{supp}(t_\alpha\zeta) = \mathrm{supp}(\zeta)$. As $\delta(t_\alpha\zeta) = 1$, it follows that $\mathcal{L}(t_\alpha\zeta) \geq \#\mathrm{supp}(\zeta) - 1 = \mathcal{L}(\zeta) - 1$ and thus $t_\alpha\zeta \lessdot \zeta$.

Let ζ be a minimal cycle and a as above, $b > a$ with $a > \zeta(b)$, and $\#\mathrm{supp}(\zeta)$ minimal having these properties. By Algorithm 2.15, $\eta := \zeta t_{y\,x} \lessdot \zeta$, where x is maximal in $\mathrm{supp}(\zeta)$ and y is minimal subject to $y \leq \zeta(x) < \zeta(y) \leq x$. Then $y \neq a$, as $y < \zeta(y)$. Since $a < x$, $\delta(\eta) = 0$. As ζ is minimal and $\eta \lessdot \zeta$, Corollary 6.3 implies that η is minimal. In $\mathcal{S}_{\pm\infty}$, η is the product of 2 cycles, so as a permutation in $\mathcal{B}_\infty$, either η is irreducible and $\mathrm{supp}(\eta) \subsetneq \mathrm{supp}(\zeta)$ or else η is the disjoint product of two minimal cycles with x in the support of one and $|y|$ in the support of the other.

If $y < 0$, then $y = \overline{a}$ as $\zeta(y) > 0$ and $s(\zeta) = 1$. Then $\eta(a) = \overline{\zeta(x)}$ and $\eta(x) = \alpha$, so $\mathrm{supp}(\eta) = \mathrm{supp}(\zeta)$ and so η is reducible with x in the support of one component and a in the other. But $x > a > \alpha = \eta(x) > \overline{\zeta(x)} = \eta(a)$, contradicting disjointness.

If $y > 0$, then $\zeta(x) > \alpha$, for otherwise $y = \overline{a} < 0$. Thus $x > b$ and so we have $b > a > \alpha > \eta(b)$ with $\eta(a) = \overline{\alpha}$. Thus the component η' of η whose support contains a also contains b and $\mathrm{supp}(\eta') \subsetneq \mathrm{supp}(\zeta)$ with $\delta(\eta') = 0$, contradicting the minimality of $\#\mathrm{supp}(\zeta)$. $\square$

6.2. **Grassmannian Bruhat order on $\mathcal{S}_\infty$.** We develop some additional combinatorics for the symmetric group $\mathcal{S}_\infty$. Recall that the Grassmannian Bruhat order on $\mathcal{S}_\infty$ is induced from that on $\mathcal{S}_{\pm\infty}$.

Covers $\eta \lessdot \zeta$ in the Grassmannian Bruhat order correspond to transpositions $(\alpha,\, \beta) = \zeta\eta^{-1}$. We construct a labeled Hasse diagram for $(\mathcal{S}_\infty,\, \lessdot\,)$, labeling such a cover with the greater of α, β. By Theorem 3.2.3 of [3], the map $\eta \mapsto \zeta\eta^{-1}$ induces an order-reversing isomorphism between $[e, \zeta]_{\lessdot}$ and $[e, \zeta^{-1}]_{\lessdot}$, preserving the edge labels. Also, if $P = \{p_1, p_2, \dots\} \subset \mathbb{N}$ and $\varepsilon_P : \mathcal{S}_{\#P} \hookrightarrow \mathcal{S}_\infty$ is the map induced by $i \mapsto p_i$ (these maps induce shape-equivalence), then ε_P induces an isomorphism $[e, \zeta]_{\lessdot} \xrightarrow{\sim} [e, \varepsilon_P(\zeta)]_{\lessdot}$, preserving the relative order of the edge labels. Specifically, an edge label i of $[e, \zeta]_{\lessdot}$ is mapped to the label p_i of $[e, \varepsilon_P(\zeta)]_{\lessdot}$. Lastly, we remark that Algorithm 2.15 restricted to $\mathcal{S}_\infty$, and with $t_{a\,b}$ replaced by the transposition $(a,\, b)$, gives a chain in the $\lessdot$-order on $\mathcal{S}_\infty$ from e to ζ.

Lemma 6.6. *Let $\zeta \in \mathcal{S}_\infty$ and suppose that x is maximal subject to $x \neq \zeta(x)$. Then, for any α*

(1) $(\alpha,\, x)\zeta \lessdot \zeta \implies \zeta^{-1}(x) \leq \zeta(x) = \alpha$.
(2) $(\alpha,\, x) \lessdot \zeta \implies \alpha = \zeta^{-1}(x) \geq \zeta(x)$.

Proof. For 1, let $\eta := (\alpha,\, x)\zeta \lessdot \zeta$ and set $a = \zeta^{-1}(x)$ and $b = \zeta^{-1}(\alpha)$. Note that $a \neq b$ and $\eta(b) = x$. We claim that $b = x$ and $a \leq \alpha$, which will establish 1.

Suppose $b \neq x = \eta(b)$. Then, by the maximality of x, $b < \eta(b)$ and so the definition of $\lessdot$ implies $\eta(b) \leq \zeta(b) = \alpha$. Since $\alpha < x$, this implies $x < x$, a contradiction. Suppose now that $a > \alpha = \eta(a)$. By the definition of $\lessdot$, this implies that $\eta(a) \geq \zeta(a) = x$, and so $a > x$, contradicting the maximality of x.

The second assertion follows from the first by applying the anti-isomorphism $\eta \mapsto \eta\zeta^{-1}$ between $[e, \zeta]_{\lessdot}$ and $[e, \zeta^{-1}]_{\lessdot}$,

$$e \;\lessdot\; (\alpha,\, x) \;\lessdot\; \zeta \iff (\alpha,\, x)\zeta^{-1} \;\lessdot\; \zeta^{-1}. \quad \square$$

We illustrate Lemma 6.6. Let $\zeta = (1, 2, 5, 3, 4) \in \mathcal{S}_5$ and consider Figure 7 in Section 6.4, which shows the interval $[e, \iota(\zeta)]_{\prec} \simeq [e, \zeta]_{\lessdot}$. Then $2 = \zeta^{-1}(5) < \zeta(5) = 3$, and $(1, 2, 3, 4) = (2, 5)\zeta \lessdot \zeta$ in accordance with (1). Similarly, for $\zeta = (1, 2, 3, 4)$, we have $3 = \zeta^{-1}(4) \geq \zeta(4) = 1$, and $(3, 4) \lessdot \zeta$, in accordance with (2).

A cycle $\zeta \in \mathcal{S}_\infty$ is *minimal* if $\mathcal{L}_{A}(\zeta) = \#\mathrm{supp}(\zeta) - 1$, that is, if its length is minimal given its support. A permutation is *minimal* if it is the disjoint product of minimal cycles. For example $\zeta = (1, 3, 5, 2, 4) \in \mathcal{S}_5$ is not minimal since $\mathcal{L}_{A}(\zeta) = 6 > 4 = \#\mathrm{supp}(\zeta) - 1$, and $(1, 5, 2)(3, 4)$ is minimal since both $(1, 5, 2)$ and $(3, 4)$ are minimal cycles of $\mathcal{S}_5$. It is worthwhile to note that if $\zeta \in \mathcal{S}_\infty$ is a minimal cycle, then under the embedding $\iota : \mathcal{S}_\infty \hookrightarrow \mathcal{B}_\infty$ the cycle $\iota(\zeta)$ is minimal in $\mathcal{B}_\infty$. Indeed, from Corollary 2.11, $\mathcal{L}(\iota(\zeta)) = (2\mathcal{L}_{A}(\zeta) + 0)/2 = \mathcal{L}_{A}(\zeta) = \#\mathrm{supp}(\zeta) - 1 = \#\mathrm{supp}(\iota(\zeta)) - \delta(\iota(\zeta))$

A maximal chain in an interval $[e, \zeta]_{\lessdot}$ is *peakless* if we do not have $a_{i-1} < a_i > a_{i+1}$ for any $i = 2, \dots, \mathcal{L}_{A}(\zeta) - 1$, where $a_1, \dots, a_{\mathcal{L}_{A}(\zeta)}$ is the sequence of labels in that chain.

Lemma 6.7. *Suppose $\zeta \in \mathcal{S}_\infty$ is a minimal cycle. Then there is a unique peakless chain in the labeled interval $[e,\, \zeta]_{\lessdot}$. If β is the smallest label in such a chain, then the transposition of that cover is (α, β) where $\alpha < \beta$ are the two smallest elements of* $\mathrm{supp}(\zeta)$.

Proof. We argue by induction on $\mathcal{L}_{A}(\zeta)$, which we assume is at least 2, as the case $\mathcal{L}_{A}(\zeta) = 1$ is immediate. Replacing ζ by a shape-equivalent permutation if necessary, we may assume that $\mathrm{supp}(\zeta) = [n]$, so that $\mathcal{L}_{A}(\zeta) = m = n - 1$.

Replacing ζ by ζ^{-1} would only reverse such a chain, so we may assume that $a := \zeta^{-1}(n) < b := \zeta(n)$. We claim that $(b,\, n)\zeta = \zeta(a,\, n) \lessdot \zeta$. Given this, the conclusion of the lemma follows. Indeed, let $\eta := (b,\, n)\zeta$. Since $\eta(n) = n$, this is an irreducible minimal permutation in $\mathcal{S}_{n-1}$. By the inductive hypothesis, $[e, \eta]_{\lessdot}$ has a unique chain with labels $\beta_1 > \cdots > \beta_k < \cdots < \beta_{n-2}$, and each $\beta_i < n$. The unique extension of this to a chain in $[e, \zeta]_{\lessdot}$ has $\beta_{n-2} < \beta_{n-1} = n$. This is the unique such chain in $[e, \zeta]_{\lessdot}$ as $\eta \lessdot \zeta$ is the unique terminal cover in $[e, \zeta]_{\lessdot}$ with edge label n, by Lemma 6.6.

Suppose the statement about the minimal cover holds for the permutation η of the previous paragraph. Since β_{n-1} is the maximum of $\mathrm{supp}(\zeta)$, the statement about the minimal cover also holds for ζ.

We prove the claim, that $\zeta(a,\, n) \lessdot \zeta$, where $a = \zeta^{-1}(n)$. By Algorithm 2.15, if y is chosen minimal so that $y \leq \zeta(n) = b < \zeta(y)$, then $\zeta(y,\, n) \lessdot \zeta$. We show that $y = a$, which will establish the claim and complete the proof.

Suppose $y \neq a$. Since $a < b < n = \zeta(a)$, the minimality of y implies that $y < a$. But $\zeta = (\cdots a\, n\, b \cdots)$, so $\zeta(y,\, n)$ consists of two cycles, which we call η and η' where we have $\eta(a) = n$ and $\eta'(y) = b$. Since $y < a < b < n$, these cycles are not disjoint, so we have

$$
\begin{aligned}
n - 2 \;=\; \mathcal{L}_{A}(\eta \cdot \eta') \;\; &> \;\; \mathcal{L}_{A}(\eta) + \mathcal{L}_{A}(\eta') \\
&\geq \;\; \#\mathrm{supp}(\eta) - 1 + \#\mathrm{supp}(\eta') - 1 \;=\; n - 2,
\end{aligned}
$$

a contradiction. ☐

6.3. **Labeled Lagrangian order.** The Pieri-type formula for $SO_{2n+1}\mathbb{C}/B$ has two formulations (Theorems A and D), which we relate. The labeled Lagrangian and 0-Bruhat orders on $\mathcal{B}_\infty$ are obtained from the Hasse diagrams of the underlying orders by labeling each cover with the integer β, where that cover is either $\zeta \prec t_\beta \zeta$ or $\zeta \prec t_{\alpha\,\beta}\zeta$ in the Lagrangian order ($u \lessdot t_\beta u$ or $u \lessdot t_{\alpha\,\beta} u$ in the 0-Bruhat order). By Corollary 5.5, the map $\iota : \mathcal{S}_\infty \hookrightarrow \mathcal{B}_\infty$ maps the Grassmannian Bruhat order on $\mathcal{S}_\infty$ isomorphically onto its image in the Lagrangian order. This map preserves edge labels, as $\iota(\alpha, \beta) = t_{\alpha\,\beta}$ and the covers $\eta \lessdot (\alpha, \beta)\eta$ in the Grassmannian Bruhat order and $\iota(\eta) \prec t_{\alpha\,\beta}\iota(\eta)$ have the same label, β. Thus ι is an inclusion of *labeled* orders.

A chain in $[e, \zeta]_{\prec}$ is *peakless* if in its sequence $\beta_1, \ldots, \beta_m$ of labels, we do not have $\beta_{i-1} < \beta_i > \beta_{i+1}$, for any $i = 2, \ldots, m-1$. Recall that $s(\zeta)$ counts the sign changes $\{a > 0 \mid \zeta(a) < 0\}$ for $\zeta \in \mathcal{B}_\infty$.

Lemma 6.8. *Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle. Then there is a unique peakless chain in the labeled interval $[e, \zeta]_{\prec}$. If $\delta(\zeta) = 0$, then the minimal label of that chain corresponds to the cover whose reflection is t_α, where α is minimal in the support of ζ.*

Proof. If $\delta(\zeta) = 1$ this is a consequence of Lemma 6.7. Suppose that $\delta(\zeta) = 0$. Replacing ζ by a shape-equivalent permutation if necessary, we may assume that $\mathrm{supp}(\zeta) = [n]$ and $n > 1$, as the case $n = 1$ is immediate. Replacing ζ by ζ^{-1} if necessary, we may assume that $a := \zeta^{-1}(n) < b := \zeta(n)$. As in the proof of Lemma 6.7, $(b, n)\zeta \lessdot \zeta$ in the Grassmannian Bruhat order on $\mathcal{S}_{\pm\infty}$. By Remark 2.3, either $t_{b\,n}\zeta \prec \zeta$ or else we have both $t_b\zeta \prec \zeta$ and $t_n\zeta \prec \zeta$. The second case implies $s(\zeta) > 1$, contradicting the minimality of ζ. Thus $\eta := t_{b\,n}\zeta \prec \zeta$.

Then η is a minimal cycle with $\delta(\eta) = 0$ and $\mathrm{supp}(\eta) = [n-1]$. Appending the cover $\eta \xrightarrow{n} \zeta$ to the unique peakless chain in $[e,\eta]_\prec$ gives a peakless chain in $[e,\zeta]_\prec$. Moreover, η is the unique permutation with $\eta \prec \zeta$ and $\eta \xrightarrow{n} \zeta$, showing the uniqueness of this chain.

Since the last label in that chain is $n > 1$, the minimal label occurs in its restriction to $[e,\eta]_\prec$. Since $\mathrm{supp}(\eta) = \mathrm{supp}(\zeta) - \{n\}$, the conclusion of the lemma follows. $\square$

We illustrate Lemma 6.8. Let $\zeta = \langle 132\overline{4}\,] \in \mathcal{B}_4$. Then $\delta(\zeta) = 0$. Figure 6 shows the labeled order $[e,\zeta]_\prec$. This has a unique peakless chain:

$$e \xrightarrow{2} \langle 12\rangle \xrightarrow{1} \langle 12\,] \xrightarrow{3} \langle 132\,] \xrightarrow{4} \langle 132\overline{4}\,] .$$

Observe that $\langle 12\rangle \prec t_1\langle 12\,]$ is the cover with minimal label in this chain.

For any $\zeta \in \mathcal{B}_\infty$, let $\Pi(\zeta)$ be the number of peakless chains in $[e,\zeta]_\prec$.

Lemma 6.9. *If $\eta,\zeta \in \mathcal{B}_\infty$ are disjoint, then*

$$\Pi(\eta\cdot\zeta) \;=\; 2\,\Pi(\eta)\cdot\Pi(\zeta) .$$

Proof. A shuffle S of two sequences A, B is a sequence consisting of elements of A and of B, in which the elements from A (respectively from B) are in the same order in S as they are in A (respectively in B). By Proposition 2.12, $[e,\eta\cdot\zeta]_\prec \simeq [e,\eta]_\prec \times [e,\zeta]_\prec$. Thus a chain in $[e,\eta\cdot\zeta]_\prec$ is a shuffle of a chain from $[e,\eta]_\prec$ with a chain from $[e,\zeta]_\prec$, and all shuffles occur.

For $\xi \in \mathcal{B}_\infty$, let $W(\xi)$ be the multiset of words formed from labels of chains in $[e,\xi]_\prec$. The alphabet of these words is a subset of $\mathrm{supp}(\xi)$. Thus $W(\eta)$ and $W(\zeta)$ have disjoint alphabets. Then $W(\eta\cdot\zeta)$ consists of all shuffles of pairs of words in $W(\eta)\times W(\zeta)$. The lemma follows from a combinatorial result concerning peakless words and shuffles proven in Lemma 6.10. $\square$

For a set A of words in an ordered alphabet $\mathcal{A}$, let $\mathrm{peak}(A)$ be the subset of peakless words from A. Suppose that A' is another set of words with a different

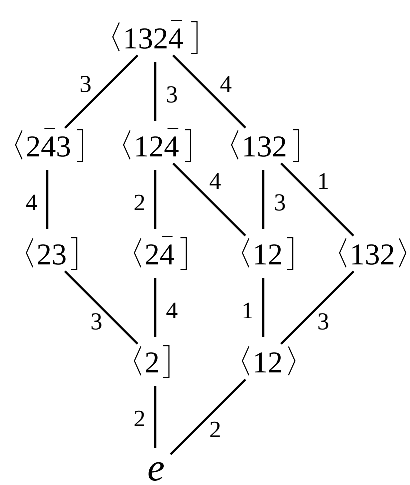


Figure 6. The interval $\big[e, \langle 132\overline{4}\,]\big]_\prec$.

alphabet $\mathcal{A}'$ and fix some total order on the disjoint union $\mathcal{A}\coprod\mathcal{A}'$ which extends the given orders on each of $\mathcal{A},\mathcal{A}'$. Let $\mathrm{sh}(A,A')$ be all shuffles of pairs of words in $A\times A'$.

Lemma 6.10. *The natural restriction map* $\mathrm{sh}(A,A')\twoheadrightarrow A\times A'$ *induces a* 2 *to* 1 *map*

$$\mathrm{peak}(\mathrm{sh}(A,A'))\ \longrightarrow\!\!\!\!\rightarrow\ \mathrm{peak}(A)\times\mathrm{peak}(A')\,.$$

Proof. It is clear that the restriction map takes a peakless word in $\mathrm{sh}(A,A')$ to a pair of peakless words in $A\times A'$. Given a pair of peakless words $(\omega,\omega')\in A\times A'$, there are exactly two shuffles of ω,ω' which are peakless: Suppose the minimal letter a in ω is greater than the minimal letter in ω'. Then these two shuffles differ only in their subwords consisting of a and u', where u' is that subword of ω' consisting of all letters less than a. Then u' is a segment of ω', as ω' is peakless. The two subwords of peakless shuffles are $a.u'$ and $u'.a$. $\square$

Lemma 6.11. *Let* $\zeta\in\mathcal{B}_\infty$ *and suppose there is a peakless chain in* $[e,\zeta]_\prec$. *Then* ζ *is minimal.*

Proof. Suppose by way of contradiction that $\zeta\in\mathcal{B}_\infty$ is irreducible and not minimal, but $\Pi(\zeta)\neq 0$. We may further assume that among all such permutations, ζ has minimal rank, and that $\mathrm{supp}(\zeta)=[n]$. Let $\beta_1>\cdots>\beta_k<\cdots<\beta_m$ be the labels of a peakless chain in $[e,\zeta]_\prec$. Replacing ζ by ζ^{-1} if necessary (which merely reverses the chain), we may assume that $\beta_m=n$ and so $\beta_1\neq n$, by Lemma 6.6 and Theorem 2.2. Let η be the penultimate member of this chain. Then $\Pi(\eta)\neq 0$, as the initial segment of this chain gives a peakless chain in $[e,\eta]_\prec$. Thus η is a minimal permutation, by our assumption on ζ, and so

$$\mathcal{L}(\eta)\ \leq\ \#\mathrm{supp}(\eta)-\delta(\eta)\ \leq\ n-\delta(\zeta)\ <\ \mathcal{L}(\zeta)\ =\ \mathcal{L}(\eta)+1\,,$$

as ζ is not minimal and $\eta\prec\zeta$ so $\delta(\eta)\geq\delta(\zeta)$. Therefore the weak inequalities are equalities and $\mathrm{supp}(\eta)=[n]$. Since $\beta_1>\cdots>\beta_k<\cdots<\beta_{m-1}$ are the labels of a chain in $[e,\eta]_\prec$ and $\beta_{m-1}<n$, we must have $\beta_1=n$, as $\mathrm{supp}(\eta)=[n]$. But this contradicts our earlier observation about β_1. $\square$

We relate the two formulations of the Pieri-type formula for $SO_{2n+1}\mathbb{C}/B$. For $\zeta\in\mathcal{B}_\infty$, define

$$\theta(\zeta)\ =\ \begin{cases} 2^{\#\{\text{irreducible factors of }\zeta\}-1} & \text{if }\zeta\text{ is minimal,}\\ 0 & \text{otherwise.}\end{cases}$$

Corollary 6.12. *For* $\zeta\in\mathcal{B}_\infty$, $\Pi(\zeta)=\theta(\zeta)$.

Proof. This is clear if ζ is minimal as both $\Pi(\zeta)$ and $\theta(\zeta)$ satisfy the same recursion, by Lemmas 6.8 and 6.9, and when ζ is not minimal, $\Pi(\zeta)=0\,(=\theta(\zeta))$ by Lemma 6.11. $\square$

6.4. **Lagrangian réseau.** The enumerative significance of the constants $c^w_{u\,v}$ is best expressed in terms of maximal chains in certain directed multigraphs associated to intervals in the Bruhat order. A cover $\eta\prec\zeta$ in the Lagrangian order corresponds to a reflection $\eta\zeta^{-1}$, which is either of the form $t_{a\,b}$ or of the form t_a. The *Lagrangian réseau* on $\mathcal{B}_\infty$ is the labeled directed multigraph where a cover $\eta\prec\zeta$ in the Lagrangian order with $\eta\zeta^{-1}=t_a$ is given a single edge $\eta\xrightarrow{a}\zeta$ and a cover with $\eta\zeta^{-1} = t_{a\,b}$ is given two edges $\eta \xrightarrow{\overline{a}} \zeta$ and $\eta \xrightarrow{b} \zeta$. We obtain the labeled Lagrangian order from this réseau by erasing those edges with negative labels.

In the Grassmannian Bruhat order on $\mathcal{S}_\infty$, there are two conventions for labeling a cover $\eta \lessdot \zeta$: This cover gives a transposition $(\alpha, \beta) := \eta\zeta^{-1}$ with $\alpha < \beta$, and we may choose either α or β. For want of a better term, we call the consistent choice of α the lower convention, and the consistent choice of β the upper convention. (We have been using the upper convention in the previous sections.) We make use of the following fact.

Proposition 6.13. *Let $\eta \in \mathcal{S}_\infty$. If there is a chain in $[e,\eta]_{\lessdot}$ with decreasing labels in the lower convention, then there is a chain in $[e,\eta]_{\lessdot}$ with decreasing labels in the upper convention, and these chains are unique. The same is true for chains with increasing labels, and in either case η is minimal.*

Proof. The map $\zeta \in \mathcal{S}_n \mapsto \omega_n\zeta\omega_n$ ($\omega_n \in \mathcal{S}_n$ is the longest element) is an isomorphism of orders [3, Theorem 3.2.3 (vii)] which takes decreasing chains in the upper (respectively lower) convention to increasing chains in the lower (respectively upper) convention. By the definition of the Grassmannian Bruhat order $\lessdot$ and Lemmas 2 and 6 in [32], $[e,\zeta]_{\lessdot}$ has an increasing chain in the upper convention if and only if $[e,\omega_n\zeta\omega_n]_{\lessdot}$ has a decreasing chain in the upper convention, and any such chain must be unique. ∎

A chain with increasing labels is an *increasing chain* and one with decreasing labels is a *decreasing chain*.

Lemma 6.14. *Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle. Then the réseau $[e,\zeta]_{\prec}$ has an increasing chain. If $\delta(\zeta) = 1$, then there are at least 2 increasing chains.*

Proof. Consider the peakless chain in the labeled order $[e,\zeta]_{\prec}$ given by Lemma 6.8:

$$e \xrightarrow{\alpha_1} \zeta_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} \zeta_m = \zeta. \tag{6.1}$$

Let α_k be the minimal label in this chain. Then $\zeta_{k-1} = \iota(\eta_{k-1})$ for some $\eta_{k-1} \in \mathcal{S}_\infty$. To see this, if $\delta(\zeta) = 0$, then by Lemma 6.8, the label α_k corresponds to the only cover whose reflection is not in $\iota(\mathcal{S}_\infty)$, and so $\delta(\zeta_{k-1}) = 1$. If $\delta(\zeta) = 1$, then $\delta(\zeta_i) = 1$, by Lemma 5.3.

The pullback of the initial segment of this chain to $[e,\eta_{k-1}]_{\lessdot}$ gives a decreasing chain (with labels $\alpha_1, \ldots, \alpha_{k-1}$) in the upper convention. Consider the unique decreasing chain

$$e \xrightarrow{\beta_1} \eta_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{k-1}} \eta_{k-1}$$

in the lower convention. Then

$$e \xrightarrow{\overline{\beta_1}} \iota(\eta_1) \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) = \zeta_{k-1}$$

is an increasing chain in the réseau $[e,\zeta_{k-1}]_{\prec}$. Concatenating the end of the peakless chain (6.1) onto this gives an increasing chain

$$e \xrightarrow{\overline{\beta_1}} \iota(\eta_1) \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) \xrightarrow{\alpha_k} \cdots \xrightarrow{\alpha_m} \zeta$$

in the réseau $[e,\zeta]_{\prec}$.

Suppose $\delta(\zeta) = 1$ and consider the middle portion of this increasing chain:

$$\iota(\eta_{k-2}) \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) = \zeta_{k-1} \xrightarrow{\alpha_k} \zeta_k\,.$$

Let $\overline{b}$ be the label of the other edge between ζ_{k-1} and ζ_k. Then we claim that $\overline{\beta_{k-1}} < \overline{b}$ so that replacing $\zeta_{k-1} \xrightarrow{\alpha_k} \zeta_k$ by $\zeta_{k-1} \xrightarrow{\overline{b}} \zeta_k$ gives a second increasing chain in the réseau.

To see this, first note that by Theorem 2.2 and [5, Equation (1.1)(4)], $\beta_{k-1} = b$ is impossible as these are consecutive covers in the Lagrangian order. Define η by $\iota(\eta) = \zeta$ and pull this chain back to $[e, \eta]_{\lessdot}$. It is the unique peakless chain in $[e, \eta]_{\lessdot}$ and α_k is the minimal label. By Lemma 6.7, $b = \min(\mathrm{supp}(\eta))$ and so $\beta_{k-1} \geq b$. ☐

Lemma 6.15. *Let $\zeta \in \mathcal{B}_\infty$ and suppose there is an increasing chain in the réseau $[e, \zeta]_\prec$. Then ζ is minimal. If ζ is a minimal cycle, then there are precisely $2^{\delta(\zeta)}$ such chains.*

Proof. Let

$$e \xrightarrow{\beta_1} \zeta_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} \zeta_m = \zeta \tag{6.2}$$

be an increasing chain in $[e, \zeta]_\prec$. Suppose that $\beta_{k-1} < 0 < \beta_k$. Then for $i < k$, $\delta(\zeta_i) = 1$. Define $\eta_i \in \mathcal{S}_\infty$ by $\iota(\eta_i) = \zeta_i$ for $i < k$. Then

$$e \xrightarrow{\overline{\beta_1}} \eta_1 \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \eta_{k-1}$$

is a decreasing chain in $[e, \eta]_{\lessdot}$, with the lower labeling convention. Let

$$e \xrightarrow{\alpha_1} \xi_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} \xi_{k-1} = \eta_{k-1}$$

be the unique decreasing chain in the upper labeling convention, by Proposition 6.13. Concatenating the image of this chain in $[e, \zeta]_\prec$ with the end of the chain (6.2) gives a peakless chain

$$e \xrightarrow{\alpha_1} \iota(\xi_1) \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} \iota(\xi_{k-1}) = \zeta_{k-1} \xrightarrow{\beta_k} \cdots \xrightarrow{\beta_m} \zeta \tag{6.3}$$

in the interval $[e, \zeta]_\prec$ in the Lagrangian order. By Lemma 6.11, ζ is necessarily minimal.

Suppose now that ζ is a minimal cycle, then the réseau $[e, \zeta]_\prec$ has an increasing chain, by Lemma 6.14, and the peakless chain (6.3) is unique. Consider another increasing chain

$$e \xrightarrow{\beta'_1} \zeta'_1 \xrightarrow{\beta'_2} \cdots \xrightarrow{\beta'_m} \zeta'_m = \zeta \tag{6.4}$$

and form η'_i, ξ'_i, α'_i,and k' as for the original chain (6.2). If $k = k'$, then the chains (6.2) and (6.4) coincide: The final segments agree, by the uniqueness of (6.3), as do their initial segments, by Proposition 6.13.

If $\delta(\zeta) = 0$, then the minimal label in the peakless chain (6.3) (either α_{k-1} or β_k) corresponds to the cover whose reflection has the form t_a. As $\delta(\zeta_{k-1}) = 1$, this must be β_k and so $k = k'$ and the chain (6.2) is the unique increasing chain in the réseau $[e, \zeta]_\prec$.

Suppose now that $\delta(\zeta) = 1$ and $k < k'$. Since $\alpha_i = \alpha'_i$ for $i < k$, $\beta_i = \beta'_i$ for $i \geq k'$, and $\alpha_1 > \cdots > \alpha_{k-1}$ and $\beta_k < \cdots < \beta_m$, we must have $k' = k + 1$. But then $\xi_i = \xi'_i$ for $i < k$ and also $\zeta_i = \zeta'_i$ for $i \geq k$, and so the two chains (6.2) and (6.4) agree except for the label of the cover $\zeta_{k-1} \prec \zeta_k$. Thus there are at most 2 increasing chains in the réseau $[e, \zeta]_\prec$ and their underlying permutations coincide. Thus by Lemma 6.14, there are exactly $2 = 2^{\delta(\zeta)}$ increasing chains. ☐

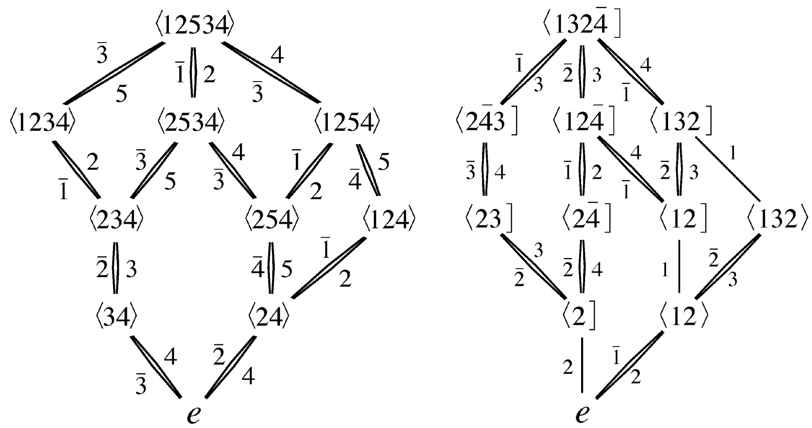


FIGURE 7. The intervals $[e, \langle 12534\rangle]_{\prec}$ and $\big[e, \langle 132\overline{4}\,]\big]_{\prec}$.

We illustrate Lemma 6.15. Let $\eta = (1,2,5,3,4) \in \mathcal{S}_5$. Let $\zeta = \langle 12534\rangle = \iota(\eta) \in \mathcal{B}_5$. Figure 7 shows the réseau $[e,\zeta]_{\prec}$. In the réseau, there are two increasing chains

$$e \xrightarrow{\overline{3}} \langle 34\rangle \xrightarrow{\overline{2}} \langle 234\rangle \xrightarrow{\overline{1}} \langle 1234\rangle \xrightarrow{5} \langle 12534\rangle,$$
$$e \xrightarrow{\overline{3}} \langle 34\rangle \xrightarrow{\overline{2}} \langle 234\rangle \xrightarrow{2} \langle 1234\rangle \xrightarrow{5} \langle 12534\rangle$$

which correspond to the unique peakless chain in $[e,\eta]_{\lessdot}$

$$e \xrightarrow{4} (3,4) \xrightarrow{3} (2,3,4) \xrightarrow{2} (1,2,3,4) \xrightarrow{5} (1,2,5,3,4)\,.$$

Now let $\zeta = \langle 132\overline{4}\,] \in \mathcal{B}_5$. The réseau $[e,\zeta]_{\prec}$ is also illustrated in Figure 7. There is a unique peakless chain in $[e,\zeta]_{\prec}$:

$$e \xrightarrow{2} \langle 12\rangle \xrightarrow{1} \langle 12\,] \xrightarrow{3} \langle 132\,] \xrightarrow{4} \langle 132\overline{4}\,]\,.$$

We enumerate the increasing chains in an interval in the Lagrangian réseau. Let $I(\zeta)$ be the number of increasing chains in the réseau $[e,\zeta]_{\prec}$.

Lemma 6.16. *If $\eta,\zeta \in \mathcal{B}_\infty$ are disjoint, then $I(\eta\cdot\zeta) = I(\eta)\cdot I(\zeta)$.*

Proof. As with Lemma 6.9, this is a consequence of the analogous bijection concerning increasing words among shuffles of words with disjoint alphabets. ☐

We relate the two formulations of the Pieri-type formula for $Sp_{2n}\mathbb{C}/B$. For $\zeta \in \mathcal{B}_\infty$, define

$$\chi(\zeta) \;=\; \begin{cases} 2^{\#\{\text{irreducible factors } \eta \text{ of } \zeta \text{ with } \delta(\eta)\,=\,1\}} & \text{if } \zeta \text{ is minimal,}\\ 0 & \text{otherwise.}\end{cases}$$

Let $D(\zeta)$ be the number of the decreasing chains in the réseau $[e,\zeta]_{\prec}$. By Proposition 2.13(6), an increasing chain in ζ becomes a decreasing chain in ζ^{-1}. By Lemmas 6.16 and 6.15, the following result is now immediate.

Corollary 6.17. *For $\zeta \in \mathcal{B}_\infty$, $I(\zeta) = \chi(\zeta) = D(\zeta)$.*

We compare the two formulas in Theorem D. Theorem D asserts that $b^w_{u\,m} = \theta(wu^{-1})$ and $c^w_{u\,m} = \chi(wu^{-1})$. Since the number of sign changes in the permutation $v(m)$ is 1, formula (3.2) states that

$$2\, b^w_{u\,m} \;=\; 2^{s(w)-s(u)}\, c^w_{u\,m}\,.$$

or (when neither coefficient is zero), $b^w_{u\,m}/c^w_{u\,m} = 2^{s(w)-s(u)-1}$. When wu^{-1} is minimal, $s(wu^{-1})$ is the number of irreducible factors η of wu^{-1} with $\delta(\eta) = 0$. Combining this with the definitions of $\theta(wu^{-1})$ and $\chi(wu^{-1})$, we see that

$$\theta(wu^{-1})/\chi(wu^{-1}) \;=\; 2^{s(w)-s(u)-1} \;=\; b^w_{u\,m}/c^w_{u\,m}$$

which shows the equivalence of the two formulas in Theorem D.

7. Proof of the Pieri-type formula

We first deduce the Pieri-type formula (Theorem D) from the case when the permutation ζ is irreducible and minimal, and then we separately treat the subcases of $\delta(\zeta) = 1$ (Section 7.1) and $\delta(\zeta) = 0$ (Section 7.2). Both cases reduce to the same calculation in the cohomology of the ordinary flag manifold, and the case when $\delta(\zeta) = 0$ is more involved.

For $\zeta \in \mathcal{B}_n$, $\delta(\zeta) = 1$ if $a > 0$ implies $\zeta(a) > 0$ and $\delta(\zeta) = 0$ otherwise. For $\zeta \in \mathcal{B}_n$, we defined

$$\chi(\zeta) \quad = \quad \begin{cases} 2^{\#\{\text{irreducible factors } \eta \text{ of } \zeta \text{ with } \delta(\eta)\,=\,1\}} & \text{if } \zeta \text{ is minimal,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta(\zeta) \quad = \quad \begin{cases} 2^{\#\{\text{irreducible factors } \zeta\}-1} & \text{if } \zeta \text{ is minimal,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.1. *Let $\zeta \in \mathcal{B}_n$ be irreducible and minimal with* $\mathrm{supp}(\zeta) = [n]$. *Set $m := \mathcal{L}(\zeta)$. Then*

$$c^\zeta_m \quad = \quad \chi(\zeta) \quad = \quad \begin{cases} 1 & \text{if } \delta(\zeta) = 0, \\ 2 & \text{if } \delta(\zeta) = 1. \end{cases}$$

We prove the case of Lemma 7.1 when $\delta(\zeta) = 1$ in Section 7.1 and the case when $\delta(\zeta) = 0$ in Section 7.2. We deduce Theorem D from Lemma 7.1.

Theorem D (Pieri-type Formula)**.** *Let $\zeta \in \mathcal{B}_\infty$ with $\mathcal{L}(\zeta) = m$. Then $c^\zeta_m = \chi(\zeta)$ and $b^\zeta_m = \theta(\zeta)$.*

We remark that by Corollaries 6.12 and 6.17, this implies Theorem A, the chain-theoretic version of the Pieri-type formula.

Proof. Let $s(\eta)$ be the number of sign changes in a permutation $\eta \in \mathcal{B}_\infty$. Since $s(v_m) = 1$ and $s(\zeta) = s(\zeta u) - s(u)$ if $u \leq_0 \zeta u$, Equation (3.2) implies that $b^\zeta_m = 2^{s(\zeta)-1} c^\zeta_m$. By Lemma 6.1, a minimal cycle ζ has $s(\zeta) + \delta(\zeta) = 1$, and so

$$\theta(\zeta) \;=\; 2^{s(\zeta)-1}\chi(\zeta)\,.$$

Thus it suffices to show $c^\zeta_m = \chi(\zeta)$.

By Theorem B(2), we may replace ζ by a shape-equivalent permutation if necessary and assume that $\mathrm{supp}(\zeta) = [n]$. Let M be a general isotropic $(n+1-m)$-plane in V. Then, by the projection formula and Kleiman's Theorem on the transversality of a general translate [19], c^ζ_m counts the Lagrangian subspaces K in $\mathcal{Y}_\zeta$ which meet M non-trivially, as q_m is represented by the set of flags $F_\bullet$ where $F_{\overline{1}}$ meets M non-trivially.

Let $\zeta = \zeta_1 \cdots \zeta_s$ be the factorization of ζ into disjoint irreducible permutations. By Lemma 5.9, each factor ζ_i with $\delta(\zeta_i) = 1$ gives quadratic form q_i which vanishes on every $K \in \mathcal{Y}_\zeta$. Let r be the number of these forms. If $K \in \mathcal{Y}_\zeta$ meets M, then $K \cap M$ lies in the common zero locus of these forms on M. Since M is in general position, we must have $r < \dim M = n + 1 - m$. By Corollary 6.4,

$$n + 1 - m \leq n + 1 - (n - \sum_i \delta(\zeta_i)) = r + 1\,,$$

with equality only if each ζ_i is a minimal cycle. Thus $c^\zeta_m = 0$ if ζ is not minimal, and for non-minimal ζ, we have $\chi(\zeta) = 0$.

Thus we need only consider the case when ζ is minimal. Let $\mathcal{W}_\zeta$ be the common zero locus of the r quadratic forms q_i, a reduced complete intersection of codimension r and degree 2^r, by Lemma 5.9. Since M is in general position, $M \cap \mathcal{W}_\zeta$ is $2^r = \chi(\zeta)$ lines. It follows that $c^\zeta_m = d \cdot \chi(\zeta)$, where d counts the Lagrangian subspaces K in $\mathcal{Y}_\zeta$ which contain a general line of $\mathcal{W}_\zeta$. By Lemma 7.1, we deduce that $d = 1$ when ζ is a minimal cycle. We now use this to show $d = 1$ for all minimal permutations.

Let $\zeta = \zeta_1 \cdots \zeta_s$ be the factorization of ζ into disjoint irreducible permutations. Then each ζ_i is a minimal cycle. Set $n_i = \#\mathrm{supp}(\zeta_i)$, and let $\zeta'_i \in \mathcal{B}_{n_i}$ be shape-equivalent to ζ_i. Let V_i be a symplectic vector space of dimension $2n_i$ and identify V with $V_1 \oplus \cdots \oplus V_s$. Then, by Theorem 3.5 the map $\Xi : \mathit{Lag}(V_1) \times \cdots \times \mathit{Lag}(V_s) \to \mathit{Lag}(V)$ defined by $\Xi(K_1, \ldots, K_s) = K_1 + \cdots + K_s$ restricts to an isomorphism $\Xi : \mathcal{Y}_{\zeta'_1} \times \cdots \times \mathcal{Y}_{\zeta'_s} \xrightarrow{\sim} \mathcal{Y}_\zeta$.

Let $0 \neq v \in \mathcal{W}_\zeta$ be a general vector. From the construction of the the forms q_i in Lemma 5.9, we see that $v = v_1 \oplus \cdots \oplus v_s$, where $0 \neq v_i \in \mathcal{W}_{\zeta'_i}$ is a general vector for each $i = 1, \ldots, s$. It follows that $v \in K$ if and only if $v_i \in K_i$. Since there is a unique such K_i for each i, K is unique, and so we have $d = 1$. □

7.1. **Lemma 7.1, case $\delta(\zeta) = 1$.** This case follows from the following lemma. In the notation of the preceding proof, we need only show there is a unique $K \in \mathcal{Y}_\zeta$ meeting a general $v \in \mathcal{W}_\zeta$, when ζ is a minimal cycle with $\delta(\zeta) = 1$.

Lemma 7.2. *Let $\zeta \in \mathcal{B}_n$ with $\mathrm{supp}(\zeta) = [n]$ and $\mathcal{L}(\zeta) = n - 1$ so that ζ is a minimal cycle with $\delta(\zeta) = 1$. Then, for a general $0 \neq v \in \mathcal{W}_\zeta$, there is a unique $K \in \mathcal{Y}_\zeta$ with $v \in K$.*

Proof. Define $\eta \in \mathcal{S}_\infty$ by $\iota(\eta) = \zeta$. Set $k := \#\{a \mid a < \eta(a)\}$. Recall the notation of Section 5.2: Let L, L^* be complementary Lagrangian subspaces of V, which are identified as linear duals. Define the map $\Phi_k : \mathbf{G}_k(L) \to \mathit{Lag}(V)$ by $H \mapsto (H + H^\perp)$, where $H^\perp \subset L^*$ is the annihilator of H. Define $\pi_k : \mathbb{F}\ell(L) \to \mathbf{G}_k(L)$ by $E_\bullet \mapsto E_k$. This gives the commutative diagram (5.6) of Section 5.2. By Corollary 5.8, $\Phi_k : \mathcal{X}_\eta \xrightarrow{\sim} \mathcal{Y}_\zeta$ where $\mathcal{X}_\eta := \pi_k(X_u E_\bullet \cap X_{(\eta u)^\vee} E'_\bullet)$.

Schubert varieties Ω_ϱ of the Grassmannian $\mathbf{G}_k(L)$ are indexed by ordinary partitions (*weakly* decreasing sequences) $\varrho : n - k \geq \varrho_1 \geq \cdots \geq \varrho_k \geq 0$ [17, p. 18]. We show

$$\Phi_k^{-1}\left(\{K \in \mathit{Lag}(V) \mid v \in K\}\right) \;=\; \Omega_{h(n,k)}\,, \tag{7.1}$$

where $h(n,k)$ is the hook-shaped partition with first row $n - k$ and first column k. It follows from the projection formula that

$$\deg\left(\mathcal{Y}_\zeta \cap \{K \mid v \in K\}\right) \;=\; \deg\left(\mathcal{X}_\eta \cap \Omega_{h(n,k)}\right)$$

which is $\deg(\mathfrak{S}_w \cdot \mathfrak{S}_{\omega_0 \eta w} \cdot \pi_k^*[\Omega_{h(n,k)}])$, the product in $H^*(\mathbb{F}\ell(L))$. By [32, Theorem 8], this counts the chains in the k-Bruhat order

$$w \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-1}} \eta w$$

with $\beta_1 > \beta_2 > \cdots > \beta_k < \beta_{k+1} < \cdots < \beta_{n-1}$. The conclusion follows by Lemma 6.7.

To show (7.1), suppose $H \in \mathbf{G}_k(L)$ and $v \in H + H^\perp$. Let v^- be the projection of v into L and v^+ its projection to L^*. Then $v = v^- \oplus v^+$ and $v^- \in H$ and $v^+ \in H^\perp$ so that $H \subset (v^+)^\perp$. Thus $\Phi_k^{-1}(\{K \mid v \in K\}) = \{H \mid v^- \in H \text{ and } H \subset (v^+)^\perp\}$, which is just the Schubert variety $\Omega_{h(n,k)}$. □

7.2. **Lemma 7.1, case $\delta(\zeta) = 0$.** We first present an example.

Example 7.3. Let $\zeta = \langle 1256\overline{4}\,\overline{3}] \in \mathcal{B}_6$. This is a minimal cycle with $\delta(\zeta) = 0$. We have $t_4\zeta = \langle 125643\rangle \prec \zeta$, and in accordance with Lemma 6.5, 4 ($= \alpha$) is unique with this property. Let $\eta = (1, 2, 5, 6, 4, 3) \in \mathcal{S}_6$, then $\iota(\eta) = t_4\zeta$. If we set $u = \overline{5}\,\overline{2}6\overline{1}34$, then $\zeta u = \overline{6}\,\overline{5}\,\overline{4}\,\overline{2}13$ is a Grassmannian permutation and $u <_0 \zeta u$. These are the permutations constructed from ζ in Remark 2.8. Then $t_4\zeta u = \overline{6}\,\overline{5}\,4\overline{2}13$. Note that $3 = \#\{a \mid 0 < a < \zeta(a)\} = \#\{a \mid a < \eta(a)\}$.

Let $e_6, \ldots, e_1, e_{\overline{1}}, \ldots, e_{\overline{6}}$ be a basis for V in which the alternating form β is given by $\beta(e_i, e_j) = 0$ unless $i + j = 0$, and $\beta(e_i, e_{\overline{\imath}}) = 1$ for $i > 0$. Define opposite flags $E'_\bullet$ and $E_\bullet$ by $E'_i = \langle e_{\overline{6}}, \ldots, e_i\rangle$ and $E_i = \langle e_6, \ldots, e_{\overline{\imath}}\rangle$. We represent a typical flag $F_\bullet \in Y_u^\circ E'_\bullet \cap Y_{w_0\zeta u}^\circ E_\bullet$ by a 6×12-matrix M, where $M(e_6, \ldots, e_1, e_{\overline{1}}, \ldots, e_{\overline{6}}) = (g_{\overline{6}}, \ldots, g_{\overline{1}})$, the vectors of Lemma 3.2:

$$\begin{array}{c} \\ g_{\overline{6}} \\ g_{\overline{5}} \\ g_{\overline{4}} \\ g_{\overline{3}} \\ g_{\overline{2}} \\ g_{\overline{1}} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} e_{\overline{6}} & e_{\overline{5}} & e_{\overline{4}} & e_{\overline{3}} & e_{\overline{2}} & e_{\overline{1}} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & b & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -c/b & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & abf & -bf & f & \cdot & \cdot & \cdot & \cdot & e & d & 1 \\
\cdot & \theta & ab & -b & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-d & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\end{array}$$

Here, $\cdot$ indicates an entry of zero, and a, b, c, d, e, f are arbitrary complex numbers with $abcde \neq 0$ and $\theta = -abe/d$. Observe that if $f = 0$, then the resulting vectors give a flag $\phi(F_\bullet)$ in $Y_u E'_\bullet \cap Y_{w_0\iota(\eta)u} E_\bullet$. Let $g_{\overline{3}}^+$ be the vector $g_{\overline{3}}$ with $f = 0$. Then $\langle g_{\overline{6}}, g_{\overline{5}}, g_{\overline{3}}^+\rangle$ annihilates $\langle g_{\overline{4}}, g_{\overline{2}}, g_{\overline{1}}\rangle$, and we can also define $\phi(F_\bullet)$ by:

$$\begin{array}{lll}
\phi(F_\bullet)_{\overline{6}} = \langle g_{\overline{6}}\rangle, & \phi(F_\bullet)_{\overline{4}} = \langle g_{\overline{6}}, g_{\overline{5}}, g_{\overline{4}}\rangle, & \phi(F_\bullet)_{\overline{2}} = \langle \langle g_{\overline{4}}, g_{\overline{2}}, g_{\overline{1}}\rangle^\perp, g_{\overline{4}}, g_{\overline{2}}\rangle, \\
\phi(F_\bullet)_{\overline{5}} = \langle g_{\overline{6}}, g_{\overline{5}}\rangle, & \phi(F_\bullet)_{\overline{3}} = \langle \langle g_{\overline{4}}, g_{\overline{2}}, g_{\overline{1}}\rangle^\perp, g_{\overline{4}}\rangle, & \phi(F_\bullet)_{\overline{1}} = \langle \langle g_{\overline{4}}, g_{\overline{2}}, g_{\overline{1}}\rangle^\perp, g_{\overline{4}}, g_{\overline{2}}, g_{\overline{1}}\rangle.
\end{array}$$

This alternative description of $\phi(F_\bullet)$ is the idea behind the proof of Lemma 7.5, which is the most technical result in this section (and perhaps in this paper). We prove the following case of Lemma 7.1 here.

Case $\delta(\zeta) = 0$ of Lemma 7.1. Let $\zeta \in \mathcal{B}_n$ with $\delta(\zeta) = 0$ and $\mathcal{L}(\zeta) = n$. Then $c_n^\zeta = 1$.

We use a geometric correspondence to reduce the computation of c_n^ζ to the same calculation in the cohomology of the classical flag manifold used in Section 7.1. We first define this correspondence, then state two lemmas concerning this correspondence. We next deduce Lemma 7.1 from these lemmas, and finally prove each one.

Throughout this section, ζ will be fixed a minimal cycle in $\mathcal{B}_n$ with $\delta(\zeta) = 0$ and $\mathcal{L}(\zeta) = n$. By Lemma 6.5, there is a unique $a > 0$ with $0 > \zeta(a)$. Set $\alpha := \overline{\zeta(a)}$. Since $c^{\zeta}_{\lambda} = c^{\zeta^{-1}}_{\lambda}$, we may replace ζ by ζ^{-1} if necessary and assume $a \geq \alpha$ so that $t_\alpha\zeta \prec \zeta$. We then have $\delta(t_\alpha\zeta) = 1$ and so there is a permutation $\eta \in \mathcal{S}_n$ with $\iota(\eta) = t_\alpha\zeta$. Define k to be $\#\{a \mid 0 < a < \zeta(a)\}$, which is also $\#\{a \mid a < \eta(a)\}$.

Let $u \in \mathcal{B}_n$ be the permutation constructed in Remark 2.8 with $u <_0 \zeta u$ and ζu a Grassmannian permutation. Define j by $u(j) < 0 < \zeta u(j) = \alpha$. Since ζu is Grassmannian, we have $j \leq k + 1$.

Let L, L^* be complementary Lagrangian subspaces of V, and identify L^* with the linear dual of L as in Section 5.2. Let $\mathcal{Y}_\zeta = \pi(Y_u E_\bullet \cap Y_{w_0\zeta u} E'_\bullet)$, where $E_\bullet$ and $E'_\bullet$ are opposite flags with $E'_{\overline{1}} = L$ and $E_{\overline{1}} = L^*$. Set

$$\mathbb{F}_k \;:=\; \{F_k \subset F_{k+1} \subset L \mid \dim F_i = i\}\,,$$

a variety of partial flags in L.

Let $\wp : \mathbb{F}\ell(L) \to \mathbb{F}_k$ be the projection. Then the projections π_k, π_{k+1} of $\mathbb{F}\ell(L)$ to $\mathbf{G}_k(L)$, $\mathbf{G}_{k+1}(L)$ factor through $\wp$. Let ρ_k, ρ_{k+1} denote the projections of $\mathbb{F}_k$ to $\mathbf{G}_k(L)$, $\mathbf{G}_{k+1}(L)$. Consider the incidence variety Γ:

$$\begin{tikzcd}[column sep=small] & \Gamma := \{(F_k, F_{k+1}, K) \mid F_k \oplus F_{k+1}^{\perp} \subsetneq K \subsetneq F_{k+1} \oplus F_k^{\perp}\} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ \mathbb{F}_k & & \mathit{Lag}(V) \end{tikzcd}$$

Then Γ is a $\mathbb{P}^1$-bundle over $\mathbb{F}_k$ and f is generically 1-1: The image of f consists of those K with $\dim K \cap L \geq k$ and $\dim K \cap L^* \geq n-k-1$, which is an intersection of Schubert varieties. Thus a generic K in this intersection determines $g(f^{-1}(K)) = (K \cap L, (K \cap L^*)^{\perp})$ uniquely. We have the diagram

$$\begin{tikzcd} \mathbb{F}\ell(L) \arrow[dd, "\pi_k"'] \arrow[dr, "\wp"] & & \Gamma \arrow[dl, "g"'] \arrow[dr, "f"] & \\ & \mathbb{F}_k \arrow[dl, "\rho_k"'] \arrow[dr, "\rho_{k+1}"] & & \mathit{Lag}(V) \\ \mathbf{G}_k(L) & & \mathbf{G}_{k+1}(L) & \end{tikzcd}$$

The value of this construction is the following lemma.

Lemma 7.4. *Let $u, E_\bullet, E'_\bullet$, and ζu be as above, and set*

$$\mathcal{Y}_\zeta \;=\; \pi(Y_u E_\bullet \cap Y_{w_0\zeta u} E'_\bullet)\,.$$

Then $\mathcal{Y}^\circ_\zeta = \pi(Y^\circ_u E_\bullet \cap Y^\circ_{w_0\zeta u} E'_\bullet) \subset \mathit{Lag}(V)$ is contained in the locus over which the map $f\colon \Gamma \to \mathit{Lag}(V)$ is injective.

Proof. Let $K \in \mathcal{Y}^\circ_\zeta$ so that $K = F_{\overline{1}}$ for some $F_\bullet$ in the intersection of Schubert cells $Y^\circ_u E_\bullet \cap Y^\circ_{w_0\zeta u} E'_\bullet$. We must show that $\dim F_{\overline{1}} \cap E_{\overline{1}} = k$ and $\dim F_{\overline{1}} \cap E'_{\overline{1}} = n-k-1$.

We have that

$$\begin{aligned}\dim F_{\overline{1}}\cap E_{\overline{1}} &= \#\{l>0 \mid u(l)<0\}\\ &= \#\{0<a<\zeta(a)\} \;=\; k\,,\end{aligned}$$

the second equality by the construction of u (see Remark 2.8). Similarly

$$\begin{aligned}\dim F_{\overline{1}}\cap E'_{\overline{1}} &= \#\{l>0 \mid w_0\zeta u(l)<0\}\\ &= \#\{l>0 \mid \zeta u(l)>0\}\,.\end{aligned}$$

Since ζ has exactly one position $a>0$ with value $\zeta(a)<0$, this is

$$n-1-\#\{l>0 \mid u(l)<0\} \;=\; n-1-k\,. \quad\square$$

For $l\in[n]$, let $h(n,l)$ be the hook-shaped partition with first row $n-l$ and first column l. For a partition $\varrho : n-l\geq\varrho_1\geq\cdots\geq\varrho_l\geq 0$, let $\sigma_\varrho\in H^*(\mathbf{G}_l(L))$ be the Schubert class associated to the partition ϱ, as in [17, p. 18]. We will show:

Lemma 7.5. *There is a permutation $w\in\mathcal{S}_n$ with $w\lhd_k \eta w$, and $w\not\lhd_{k+1}\eta w$ so that we have $gf^{-1}\mathcal{Y}_\zeta \;=\; \wp\left(X_w\cap X'_{(\eta w)^\vee}\right)$.*

Lemma 7.6. $g_*f^*q_n \;=\; \rho_k^*\sigma_{h(n,k)}+\rho_{k+1}^*\sigma_{h(n,k+1)}$.

Proof of Lemma 7.1 with $\delta(\zeta)=0$. Let $w\in\mathcal{S}_n$ be the permutation of Lemma 7.5. Set $\mathcal{Z}:=\wp\left(X_w\cap X'_{(\eta w)^\vee}\right)$. Then $\rho_k(\mathcal{Z})=\mathcal{X}_\eta$ and $\mathcal{Z}$ is irreducible of dimension $n-1=\mathcal{L}_A(\eta)$, as both $X_w\cap X'_{(\eta w)^\vee}$ and $\mathcal{X}_\eta$ are irreducible of dimension $n-1$.

By Lemma 7.5, $g^{-1}(\mathcal{Z})\supset f^{-1}(\mathcal{Y}_\zeta)$. Since both of these are irreducible of dimension $n=\mathcal{L}(\zeta)$, we have equality, and so $fg^{-1}(\mathcal{Z})=\mathcal{Y}_\zeta$. The map f is generically 1-1 on $f^{-1}(\mathcal{Y}_\zeta)$, as a generic $K\in\mathcal{Y}_\zeta$ has $\dim K\cap L=k$ and $\dim K\cap L^*=n-k-1$. Thus $f_*g^*[\mathcal{Z}]=[\mathcal{Y}_\zeta]$. We now compute c_n^ζ:

$$\begin{aligned}c_n^\zeta \;=\; \deg([\mathcal{Y}_\zeta]\cdot q_n) &= \deg(f_*g^*[\mathcal{Z}]\cdot q_n)\\ &= \deg([\mathcal{Z}]\cdot g_*f^*q_n)\,,\end{aligned}$$

by the projection formula applied to the maps f,g and Lemma 7.5, as $\mathcal{Z}=\wp(X_w\cap X'_{(\eta w)^\vee})$. By Lemma 7.6, and the projection formula applied to the map $\wp$, noting that $\wp^*\rho_k^*=\pi_k^*$, and also $\wp^*\rho_{k+1}^*=\pi_{k+1}^*$, we see that c_n^ζ is

$$\deg\left(\mathfrak{S}_w\cdot\mathfrak{S}_{\eta w^\vee}\cdot(\pi_k^*\sigma_{h(n,k)}+\pi_{k+1}^*\sigma_{h(n,k+1)})\right)\,.$$

Since $\eta w\not\lhd_{k+1} w$, only the first term is non-zero. By [32, Theorem 8] and by Lemma 6.7, this degree is 1, as $[e,\eta]_{\lhd}$ has a unique peakless chain. $\square$

Proof of Lemma 7.6. The class $q_n\in H^*(Lag(V))$ is represented by the Schubert variety

$$\Upsilon_v \;:=\; \{K\in Lag(V)\mid v\in K\} \;=\; \{K\mid \beta(v,K)\equiv 0\}\,,$$

where $0\neq v\in V$ and β is the alternating form. Then $g_*f^*q_n$ is represented by

$$g(f^{-1}\Upsilon_v) \;=\; \{F_k\subset F_{k+1}\mid \exists K \text{ with } v\in K \text{ and } F_k\oplus F_{k+1}^\perp\subsetneq K\subsetneq F_{k+1}\oplus F_k^\perp\}\,.$$

We may write a general $v\in V=L\oplus L^*$ uniquely as $v=w\oplus u$ with $w\in L$ and $u\in L^*$ and so $g(f^{-1}\Upsilon_v)$ is a subset of

$$\{F_k\subset F_{k+1}\mid w\in F_{k+1}\}\;\cap\;\{F_k\subset F_{k+1}\mid F_k\subset u^\perp\}\,.$$

This is an intersection of Schubert varieties (in general position if v is general) of codimensions $n-k-1$ and k, respectively. These Schubert varieties have classes

$\rho_{k+1}^*\sigma_{(n-k-1)}$ and $\rho_k^*\sigma_{(1^k)}$. Since Υ_v has codimension n, $g(f^{-1}\Upsilon_v)$ equals this intersection if the map $g : f^{-1}\Upsilon_v \to g(f^{-1}\Upsilon_v)$ is finite. Thus

$$g_*f^*q_n \;=\; d(\rho_{k+1}^*\sigma_{(n-k-1)}\cdot\rho_k^*\sigma_{(1^k)})\,,$$

where d is the degree of the map $g : f^{-1}\Upsilon_v \to g(f^{-1}\Upsilon_v)$ (which is 0 if the map is not finite).

To compute d, let $F_k \subset F_{k+1}$ satisfy $w \in F_{k+1}$ and $F_k \subset u^\perp$ with $F_{k+1} \not\subset u^\perp$ and $w \notin F_k$. Then $F_{k+1}\oplus F_k^\perp \not\subset v^\perp$, and so $f(g^{-1}(F_k, F_{k+1})) = F_k \oplus F_k^\perp \cap v^\perp$, which shows $d = 1$. Lastly, the Pieri-type formula [32] in $H^*(\mathbb{F}_k)$ shows

$$\rho_{k+1}^*\sigma_{(n-k-1)}\cdot\rho_k^*\sigma_{(1^k)} \;=\; \rho_{k+1}^*\sigma_{h(n,k+1)} + \rho_k^*\sigma_{h(n,k)}\,.$$

(While the Pieri-type formula of [22, 32] is formulated in terms of the cohomology of the complete flag manifold, it gives valid formulas in the cohomology of any partial flag manifold.) □

Proof of Lemma 7.5. Since $\rho_k : \mathcal{Z} \to \mathcal{X}_\eta$ is generically 1-1, it suffices to show that $\rho_k \circ g \circ f^{-1}(\mathcal{Y}_\zeta) = \mathcal{X}_\eta$. As in Section 5.2 (see diagram (5.6)), given a k-plane H in L, let $H^\perp$ be its annihilator in L^* and define $\Phi_k(H) = H + H^\perp \in Lag(V)$. By Corollary 5.8, $\Phi_k(\mathcal{X}_\eta) = \mathcal{Y}_{\iota(\eta)}$. Since the map Φ_k is 1-1, it will suffice to show that $\Phi_k \circ \rho_k \circ g \circ f^{-1}(\mathcal{Y}_\zeta) = \mathcal{Y}_{\iota(\eta)}$.

Let $K \in \mathcal{Y}_\zeta$ satisfy $\dim K \cap L = k$. Then

$$\begin{aligned}\Phi_k\circ\rho_k\circ g\circ f^{-1}(K) &= \Phi_k\circ\rho_k(K\cap L \;\subset\; (K\cap L^*)^\perp)\\ &= \Phi_k(K\cap L) \quad = \quad (K\cap L)\oplus(K\cap L)^\perp\,.\end{aligned}$$

We claim that $(K\cap L)\oplus(K\cap L)^\perp \in \mathcal{Y}_{\iota(\eta)}$. From this it follows that

$$\Phi_k\circ\rho_k\circ g\circ f^{-1}(\mathcal{Y}_\zeta) \;\subset\; \mathcal{Y}_{\iota(\eta)}\,.$$

Equality follows as both cycles are irreducible, and the dimension of $\Phi_k \circ \rho_k \circ g \circ f^{-1}(\mathcal{Y}_\zeta)$ is at least $\dim\mathcal{Y}_\zeta - 1 = \dim\mathcal{Y}_{\iota(\eta)}$. This is because the map g has 1-dimensional fibres, while f, ρ_k, and Φ_k are generically 1-1 on the relevant images of $\mathcal{Y}_\zeta$ in this composition.

To show $(K\cap L)\oplus(K\cap L)^\perp \in \mathcal{Y}_{\iota(\eta)}$, we define a map

$$\phi \;:\; Y_u^\circ E_\bullet \cap Y_{w_0\zeta u}^\circ E'_\bullet \;\longrightarrow Y_u E_\bullet \cap Y_{w_0\iota(\eta)u}E'_\bullet$$

so that for $F_\bullet \in Y_u^\circ E_\bullet \cap Y_{w_0\zeta u}^\circ E'_\bullet$ we have $\phi(F_\bullet)_{\overline{1}} = (F_{\overline{1}}\cap L)\oplus(F_{\overline{1}}\cap L)^\perp$.

Let $F_\bullet \in Y_u^\circ E_\bullet \cap Y_{w_0\zeta u}^\circ E'_\bullet$ and let $g_{\overline{n}},\ldots,g_{\overline{1}}$ be the vectors constructed in Lemma 3.2 with $F_{\overline{\imath}} = \langle g_{\overline{n}},\ldots,g_{\overline{\imath}}\rangle$ and $g_{\overline{\imath}} \in E_{\zeta u(i)}\cap E'_{\overline{u(i)}}$ for $i = 1,\ldots,n$. It follows from the construction of u that $g_{\overline{n}},\ldots,g_{\overline{k+2}} \in L^*$ and $g_{\overline{k+1}},\ldots,\widehat{g_{\overline{\jmath}}},\ldots,g_{\overline{1}} \in L$.

Since $V = L\oplus L^*$, we may write $g_{\overline{\jmath}} = g_{\overline{\jmath}}^+ \oplus g_{\overline{\jmath}}^-$ with $g_{\overline{\jmath}}^+ \in L^*$ and $g_{\overline{\jmath}}^- \in L$. If $g_{\overline{\jmath}}^+ \in F_{\overline{1}}\cap L^* = \langle g_{\overline{n}},\ldots,g_{\overline{k+1}}\rangle$, then $F_{\overline{1}}\cap L = \langle g_{\overline{k}},\ldots,g_{\overline{1}}\rangle$ and so has dimension $k+1$, contradicting $F_\bullet \in Y_u^\circ E_\bullet$. Define

$$\phi(F_\bullet)_{\overline{\imath}} \;:=\; \begin{cases}\langle g_{\overline{n}},\ldots,g_{\overline{\imath}}\rangle & \text{if } i > j,\\ \langle g_{\overline{n}},\ldots,\widehat{g_{\overline{\jmath}}},\ldots,g_{\overline{\imath}}, g_{\overline{\jmath}}^+\rangle & \text{if } i \le j.\end{cases}$$

Since $g_{\overline{\jmath}}^+$ annihilates $F_{\overline{1}}\cap L$ we see that $(F_{\overline{1}}\cap L)^\perp = \langle g_{\overline{n}},\ldots,g_{\overline{k+2}}, g_{\overline{\jmath}}^+\rangle$. Thus

$$\begin{aligned}\phi(F_\bullet)_{\overline{1}} &= \langle g_{\overline{k+1}},\ldots,\widehat{g_{\overline{\jmath}}},\ldots,g_{\overline{1}}\rangle \oplus \langle g_{\overline{n}},\ldots,g_{\overline{k+2}}, g_{\overline{\jmath}}^+\rangle\\ &= (F_{\overline{1}}\cap L)\oplus(F_{\overline{1}}\cap L)^\perp\,.\end{aligned}$$

Since $g_{\overline{\jmath}}^{-} \in E'_{\overline{1}} \subset E'_{\overline{u(j)}}$, we have $g_{\overline{\jmath}}^{+} \in E'_{\overline{u(j)}}$. As $g_{\overline{\imath}} \in E'_{\overline{u(i)}}$ and $\phi(F_\bullet)_{\overline{\imath}}$ is the span of the first $n-i$ of the vectors $g_{\overline{n}}, \ldots, g_{\overline{\jmath}}^{+}, \ldots, g_{\overline{1}}$, it follows that $\phi(F_\bullet)_{\overline{\imath}} \in Y_u E_\bullet$.

We use the definition of Schubert variety to show that $\phi(F_\bullet)_{\overline{\imath}} \in Y_{w_0\iota(\eta)u} E'_\bullet$. That is, for all $a \in [n]$ and $b \in \pm[n]$, we show

$$\dim \phi(F_\bullet)_{\overline{a}} \cap E_b \;\geq\; \#\{n \geq l \geq a \mid \iota(\eta)u(l) \geq \overline{b}\}\,. \tag{7.2}$$

Since $F_\bullet \in Y^\circ_{w_0\zeta u} E'_\bullet$, we have

$$\dim F_{\overline{a}} \cap E_b \;=\; \#\{n \geq l \geq a \mid \zeta u(l) \geq \overline{b}\}\,. \tag{7.3}$$

Since $\iota(\eta)u(l) = \zeta u(l)$ unless $l = j$, and $\iota(\eta)u(j) = \alpha = \overline{\zeta u(j)}$, the right hand sides of (7.2) and of (7.3) differ only when $a \leq j$ and $\overline{\alpha} \leq b < \alpha$, and in that case, the right hand side of (7.2) is larger by 1.

Similarly, we have $F_{\overline{a}} = \phi(F_\bullet)_{\overline{a}}$ if $a > j$, and if $a \leq j$, then we have

$$F_{\overline{a}} \;=\; \langle g_{\overline{n}}, \ldots, g_{\overline{\jmath}}, \ldots, g_{\overline{a}}\rangle \qquad \text{and} \qquad \phi(F_\bullet)_{\overline{a}} \;=\; \langle g_{\overline{n}}, \ldots, g_{\overline{\jmath}}^{+}, \ldots, g_{\overline{a}}\rangle\,.$$

Since $g_{\overline{\jmath}} \in E_{\overline{\zeta u(j)}}$ and $g_{\overline{\jmath}}^{+} \in L \subset E_{\overline{\zeta u(j)}}$, we see that the left hand sides of (7.2) and of (7.3) are the same if $a > j$ or if $b \geq \alpha$. Finally, since $g_{\overline{\imath}} \in E_{\overline{\zeta u(i)}}$, if $b < \overline{\alpha}$, then $\phi(F_\bullet)_{\overline{a}} \cap E_b \subset F_{\overline{a}} \cap E_b$.

Thus it suffices to show the following statement: If $a \leq j$, then

$$\dim \phi(F_\bullet)_{\overline{a}} \cap E_{\overline{\alpha}} \;=\; \dim F_{\overline{a}} \cap E_{\overline{\alpha}} + 1\,. \tag{7.4}$$

When $k+1 \leq i$ and $i \neq j$, we have $g_{\overline{\imath}} \in L$. Since $L \cap E_{\overline{\alpha}} = \emptyset$, $\phi(F_\bullet)_{\overline{a}} \cap E_{\overline{\alpha}}$ equals $\phi(F_\bullet)_{\overline{1}} \cap E_{\overline{\alpha}}$, which is just $\langle (F_{\overline{1}} \cap L)^\perp \cap E_{\overline{\alpha}}\rangle$. Thus the left hand side of (7.4) is

$$\dim (F_{\overline{1}} \cap L)^\perp \cap E_{\overline{\alpha}} \;=\; \dim \langle g_{\overline{k+1}}, \ldots, \widehat{g_{\overline{\jmath}}}, \ldots, g_{\overline{1}}\rangle^\perp \cap E_{\overline{\alpha}}\,.$$

Since $E_{\overline{\alpha}}$ annihilates $E_{\alpha-1} \cap L^*$, this is

$$n - k - (\alpha - 1) + \dim \langle g_{\overline{k+1}}, \ldots, \widehat{g_{\overline{\jmath}}}, \ldots, g_{\overline{1}}\rangle \cap E_{\alpha-1}\,.$$

Since $F_\bullet \in Y^\circ_{w_0\zeta u} E'_\bullet$, we have $g_{\overline{\jmath}} \in E_{\zeta u(j)} - E_{\zeta u(j)-1}$. As $\zeta u(1) < \cdots < \zeta u(k+1)$ and $\zeta u(j) = \alpha$, we see this intersection is $\langle g_{\overline{k+1}}, \ldots, g_{\overline{j+1}}\rangle$, and so the left side of (7.4) is $n - k - \alpha + 1 + k - j + 1 = n + 2 - j - \alpha$.

We compute the right hand side of (7.4). Since, for $i \leq k+2$, $g_{\overline{\imath}} \notin E_{\overline{1}} \supset E_{\overline{\alpha}}$, we see that $F_{\overline{\jmath}} \cap E_{\overline{\alpha}} = F_{\overline{1}} \cap E_{\overline{\alpha}}$, and so the right hand side of (7.4) is

$$\begin{aligned} 1 + \dim F_{\overline{1}} \cap E_{\overline{\alpha}} &= 1 + \#\{l \mid \zeta u(l) \geq \alpha\} \\ &= n + 2 - \alpha - \#\{l \mid \zeta u(j) < \overline{\alpha}\} \\ &= n + 2 - \alpha - j\,, \end{aligned}$$

as $\zeta u(j) = \overline{\alpha}$ and $\zeta u(1) < \cdots < \zeta u(n)$. This completes the proof. ∎

8. Consequences of the Pieri-type formula

The identity of Theorem B(2)′ was crucial in our proof of the Pieri-type formula. We show below that it is a consequence of the Pieri-type formula. The identities we have established allow us to determine many of the constants $c^w_{u\,\lambda}$, and we make that precise below. Lastly, the identities of Theorem C have non-trivial implications for enumerating chains in the Bruhat order.

For any composition $\alpha = (\alpha_1, \ldots, \alpha_s)$ with each $\alpha_i > 0$, let $p_\alpha := p_{\alpha_1} \cdots p_{\alpha_s}$, $q_\alpha := q_{\alpha_1} \cdots q_{\alpha_s}$, and $I(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{s-1}\}$. The *peak set* of a (maximal) chain in a labeled order is the set of indices of peaks in the chain.

Given a chain in a labeled réseau, its *descent set* (respectively *ascent set*) is the set of indices of descents (respectively ascents) in the chain.

Theorem 8.1. *Let $u, w, x, z \in \mathcal{B}_n$.*

(1) *Let α be any composition. Then the coefficient of $\mathfrak{B}_w$ in the product $\mathfrak{B}_u \cdot p_\alpha$ is the number of chains in the interval $[u, w]_0$ in the labeled 0-Bruhat order with peak set contained in $I(\alpha)$.*
(2) *Let α be any composition. Then the coefficient of $\mathfrak{C}_w$ in the product $\mathfrak{C}_u \cdot q_\alpha$ is the number of chains in the interval $[u, w]_0$ in the labeled 0-Bruhat réseau with descent set contained in $I(\alpha)$. This is also the number with ascent set contained in $I(\alpha)$.*
(3) *Suppose the Pieri-type formula (Theorem A) holds. Then the intervals $[u, w]_0$ and $[x, z]_0$ have the same number of chains with peak set $I(\alpha)$ for every composition α if and only if for every strict partition λ, $b^w_{u\,\lambda} = b^z_{x\,\lambda}$. The same statement holds for ascent/descent sets for chain in the réseaux and the coefficients $c^w_{u\,\lambda}, c^z_{x\,\lambda}$. In particular, Theorem B(1) implies Theorem B(2).*

Moreover, the numbers in (1) and (2) depend only upon the multiset $\{\alpha_1, \ldots, \alpha_s\}$.

Statements (1) and (2) follow from Theorem A. For (3), note that the Schur P-polynomials (respectively Q-polynomials) are linear combinations of the p_α (respectively the q_α) [24, III.8.6]. This linear combination gives a formula for $b^w_{u\,\lambda}$ (respectively $c^w_{u\,\lambda}$) in terms of chains with given peak sets (respectively, given ascent/descent sets).

Theorems B′ and C allow us to determine many of the constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$, showing they equal certain Littlewood-Richardson coefficients $b^\kappa_{\mu\,\lambda}$ and $c^\kappa_{\mu\,\lambda}$ for Schur P- and Q-functions. These are defined by the identities

$$P_\mu \cdot P_\lambda \;=\; \sum_\kappa b^\kappa_{\mu\,\lambda}\, P_\kappa \qquad \text{and} \qquad Q_\mu \cdot Q_\lambda \;=\; \sum_\kappa c^\kappa_{\mu\,\lambda}\, Q_\kappa\,.$$

A combinatorial formula for these coefficients was given by Stembridge [34].

Definition 8.2. Let μ, κ be strict partitions with $\mu \subset \kappa$. We say that a permutation $\zeta \in \mathcal{B}_n$ has *skew shape* κ/μ if

(1) Either ζ or $\rho\zeta\rho$ is shape-equivalent to $v(\kappa)v(\mu)^{-1}$, or
(2) If $a \cdot \zeta(a) > 0$ for all a, and one of $\zeta, \gamma\zeta\gamma^{-1}, \gamma^2\zeta\gamma^{-2}, \ldots, \gamma^{n-1}\zeta\gamma^{1-n}$ is shape-equivalent to $v(\kappa)v(\mu)^{-1}$.

Corollary 8.3. *If $u \leq_0 w$ are permutations in $\mathcal{B}_n$ and wu^{-1} has a skew shape κ/μ, then for any strict partition λ we have*

$$b^w_{u\,\lambda} \;=\; b^\kappa_{\mu\,\lambda} \qquad \textit{and} \qquad c^w_{u\,\lambda} \;=\; c^\kappa_{\mu\,\lambda}\,.$$

By Theorem 8.1(3), Theorem C has a purely enumerative corollary.

Corollary 8.4. *For any $\zeta \in \mathcal{B}_n$,*

(1) *For any subset S of $\{2, \ldots, \mathcal{L}(\zeta) - 1\}$, the intervals $[e, \zeta]_\prec$ and $[e, \rho\zeta\rho]_\prec$ in the Lagrangian order have the same number of chains with peak set S.*
(2) *For any subset S of $\{1, \ldots, \mathcal{L}(\zeta) - 1\}$, the intervals $[e, \zeta]_\prec$ and $[e, \rho\zeta\rho]_\prec$ in the Lagrangian réseau order have the same number of chains with descent set S and the same number of chains with ascent set S, and these two numbers are equal.*
(3) *If $\delta(\zeta) = 1$, then the same is true for $[e, \zeta]_\prec$ and $[e, \gamma\zeta\gamma^{-1}]_\prec$.*

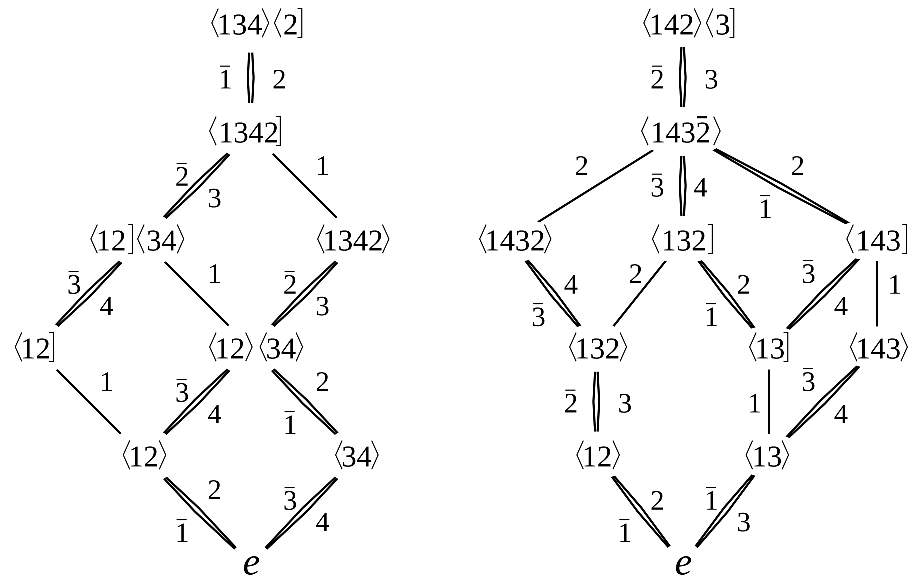


FIGURE 8. Conjugation by ρ on labeled intervals.

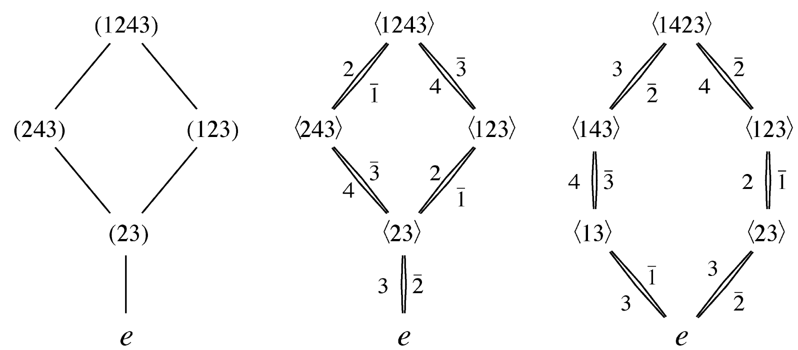


FIGURE 9. Conjugation by γ on labeled intervals.

These assertions are non-trivial, as in general, $[e,\zeta]_\prec \not\simeq [e,\rho\zeta\rho]_\prec$, and if $\delta(\zeta) = 1$, then in general, $[e,\zeta]_\prec \not\simeq [e,\gamma\zeta\gamma^{-1}]_\prec$. We close with two examples of this phenomenon which also serve to illustrate Corollary 8.4.

Example 8.5. In $\mathcal{B}_4$, let $\zeta = \langle 134\rangle\langle 2]$. Then $\rho\zeta\rho = \langle 142\rangle\langle 3]$. Consider the intervals $[e,\zeta]_\prec$ and $[e,\rho\zeta\rho]_\prec$ in the labeled Lagrangian réseau displayed in Figure 8.

While they are not isomorphic, they have the same rank, the same number of maximal chains, 80, and the underlying orders each have 5 chains. Moreover, they each have 2 chains with peak set $\{3\}$, and one each with peak sets $\{2\}, \{4\}$, and $\{2,4\}$. The réseaux have the same number of chains with fixed descent sets. The jth component of the following vector records the number of chains with descent set equal to the position of the 1's in the binary representation of $j-1$:

$$(0, 2, 6, 4, 6, 12, 8, 2, 2, 8, 12, 6, 4, 6, 2, 0)$$

The symmetry in this vector results from the identity $c^{\zeta}_{\lambda} = c^{\zeta^{-1}}_{\lambda}$ and Theorem 8.1(3).

Example 8.6. Let $\eta = (1, 2, 4, 3)$. Then $\zeta = \iota(\eta) = \langle 1243\rangle$ and $\gamma\zeta\gamma^{-1} = \langle 1423\rangle$. The labeled intervals $[e, \eta]_{\lhd}$ and $[e, \zeta]_{\prec}$ are isomorphic. Consider the intervals $[e, \zeta]_{\prec}$ and $[e, \gamma\zeta\gamma^{-1}]_{\prec}$ in the labeled Lagrangian réseau displayed in Figure 9. While they are not isomorphic, they have the same rank, the same number of maximal chains, 16, and the underlying orders each have 2 maximal chains. Moreover, they each have a peakless chain and one with peak set $\{2\}$. The réseaux each have 2 increasing chains, 2 decreasing chains, 6 with descent set $\{1\}$, and 6 with descent set $\{2\}$.

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