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APPLICATIONS OF LANGLANDS' FUNCTORIAL LIFT OF ODD ORTHOGONAL GROUPS

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Abstract. Together with Cogdell, Piatetski-Shapiro and Shahidi, we proved earlier the existence of a weak functorial lift of a generic cuspidal representation of SO_{2n+1} to GL_{2n}. Recently, Ginzburg, Rallis and Soudry obtained a more precise form of the lift using their integral representation technique, namely, the lift is an isobaric sum of cuspidal representations of GL_{n_i} (more precisely, cuspidal representations of GL_{2n_i} such that the exterior square L-functions have a pole at $s = 1$). One purpose of this paper is to give a simpler proof of this fact in the case that a cuspidal representation has one supercuspidal component. In a separate paper, we prove it without any condition using a result on spherical unitary dual due to Barbasch and Moy. We give several applications of the functorial lift: First, we parametrize square integrable representations with generic supercuspidal support, which have been classified by Moeglin and Tadic. Second, we give a criterion for cuspidal reducibility of supercuspidal representations of $GL_m \times SO_{2n+1}$. Third, we obtain a functorial lift from generic cuspidal representations of SO_5 to automorphic representations of GL_5, corresponding to the L-group homomorphism $Sp_4(\mathbb{C}) \longrightarrow GL_5(\mathbb{C})$, given by the second fundamental weight.

1. Introduction

The purpose of this note is to give several applications of Langlands' functorial lift of odd-orthogonal groups. Recall the L-group homomorphism ${}^L SO^\circ_{2n+1} = Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^L GL^\circ_{2n}$. Langlands' functoriality predicts that there is a map from cuspidal representations of $SO_{2n+1}(\mathbb{A}_F)$ to automorphic representations of $GL_{2n}(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of a number field F. Throughout this note, cuspidal representations mean unitary ones.

In [C-Ki-PS-S], it is proved that given a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$, there exists a weak lift (see Definition 2.1 for the notion). We prove here the existence of a strong lift. Especially, a generic cuspidal representation with one supercuspidal component has the strong lift which is of the expected form, namely, if π is a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$ with one supercuspidal component, then the lift Π is an automorphic representation of $GL_{2n}(\mathbb{A}_F)$, which is of the form $Ind\, \sigma_1 \otimes \cdots \otimes \sigma_p$, where the σ_i's are (unitary) self-contragredient cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. In a separate paper [Ki6], we prove that it is

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true without any condition, namely, the lift of any generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$ is of the form $Ind\,\sigma_1 \otimes \cdots \otimes \sigma_p$, where the σ_i's are (unitary) self-contragredient cuspidal representations of $GL_{2n_i}(\mathbb{A}_F)$ such that $L(s, \sigma_i, \wedge^2)$ has a pole at $s = 1$. In order to do so, we need a deep result on spherical unitary dual due to Barbasch and Moy [B-Mo]. One purpose of this paper is to give a simpler proof in the case that a cuspidal representation has one supercuspidal component. Recently Ginzburg, Rallis and Soudry also proved that the lift is of the above form without any condition, using their integral representation technique. They can also obtain the backward lifting.

We give several applications of the functorial lift to local problems. First, we parametrize square integrable representations with generic supercuspidal support, which has been classified by Moeglin and Tadic [M-Ta]. More precisely, let $G = SO_{2n+1}(k)$, where k is a p-adic field of characteristic zero and let W_k be the Weil group of k. The local Langlands' correspondence predicts that an admissible homomorphism $\phi : W_k \times SL_2(\mathbb{C}) \longrightarrow {}^LG^\circ = Sp_{2n}(\mathbb{C})$, parametrizes a finite set Π_ϕ, called L-packet, of isomorphism classes of irreducible admissible representations of G, and every admissible irreducible representation of G belongs to Π_ϕ for a unique ϕ. Since the local Langlands' correspondence is available for $GL_n(k)$ [H-T], [He], we will use the L-group homomorphism ${}^LSO^\circ_{2n+1} = Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^LGL^\circ_{2n}$, to obtain the parametrization for square integrable representations with generic supercuspidal support. We should note that given an admissible homomorphism $\phi : W_k \times SL_2(\mathbb{C}) \longrightarrow {}^LG^\circ = Sp_{2n}(\mathbb{C})$, which parametrizes a non-supercuspidal square integrable representation with generic supercuspidal support, the L-packet Π_ϕ contains other representations besides the non-supercuspidal square integrable representations with generic supercuspidal support. We speculate that the remaining ones are non-generic supercuspidal representations and non-supercuspidal square integrable representations with non-generic supercuspidal support. Also for the parametrization, we need an assumption (Assumption 5.1) that the local Artin exterior square (symmetric square, resp.) L-function has a pole at $s = 0$ if and only if the local Shahidi's exterior square (symmetric square, resp.) L-function has a pole at $s = 0$. This assumption is the same as the assertion that the Shahidi's L-functions are Artin L-functions. It may be even more difficult to show the equality of their ϵ-factors.

Second, we give a criterion for cuspidal reducibility of supercuspidal representations of $GL_m \times SO_{2n+1}$. More precisely, let ρ be a self-contragredient supercuspidal representation of $GL_m(k)$ and τ be a generic supercuspidal representation of $SO_{2n+1}(k)$. Then there exists a unique $s_0 \geq 0$ such that the induced representation $Ind\,|det|^s\rho \otimes \tau$ is reducible at $s = s_0, -s_0$ and irreducible at all other points. The deep result of Shahidi [Sh1] is that $s_0 \in \{0, \frac{1}{2}, 1\}$. We say that (ρ, τ) satisfies (Ci) if $Ind\ |det|^s\rho \otimes \tau$ is reducible at $s = i$. We give a precise criterion of when (ρ, τ) satisfies (Ci) in terms of the functorial lift of τ.

Third, we obtain a functorial lift from cuspidal representations of $SO_5(\mathbb{A}_F)$ to automorphic representations of $GL_5(\mathbb{A}_F)$, corresponding to the L-group homomorphism $Sp_4(\mathbb{C}) \longrightarrow GL_5(\mathbb{C})$, given by the second fundamental weight: Given a generic cuspidal representation π of $SO_5(\mathbb{A}_F)$, first we obtain a strong lift Π to $GL_4(\mathbb{A}_F)$. In [Ki4], we showed that the exterior square lift $\bigwedge^2 \Pi$ is an automorphic representation of $GL_6(\mathbb{A}_F)$. We show that $\bigwedge^2 \Pi = \tau \boxplus 1$, where τ is an automorphic representation of $GL_5(\mathbb{A}_F)$; τ is the desired functorial lift.

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2. Strong lift from SO_{2n+1} to GL_{2n}

Throughout this paper, let F be a number field and $\mathbb{A}_F$ be the ring of adeles. We fix an additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}_F/F$.

Definition 2.1. *Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$. We say that an automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $GL_{2n}(\mathbb{A}_F)$ is a strong lift of π if every Π_v is a lift of π_v for all v, in the sense that*

$$\gamma(s, \sigma_v \times \pi_v, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v),$$
$$L(s, \sigma_v \times \pi_v) = L(s, \sigma_v \times \Pi_v),$$

for all generic irreducible representation σ_v of $GL_m(F_v)$, $1 \le m \le 2n-1$. Here the left-hand side is the γ-factor and L-factor defined in [Sh1, section 7] *and the right-hand side is the one defined in* [J-PS-S]. *Due to local Langlands' correspondence, they are the Artin γ- and L-factors.*

If the above equality holds for almost all v, then Π is called the weak lift of π.

Recall the equalities

$$\gamma(s, \sigma_v \times \pi_v, \psi_v) = \epsilon(s, \sigma_v \times \pi_v, \psi_v) \frac{L(1-s, \tilde{\sigma}_v \times \tilde{\pi}_v)}{L(s, \sigma_v \times \pi_v)},$$
$$\gamma(s, \sigma_v \times \Pi_v, \psi_v) = \epsilon(s, \sigma_v \times \Pi_v, \psi_v) \frac{L(1-s, \tilde{\sigma}_v \times \tilde{\Pi}_v)}{L(s, \sigma_v \times \Pi_v)}.$$

Hence the equalities of γ- and L-factors imply the equality of ϵ-factors. Note that if Π_v is generic for all v, the above identities uniquely determine Π. Recall how L- and ϵ-factors are defined from [Sh1, section 7]. From the theory of local coefficients, which is defined through intertwining operators, a γ-factor $\gamma(s, \pi_v, r_i, \psi_v)$ is defined for every irreducible generic admissible representation π_v and certain finite dimensional representation r_i. If π_v is tempered, $L(s, \pi_v, r_i)$ is defined to be

$$L(s, \pi_v, r_i) = P_{\pi_v, i}(q_v^{-s})^{-1},$$

where $P_{\pi_v,i}$ is the unique polynomial satisfying $P_{\pi_v,i}(0) = 1$ such that $P_{\pi_v,i}(q_v^{-s})$ is the numerator of $\gamma(s, \pi_v, r_i, \psi_v)$. We define the ϵ-factor using the identity

$$\gamma(s, \pi_v, r_i, \psi_v) = \epsilon(s, \pi_v, r_i, \psi_v) \frac{L(1-s, \tilde{\pi}_v, r_i)}{L(s, \pi_v, r_i)}.$$

Hence if π_v is tempered, then the γ-factor canonically defines both the L-factor and the ϵ-factor. If π_v is non-tempered, write it as a Langlands' quotient of an induced representation formed from a twist of tempered representations (actually by standard module conjecture, π_v is the full induced representation), and we can write the corresponding intertwining operator as a product of rank-one operators. For these rank-one operators, there correspond L-factors and we define $L(s, \pi_v, r_i)$ to be the product of these rank-one L-factors. We then define ϵ-factor to satisfy the above relation.

Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$. Recall

Theorem 2.2 ([C-Ki-PS-S]). *There exists a weak lift $\Pi = \bigotimes_v \Pi_v$. It is an automorphic representation of $GL_{2n}(\mathbb{A}_F)$ and if π_v is unramified, Π_v is generic. Moreover, Π_v is a lift of π_v for all $v = \infty$ and any finite place v where π_v is unramified.*

We want to show that there exists a strong lift. In order to prove the existence of a strong lift, first we need to construct a local lift for all v. Then we apply the converse theorem twice. For the converse theorem and its applications to the lifting, we will freely use the fundamental paper [C-Ki-PS-S].

First, we show that a generic supercuspidal representation of $SO_{2n+1}(F_v)$ has a local lift to $GL_{2n}(F_v)$, which is tempered.

We use the following setup. Let k be a local p-adic field of characteristic zero. Let ρ be a generic supercuspidal representation of $SO_{2n+1}(k)$. Then by [Sh1, Proposition 5.1], there exists a number field F and a generic cuspidal automorphic representation $\pi = \bigotimes_w \pi_w$ of $SO_{2n+1}(\mathbb{A}_F)$ such that $F_v = k$, $\pi_v = \rho$ and π_w is unramified for all $w < \infty$ and $w \neq v$.

Let Π be a weak lift of π such that Π_w is a lift of π_w for $w \neq v$. By the classification of automorphic representations [J-S], it is equivalent to a subquotient of

$$\Xi = Ind\, |det|^{r_1}\sigma_1 \otimes \cdots \otimes |det|^{r_p}\sigma_p,$$

where the σ_i's are unitary cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. Then Π_v is a subquotient of Ξ_v.

Lemma 2.3. $|r_i| < \frac{1}{2}$, $i = 1, ..., p$.

Proof. Consider an unramified local component π_w of $SO_{2n+1}(F_w)$. Since it is generic and unitary, it has the form

$$\pi_w = Ind\, |\ |^{a_1}\mu_1 \otimes \cdots \otimes |\ |^{a_n}\mu_n,$$

where the $\mu_1, ..., \mu_n$'s are unramified unitary characters of $F_w^\times$ and $0 \le a_n \le \cdots \le a_1 < \frac{1}{2}$ (see [Yo, Theorem B]). Hence the lift Π_w is of the form

$$\Pi_w = Ind\, |\ |^{a_1}\mu_1 \otimes \cdots \otimes |\ |^{a_n}\mu_n \otimes |\ |^{-a_n}\mu_n^{-1} \otimes \cdots \otimes |\ |^{-a_1}\mu_1^{-1}.$$

Note that $\Xi_w = Ind\, |det|^{r_1}\sigma_{1w} \otimes \cdots \otimes |det|^{r_p}\sigma_{pw}$ and σ_{iw} is of the form $Ind\, |\ |^{b_1}\nu_1 \otimes |\ |^{b_2}\nu_2 \otimes \cdots \otimes |\ |^{-b_1}\nu_1$, where $\frac{1}{2} > b_1 \ge b_2 \ge \cdots$ and ν_j's are unitary characters.

Since Π_w is unramified, it is the unique unramified subquotient of Ξ_w. Since Π_w is generic, by [Li, Theorem 2.2], $\Pi_w = \Xi_w$. Comparing the a_i, b_j, r_l's, we obtain our result. $\square$

Proposition 2.4. $r_1 = \cdots = r_p = 0$. *Hence $\Pi_v = \Xi_v$ and it is a local lift of π_v. Moreover, it is tempered.*

Proof. Let Π'_v be an another constituent of Ξ_v and let $\Pi' = \bigotimes_{w \neq v} \Pi_w \otimes \Pi'_v$. Then by the result of Langlands [La3], it is an automorphic representation of $GL_{2n}(\mathbb{A}_F)$ which is a weak lift of π. We first show that, for discrete series σ_v of $GL_m(F_v)$ (any m),

$$\gamma(s, \sigma_v \times \pi_v, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v) = \gamma(s, \sigma_v \times \Pi'_v, \psi_v).$$

By [Ro], we can find a cuspidal representation σ of $GL_m(\mathbb{A}_F)$ whose local component at v is σ_v. Consider the three Rankin-Selberg L-functions $L(s,\sigma\times\pi)$, $L(s,\sigma\times\Pi)$ and $L(s,\sigma\times\Pi')$. All three have the functional equations:

$$L(s,\sigma\times\pi)=\epsilon(s,\sigma\times\pi)L(1-s,\tilde{\sigma}\times\tilde{\pi}),$$
$$L(s,\sigma\times\Pi)=\epsilon(s,\sigma\times\Pi)L(1-s,\tilde{\sigma}\times\tilde{\Pi}),$$
$$L(s,\sigma\times\Pi')=\epsilon(s,\sigma\times\Pi')L(1-s,\tilde{\sigma}\times\tilde{\Pi}').$$

From these equalities, we obtain

$$\gamma(s,\sigma_v\times\pi_v,\psi_v)=\gamma(s,\sigma_v\times\Pi_v,\psi_v)$$

and

$$\gamma(s,\sigma_v\times\pi_v,\psi_v)=\gamma(s,\sigma_v\times\Pi'_v,\psi_v).$$

We only prove the first equality: Since π_w is unramified for $w\neq v, w<\infty$, Π_w is the lift of π_w for all $w\neq v$. Hence

$$L(s,\sigma_w\times\pi_w)=L(s,\sigma_w\times\Pi_w),\quad \epsilon(s,\sigma_w\times\pi_w,\psi_w)=\epsilon(s,\sigma_w\times\Pi_w,\psi_w),$$

for all $w\neq v$. The functional equations above can be written in the form

$$\gamma(s,\sigma_v\times\pi_v,\psi_v)=\prod_{w\neq v}\frac{L(s,\sigma_w\times\pi_w)}{\epsilon(s,\sigma_w\times\pi_w,\psi_w)L(1-s,\tilde{\sigma}_w\times\tilde{\pi}_w)},$$
$$\gamma(s,\sigma_w\times\Pi_w,\psi_w)=\prod_{w\neq v}\frac{L(s,\sigma_w\times\Pi_w)}{\epsilon(s,\sigma_w\times\Pi_w,\psi_w)L(1-s,\tilde{\sigma}_w\times\tilde{\Pi}_w)}.$$

Hence $\gamma(s,\sigma_v\times\pi_v,\psi_v)=\gamma(s,\sigma_v\times\Pi_v,\psi_v)$.

We write Ξ_v to be in the Langlands' situation and take Π'_v to be the Langlands' quotient of Ξ_v, namely, the quotient of

$$\Xi_v=Ind\,|det|^{a_1}\tau_{1v}\otimes\cdots\otimes|det|^{a_l}\tau_{lv},$$

where the τ_{iv}'s are discrete series of $GL_{m_i}(F_v)$, $a_1\geq\cdots\geq a_l$, and $|a_i|<1$. The last inequality comes from the fact that $|r_i|<\frac{1}{2}$ by Lemma 2.3 and σ_{iv} are unitary representations of $GL_{n_i}(F_v)$. Suppose one of the a_i's is not zero. Since Π'_v has a trivial central character, $a_l<0$.

By [Ro], let τ_l be a cuspidal representation of $GL_{m_i}(\mathbb{A}_F)$ whose local component at v is τ_{lv}. Then

$$\gamma(s,\tilde{\tau}_{lv}\times\pi_v,\psi_v)=\gamma(s,\tilde{\tau}_{lv}\times\Pi'_v,\psi_v)=\prod_{i=1}^{l}\gamma(s+a_i,\tilde{\tau}_{lv}\times\tau_{iv},\psi_v).$$

In order to proceed, we need to use the fact that π_v is supercuspidal: Suppose τ_{lv} is a Steinberg representation, given as the subrepresentation of $Ind\,|det|^{\frac{p-1}{2}}\eta\otimes|det|^{\frac{p-1}{2}-1}\eta\otimes\cdots\otimes|det|^{-\frac{p-1}{2}}\eta$. Then the left-hand side has possible poles only at $Re\,s=\frac{p+1}{2}$, and possible zeros only at $Re\,s=-\frac{p-1}{2}$ (see, for example, [Ki7]).

On the other hand, suppose τ_{iv} is a Steinberg representation, given as the subrepresentation of

$$Ind\,|det|^{\frac{p_i-1}{2}}\eta_i\otimes|det|^{\frac{p_i-1}{2}-1}\eta_i\otimes\cdots\otimes|det|^{-\frac{p_i-1}{2}}\eta_i.$$

Then (we suppose $p\geq p_i$) $\gamma(s,\tilde{\tau}_{lv}\times\tau_{iv},\psi_v)$ has possible poles at $Re\,s=\frac{p+1}{2}+\frac{p_i+1}{2}-1,\frac{p+1}{2}+\frac{p_i+1}{2}-2,...,\frac{p+1}{2}+\frac{p_i+1}{2}-p_i$; possible zeros at $Re\,s=$

$-(\frac{p-1}{2}+\frac{p_i-1}{2}), -(\frac{p-1}{2}+\frac{p_i-1}{2}-1), ..., -(\frac{p-1}{2}-\frac{p_i-1}{2})$ (if $p \le p_i$, then exchange p_i and p; see [Ki7]).

Especially, $\gamma(s, \tilde{\tau}_{lv} \times \tau_{lv}, \psi_v)$ has poles at $Re\, s = p, p-1, ..., 1$ and zeros at $Re\, s = -(p-1), ..., -1, 0$. Consider the pole at $Re\, s = p$. Since $|a_i| < 1$, the only possible way to match poles is when $p = 1$, namely, τ_{lv} is supercuspidal. In that case, $\gamma(s+a_l, \tilde{\tau}_{lv} \times \tau_{lv}, \psi_v)$ has poles at $Re\, s = 1 - a_l$ and zeros at $Re\, s = -a_l$. The poles at $Re\, s = 1 - a_l$ cannot be cancelled, since $a_l < 0$. Contradiction.

Hence $a_i = 0$ for all i and this implies that $r_1 = \cdots = r_p = 0$. Therefore, $\Xi_v = Ind\, \tau_{1v} \otimes \cdots \otimes \tau_{lv}$, where τ_{iv}'s are discrete series of $GL_{m_i}(F_v)$. By a well-known result (see, for example, [Ta]), Ξ_v is irreducible and is tempered. Hence $\Xi_v = \Pi_v$. □

We can give a more precise form of the lift Π_v.

Lemma 2.5. *Let π_v be a generic supercuspidal representation of $SO_{2n+1}(F_v)$. Then the lift Π_v is a tempered representation, of the form $\Pi_v = \sigma_1 \boxplus \cdots \boxplus \sigma_p$, where the σ_i's are supercuspidal representations of $GL_{n_i}(F_v)$ such that $L(s, \sigma_i, \wedge^2)$ has a pole at $s = 0$. (Here $L(s, \sigma_i, \wedge^2)$ is the Shahidi's L-function given by the Langlands-Shahidi method* [Sh1]. *More precisely, $L(s, \sigma, \wedge^2)$ comes from the theory of Eisenstein series relative to $GL_{2n} \subset SO_{4n}$.) In particular, n_i is even, and the σ_i's are self-contragredient and have the trivial central character. Moreover, the σ_i's are inequivalent.*

Proof. We showed above that Π_v is tempered. Any tempered representation of $GL_n(F_v)$ is unitarily induced from the discrete series of GL, i.e.,

$$\Pi_v = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_p,$$

where the σ_i's are discrete series of GL. Suppose one of them, say σ_1, is not supercuspidal. Then compare the two L-functions

$$L(s, \tilde{\sigma}_1 \times \pi_v) = L(s, \tilde{\sigma}_1 \times \Pi_v).$$

Suppose σ_1 is given as the subrepresentation of $|det|^{\frac{a-1}{2}}\rho \otimes \cdots \otimes |det|^{-\frac{a-1}{2}}\rho$, where a is a positive integer with $a > 1$ and ρ is a supercuspidal representation of GL. Then

$$L(s, \tilde{\sigma}_1 \times \Pi_v) = \prod_{i=1}^{p} L(s, \tilde{\sigma}_1 \times \sigma_i).$$

Here $L(s, \tilde{\sigma}_1 \times \sigma_1) = \prod_{i=0}^{a-1} L(s+i, \tilde{\rho} \times \rho)$ and $L(s, \tilde{\rho} \times \rho)$ has a pole at $s = 0$. Since the local L-functions have no zeros, $L(s, \tilde{\sigma}_1 \times \Pi_v)$ has a pole at $s = -(a-1)$. On the other hand, $L(s, \tilde{\sigma}_1 \times \pi_v) = L(s + \frac{a-1}{2}, \tilde{\rho} \times \pi_v)$. Since $L(s, \tilde{\rho} \times \pi_v)$ has a possible pole only at $Re\, s = 0$, $L(s, \tilde{\sigma}_1 \times \pi_v)$ has no pole at $s = -(a-1)$. We obtain a contradiction.

Hence $\Pi_v = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_p$, where the σ_i's are supercuspidal representations of GL. If σ_1 is not self-contragredient, then consider

$$L(s, \tilde{\sigma}_1 \times \pi_v) = L(s, \tilde{\sigma}_1 \times \Pi_v) = \prod_{i=1}^{k} L(s, \tilde{\sigma}_1 \times \sigma_i).$$

The left-hand side does not have a pole at $s = 0$ [Sh1, Corollary 7.6], while the right-hand side has a pole at $s = 0$. Contradiction. In the same way, we show that the σ_i's are self-contragredient.

Now consider two L-functions: $L(s, \sigma_1 \times \pi_v)$ and $L(s, \sigma_1, Sym^2)$. (Here $L(s, \sigma_1, Sym^2)$ is the Shahidi's L-function given by the Langlands-Shahidi method [Sh1]. More precisely, $L(s, \sigma_1, Sym^2)$ comes from the theory of Eisenstein series relative to $GL_{2n} \subset SO_{4n+1}$.)

If we consider the supercuspidal representation $\sigma_1 \otimes \pi_v$ for $M(F_v) = GL_m(F_v) \times SO_{2n+1}(F_v) \subset SO_{2(m+n)+1}(F_v)$, the local coefficient attached to $(M, \sigma_1 \otimes \pi_v)$ has as its denominator $L(s, \sigma_1 \times \pi_v)L(2s, \sigma_1, Sym^2)$ (see [Sh1]). So by [Sh1, Corollary 7.6], only one of the two L-functions can have a simple pole at $s = 0$. Since $L(s, \sigma_1 \times \pi_v)$ has a pole at $s = 0$, $L(s, \sigma_1, Sym^2)$ is holomorphic at $s = 0$. Consider the identity $L(s, \sigma_1 \times \sigma_1) = L(s, \sigma_1, \wedge^2)L(s, \sigma_1, Sym^2)$. Since σ_1 is self-contragredient, the left-hand side has a pole at $s = 0$. Hence $L(s, \sigma_1, \wedge^2)$ has a pole at $s = 0$. So by [Sh1, Lemma 7.4], n_1 is even. (It comes from the fact that if n is odd, the parabolic subgroup $P = MN$, $M = GL_n$, is not self-conjugate in SO_{2n}.) We prove the same thing for all σ_i, $i = 2, ..., p$.

Suppose $\sigma_1 \simeq \sigma_2$. Then $L(s, \sigma_1 \times \pi_v)$ has a double pole at $s = 0$, which is a contradiction (see [Sh1, Corollary 7.6]).

It remains to prove that the central character of the σ_i's is trivial. As in [Ki6, Proposition 3.8], we use the similitude group $GSO_{4n}(F_v)$ [Go], namely,

$$GSO_{4n} = \{g \in GL_{4n} | {}^t g J_{4n} g = \lambda(g) J_{4n}, \quad det(g)\lambda(g)^{-2n} = 1; \lambda(g) \in GL_1\}.$$

Let $P = MN$ with $M = GL_{2n} \times GL_1$. Let σ be a supercuspidal representation of $GL_{2n}(F_v)$ with the central character ω_σ and let χ be a unitary character. Then $\sigma \otimes \chi$ is a supercuspidal representation of $M(F_v)$. [Go, 2.8] shows that $w_0(\sigma \otimes \chi) \simeq \sigma \otimes \chi$ if and only if $\sigma \simeq \tilde{\sigma}$ and $\omega_\sigma = 1$. Hence if $\sigma \simeq \tilde{\sigma}$ and $\omega_\sigma \neq 1$, then $w_0(\sigma \otimes \chi) \not\simeq \sigma \otimes \chi$. The L-function which shows up as a normalizing factor of the intertwining operator [Sh1, section 7], is $L(s, \sigma, \wedge^2)$. Hence by [Sh1, Corollary 7.6], we can see that $L(s, \sigma, \wedge^2)$ has no pole at $s = 0$. □

Next we show that a non-supercuspidal generic square integrable representation of $SO_{2n+1}(F_v)$ has a lift to $GL_{2n}(F_v)$. First, we need the following which is well-known [Ze].

Lemma. *In the language of* [Ca-Sh]*, any standard module of $GL_n(F_v)$ satisfies injectivity, namely, the following induced representation has the unique irreducible subrepresentation which is generic;*

$$Ind\, |det|^{a_1} \tau_1 \otimes \cdots \otimes |det|^{a_m} \tau_m \otimes \sigma_1 \otimes \cdots \otimes \sigma_l \otimes |det|^{-a_1} \tilde{\tau}_1 \otimes \cdots \otimes |det|^{-a_m} \tilde{\tau}_m,$$

where $\tau_1, ..., \tau_m, \sigma_1, ..., \sigma_l$ are discrete series of GL and $a_1 \geq \cdots \geq a_m > 0$.

Let π_v be a non-supercuspidal generic square integrable representation of $SO_{2n+1}(F_v)$. (The discussion below would be true for any non-supercuspidal square integrable representations with generic supercuspidal support, if we assume multiplicativity of the γ- and L-factors.) Then by the classification of discrete series for odd-orthogonal groups [Ja2], [M-Ta], π_v is a subrepresentation of an induced representation (see section 5 for more precise parametrization)

$$Ind\, |det|^{a_1} \tau_1 \otimes \cdots \otimes |det|^{a_m} \tau_m \otimes \tau_0,$$

where $\tau_1, ..., \tau_m$ are discrete series representations of GL and τ_0 is a generic supercuspidal representation of $SO_{2l+1}(F_v)$. Let Π_0 be the local lift of τ_0. By the above lemma, the induced representation

$$Ind\, |det|^{a_1} \tau_1 \otimes \cdots \otimes |det|^{a_m} \tau_m \otimes \Pi_0 \otimes |det|^{-a_1} \tilde{\tau}_1 \otimes \cdots \otimes |det|^{-a_m} \tilde{\tau}_m,$$

has the unique generic irreducible subrepresentation. We denote it by Π_v. Then

Proposition 2.6. *Π_v is the local lift of π_v. It is unique. Moreover, Π_v is tempered.*

Proof. Let σ_v be a discrete series of $GL_p(F_v)$. Then

$$\begin{aligned}\gamma(s,\sigma_v\times\pi_v,\psi_v) &= \gamma(s,\sigma_v\times\tau_0,\psi_v)\prod_{i=1}^{m}\gamma(s+a_i,\sigma_v\times\tau_i,\psi_v)\gamma(s-a_i,\sigma_v\times\tilde{\tau}_i,\psi_v)\\ &= \gamma(s,\sigma_v\times\Pi_0,\psi_v)\prod_{i=1}^{m}\gamma(s+a_i,\sigma_v\times\tau_i,\psi_v)\gamma(s-a_i,\sigma_v\times\tilde{\tau}_i,\psi_v)\\ &= \gamma(s,\sigma_v\times\Pi_v,\psi_v).\end{aligned}$$

By multiplicativity of L-factors ([Sh5, Theorem 5.2]; we note that the assumption in the statement of the theorem has been verified by [Ca-Sh] and [Mu].), the same equality holds for L-factors also, namely,

$$L(s,\sigma_v\times\pi_v)=L(s,\sigma_v\times\Pi_v).$$

The temperedness of Π_v follows easily from the above identity, by comparing poles of both sides. More precisely, let Π_v be a Langlands' quotient of

$$Ind\,|det|^{a_1}\eta_1\otimes\cdots\otimes|det|^{a_l}\eta_l,$$

where the η_i's are discrete series of $GL_{m_i}(F_v)$, $a_1\geq\cdots\geq a_l$. Suppose one of the a_i's is not zero. Since Π_v has a trivial central character, $a_l<0$. Then

$$L(s,\tilde{\eta}_l\times\pi_v)=\prod_{i=1}^{l}L(s+a_i,\tilde{\eta}_l\times\eta_i).$$

The left-hand side has no poles for $Re\,s>0$. But the right-hand side has a pole at $Re\,s=-a_l>0$.

Since Π_v is generic, the equality of γ- and L-factors implies that it is unique. □

Proposition 2.7. *Any generic tempered representation of $SO_{2n+1}(F_v)$ has the local lift, which is tempered. Hence any generic irreducible representation of $SO_{2n+1}(F_v)$ has a local lift.*

Proof. Any tempered representation π_v of $SO_{2n+1}(F_v)$ is a subrepresentation of

$$Ind\,\sigma_1\otimes\cdots\otimes\sigma_m\otimes\tau,$$

where $\sigma_1,...,\sigma_m$ are discrete series of GL and τ is a discrete series of $SO_{2l+1}(F_v)$. Let Π_0 be the local lift of τ. Then the following induced representation is irreducible (see, for example, [Ta]) and tempered. We denote it by Π_v;

$$Ind\,\sigma_1\otimes\cdots\otimes\sigma_k\otimes\Pi_0\otimes\tilde{\sigma}_m\otimes\cdots\otimes\tilde{\sigma}_1.$$

It is easy to show that Π_v is the local lift of π_v by showing the equality of γ-factors from multiplicativity of γ-factors.

By standard module conjecture, proved by Muić [Mu, Theorem 0.4], any generic irreducible representation π_v can be written as a full induced representation

$$Ind\,|det|^{r_1}\sigma_1\otimes\cdots\otimes|det|^{r_m}\sigma_m\otimes\tau,$$

where $\sigma_1,...,\sigma_m$ are discrete series of GL and τ is a tempered representation of $SO_{2l+1}(F_v)$. Let Π_0 be the local lift of τ and let Π_v be the unique quotient of

$$\Xi_v=Ind\,|det|^{r_1}\sigma_1\otimes\cdots\otimes|det|^{r_m}\sigma_m\otimes\Pi_0\otimes|det|^{-r_m}\tilde{\sigma}_m\otimes\cdots\otimes|det|^{-r_1}\tilde{\sigma}_1.$$

Note that if $r_i = \frac{1}{2}$ for some i, Ξ_v is reducible and hence Π_v is not generic. Then we can show from multiplicativity of γ- and L-factors that for any discrete series σ_v of GL,

$$\gamma(s, \sigma_v \times \pi_v, \psi_v) = \gamma(s, \sigma_v \times \Pi_v, \psi_v), \quad L(s, \sigma_v \times \pi_v) = L(s, \sigma_v \times \Pi_v).$$

Namely, Π_v is a local lift of π_v. □

Remark. Note that in the above, when Ξ_v is reducible, it has the unique irreducible subrepresentation Π'_v which is generic. In that case, by multiplicativity of γ-factors, we have

$$\gamma(s, \sigma_v \times \pi_v, \psi_v) = \gamma(s, \sigma_v \times \Pi'_v, \psi_v).$$

However, $L(s, \sigma_v \times \pi_v) \neq L(s, \sigma_v \times \Pi'_v)$. Let us give an example. Let k be a p-adic field of characteristic zero and ρ be a discrete series of $GL_2(k)$ with the trivial central character. Then ρ can be considered a representation of $PGL_2(k) \simeq SO_3(k)$. Consider $\tau = Ind_{GL_2 \times SO_3}^{SO_7} |det|^{\frac{1}{2}} \rho \otimes \rho$. Then τ is irreducible and hence it is a generic representation of $SO_7(k)$. However, its lift Π to $GL_6(k)$ is not generic. It is the unique quotient of

$$Ind_{GL_2 \times GL_2 \times GL_2}^{GL_6} |det|^{\frac{1}{2}} \rho \otimes \rho \otimes |det|^{-\frac{1}{2}} \rho.$$

Let Π' be the unique irreducible subrepresentation. Then

$$\gamma(s, \rho \times \tau, \psi) = \gamma(s, \rho \times \Pi, \psi) = \gamma(s, \rho \times \Pi', \psi),$$

but

$$L(s, \rho \times \tau) = L(s - \frac{1}{2}, \rho \times \rho) L(s, \rho \times \rho) L(s + \frac{1}{2}, \rho \times \rho);$$
$$L(s, \rho \times \Pi') = L(s, \rho \times \rho) L(s + \frac{1}{2}, \rho \times \rho).$$

Before proceeding, we recall here the converse theorem of Cogdell and Piatetski-Shapiro (see [C-Ki-PS-S] for details):

Theorem ([Co-PS]). *Suppose $\Pi = \bigotimes_v \Pi_v$ is an irreducible admissible representation of $GL_N(\mathbb{A}_F)$ such that $\omega_\Pi = \bigotimes_v \omega_{\Pi_v}$ is a grössencharacter of F. Let S be a finite set of finite places and let $\mathcal{T}^S(m)$ be a set of cuspidal representations of $GL_m(\mathbb{A}_F)$ that are unramified at all places $v \in S$. Suppose $L(s, \sigma \times \Pi)$ is nice (i.e., entire, bounded in vertical strips and satisfies a functional equation) for all cuspidal representations $\sigma \in \mathcal{T}^S(m)$, $m < N$. Then there exists an automorphic representation Π' of $GL_N(\mathbb{A}_F)$ such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.*

Theorem 2.8. *Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$. Then a strong lift exists. It is an automorphic representation of $GL_{2n}(\mathbb{A}_F)$.*

Proof. Since π is generic, each π_v is generic. For each π_v, we found a local lift Π_v. Let $\Pi = \bigotimes_v \Pi_v$. It is an irreducible admissible representation of $GL_{2n}(\mathbb{A}_F)$. Pick two finite places v_1, v_2, where both π_{v_1}, π_{v_2} are unramified. Let $S_i = \{v_i\}$, $i = 1, 2$. We apply the above converse theorem twice to Π with S_1 and S_2, to obtain two automorphic representations Π_1, Π_2 of $GL_{2n}(\mathbb{A}_F)$ such that $\Pi_{1v} \simeq \Pi_v$ for $v \neq v_1$, and $\Pi_{2v} \simeq \Pi_v$ for $v \neq v_2$. Hence $\Pi_{1v} \simeq \Pi_{2v} \simeq \Pi_v$ for all $v \neq v_1, v_2$.

By the classification of automorphic representations [J-S], Π_1 and Π_2 are equivalent to subquotients of Ξ_1, Ξ_2, resp. which are of the form

$$Ind\, |det|^{r_1} \sigma_1 \otimes \cdots \otimes |det|^{r_m} \sigma_m,$$

where the σ_i's are (unitary) cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$.

If π_v, $v \neq v_1, v_2$ is unramified, Π_{1v}, Π_{2v} are the unique unramified subquotient of Ξ_{1v}, Ξ_{2v}, resp. But since $\Pi_{1v} \simeq \Pi_{2v} \simeq \Pi_v$ is generic by [Li, Theorem 2.2], $\Xi_{1v} \simeq \Pi_{1v} \simeq \Pi_{2v} \simeq \Xi_{2v}$ if π_v is unramified and $v \neq v_1, v_2$. Hence by strong multiplicity one theorem [J-S], $\Xi_{1w} \simeq \Xi_{2w}$ for all w. Especially, $\Xi_{1v_i} \simeq \Xi_{2v_i}$ for $i = 1, 2$. Since π_{v_i}, $i = 1, 2$, is unramified, $\Pi_{1v_i} \simeq \Xi_{1v_i}$ and $\Pi_{2v_i} \simeq \Xi_{2v_i}$ for $i = 1, 2$. Hence Ξ_{1v_i} and Ξ_{2v_i} are irreducible, and $\Pi_{1v} \simeq \Xi_{1v} \simeq \Xi_{2v} \simeq \Pi_{2v} \simeq \Pi_v$ for $v = v_1, v_2$. Therefore $\Pi \simeq \Pi_1 \simeq \Pi_2$. This proves that $\Pi = \bigotimes_v \Pi_v$ is an automorphic representation of $GL_{2n}(\mathbb{A}_F)$. $\square$

If $\pi = \bigotimes_v \pi_v$ has one supercuspidal component, one has a more precise form of the lift.

Theorem 2.9. *Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$ such that π_{v_0} is supercuspidal. Then the strong lift exists and it is of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_k$, where σ_i is a self-contragredient cuspidal representation of $GL_{n_i}(\mathbb{A}_F)$.*

Proof. Let Π be a strong lift of π, constructed in Theorem 2.8. It is equivalent to a subquotient of

$$\Xi = Ind\, |det|^{r_1} \sigma_1 \otimes \cdots \otimes |det|^{r_p} \sigma_p,$$

where the σ_i's are (unitary) cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$. In particular, Π_{v_0} is a subquotient of Ξ_{v_0}. By repeating the arguments of Lemma 2.3 and Proposition 2.4, we can show that $r_1 = \cdots = r_p = 0$ and $\Xi_{v_0} = \Pi_{v_0}$ is tempered. Hence $\Xi = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_p$. Since Ξ_v is irreducible for all v [Ta], Ξ is irreducible. Therefore, $\Pi = \Xi = Ind\, \sigma_1 \otimes \cdots \otimes \sigma_p$.

Suppose σ_1 is not self-contragredient. Then consider $L(s, \tilde{\sigma}_1 \times \pi)$. It is holomorphic at $s = 1$ since σ_1 is not self-contragredient ([Ki6, Proposition 3.1] or [C-Ki-PS-S, Proposition 3.1]). But $L(s, \tilde{\sigma}_1 \times \pi) = \prod_{i=1}^p L(s, \tilde{\sigma}_1 \times \sigma_i)$ has a pole at $s = 1$. Contradiction. $\square$

Remark. It is expected that $L(s, \sigma_i, \wedge^2)$ has a pole at $s = 1$ for all i (hence n_i is even [Ki1]), and σ_{iv_0} is supercuspidal. In fact, we prove in a separate paper [Ki6] that the above theorem is true in general without supercuspidal component condition, and also that $L(s, \sigma_i, \wedge^2)$ has a pole at $s = 1$ for all i. The only thing we need to show is that the Rankin-Selberg L-function $L(s, \sigma \times \pi)$ is holomorphic for $Re\, s > 1$ for any cuspidal representation σ of $GL_m(\mathbb{A}_F)$. The proof requires a deep result on the spherical unitary dual due to Barbasch and Moy [B-Mo]. Our purpose in this paper is to give a simpler proof of the above theorem without using unitary dual. Recently Ginzburg, Rallis and Soudry proved that the Rankin-Selberg L-function $L(s, \sigma \times \pi)$ is holomorphic for $Re\, s > 1$ for any cuspidal representation σ of $GL_m(\mathbb{A}_F)$, and hence the above theorem without any condition. They can also obtain the backward lifting.

By combining Theorem 2.8 and Corollary 3 in section 6 of [C-Ki-PS-S], we obtain

Theorem 2.10. *Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$ which has one unramified tempered local component. Then the strong lift is of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_p$, where σ_i is a cuspidal representation of $GL_{n_i}(\mathbb{A}_F)$.*

In the next two corollaries, let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$ such that π contains either a supercuspidal component, or tempered unramified component.

Corollary 2.11. *For any cuspidal representation σ of $GL_m(\mathbb{A}_F)$, the Rankin-Selberg L-function $L(s, \sigma \times \pi)$ is holomorphic, except possibly at $s = 0, 1$. It is entire if $m > 2n$.*

Proof. By Theorems 2.9 and 2.10, the strong lift Π of π is given by $\sigma_1 \boxplus \cdots \boxplus \sigma_p$, where the σ_i's are cuspidal representations of GL. Hence

$$L(s, \sigma \times \pi) = L(s, \sigma \times \Pi) = \prod_{i=1}^{p} L(s, \sigma \times \sigma_i).$$

Our assertion follows from the well-known fact on the Rankin-Selberg L-functions of $GL_a \times GL_b$. $\square$

Corollary 2.12. *Suppose $\pi_v = Ind\, |det|^{r_1} \tau_1 \otimes \cdots \otimes |det|^{r_a} \tau_a \otimes \tau_0$, where $0 < r_a \leq \cdots \leq r_1$, $\tau_1, ..., \tau_p$ are discrete series of GL, and τ_0 is a tempered representation of $SO_{2l+1}(F_v)$. This is the case due to standard module conjecture, proved by* [Mu, *Theorem* 0.4]. *Then $r_1 < \frac{1}{2}$.*

Proof. Let $\Pi = \bigotimes_v \Pi_v$ be the strong lift of π, which is of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_p$. In particular, Π_v is generic and unitary. Now Π_v is the unique quotient of

$$Ind\, |det|^{r_1} \tau_1 \otimes \cdots \otimes |det|^{r_a} \tau_a \otimes \Pi_0 \otimes |det|^{-r_a} \tilde{\tau}_a \otimes \cdots \otimes |det|^{-r_1} \tilde{\tau}_1,$$

where Π_0 is the local lift of τ_0. Our assertion follows from the classification of unitary dual of GL_n [Ta]. $\square$

3. Reducibility criterion

In this section, let k be a p-adic field of characteristic zero. Let τ be a generic supercuspidal representation of $SO_{2n+1}(k)$. Let Π be its lift to $GL_{2n}(k)$. By Lemma 2.5, it is of the form

$$\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_p,$$

where the σ_i's are (unitary) supercuspidal representations of $GL_{n_i}(k)$ and $L(s, \sigma_i, \wedge^2)$ has a pole at $s = 0$. In particular, n_i is even, and σ_i's are self-contragredient and have the trivial central character.

Let σ be a self-contragredient supercuspidal representation of $GL_m(k)$. Then there exists a unique $s_0 \geq 0$ such that the induced representation $Ind\, |det|^s \sigma \otimes \tau$ is reducible at $s = s_0, -s_0$ and irreducible at all other points. The deep result of Shahidi [Sh1] is that $s_0 \in \{0, \frac{1}{2}, 1\}$.

Definition 3.1. *(σ, τ) satisfies (Ci) if $Ind\, |det|^s \sigma \otimes \tau$ is reducible at $s = i$.*

Note that if $l = 0$, we set the convention that $\tau = 1$. In that case, $(\sigma, 1)$ satisfies either $(C\frac{1}{2})$ or $(C0)$. We give a precise criterion of when (σ, τ) satisfies (Ci) in terms of the functorial lift of τ. By [Sh1, Corollary 7.6],

(1) (σ, τ) satisfies $(C1)$ if and only if $L(s, \sigma \times \tau)$ has a pole at $s = 0$.
(2) (σ, τ) satisfies $(C\frac{1}{2})$ if and only if $L(s, \sigma, Sym^2)$ has a pole at $s = 0$.
(3) (σ, τ) satisfies $(C0)$ if and only if $L(s, \sigma \times \tau) L(s, \sigma, Sym^2)$ is holomorphic at $s = 0$. This means that $L(s, \sigma \times \tau)$ is holomorphic at $s = 0$ but $L(s, \sigma, \wedge^2)$ has a pole at $s = 0$.

We prove

Proposition 3.2. *Let σ, τ be as above. Then*

(1) *(σ, τ) satisfies $(C1)$ if and only if $\sigma \simeq \sigma_i$ for some i.*

(2) (σ, τ) *satisfies* $(C\frac{1}{2})$ *if and only if* $L(s, \sigma, \wedge^2)$ *is holomorphic at* $s = 0$.
(3) (σ, τ) *satisfies* $(C0)$ *if and only if* $\sigma \not\simeq \sigma_i$ *for all* i *and* $L(s, \sigma, \wedge^2)$ *has a pole at* $s = 0$.

Proof. (1) Since $\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_p$,

$$L(s, \sigma \times \tau) = L(s, \sigma \times \Pi) = \prod_{i=1}^{p} L(s, \sigma \times \sigma_i).$$

Note that the Rankin-Selberg L-function $L(s, \rho_1 \times \rho_2)$ of $GL_a \times GL_b$, has a pole at $s = 0$ if and only if $\rho_1 \simeq \tilde{\rho}_2$, where ρ_1, ρ_2 are supercuspidal representations of GL_a, GL_b, resp. Hence our assertion follows.

(2) and (3) follows from the identity $L(s, \sigma \times \sigma) = L(s, \sigma, Sym^2) L(s, \sigma, \wedge^2)$. ☐

Corollary 3.3. *Suppose* (σ, τ) *satisfies (C1). Then* $L(s, \sigma \times \sigma)^{-1}$ *divides* $L(s, \sigma \times \tau)^{-1}$ *as polynomials in* q^{-s}, *namely,*

$$L(s, \sigma \times \tau) = L(s, \sigma \times \sigma) \prod_j (1 - \alpha_j q^{-s})^{-1},$$

where $\alpha_j \in \mathbb{C}$ *is of absolute value 1.*

Proof. It is clear from the equation in the above proof. ☐

The following is expected (cf. [Ki6, Conjecture 8.3]).

Conjecture 3.4. *Let* F *be a number field and* $\mathbb{A}_F$ *its ring of adeles. Let* $\tau = \bigotimes_v \tau_v$ *be a generic cuspidal representation of* $SO_{2n+1}(\mathbb{A}_F)$ *such that* τ_{v_0} *is supercuspidal. Suppose the lift of* τ_{v_0} *to* $GL_{2n}(F_{v_0})$ *is of the form* $\sigma_{1v_0} \boxplus \cdots \boxplus \sigma_{kv_0}$, *where* σ_{iv_0} *is a supercuspidal representation of* $GL_{n_i}(F_{v_0})$ *such that* $L(s, \sigma_{iv_0}, \wedge^2)$ *has a pole at* $s = 0$. *Then the lift of* τ *is of the form* $\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k$, *where the* σ_i*'s are cuspidal representations of* $GL_{n_i}(\mathbb{A}_F)$.

4. Review of the local Langlands' correspondence for GL_n

In this section we follow [Ku]. Let k be a p-adic field of characteristic zero. Let $\mathcal{G}_k(n)$ be the set of isomorphism classes of admissible representations of $W_k \times SL_2(\mathbb{C})$ of degree n, i.e., $\phi : W_k \times SL_2(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$. Here two representations into $GL_n(\mathbb{C})$ are said to be equivalent if they are conjugate by an element of $GL_n(\mathbb{C})$. We identify the element of the set $\mathcal{G}_k(1)$, the characters of W_k, with characters of $k^\times$ via the reciprocity isomorphism $r_k : W_k^{ab} \longleftarrow k^\times$. Let $\mathcal{A}_k(n)$ be the set of inequivalent classes of admissible representations of $GL_n(k)$. The following theorem is called the local Langlands' correspondence for GL_n.

Theorem 4.1 ([H-T], [He])**.** *For each* $n \geq 1$, *there is a canonical bijection*

$$\pi_k : \mathcal{G}_k(n) \longrightarrow \mathcal{A}_k(n), \quad \rho \longmapsto \pi_k(\rho),$$

such that

(1) $\pi_k(\rho(\chi)) = \pi_k(\rho)(\chi)$ *for any character* χ *of* $k^\times$.
(2) $det(\rho)$ *corresponds to* $\omega_{\pi_k(\rho)}$, *the central character of* $\pi_k(\rho)$ *via the isomorphism of local class field theory.*
(3) $\widetilde{\pi_k(\rho)} = \pi_k(\tilde{\rho})$.
(4) $L(s, \rho_1 \otimes \rho_2) = L(s, \pi_k(\rho) \times \pi_k(\rho))$.
(5) $\epsilon(s, \rho_1 \otimes \rho_2, \psi) = \epsilon(s, \pi_k(\rho) \times \pi_k(\rho), \psi)$.

(6) π_k *preserves conductors.*
(7) *If* K *is a finite Galois extension of a field* k*, then* π_K *is compatible with the natural actions of* $Gal(K/k)$ *on* $\mathcal{G}_K$ *and* $\mathcal{A}_K$*.*

Theory of Jordan normal forms implies that a unipotent matrix in $GL_n(\mathbb{C})$ is conjugate to $J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s)$, $p_1 \geq p_2 \geq \cdots \geq p_s$, $p_1 + p_2 + \cdots + p_s = n$, where $J(p)$ is the $p \times p$ Jordan matrix with entries 1 just above the diagonal and zero everywhere else. Therefore unipotent classes in $GL_n(\mathbb{C})$ are in 1 to 1 correspondence with partitions λ of N. We use the following standard notation for λ: $\lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \cdots)$, where r_j is the number of p_i equal to j.

We say that a unipotent element u is distinguished in G if all maximal tori of $Cent(u, G)$ are contained in the center of G°, the connected component of the identity. This is equivalent to the fact that the unipotent orbit O (conjugacy classes) of u does not meet any proper Levi subgroup of G. In the case of $GL_n(\mathbb{C})$, there is only one distinguished unipotent orbit, namely, the one attached to the partition (n). (See [Ki5, section 3] for detail.)

Let O be the distinguished unipotent orbit in $GL_p(\mathbb{C})$. There is a homomorphism $\phi_2 : SL_2(\mathbb{C}) \longrightarrow GL_p(\mathbb{C})$, given by O. Given a homomorphism $\phi_1 : W_k \longrightarrow GL_m(\mathbb{C})$, which parametrizes a supercuspidal representation σ of $GL_m(k)$, we attach a homomorphism

$$\phi_1 \otimes \phi_2 : W_k \times SL_2(\mathbb{C}) \longrightarrow GL_{mp}(\mathbb{C}).$$

It parametrizes the Steinberg representation, denoted by $St(\sigma, p)$. It is the unique subrepresentation of

$$Ind\, |det|^{\frac{p-1}{2}} \sigma \otimes |det|^{\frac{p-1}{2}-1} \sigma \cdots \otimes |det|^{-\frac{p-1}{2}} \sigma.$$

We note that the unique quotient of the above induced representation is parametrized by

$$\begin{aligned} &|det|^{\frac{p-1}{2}} \phi_1 \oplus |det|^{\frac{p-1}{2}-1} \phi_1 \oplus \cdots \oplus |det|^{-\frac{p-1}{2}} \phi_1 : \\ &\quad W_k \times SL_2(\mathbb{C}) \longrightarrow GL_m(\mathbb{C}) \times \cdots \times GL_m(\mathbb{C}) \subset GL_{mp}(\mathbb{C}). \end{aligned}$$

5. Local Langlands' correspondence for odd-orthogonal groups

Recall the formulation of the local Langlands' correspondence for odd-orthogonal groups; Let k be a p-adic field of characteristic zero and $G = SO_{2n+1}(k)$. Then ${}^L G^\circ = Sp_{2n}(\mathbb{C})$. Let $\phi : W_k \times SL_2(\mathbb{C}) \longrightarrow Sp_{2n}(\mathbb{C})$ be an admissible representation.

Then the local Langlands' correspondence predicts that ϕ parametrizes a finite set Π_ϕ, called L-packet, of isomorphism classes of irreducible admissible representations of G, and every admissible representation of G belongs to Π_ϕ for a unique ϕ. Two elements π, π' in the same set Π_ϕ would have the same L- and ϵ-factors, and are hence called L-indistinguishable. We have [Bo]

(1) the elements of Π_ϕ are square integrable if and only if $Im(\phi)$ is not contained in any proper parabolic subgroup of $Sp_{2n}(\mathbb{C})$.
(2) the elements of Π_ϕ are tempered if and only if $\phi(W_k)$ is bounded.

The representations in the L-packet Π_ϕ are parametrized by the component group $C_\phi = S_\phi / Z_{{}^L G^\circ} S_\phi^\circ$, where S_ϕ is the centralizer of $Im(\phi)$ in ${}^L G^\circ$, S_ϕ° is the connected component of the identity, and $Z_{{}^L G^\circ}$ is the center of ${}^L G^\circ$.

Here we note that for $G = GL_n(\mathbb{C})$, the centralizer $Z_G(S)$ is connected for any subset S of G. Hence the component group C_ϕ is trivial for any ϕ. This justifies the formulation of the local Langlands' correspondence in the case of GL_n in section 4 without the L-packets.

In this section, we parametrize all square integrable representations of $SO_{2n+1}(k)$ with generic supercuspidal support by admissible homomorphisms of $W_k \times SL_2(\mathbb{C})$. We cannot construct all the elements of the L-packet Π_ϕ for a given admissible homomorphism ϕ, since we expect that it should contain certain non-generic supercuspidal representations.

First we need the following: Let σ be a self-dual supercuspidal representation of $GL_{2n}(k)$ and let $\phi : W_k \longrightarrow GL_{2n}(\mathbb{C})$ be a self-dual admissible homomorphism, corresponding to a supercuspidal representation σ. Let $L(s, Sym^2(\phi)), L(s, \wedge^2(\phi))$ be the Artin L-functions attached to the symmetric square and the exterior square map. More precisely, let Sym^2 (resp. $\wedge^2$) be the finite dimensional representation

$$g : X \longmapsto {}^t g X g, \quad g \in GL_{2n}(\mathbb{C}),$$

of $GL_{2n}(\mathbb{C})$ on the space of symmetric (resp. skew-symmetric) $2n \times 2n$-matrices. Then $L(s, Sym^2(\phi)), L(s, \wedge^2(\phi))$ are the Artin L-functions attached to $Sym^2 \circ \phi, \wedge^2 \circ \phi$, resp. Let $L(s, \sigma, Sym^2), L(s, \sigma, \wedge^2)$ be the Shahidi's L-functions given by the Langlands-Shahidi method [Sh1]. More precisely, $L(s, \sigma, Sym^2)$ comes from the theory of Eisenstein series relative to $GL_{2n} \subset SO_{4n+1}$; $L(s, \sigma, \wedge^2)$ comes from the theory of Eisenstein series relative to $GL_{2n} \subset SO_{4n}$.

Assumption 5.1. (1) *$L(s, Sym^2(\phi))$ has a pole at $s = 0$ if and only if $L(s, \sigma, Sym^2)$ has a pole at $s = 0$.*

(2) *$L(s, \wedge^2(\phi))$ has a pole at $s = 0$ if and only if $L(s, \sigma, \wedge^2)$ has a pole at $s = 0$.*

Remark. In [P-R, Proposition 5.2], (2) is shown to be true for $n = 2$. The above assumption is the same as the assertion that the Shahidi's L-factors are Artin L-factors. It is an observation due to F. Shahidi and it is true for any L-function as follows; Let σ be an irreducible supercuspidal representation and $L(s, \sigma, r_i)$ be the Shahidi's L-function defined in [Sh1, section 7]. By [Sh1, Proposition 7.3], it is a product of $(1 - \alpha q^{-s})^{-1}$, where $\alpha \in \mathbb{C}$ is of absolute value one. Hence if s_0 is a pole of $L(s, \sigma, r_i)$, then there exists an unramified character χ such that $L(s, \sigma \otimes \chi, r_i)$ has a pole at $s = 0$. Therefore, if Shahidi's L-function is the same as the Artin L-function at $s = 0$ for any unitary supercuspidal representation, then so is the case everywhere else.

It may be much harder to prove that Shahidi's ϵ-factors are Artin ϵ-factors.

Lemma 5.2. *Let $\phi : W_k \times SL_2(\mathbb{C}) \longrightarrow GL_{2n}(\mathbb{C})$ be an admissible homomorphism, corresponding to an irreducible admissible representation σ. Under Assumption 5.1, ϕ factors through $Sp_{2n}(\mathbb{C})$ ($O_{2n}(\mathbb{C})$, resp.) if and only if $L(s, \sigma, \wedge^2)$ ($L(s, \sigma, Sym^2)$, resp.) has a pole at $s = 0$.*

Proof. Note that ϕ factors through $Sp_{2n}(\mathbb{C})$ ($O_{2n}(\mathbb{C})$, resp.) if and only if the representation $\wedge^2 \circ \phi$ ($Sym^2 \circ \phi$, resp.) contains the trivial representation. This is the case if and only if the Artin L-function $L(s, \wedge^2(\phi))$ ($L(s, Sym^2(\phi))$, resp.) has a pole at $s = 0$. Our result follows from Assumption 5.1 if σ is supercuspidal. If σ is arbitrary, it is a subquotient of an induced representation, induced from a supercuspidal representation. Hence by multiplicativity of L-factors, we can see easily that Shahidi's exterior square L-function $L(s, \sigma, \wedge^2)$ is the Artin L-function under Assumption 5.1. The same is true for the symmetric square L-function $L(s, \sigma, Sym^2)$. □

From this section on, we make Assumption 5.1.

5.1 Parametrization for generic supercuspidal representations. Let τ be a generic supercuspidal representation of $SO_{2n+1}(k)$. Let Π be the lift of τ. It is of the form $\rho_1 \boxplus \cdots \boxplus \rho_p$, where the ρ_i's are supercuspidal representations of $GL_{2n_i}(k)$ such that $L(s, \rho_i, \wedge^2)$ has a pole at $s = 0$.

Let $\phi_i : W_k \longrightarrow GL_{2n_i}(\mathbb{C})$ be the admissible homomorphism corresponding to ρ_i. Then it factors through $Sp_{2n_i}(\mathbb{C})$ by Lemma 5.2. Hence we have a homomorphism

$$\phi = \phi_1 \oplus \cdots \oplus \phi_p : W_k \longrightarrow Sp_{2n_1}(\mathbb{C}) \times \cdots \times Sp_{2n_p}(\mathbb{C}) \hookrightarrow Sp_{2n}(\mathbb{C}),$$

which parametrizes the given generic supercuspidal representation τ. We set ϕ to be trivial on $SL_2(\mathbb{C})$. Now, since $Sp_{2n}(\mathbb{C})$ is connected, S_ϕ, the centralizer of $Im(\phi)$ in ${}^LG^\circ$, is connected. Hence $C_\phi = 1$. Namely, the L-packet Π_ϕ consists of only one element τ.

5.2 Parametrization for discrete series representations. Let us recall the recent classification of discrete series due to Moeglin and Tadic [M-Ta]. Let

$$\pi = \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{u_1} \otimes \cdots \otimes \underbrace{\sigma_t \otimes \cdots \otimes \sigma_t}_{u_t} \otimes \tau,$$

be a generic supercuspidal representation of a Levi subgroup of $SO_{2n+1}(k)$, where $\sigma_1, ..., \sigma_t$ are non-equivalent, self-contragredient supercuspidal representations of $GL_{m_i}(k)$ and τ is a generic supercuspidal representation of $SO_{2l+1}(k)$.

(1) If (σ_i, τ) satisfies $(C1)$, then we attach a distinguished unipotent orbit of $O_{2u_i+1}(\mathbb{C})$.
(2) If (σ_i, τ) satisfies $(C\frac{1}{2})$, then we attach a distinguished unipotent orbit of $Sp_{2u_i}(\mathbb{C})$.
(3) If (σ_i, τ) satisfies $(C0)$, then we attach a distinguished unipotent orbit of $O_{2u_i}(\mathbb{C})$ $(u_i \geq 2)$.

Note that a distinguished unipotent orbit of $SO_{2n+1}(\mathbb{C}), O_{2n}(\mathbb{C})$ is given by a partition $(p_1, ..., p_r)$, where the p_i's are distinct odd positive integers such that $p_1 + \cdots + p_r = 2n + 1$ or $2n$. A distinguished unipotent orbit of $Sp_{2n}(\mathbb{C})$ is given by $(p_1, ..., p_r)$, where the p_i's are distinct even positive integers such that $p_1 + \cdots + p_r = 2n$.

Given a distinguished unipotent orbit $O = (p_1, ..., p_r)$, we form a set $P(O)$ of ordered partitions as follows [M1]: $\mathfrak{p} = (; p_1, ..., p_r) \in P(O)$ if and only if

(1) $(p_1, ..., p_r)$ is O if we ignore the order.
(2) For all $1 \leq j \leq [\frac{r+1}{2}]$, $p_{2j-1} > p_{2j}$ and there does not exist $1 \leq k \leq [\frac{r+1}{2}]$ such that $p_{2j-1} > p_{2k-1} > p_{2j} > p_{2k}$.
(3) If there exists a $1 \leq k \leq r$ such that $p_{2j-1} > p_k > p_{2j}$, then $k < 2j - 1$.

We set $p_{r+1} = 0$ if r is odd.

Let $A(O)$ be a finite abelian group generated by the order two elements $a(p_1), ..., a(p_r)$. Let $\bar{A}(\mathfrak{p}) = A(O)/K_\mathfrak{p}$, where $K_\mathfrak{p}$ is generated by $a(p_{2i-1})a(p_{2i})^{-1}$ for all $1 \leq i \leq [\frac{r+1}{2}]$. We set $a(p_{r+1}) = 1$ if r is odd. We note that $|\bar{A}(\mathfrak{p})| = 2^{[\frac{r}{2}]}$.

Then (see [M1]) $Springer(O) \simeq \bigcup_{\mathfrak{p} \in P(O)} \widehat{\bar{A}(\mathfrak{p})}$, where $\widehat{\bar{A}(\mathfrak{p})}$ is the character group of $\bar{A}(\mathfrak{p})$. We recall that the Springer correspondence is an injective map from the characters of W, the Weyl group of ${}^LG^\circ$ into the set of pairs (O, η), where O is a unipotent orbit in ${}^LG^\circ$ and η is a character of $A(O)$. Given a unipotent orbit O in ${}^LG^\circ$, $Springer(O)$ is the set of characters of $A(O)$ which are in the image of the Springer correspondence. Also recall that if O is a unipotent orbit in ${}^LG^\circ$, $A(O) = C(u)/C(u)^0$, where $C(u) = Cent(u, {}^LG^\circ)$, $u \in O$.

Let $\mathfrak{p} = (;p_1, ..., p_r) \in P(O)$. Suppose r is odd, and write

$$\mathfrak{p} = (;a_1, b_1, ..., a_s, b_s, a_{s+1}).$$

Then we can form a chain

$$\lambda_{\mathfrak{p}} = (\frac{a_1-1}{2}, \frac{a_1-3}{2}, ..., -\frac{b_1-1}{2}, ..., \frac{a_s-1}{2}, \frac{a_s-3}{2}, ..., \\ -\frac{b_s-1}{2}, \frac{a_{s+1}-1}{2}, \frac{a_{s+1}-3}{2}, ..., \frac{a_{s+1}+1}{2} - [\frac{a_{s+1}}{2}]).$$

Notice that

$$\frac{a_{s+1}+1}{2} - [\frac{a_{s+1}}{2}] = \begin{cases} \frac{1}{2}, & \text{if } a_{s+1} \text{ is even,} \\ 1, & \text{if } a_{s+1} \text{ is odd.} \end{cases}$$

Note that if $\mathfrak{p} = (;a, b)$ and $\pi = \sigma \otimes \cdots \otimes \sigma$, where σ is a supercuspidal representation of GL, the following induced representation of GL,

$$Ind^{GL}\, \lambda_{\mathfrak{p}} \otimes \pi = Ind\, |det|^{\frac{a-1}{2}} \sigma \otimes |det|^{\frac{a-3}{2}} \sigma \otimes \cdots \otimes |det|^{-\frac{b-1}{2}} \sigma,$$

has a unique subrepresentation, which is $|det|^{\frac{a-b}{4}} St(\sigma, \frac{a+b}{2})$.

We first consider a special case when $\pi = \sigma \otimes \cdots \otimes \sigma \otimes \tau$. Let O be a distinguished unipotent orbit, determined by (σ, τ). We write $\mathfrak{p} \in P(O)$ as $\mathfrak{p} = (;a_1, b_1, ..., a_s, b_s, a_{s+1})$. For simplicity, assume that a_{s+1} is even. By inducing in stages, we see that $Ind^G\, \lambda_{\mathfrak{p}} \otimes \pi$ has a subrepresentation, namely,

$$Ind^G\, |det|^{\frac{a_1-b_1}{4}} St(\sigma, \frac{a_1+b_1}{2}) \otimes \cdots \otimes |det|^{\frac{a_s-b_s}{4}} St(\sigma, \frac{a_s+b_s}{2}) \\ \otimes |det|^{\frac{a_{s+1}}{4}} St(\sigma, \frac{a_{s+1}}{2}) \otimes \tau.$$

Let $Unip(\mathfrak{p})$ be the set of direct summands of the maximal completely reducible subrepresentation of the above induced representation. It is parametrized by $\widehat{\bar{A}(\mathfrak{p})}$. Let $Unip(O)$ be the union of $Unip(\mathfrak{p})$ as $\mathfrak{p}$ runs through $P(O)$. Then it is parametrized by $Springer(O)$. The result of Moeglin-Tadic [M-Ta] is that they are all non-supercuspidal square integrable representations with supercuspidal support (σ, τ). More generally, if $\pi = \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{u_1} \otimes \cdots \otimes \underbrace{\sigma_t \otimes \cdots \otimes \sigma_t}_{u_t} \otimes \tau$, the non-supercuspidal square integrable representations with supercuspidal support π, are parametrized by $Springer(O_1) \times \cdots \times Springer(O_t)$, where the O_i's are attached to (σ_i, τ). In summary,

Theorem 5.3 ([M-Ta])**.** *Let*

$$\pi = \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{u_1} \otimes \cdots \otimes \underbrace{\sigma_t \otimes \cdots \otimes \sigma_t}_{u_t} \otimes \tau$$

be a generic supercuspidal representation of a Levi subgroup of G. All non-supercuspidal square integrable representations with supercuspidal support π are parametrized by $(O_1, ..., O_t, \eta)$, where O_i is a distinguished unipotent orbit determined by (σ_i, τ), and $\eta \in Springer(O_1) \times \cdots \times Springer(O_t)$.

In this paper, for simplicity, we restrict ourselves to the special case when $\pi = \underbrace{\sigma \otimes \cdots \otimes \sigma}_{u} \otimes \tau$, where σ is a supercuspidal representation of $GL_m(k)$ and τ is a generic supercuspidal representation of $SO_{2l+1}(k)$, and O be a distinguished unipotent orbit, determined by (σ, τ). Hence $mu + l = n$. All the square integrable representations parametrized by the unipotent orbit O will be in the same L-packet. We now give parametrization:

Given a positive integer $p \in O$, we have a discrete series of $GL_{mp}(k)$, given as the subrepresentation of $|det|^{\frac{p-1}{2}}\sigma \otimes |det|^{\frac{p-1}{2}-1} \otimes \cdots \otimes |det|^{-\frac{p-1}{2}}\sigma$. We denoted it by $St(\sigma, p)$ in section 4. Let $\phi_p : W_k \times SL_2(\mathbb{C}) \longrightarrow GL_{mp}(\mathbb{C})$ be the homomorphism attached to the discrete series $St(\sigma, p)$.

Lemma 5.4. *ϕ_p factors through $Sp_{mp}(\mathbb{C})$.*

Proof. First of all, we emphasize that mp is even. It will be clear in the proof. We need to show that $L(s, St(\sigma, p), \wedge^2)$ has a pole at $s = 0$. If (σ, τ) satisfies either $(C1)$ or $(C0)$, then p is odd and m is even. By looking at [Sh5, Proposition 8.1], the L-function contains the factor $L(s, \sigma, \wedge^2)$, which has a pole at $s = 0$.

Suppose (σ, τ) satisfies $(C\frac{1}{2})$. Then p is even. Then the L-function contains the factor $L(s, \sigma, Sym^2)$, which has a pole at $s = 0$. ☐

First consider the case when (σ, τ) satisfies $(C1)$. Then the lift of τ is given by $Ind\, \sigma \otimes \sigma_1 \otimes \cdots \sigma_a$ by Proposition 3.2. Hence m is even. Let $O = (p_1, ..., p_r)$ be a distinguished unipotent orbit in $SO_{2u+1}(\mathbb{C})$. Hence $p_1 + \cdots + p_r = 2u + 1$. Then p_i and r are odd. In this case, Let

$$\phi_0 : W_k \longrightarrow Sp_{2l-m}(\mathbb{C})$$

be the homomorphism which parametrizes $Ind\, \sigma_1 \otimes \cdots \otimes \sigma_a$. Let

$$\phi_i : W_k \times SL_2(\mathbb{C}) \longrightarrow Sp_{mp_i}(\mathbb{C}),$$

be the homomorphism which parametrizes $St(\sigma, p_i)$. Then

$$\begin{aligned}\phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_r : W_k \times SL_2(\mathbb{C}) &\longrightarrow Sp_{2l-m}(\mathbb{C}) \times Sp_{mp_1}(\mathbb{C}) \times \cdots \times Sp_{mp_r}(\mathbb{C}) \\ &\hookrightarrow Sp_{2n}(\mathbb{C}),\end{aligned}$$

parametrizes the discrete series attached to (σ, τ) and O.

Next, let (σ, τ) satisfy $(C\frac{1}{2})$. In this case, m may not be even. Let $O = (p_1, ..., p_r)$ be a distinguished unipotent orbit of $Sp_{2u}(\mathbb{C})$ with $p_1 + \cdots + p_r = 2u$. Then p_i is even. Let

$$\phi_0 : W_k \longrightarrow Sp_{2l}(\mathbb{C})$$

be the homomorphism which parametrizes τ as in §5.1.

For each $p_i \in O$, let

$$\phi_i : W_k \times SL_2(\mathbb{C}) \longrightarrow Sp_{mp_i}(\mathbb{C}),$$

be the homomorphism which parametrizes $St(\sigma, p_i)$. Then

$$\begin{aligned}\phi_0 \oplus \cdots \oplus \phi_r : W_k \times SL_2(\mathbb{C}) &\longrightarrow Sp_{2l}(\mathbb{C}) \times Sp_{mp_1}(\mathbb{C}) \times \cdots \times Sp_{mp_r}(\mathbb{C}) \\ &\hookrightarrow Sp_{2n}(\mathbb{C})\end{aligned}$$

parametrizes the discrete series attached to (σ, τ) and O.

Finally, let (σ, τ) satisfy $(C0)$. By Proposition 3.2, $L(s, \sigma, \wedge^2)$ has a pole at $s = 0$. Hence m is even. Let $O = (p_1, ..., p_r)$ be a distinguished unipotent orbit in $O_{2u}(\mathbb{C})$

with $p_1 + \cdots + p_r = 2u$. Then p_i is odd. The parametrization is exactly the same as in $(C\frac{1}{2})$ case.

Note that $C_\phi \simeq (\mathbb{Z}/2\mathbb{Z})^r$. Hence there are 2^r representations in the L-packet attached to the distinguished unipotent orbit $O = (p_1, ..., p_r)$. But the discrete series in the L-packet are parametrized by the subset $Springer(O) \subset C_\phi$ whose cardinality is $|Springer(O)| = {}_rC_{[\frac{r}{2}]}$. (See [Ki5, section 3] for more details.) A conjecture might be that the remaining representations in the L-packet are non-generic supercuspidal representations and non-supercuspidal square integrable representations with non-generic supercuspidal support.

6. Functorial lift from SO_5 to GL_5

Let F be a number field and $\mathbb{A}_F$ its ring of adeles. Let π be a generic cuspidal representation of $SO_5(\mathbb{A}_F)$. Let Π be a strong lift of π to $GL_4(\mathbb{A}_F)$. Then we prove in [Ki6] that Π is either cuspidal, or of the form $\sigma_1 \boxplus \sigma_2$, where σ_i's are inequivalent self-contragredient cuspidal representations of $GL_2(\mathbb{A}_F)$. We also prove that if Π is cuspidal, then $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$, and if $\Pi = \sigma_1 \boxplus \sigma_2$, then σ_i's have the trivial central character. The proof uses the integral representation technique due to Ginzburg-Rallis-Soudry [G-R-S]. However, in the following special case, we can prove the following proposition without the integral representation technique.

Proposition 6.1. *Suppose π_{v_0} is supercuspidal and Π is of the form $\sigma_1 \boxplus \sigma_2$. Then the σ_i's have the trivial central character and σ_{iv_0}, $i = 1, 2$, is supercuspidal.*

Proof. By Lemma 2.5, Π_{v_0} has to be either supercuspidal, or of the form $\rho_1 \boxplus \rho_2$, where ρ_i is a supercuspidal representation of $GL_2(F_{v_0})$ with the trivial central character. Since Π is not cuspidal, Π_{v_0} cannot be supercuspidal. Hence $\sigma_{iv_0} = \rho_i$, $i = 1, 2$.

Suppose that the central character of σ_i is not trivial. Since σ_i is self-contragredient, it is monomial. Hence $L(s, \sigma_i, Sym^2)$ has a pole at $s = 1$. Consider the situation $M = GL_2 \subset SO_5$, and the global induced representation $Ind_{GL_2}^{SO_5} |det|^{\frac{s}{2}} \sigma_i$. Since $L(s, \sigma_i, Sym^2)$ has a pole at $s = 1$, (M, σ_i) contributes to the residual (see spectrum [Ki2] for details) and the residual automorphic representation is the (global) Langlands' quotient of $Ind_{GL_2}^{SO_5} |det|^{\frac{s}{2}} \sigma_i$, and it is unitary. In particular, at $v = v_0$, the induced representation $Ind_{GL_2}^{SO_5} |det|^{\frac{s}{2}} \sigma_{iv_0}$ is reducible at $s = 1$, because otherwise it cannot be unitary. Hence by reducibility criterion [Ca-Sh], $L(s, \rho_i, Sym^2)$ has a pole at $s = 0$. It means that by the identity $L(s, \rho_i \times \rho_i) = L(s, \rho_i, Sym^2) L(s, \rho_i, \wedge^2)$, $L(s, \rho_i, \wedge^2)$ does not have a pole at $s = 0$. It contradicts Lemma 2.5. $\square$

Remark. To put it in another way, we have shown that a self-contragredient monomial cuspidal representation of $GL_2(\mathbb{A})$ cannot have a supercuspidal component which has the trivial central character.

Remark. In an unpublished note, Jacquet, Piatetski-Shapiro and Shalika established a lift from GSp_4 to GL_4 and gave a criterion for the image of the lift. Since $SO_5 \simeq PGSp_4$, our result is a very special case.

The last part of the proof of Proposition 6.1 is quite general. We record it as

Proposition 6.2. *Let $P = MN$ be a maximal parabolic subgroup of a quasi-split group G, defined over a number field F and $\sigma = \bigotimes_v \sigma_v$ be a cuspidal representation of $M(\mathbb{A})$ such that σ_{v_0} is supercuspidal. If the induced representation $I(s, \sigma_{v_0}) =$*

$Ind_P^G\,\sigma_{v_0}\otimes exp^{s\tilde{\alpha},H_P(\ )}$ is irreducible at $s=1$, then the Eisenstein series attached to (M,σ) is holomorphic for $s>\frac{1}{2}$, especially at $s=1$.

Proof. Suppose the Eisenstein series attached to (M,σ) has a pole at $s=s_0>\frac{1}{2}$. Then the residue at $s=s_0$ spans a residual automorphic representation $\bigotimes_v J(s_0,\sigma_v)$, where $J(s_0,\sigma_v)$ is the unique quotient of $I(s_0,\sigma_v)$. In particular, it is unitary. Since σ_{v_0} is supercuspidal, the only reducibility points for $I(s,\sigma_{v_0})$ for $Re\, s\geq 0$ are $\{0,\frac{1}{2},1\}$. Hence if $I(s,\sigma_{v_0})$ is irreducible at $s=1$, it cannot be unitary at least for $s>\frac{1}{2}$. We obtain a contradiction. □

Corollary 6.3. *Let $\sigma=\bigotimes_v\sigma_v$ be a cuspidal representation of $GL_n\subset SO_{2n+1}$ such that σ_{v_0} is supercuspidal. If $I(s,\sigma_{v_0})=Ind_{GL_n}^{SO_{2n+1}}\sigma_{v_0}|det|^{\frac{s}{2}}$ is irreducible at $s=1$ (this is equivalent to the fact that $I(0,\sigma_{v_0})$ is reducible), then $L(s,\sigma,Sym^2)$ is entire.*

Proof. Apply Proposition 6.2 to $M=GL_n\subset G=SO_{2n+1}$. Then the constant term of the Eisenstein series contains only one L-function $L(s,\sigma,Sym^2)$. Hence the only reducibility points for $I(s,\sigma_{v_0})$ are $\{0,1\}$. Hence if $I(s,\sigma_{v_0})$ is irreducible at $s=1$, then $I(s,\sigma_{v_0})$ is irreducible for $s>0$ and not unitary. Hence the Eisenstein series attached to (M,σ) is holomorphic for $s>0$. By following [Ki2, Theorem 3.1], we can see that $L(s,\sigma,Sym^2)$ is entire. □

Let π be a generic cuspidal representation of $SO_5(\mathbb{A}_F)$, and let Π be its lift to $GL_4(\mathbb{A}_F)$. Let $r_5: Sp_4(\mathbb{C})\longrightarrow GL_5(\mathbb{C})$ be the 5 dimensional representation, corresponding to the second fundamental weight. It is given by

$$diag(a,b,b^{-1},a^{-1})\longmapsto diag(ab,ab^{-1},1,ba^{-1},b^{-1}a^{-1}).$$

Under the identification $SO_5\simeq PGSp_4$, $L(s,\pi,r_5)$ appears in the constant term of Eisenstein series for GSp_6 (C_3-1 case of [Sh3]). More precisely, let $M=GL_1\times GSp_4$ be a maximal Levi subgroup of GSp_6. Then by considering π as a cuspidal representation of $M(\mathbb{A})$, we obtain $L(s,\pi,r_5)$ as a normalizing factor in the constant term of the Eisenstein series attached to (M,π). Hence

Proposition 6.4 ([Sh3])**.** *$L(s,\pi,r_5)$ has meromorphic continuation and satisfies the standard functional equation. It has no zeros for $Re\, s=1$.*

Consider the composition of two maps

$$Sp_4(\mathbb{C})\hookrightarrow GL_4(\mathbb{C})\longrightarrow GL_6(\mathbb{C}).$$

Then by a well-known identity,

$$\wedge^2\circ i=r_5\oplus 1.$$

Hence if Π is cuspidal, $L(s,\Pi,\wedge^2)=L(s,\pi,r_5)L(s,1)$. If $\Pi=\sigma_1\boxplus\sigma_2$, where σ_1,σ_2 are cuspidal representations of $GL_2(\mathbb{A}_F)$ with the trivial central character, then $L(s,\Pi,\wedge^2)=L(s,\sigma_1\times\sigma_2)L(s,1)L(s,1)$. Hence

By Proposition 6.4, we have

Proposition 6.5. *Suppose Π is cuspidal. Then $L(s,\Pi,\wedge^2)$ has a pole at $s=1$ and $L(s,\pi,r_5)$ is holomorphic at $s=1$. If $\Pi=\sigma_1\boxplus\sigma_2$, then $L(s,\Pi,\wedge^2)$ has a double pole at $s=1$, and hence $L(s,\pi,r_5)$ has a pole at $s=1$.*

Consider the exterior square lift $\wedge^2\Pi$ [Ki4]. First, we assume that Π is cuspidal. Then by the main result in [Ki4], $\wedge^2\Pi$ is an automorphic representation of $GL_6(\mathbb{A}_F)$, which is of the form $\pi_1 \boxplus \cdots \boxplus \pi_k$, where the π_i's are (unitary) cuspidal representations of $GL_{n_i}(\mathbb{A})$. Since $L(s,\Pi,\wedge^2)$ has a pole at $s=1$,

$$\wedge^2\Pi = \tau \boxplus 1,$$

where τ is an automorphic representation of $GL_5(\mathbb{A}_F)$ which is a functorial lift corresponding to r_5.

Suppose $\Pi = \sigma_1 \boxplus \sigma_2$, where σ_i's are inequivalent self-contragredient cuspidal representations of $GL_2(\mathbb{A}_F)$ with the trivial central character. Then

$$\wedge^2\Pi = (\sigma_1 \boxtimes \sigma_2) \boxplus 1 \boxplus 1,$$

where $\sigma_1 \boxtimes \sigma_2$ is the functorial product whose existence was proved in [Ra]. (See [Ki4] for a different proof.) It is an automorphic representation of $GL_4(\mathbb{A}_F)$. Hence $\tau = (\sigma_1 \boxtimes \sigma_2) \boxplus 1$.

In conclusion, we have shown

Theorem 6.6. *There exists a functional lift from cuspidal representations of*

$$SO_5(\mathbb{A}) \simeq PGSp_4(\mathbb{A})$$

to automorphic representations of $GL_5(\mathbb{A})$, corresponding to r_5.

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