

VERTICES FOR CHARACTERS OF p -SOLVABLE GROUPS

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ABSTRACT. Suppose that G is a finite p -solvable group. We associate to every irreducible complex character $\chi \in \text{Irr}(G)$ of G a canonical pair (Q, δ) , where Q is a p -subgroup of G and $\delta \in \text{Irr}(Q)$, uniquely determined by χ up to G -conjugacy. This pair behaves as a Green vertex and partitions $\text{Irr}(G)$ into “families” of characters. Using the pair (Q, δ) , we give a canonical choice of a certain p -radical subgroup R of G and a character $\eta \in \text{Irr}(R)$ associated to χ which was predicted by some conjecture of G. R. Robinson.

1. INTRODUCTION

Let p be a prime and let G be a finite p -solvable group. In this paper, we associate to every ordinary irreducible character $\chi \in \text{Irr}(G)$ a canonical pair (Q, δ) , where Q is a p -subgroup of G and $\delta \in \text{Irr}(Q)$, which is uniquely determined by χ up to G -conjugacy. We say that (Q, δ) is a **vertex** of χ , and we denote by

$$\text{Irr}(G|Q, \delta)$$

the set of all $\chi \in \text{Irr}(G)$ with vertex (Q, δ) . (See Section 3 for this construction.)

If G is a finite group and S is a certain complete discrete valuation ring or a field of characteristic p , we know by work of J. A. Green that each indecomposable SG -module V has an associated G -conjugacy class of p -subgroups Q of G called the *vertices* of V . Given an irreducible character $\chi \in \text{Irr}(G)$, it is possible to find a (necessarily indecomposable) SG -module V affording χ , and therefore we might associate to χ a set of vertices in the Green sense. As is well-known, however, χ does not uniquely determine V , and different SG -modules affording χ can have non-isomorphic vertices. (Of course, this is not the case with the irreducible Brauer characters of G . Given $\varphi \in \text{IBr}(G)$, we have that φ uniquely determines up to isomorphism a simple FG -module V , where F is an algebraically closed field of characteristic p , and therefore, irreducible Brauer characters do have vertices associated. If Q is a p -subgroup of G , we denote by $\text{IBr}(G|Q)$ the set of irreducible Brauer characters of G with vertex Q .)

The pairs (Q, δ) that we associate to the irreducible character of the p -solvable groups behave, in certain aspects, as the Green vertices. In order to state their main properties, we remind the reader of some definitions and notation. First of all, the **defect** $d(\chi)$ of any character $\chi \in \text{Irr}(G)$ is defined by the equation

$$p^{d(\chi)}\chi(1)_p = |G|_p.$$

Received by the editors March 10, 2001 and, in revised form, October 10, 2001.
2000 *Mathematics Subject Classification*. Primary 20C15.
Research partially supported by DGICYT and MEC.

Recall that a p -subgroup Q of G is p -radical (in G) if $Q = \mathbf{O}_p(\mathbf{N}_G(Q))$. Also, every p -subgroup Q of G is contained in a p -radical subgroup of G uniquely determined by Q and G . This is called the **radical closure** of Q in G and is the last term R of the chain $P_{i+1} = \mathbf{O}_p(\mathbf{N}_G(P_i))$ starting with $Q = P_0$. If Q is a p -subgroup of G and $\delta \in \text{Irr}(Q)$, we denote by $\mathbf{N}_G(Q, \delta)$ the elements of $\mathbf{N}_G(Q)$ which stabilize δ .

Theorem A. *Let Q be a p -subgroup of a p -solvable group G and let $\delta \in \text{Irr}(Q)$.*

(a) *Suppose that $\delta = 1_Q$ is the principal character of Q . Then restriction to p -regular elements defines a canonical bijection*

$$\text{Irr}(G|Q, \delta) \rightarrow \text{IBr}(G|Q).$$

(b) *If $\chi \in \text{Irr}(G)$ has vertex (Q, δ) , then $d(\chi) = d(\delta)$.*

(c) *If $\chi \in \text{Irr}(G)$ has vertex (Q, δ) , then $\chi(g) = 0$ whenever the p -part of g lies in no G -conjugate of Q .*

(d) *If $\chi \in \text{Irr}(G)$ has vertex (Q, δ) , then*

$$\mathbf{O}_p(\mathbf{N}_G(Q, \delta)) = Q.$$

(e) *Suppose that (Q, δ) is a vertex of $\chi \in \text{Irr}(G)$. Let R be the radical closure of Q in G . Then there is a defect group D of the p -block of χ such that $Q \subseteq R \subseteq D$ and*

$$\mathbf{C}_D(Q) = \mathbf{Z}(Q).$$

(f) *Suppose that (Q, δ) is a vertex of $\chi \in \text{Irr}(G)$, and let R be the radical closure of Q in G . Then the induced character $\eta = \delta^R$ is irreducible.*

(g) *Suppose that $Q \triangleleft G$ and that δ is G -invariant. Then $\chi \in \text{Irr}(G)$ has vertex (Q, δ) if and only if the restriction χ_Q contains δ and*

$$(\chi(1)/\delta(1))_p = |G : Q|_p.$$

In the special case where $\delta = 1_Q$, Theorem A recovers several well-known results on the irreducible Brauer characters of the p -solvable groups. Of course, Theorem A(a) gives a strong form of the Fong-Swan theorem that irreducible Brauer characters of p -solvable groups are liftable, or Theorem A(d) reproves that the vertices of the Brauer characters are p -radical (in p -solvable groups). Concerning part (e), the fact that vertices are “big” (that is, containing its own centralizer) in the defect groups is a celebrated general result of R. Knörr ([7]), although the part on the radical closure seems not have been noticed up to now.

When introducing a “new” object it is always pleasant to find applications, and we do have one for our pairs (Q, δ) . A consequence drawn by G. R. Robinson of the well-known conjectures on representation theory of groups by E. C. Dade and himself is that, given $\chi \in \text{Irr}(G)$, there always can be found a radical p -subgroup R of G which is “big” in a defect group of the p -block of χ , and which has an irreducible character $\eta \in \text{Irr}(R)$ with $d(\chi) = d(\eta)$. For p -solvable groups this is now a fact (see Theorem 2 of [11], or [1] for a partial result). Our Theorem A, parts (e) and (f), gives a canonical choice for Robinson’s predicted R and η : if (Q, δ) is a vertex of χ , it suffices to take R the radical closure of Q in G and $\eta = \delta^R$.

Our aim when introducing the vertices (Q, δ) for the irreducible characters of the p -solvable groups was to explore the relationship between the characters of G and the characters of certain local subgroups of G . In fact, our main concern is to find any connection between the sets $\text{Irr}(G|Q, \delta)$ and $\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)$. For instance,

by Theorem A(g), notice that when $\delta = 1_Q$, the set $\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)$ consists of the Alperin weights. Hence, for $\delta = 1_Q$, we have that

$$|\text{Irr}(G|Q, \delta)| = |\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)|$$

(by Theorem A, (a) and (g), and the main result in [5]). This proves, of course, the Alperin weight conjecture for p -solvable groups.

Suppose now that $\chi \in \text{Irr}(G)$ and let (Q, δ) be a vertex of χ . By Theorem A(b), it is easy to check that a character $\chi \in \text{Irr}(G)$ has p' -degree if and only if Q is a Sylow p -subgroup of G and δ is linear. If this is the case, notice that $\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)$ consists exactly of those characters of $\mathbf{N}_G(Q, \delta)$ lying over δ (by Theorem A(g)). Hence, by the Clifford correspondence, we have that $|\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)|$ is the number of irreducible characters of $\mathbf{N}_G(Q)$ lying over δ . By using the main result of [6] it is possible to prove that whenever $Q \in \text{Syl}_p(G)$ and $\delta \in \text{Irr}(Q)$ is linear, again we have

$$|\text{Irr}(G|Q, \delta)| = |\text{Irr}(\mathbf{N}_G(Q, \delta)|Q, \delta)|,$$

proving a strong form of the McKay conjecture.

Our construction of the vertex pair (Q, δ) for $\chi \in \text{Irr}(G)$ heavily uses the p -solvability of G . We do not know if it is possible to associate a similar canonical pair (Q, δ) to every irreducible character χ of every finite group G .

In order to prove some of the previous results, it is essential to establish the following property of the radical closure (which we find interesting on its own).

Theorem B. *Suppose that G is p -solvable and let $N \triangleleft G$. Suppose that $\theta \in \text{IBr}(N)$ has p' -degree. Let Q be any p -subgroup of G containing a Sylow p -subgroup of N , and let R be its radical closure in G . If θ is Q -invariant, then θ is R -invariant.*

I would like to thank M. Isaacs for many useful conversations on this subject.

2. INDUCING CHARACTERS

In this section, instead of restricting ourselves to a single prime p , we work with a set of primes π with the same amount of work. Our goal now is to associate to every irreducible $\chi \in \text{Irr}(G)$ of a π -separable group G a unique (up to G -conjugacy) pair (W, γ) , where $W \subseteq G$ and $\gamma \in \text{Irr}(W)$, satisfying certain properties. We will obtain (W, γ) by repeated Clifford induction from the “right” normal subgroups. We remind the reader that a group G is π -separable if it has a normal series $1 = G_k < \dots < G_0 = G$ such that the factors G_i/G_{i+1} are either π - or π' -groups. Of course, the p -solvable groups are the p -separable groups. Also, π -separable groups have Hall π -subgroups, and any two of them are G -conjugate.

To find the pairs (W, γ) , we use π -special characters, and next we remind the reader of their definition and main properties. If G is π -separable, an irreducible $\chi \in \text{Irr}(G)$ is said to be π -**special** if $\chi(1)$ is a π -number and for every subnormal subgroup $N \triangleleft \triangleleft G$ and $\theta \in \text{Irr}(N)$ under χ , the determinantal order of θ is a π -number. (Recall that the determinantal order of θ is the order of the linear character $\det \mathcal{X}$, where \mathcal{X} is any representation affording θ .) Of course, if G is a π -group, then all irreducible characters of G are π -special. Also, by the definition, normal irreducible constituents of π -special characters are again π -special.

Let us write down for the reader’s convenience the properties of the π -special characters that we are going to use later on. If p is a prime and $\chi \in \text{Irr}(G)$, we denote by χ^0 the restriction of χ to the set G^0 of p -regular elements of G .

(2.1) Theorem. *Suppose that G is π -separable, let H be a Hall π -subgroup of G , and let $\mathcal{X}_\pi(G)$ be the set of irreducible π -special characters of G .*

(a) If $N \triangleleft G$ is a π' -group and $\chi \in \mathcal{X}_\pi(G)$, then $N \subseteq \ker(\chi)$.

(b) If G/M is a π' -group, then restriction defines a bijection between $\mathcal{X}_\pi(G)$ and the set of π -special G -invariant characters of M .

(c) If $J \subseteq G$ has π' -index, then restriction defines a one-to-one map between $\mathcal{X}_\pi(G)$ and $\mathcal{X}_\pi(J)$.

(d) If K is a π -complement of G , then

$$|\mathcal{X}_\pi(G)| = |\text{Irr}(\mathbf{N}_G(K)/K)|.$$

(e) If G is p -solvable and $\text{IBr}_{p'}(G)$ is the set of irreducible p -Brauer characters of G of p' -degree, then restriction to p -regular elements defines a bijection

$$\mathcal{X}_{p'}(G) \rightarrow \text{IBr}_{p'}(G).$$

(f) Suppose that G is p -solvable and let $N \triangleleft G$. Suppose that $\chi \in \text{Irr}(G)$ and $\theta \in \text{Irr}(N)$ are p' -specials. Then $\chi^0 \in \text{IBr}(G)$ lies over $\theta^0 \in \text{IBr}(N)$ iff χ lies over θ .

Proof. Part (a) follows from Corollary (4.2) of [2]. Part (b) is Proposition (4.3) of [2]. Part (c) follows from Proposition (6.1) of [2]. Part (d) is deeper and can be found as Corollary (1.16) of [12]. Part (e) easily follows from Lemma (5.4) and Corollary (10.3) of [3]. Now, we prove part (f). If θ lies over χ , it is clear that χ^0 lies under θ^0 . Conversely, suppose that χ^0 lies over θ^0 . Let H be a p -complement of G . Then $H \subseteq G^0$, and we have that $\chi_H \in \text{Irr}(H)$ lies over $\theta_{H \cap N} \in \text{Irr}(N \cap H)$. Therefore, some irreducible constituent η of χ_N must lie over $\theta_{N \cap H}$. However, η is p' -special. By part (c), we have that $\eta_{N \cap H} = \theta_{N \cap H}$, and therefore $\eta = \theta$, as desired. □

Another remarkable property of special characters is that if $\alpha \in \text{Irr}(G)$ is π -special and $\beta \in \text{Irr}(G)$ is π' -special, then $\alpha\beta \in \text{Irr}(G)$ (Proposition (7.1) of [2]). In fact, this factorization is unique. The irreducible characters of G of the form $\alpha\beta$ are called π -**factorable**. Sometimes if $\chi \in \text{Irr}(G)$ is π -factorable, we write

$$\chi = \chi_\pi \chi_{\pi'},$$

where χ_π is π -special and $\chi_{\pi'}$ is π' -special. Since normal irreducible constituents of π -specials are π -special, the same happens with π -factorable characters.

(2.2) Theorem. *Suppose that G is π -separable, and let $N, M \triangleleft G$. If the irreducible constituents of χ_N and of χ_M are π -factorable, then the irreducible constituents of χ_{NM} are also π -factorable.*

Proof. We argue by induction, first on $|G|$, and then on $|G : N| + |G : M|$. If δ is an irreducible constituent of χ_{NM} , we have that the irreducible constituents of δ_M and of δ_N are π -factorable. So by induction, we may assume that $G = NM$, and we should prove that χ is π -factorable. Now suppose that $N \subseteq K \triangleleft G$ is a proper normal subgroup of G , and let ϵ lie under χ . Then the irreducible constituents of ϵ_N are irreducible constituents of χ_N , and therefore they are π -factorable. On the other hand, the irreducible constituents of $\epsilon_{K \cap M}$ lie under irreducible constituents of χ_M , and therefore, they are also π -factorable. Arguing by induction, we obtain that ϵ is π -factorable, and again by induction, that χ is π -factorable. So we may assume that N and M are maximal normal subgroups of G . Hence, each G/N and G/M is a π -group or a π' -group. Now, let $\theta \in \text{Irr}(N)$ be under χ , and let

$\eta \in \text{Irr}(N \cap M)$ be under θ . Hence η lies under χ , and therefore there is some $\varphi \in \text{Irr}(M)$ over η and under χ . The theorem now follows from Corollary (2.8) of [3]. \square

A **pair** (G, χ) is just a group G with an irreducible character $\chi \in \text{Irr}(G)$. Also, if $H \subseteq G$, $\alpha \in \text{Irr}(H)$ and $g \in G$, then $(H, \alpha)^g = (H^g, \alpha^g)$, where $\alpha^g \in \text{Irr}(H^g)$ is the character satisfying

$$\alpha^g(h^g) = \alpha(h)$$

for $h \in H$.

(2.3) Corollary. *Suppose that G is π -separable and let $\chi \in \text{Irr}(G)$. Then there is, up to G -conjugacy, a unique pair (N, θ) which is maximal subject to N being a normal subgroup of G and θ being a π -factorable irreducible character of N lying under χ .*

Proof. If (M, φ) is another π -factorable maximal pair under (G, χ) , then the irreducible constituents of χ_{NM} are π -factorable by Theorem (2.2). By maximality, we have that $N = M$ and therefore, by Clifford's theorem, $(N, \theta)^g = (M, \varphi)$. \square

(2.4) Corollary. *Suppose that G is π -separable, and let $\chi \in \text{Irr}(G)$. Let (N, θ) be a maximal normal π -factorable pair under χ . If θ is G -invariant, then $G = N$ and $\chi = \theta$.*

Proof. Suppose that N is proper in G . Let M/N be a chief factor of G . Also, let $\eta \in \text{Irr}(M)$ lie under χ and over θ . Now, by the uniqueness of the decomposition of a character as a π -special times a π' -special, it follows that the π -part and the π' -part of θ are G -invariant. Since M/N is a π -group or a π' -group, it follows by Proposition (2.7) of [3] that η is π -factorable. This contradicts the maximality of (N, θ) . \square

Given a π -separable group G and $\chi \in \text{Irr}(G)$, we define a canonical (up to G -conjugacy) pair (W, γ) satisfying $W \subseteq G$, $\gamma \in \text{Irr}(W)$ is π -factorable, and $\gamma^G = \chi$. We do this inductively on $|G|$. If χ is π -factorable, then we let $(W, \gamma) = (G, \chi)$. If χ is not π -factorable, then let (N, θ) be a maximal π -factorable normal pair under χ . Let $T = I_G(\theta)$ be the stabilizer of θ in G and let $\psi \in \text{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ . By Corollary (2.4), we have that T is proper in G . Then, by induction, we have defined a pair (W, γ) for (T, ψ) . We call every G -conjugate of every pair (W, γ) arising this way a **nucleus** via normal pairs for χ . By definition and Theorem (2.3), notice that if (W, γ) and (U, η) are nuclei for χ , then there is $g \in G$ such that

$$(W, \gamma)^g = (U, \eta).$$

(The analogous construction of the nucleus of a character via subnormal pairs is done in [3]. Our normal construction is, however, essential for our purposes here.) Since there will be no other nuclei in this paper, we will simply refer to (W, γ) as a nucleus for χ .

3. VERTICES FOR CHARACTERS

Suppose now that G is p -solvable, and let $\chi \in \text{Irr}(G)$. Let Q be a p -subgroup of G and let $\delta \in \text{Irr}(Q)$. We say that (Q, δ) is a **vertex** of χ if there is a nucleus (W, γ) for χ such that $Q \in \text{Syl}_p(W)$ and $\beta_Q = \delta$, where $\beta = \gamma_p$ is the p -part of γ .

Suppose now that (E, η) is another vertex of χ . We claim that there is $x \in G$ such that

$$(Q, \delta)^x = (E, \eta).$$

We know that there is a $g \in G$ such that E is a Sylow p -subgroup of W^g and $\eta = ((\gamma^g)_p)_E$. Now, Q^g is a Sylow p -subgroup of W^g , and therefore $Q^g = E^{w^g}$ for some $w \in W$. Hence $Q^{w^{-1}g} = E$. Also,

$$\begin{aligned} \delta^{w^{-1}g} &= ((\gamma_p)_Q)^{w^{-1}g} = ((\gamma_p)^{w^{-1}g})_{Q^{w^{-1}g}} \\ &= ((\gamma_p)^{w^{-1}g})_E = ((\gamma_p)^g)_E = ((\gamma^g)_p)_E = \eta, \end{aligned}$$

as desired.

We denote by $\text{Irr}(G|Q, \delta)$ the set of irreducible characters χ with vertex (Q, δ) .

The following easily follows from the definition of the defect of a character.

(3.1) Lemma. *Suppose that $\psi^G = \chi \in \text{Irr}(G)$. Then $d(\psi) = d(\chi)$.*

Proof. Suppose that $\psi \in \text{Irr}(H)$. Then

$$p^{d(\chi)} = |G|_p/\chi(1)_p = |G|_p/|G : H|_p\psi(1)_p = |H|_p/\psi(1)_p = p^{d(\psi)},$$

as desired. □

Next is Theorem A(b).

(3.2) Theorem. *Let G be p -solvable and suppose that $\chi \in \text{Irr}(G)$ has vertex (Q, δ) . Then $d(\chi) = d(\delta)$.*

Proof. Let (W, γ) be a nucleus for χ such that $Q \in \text{Syl}_p(W)$ and $(\gamma_p)_Q = \delta$. Hence by Lemma (3.1) we have

$$p^{d(\chi)} = p^{d(\gamma)} = p^{d(\gamma_p)} = |W|_p/\gamma_p(1) = |Q|/\delta(1) = p^{d(\delta)},$$

as desired. □

Next is Theorem A(c).

(3.3) Theorem. *Let G be p -solvable and suppose that $\chi \in \text{Irr}(G)$ has vertex (Q, δ) . Let $g \in G$. If no G -conjugate of g_p lies in Q , then $\chi(g) = 0$.*

Proof. Let (W, γ) be a nucleus for χ such that $Q \in \text{Syl}_p(W)$ and $(\gamma_p)_Q = \delta$. By hypothesis, no G -conjugate of g lies in W . Since $\chi = \gamma^G$, we have $\chi(g) = 0$, by the induction formula. □

In order to understand a bit more about the vertices (Q, δ) , we proceed to prove Theorem A(g). For that we need a few easy lemmas.

(3.4) Lemma. *Let $N \triangleleft G$, let $\theta \in \text{Irr}(N)$ and let $\chi \in \text{Irr}(G|\theta)$. Suppose that $d(\chi) = d(\theta)$. If $N \subseteq M \triangleleft G$ and $\eta \in \text{Irr}(M)$ lies under χ , then $d(\eta) = d(\chi)$.*

Proof. Since η lies over some G -conjugate of θ , it is no loss to assume that η lies over θ . Now, $\chi(1)_p/\eta(1)_p$ divides $|G : M|_p$ and $\eta(1)_p/\theta(1)_p$ divides $|M : N|_p$. Since

$$(\chi(1)_p/\eta(1)_p)(\eta(1)_p/\theta(1)_p) = \chi(1)_p/\theta(1)_p = |G : N|_p = |G : M|_p|M : N|_p,$$

the proof of the lemma easily follows. □

(3.5) Lemma. *Let $M \triangleleft G$. Suppose that $\gamma = \alpha\beta \in \text{Irr}(M)$, where α is p' -special, and β is p -special and invariant under some Sylow p -subgroup P of G . Suppose that $\chi \in \text{Irr}(G)$ lies over γ with $d(\chi) = d(\gamma)$. Then there is a nucleus (W, ρ) of χ where W/M is a p' -group and $(\rho_p)_M = \beta$. In particular, if $Q \in \text{Syl}_p(M)$, then (Q, β_Q) is a vertex for χ .*

Proof. We argue by induction on $|G|$. Let (N, θ) be a maximal normal pair over (M, γ) . Also, let $Q = P \cap N \in \text{Syl}_p(N)$. We have that β is Q -invariant. Therefore, so is $\beta_{Q \cap M}$. By Lemma (3.4), $d(\gamma) = d(\theta)$. This implies that

$$\theta_p(1)/\beta(1) = |N : M|_p = |Q : Q \cap M|.$$

Now, θ_p lies over β , and therefore $(\theta_p)_Q \in \text{Irr}(Q)$ lies over $\beta_{Q \cap M}$. By degrees, we see that $\beta_{Q \cap M}$ induces $(\theta_p)_Q$. Since $\beta_{Q \cap M}$ is Q -invariant, this implies that $Q \cap M = Q$, by Problem (6.1) of [4], for instance. Hence we have that N/M is a p' -group.

Now, let ψ be the Clifford correspondent of χ over θ . By Lemma (3.1), we have $d(\psi) = d(\gamma)$. If $T < G$, by induction, we are done. If $T = G$, by Corollary (2.4), $N = G$, and in this case the proof of the lemma is clear. \square

The following is a restatement of Theorem A(g).

(3.6) Theorem. *Let G be a p -solvable group, and fix a pair (Q, δ) with $d(\delta) = d$. Then the set $\text{Irr}(\mathbf{N}_G(Q, \delta) | Q, \delta)$ consists exactly of those $\chi \in \text{Irr}(\mathbf{N}_G(Q, \delta))$ lying over δ such that $d(\chi) = d = d(\delta)$.*

Proof. We may assume that $\mathbf{N}_G(Q, \delta) = G$. So we have that $Q \triangleleft G$ and δ is G -invariant. Let $\chi \in \text{Irr}(G | Q, \delta)$. Then there is a nucleus (W, γ) such that $Q \in \text{Syl}_p(W)$ and $(\gamma_p)_Q = \delta$. Since $Q \triangleleft G$, we have that $Q \subseteq \ker(\gamma_{p'})$, and therefore γ lies over δ . Hence χ lies over δ . By Theorem (3.2), we know that $d(\chi) = d$.

Conversely, suppose that $\chi \in \text{Irr}(G | \delta)$ has defect d . By Lemma (3.5), we deduce that (Q, δ) is a vertex of χ , and the theorem is proven. \square

4. CHARACTERS AND RADICAL SUBGROUPS

This is one of our key results.

(4.1) Theorem. *Suppose that G is p -solvable and let $N \triangleleft G$. Suppose that $\theta \in \mathcal{X}_{p'}(N)$ is Q -invariant, where Q is a p -subgroup of G containing a Sylow p -subgroup of N . Suppose that $Q \subseteq R$ is a p -subgroup of G such that $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$. Then θ is R -invariant.*

Proof. We argue by induction on $|R : Q|$. If $Q = R$, there is nothing to prove. So we assume that $Q < R$. Since $Q \subseteq R \subseteq NR$, $N \triangleleft NR$ and $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$, it is no loss to assume that $NR = G$.

Write $D = Q \cap N \in \text{Syl}_p(N)$. We have that $D \triangleleft \mathbf{N}_N(Q)$. Let $E = Q^R \triangleleft R$. Since $\mathbf{N}_N(Q)$ normalizes Q and R , it follows that $\mathbf{N}_N(Q)$ normalizes E . Since $E < R$, we have that $|E : Q| < |R : Q|$, and by induction, we conclude that θ is E -invariant. We claim that

$$\mathbf{N}_N(E) = \mathbf{N}_N(Q).$$

We already have seen that $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(E)$. Now, since $D = E \cap N$, we have that $\mathbf{N}_G(E) \subseteq \mathbf{N}_G(D)$. Hence $D \triangleleft \mathbf{N}_G(E)$. Now, $D = N \cap Q \subseteq \mathbf{N}_N(Q) \subseteq \mathbf{N}_N(E) \subseteq N$, and therefore $\mathbf{N}_N(E)/D$ is a normal p' -subgroup of $\mathbf{N}_G(E)/D$. Since E/D is a

normal p -subgroup of $\mathbf{N}_G(E)/D$, we conclude that $[\mathbf{N}_N(E), E] \subseteq D \subseteq Q$. Hence $[\mathbf{N}_N(E), Q] \subseteq Q$, and we deduce that $\mathbf{N}_N(E) \subseteq \mathbf{N}_N(Q)$. This proves the claim.

Now, we have that $\mathbf{N}_N(E) = \mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$. If $|R : E| < |R : Q|$, since θ is E -invariant, by induction we will have that θ is R -invariant, and the theorem would follow in this case. So we may assume that $Q \triangleleft R$.

Now, let $M = NQ \triangleleft NR = G$. Notice that $Q \in \text{Syl}_p(M)$ and that $R \cap M = Q$. Thus $R \in \text{Syl}_p(G)$. We claim that

$$\mathbf{N}_G(R) = \mathbf{N}_G(Q).$$

Since $R \cap M = Q$, we have that $\mathbf{N}_G(R) \subseteq \mathbf{N}_G(Q)$. Hence, $\mathbf{N}_N(R) \subseteq \mathbf{N}_N(Q)$, and by hypothesis we have that $\mathbf{N}_N(R) = \mathbf{N}_N(Q)$. Now, since $G = NR$, then $\mathbf{N}_G(R) = R\mathbf{N}_N(R) = R\mathbf{N}_N(Q)$. Also, since $Q \triangleleft R$, we have that $R \subseteq \mathbf{N}_G(Q)$ and again $\mathbf{N}_G(Q) = R\mathbf{N}_N(Q) = \mathbf{N}_G(R)$, as claimed.

Now, we have

$$\mathbf{N}_G(R)/R \cong \mathbf{N}_M(Q)/Q.$$

By Theorem (2.1.d), we have

$$|\mathcal{X}_{p'}(G)| = |\text{Irr}(\mathbf{N}_G(R)/R)| = |\text{Irr}(\mathbf{N}_M(Q)/Q)| = |\mathcal{X}_{p'}(M)|.$$

Now, by Theorem (2.1.b), we have

$$|\mathcal{X}_{p'}(G)| = |\mathcal{X}_{p',R}(M)|,$$

where $\mathcal{X}_{p',R}(M)$ is the set of p' -special characters of M which are R -invariant. Hence, we conclude that all p' -special characters of M are R -invariant. Now, since θ is Q -invariant, it follows that θ is M -invariant. Since $N \triangleleft M$ and M/N is a p -group, it follows by Theorem (2.1.b) that there exists $\eta \in \mathcal{X}_{p'}(M)$ extending θ . Since η is p' -special, we conclude that η is R -invariant. Hence, θ is R -invariant, as desired. □

Next is Theorem A(d) of the Introduction.

(4.2) Theorem. *Let G be p -solvable and suppose that (Q, δ) is a vertex of $\chi \in \text{Irr}(G)$. Then*

$$\mathbf{O}_p(\mathbf{N}_G(Q, \delta)) = Q.$$

Proof. We argue by induction on $|G|$. If χ is p -factorable, then (G, χ) is a nucleus of χ , $Q \in \text{Syl}_p(G)$, and therefore $\mathbf{O}_p(\mathbf{N}_G(Q, \delta)) = Q$, in this case.

Now, we may find a nucleus (W, γ) for χ and a normal p -factorable pair (N, θ) such that if $T = I_G(\theta)$, we have that $N \subseteq W \subseteq T$, $Q \in \text{Syl}_p(W)$, γ lies over θ , $(\gamma_p)_Q = \delta$ and (W, γ) is a nucleus for $\psi = \gamma^T$ (the Clifford correspondent of χ over θ). Also, notice that $Q \cap N \in \text{Syl}_p(N)$, $Q \in \text{Syl}_p(QN)$, and $(\gamma_p)_{NQ}$ is the unique p -special extension of δ to NQ (by Theorem (2.1.c)). Since $\psi \in \text{Irr}(T|Q, \delta)$, by induction, we have that

$$\mathbf{O}_p(\mathbf{N}_T(Q, \delta)) = Q.$$

Let $R = \mathbf{O}_p(\mathbf{N}_G(Q, \delta))$, so that $Q \subseteq R$.

Now, notice that $\mathbf{N}_N(Q, \delta) = \mathbf{N}_N(Q)$. This is because δ extends to NQ , and then every element in NQ normalizing Q fixes δ . Now, $\mathbf{N}_N(Q) \subseteq \mathbf{N}_G(Q, \delta)$, and since $R \triangleleft \mathbf{N}_G(Q, \delta)$, we have that $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$. By Theorem (4.1), $\theta_{p'}$ is R -invariant. Now, if $\beta = \gamma_p$, we already know that β_{NQ} is the unique p -special extension of δ to NQ , and that $\beta_N = e\theta_p$ for some integer p . If $r \in R$, we have

that r normalizes Q and fixes δ . Therefore, r normalizes NQ . Then $(\beta_{NQ})^r$ is another p -special extension of δ to NQ . By uniqueness, β_{NQ} is r -invariant. Since $(\beta_{NQ})_N = e\theta_p$, we deduce that θ_p is r -invariant. Hence, we have seen that R fixes θ_p and $\theta_{p'}$. Therefore $r \in T$, and we conclude that $R \subseteq T$. In particular, $R \triangleleft \mathbf{N}_T(Q, \delta)$. Since $Q = \mathbf{O}_p(\mathbf{N}_T(Q, \delta))$, we deduce that $Q = R$, as desired. \square

Now, we draw some consequences of Theorem (4.1). We remind the reader that if Q is any p -subgroup of G , the radical closure of Q in G is the last term R of the chain $P_{i+1} = \mathbf{O}_p(\mathbf{N}_G(P_i))$ starting with $Q = P_0$. Since the radical closure of Q in G is uniquely determined by Q and G , it follows that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(R)$.

(4.3) Corollary. *Suppose that G is p -solvable and let $N \triangleleft G$. Suppose that $\theta \in \mathcal{X}_{p'}(N)$ is Q -invariant, where Q is a p -subgroup of G containing a Sylow p -subgroup of N . If R is the p -radical closure of Q in G , then θ is R -invariant.*

Proof. We have that $Q \subseteq R$ and $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(R)$. Hence, $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$, and Theorem (4.1) applies. \square

Next is Theorem B of the introduction.

(4.4) Corollary. *Suppose that G is p -solvable and let $N \triangleleft G$. Suppose that $\theta \in \text{IBr}(N)$ has p' -degree and is Q -invariant, where Q is a p -subgroup of G containing a Sylow p -subgroup of N . If R is the p -radical closure of Q in G , then θ is R -invariant.*

Proof. By Theorem (2.1.e), we can find a unique $\eta \in \mathcal{X}_{p'}(N)$ such that $\eta^0 = \theta$, where η^0 is the restriction of η to the p -regular elements of N . By uniqueness in Theorem (2.1.e), we have that η is Q -invariant. Hence, by Corollary (4.3), we have that η is R -invariant. Hence, $\theta = \eta^0$ is also R -invariant. \square

We find the following result a little surprising.

(4.5) Theorem. *Suppose that G is p -solvable and let $\chi \in \text{Irr}(G)$. Suppose that (N, θ) is a normal maximal p -factorable pair under χ . If $Q \in \text{Syl}_p(N)$, then $Q = \mathbf{O}_p(\mathbf{N}_G(Q))$.*

Proof. Write $\theta = \theta_p\theta_{p'}$, where θ_p is p -special and $\theta_{p'}$ is p' -special. Let $R = \mathbf{O}_p(\mathbf{N}_G(Q))$. By Corollary (4.3), we know that $\theta_{p'}$ is R -invariant. By the Frattini argument, we have that $G = NN_G(Q)$. Then we have that $\mathbf{N}_N(Q)/Q$ is a p' -subgroup of $\mathbf{N}_G(Q)/Q$. Since R/Q is a p -subgroup of $\mathbf{N}_G(Q)/Q$, we conclude that $R \cap \mathbf{N}_N(Q) = Q$. Also, we have that $RN \triangleleft G$. Now, RN/N is a p -group, and by Proposition (2.7) of [3], all irreducible constituents of θ^{RN} are p -factorable. By maximality, we conclude that $NR = N$, and thus $R = Q$, as required. \square

(4.6) Corollary. *Suppose that G is p -solvable and let $\chi \in \text{Irr}(G)$ with vertex (Q, δ) . Then there is $N \triangleleft G$ such that $Q \cap N$ is p -radical in G .*

Proof. Let (W, γ) be a nucleus for χ such that $Q \in \text{Syl}_p(W)$ and (W, γ) lies over (N, θ) , a normal maximal factorable pair under χ . Then $Q \cap N \in \text{Syl}_p(N)$, and the result follows from Theorem (4.5). \square

From Corollary (4.6), it easily follows that vertices contain $\mathbf{O}_p(G)$, although this is immediate from their definition.

Suppose that G is p -solvable, let $\chi \in \text{Irr}(G)$ and let (Q, δ) be a vertex of χ . In view of Theorems (4.1) and (4.5), it is natural to ask if Q needs to be p -radical in G . The answer is no.

(4.7) Example. Set $p = 2$. Consider $H = NP$, where $N \triangleleft H$ is cyclic of order 5 and P is cyclic of order 4. For instance, suppose that $N = \langle x \rangle$ and $P = \langle y \rangle$, where $x = (12345)$ and $y = (2354)$. Hence $x^y = x^2$ and $H \subseteq S_5$. Also, $y^2 = (25)(34)$. Now, take H acting on $V = C_2 \times C_2 \times C_2 \times C_2 \times C_2$, and let $G = VH$ be the semidirect product. Now, let $\theta = 1 \times \lambda \times 1 \times 1 \times \lambda \in \text{Irr}(V)$, where $\lambda \neq 1$. Hence, the inertia group of θ in G is $T = V \langle y^2 \rangle$. If $\mu \in \text{Irr}(T)$ lies over θ , then $\mu^G = \chi \in \text{Irr}(G)$, (T, μ) is a normal nucleus of χ , and T is not p -radical.

5. PROOF OF THEOREM A (e) AND (f)

We start with the following result.

(5.1) Theorem. *Let $\chi \in \text{Irr}(G)$ and let (W, γ) be a nucleus for χ . Let $Q \in \text{Syl}_p(W)$ and assume that $Q \subseteq R$ is a p -subgroup of G such that $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R)$. Write $\gamma = \alpha\beta$, where α is p' -special and β is p -special. Then $(\beta_Q)^R \in \text{Irr}(R)$.*

Proof. We argue by induction on $|G|$. Write $\delta = \beta_Q$. By Theorem (2.1.c), we know that $\delta \in \text{Irr}(Q)$. Let (N, θ) be a maximal p -factorable pair such that (W, γ) is a nucleus for ψ , the Clifford correspondent of χ over θ , and γ lies over θ . Write $T = I_G(\theta)$. Also, write $\theta = \theta_p \theta_{p'}$, where θ_p is p -special and $\theta_{p'}$ is p' -special. In particular, we have that $\theta_{p'}$ is Q -invariant. If $T = G$, then by Corollary (2.4), we have that $W = G$. Hence, $Q = R$, and in this case we already know that δ is irreducible. So we may assume that $T < G$.

We claim that $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R \cap T)$. Since $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R)$, we have that $\mathbf{N}_W(Q)$ normalizes R . Since $\mathbf{N}_W(Q) \subseteq W \subseteq T$, it follows that $\mathbf{N}_W(Q)$ normalizes $R \cap T$. Hence $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R \cap T)$, and the claim follows.

Now, by induction, we have that $\delta^{R \cap T}$ is irreducible. We claim that this character lies over the irreducible character $\mu = (\theta_p)_{N \cap Q}$. First of all, $N \cap Q \in \text{Syl}_p(N)$, because $Q \in \text{Syl}_p(W)$ and $N \triangleleft W$. Hence μ is irreducible by Theorem (2.1.c). Now, since γ lies over θ , by the uniqueness in the factorization of p -factorable characters, it follows that θ_p lies under β . Hence, μ lies under β , and therefore under $\beta_Q = \delta$. Hence, μ lies under $\delta^{R \cap T}$, as claimed.

Now, observe that $N \cap Q = N \cap R \triangleleft R$. We claim that $R \cap T = I_R(\mu)$ (the stabilizer of μ in R). If $x \in R \cap T$, we have that x fixes θ . By the uniqueness of the decomposition of θ , we also have that x fixes θ_p . Since $x \in R \subseteq \mathbf{N}_G(Q \cap N)$, it follows that x fixes $(\theta_p)_{Q \cap N} = \mu$. On the other hand, if $x \in R$ fixes μ , by the uniqueness in Theorem (2.1.c), we have that x fixes θ_p . Now, since $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R)$, we have that $\mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R)$. Since $\theta_{p'}$ is Q -invariant, by Theorem (4.1) we have that $\theta_{p'}$ is R -invariant. Hence x fixes $\theta_p \theta_{p'} = \theta$ and thus $x \in R \cap T$. This proves the claim.

Finally, by the Clifford correspondence (Theorem (6.11) of [4]) we have that

$$\delta^R = (\delta^{R \cap T})^R$$

is irreducible, as desired. □

The following is Theorem A(f) of the introduction.

(5.2) Corollary. *Let G be p -solvable and suppose that (Q, δ) is a vertex of $\chi \in \text{Irr}(G)$. Let R be the radical closure of Q in G . Then $\eta = \delta^R \in \text{Irr}(R)$. In particular, $d(\eta) = d(\chi)$.*

Proof. By definition, there is a nucleus (W, γ) of χ such that $Q \in \text{Syl}_p(W)$, and $(\gamma_p)_Q = \delta$. Since R is the radical closure of Q in G , we know that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(R)$. In particular, $\mathbf{N}_W(Q) \subseteq \mathbf{N}_W(R)$. By Theorem (5.1), we conclude that $\eta = \delta^R \in \text{Irr}(R)$. By Lemma (3.1) and Theorem (3.2), we have that $d(\chi) = d(\delta) = d(\eta)$, as desired. \square

Next, we work towards proving Theorem A(e). We will use the following (which is the essence of the main results in [9]).

(5.3) Theorem. *Suppose that G is p -solvable, let $N = \mathbf{O}_{p'}(G)$ and let $\chi \in \text{Irr}(G)$. Suppose that $\chi = (\alpha\beta)^G$, where $\alpha \in \text{Irr}(U)$ has p' -degree and $\beta \in \text{Irr}(U)$ is p -special. Suppose that χ_N is homogeneous. If $Q \in \text{Syl}_p(U)$, then*

$$\mathbf{C}_{G/N}(Q) = \mathbf{Z}(Q)N/N.$$

Proof. This is Theorem (3.5) of [9]. \square

(5.4) Theorem. *Let G be p -solvable and let $\chi \in \text{Irr}(G)$. Suppose that $\chi = \gamma^G$ for some $\gamma \in \text{Irr}(W)$, and suppose that $\gamma = \alpha\beta \in \text{Irr}(W)$, where α has p' -degree and β is p -special. Let $Q \in \text{Syl}_p(W)$. Suppose that $Q \subseteq R$ is a p -group such that whenever R normalizes a p' -subgroup N of G , then $\mathbf{C}_N(R) = \mathbf{C}_N(Q)$. Then there is a defect group D of the p -block of χ such that $R \subseteq D$ and $\mathbf{C}_D(Q) \subseteq Q$.*

Proof. We argue by induction on $|G : \mathbf{O}_{p'}(G)|$ and $|G : W|$. Let $N = \mathbf{O}_{p'}(G)$. By Theorem (2.1.a), we have that $N \cap W \subseteq \ker(\beta)$. Hence, it easily follows that β has a p -special extension $\hat{\beta} \in \text{Irr}(WN/N)$. Then

$$\gamma^{WN} = (\alpha\hat{\beta}_W)^{WN} = \alpha^{WN}\hat{\beta};$$

notice that here $\alpha^{WN} \in \text{Irr}(WN)$ has p' -degree. Also, $Q \in \text{Syl}_p(WN)$. If $WN > W$, then by induction, we are done. So we may assume that $N \subseteq W$. Now, let $\theta \in \text{Irr}(N)$ be an irreducible constituent of γ_N and let T be the stabilizer of θ in G . Since $N \subseteq \ker(\beta)$, we have that θ is an irreducible constituent of α_N . Since α has p' -degree, we may assume that θ is Q -invariant (since Q permutes the irreducible constituents of α_N). Now, R normalizes N , and by hypothesis, we have that $\mathbf{C}_N(R) = \mathbf{C}_N(Q)$. By Theorem (4.1), we conclude that θ is R -invariant. Thus $R \subseteq T$. Now, $Q \subseteq T \cap W$, and then $T \cap W$ has p' -index in W . Therefore $\beta_{T \cap W}$ is irreducible and p -special by Theorem (2.1.c). Now, let η be the Clifford correspondent of α over θ , which has p' -degree. We have

$$(\eta\beta_{T \cap W})^W = \eta^W\beta = \alpha\beta$$

and therefore

$$(\eta\beta_{T \cap W})^G = \chi.$$

In particular, $(\beta_{T \cap W}\eta)^T = \mu$ is the Clifford correspondent of χ over θ . If $T < G$, by induction, there is a defect group D of the block of μ satisfying $R \subseteq D$ and $\mathbf{C}_D(Q) \subseteq Q$. Now, by Theorem (9.14) of [8], D is a defect group of the block of χ , and we are done in this case. So we may assume that $T = G$, and that the defect groups of the block of χ are the Sylow p -subgroups of G (by Theorem (10.20) of [8]). Now, by Theorem (5.3), we know that

$$\mathbf{C}_{G/N}(Q) = \mathbf{Z}(Q)N/N.$$

In particular, $\mathbf{C}_G(Q) \subseteq W$. Now, let $Q \subseteq R \subseteq D \in \text{Syl}_p(G)$. Then $\mathbf{C}_D(Q) \subseteq W$. Since $Q\mathbf{C}_D(Q)$ is a p -subgroup of W containing $Q \in \text{Syl}_p(W)$, we conclude that $\mathbf{C}_D(Q) \subseteq Q$, as desired. \square

This is Theorem A(e) of the Introduction.

(5.5) Theorem. *Let G be p -solvable and suppose that (Q, δ) is a vertex of χ . Let R be the radical closure of Q in G . Then there is a defect group D of the p -block of χ such that $Q \subseteq R \subseteq D$ and*

$$\mathbf{C}_D(Q) = \mathbf{Z}(Q).$$

Proof. Let (W, γ) be a nucleus for χ with $Q \in \text{Syl}_p(W)$. Now, since $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(R)$, it follows that if R normalizes any p' -subgroup N of G , then $\mathbf{C}_N(Q) = \mathbf{N}_N(Q) \subseteq \mathbf{N}_N(R) = \mathbf{C}_N(R)$. Now, Theorem (5.4) applies. \square

6. LIFTING BRAUER CHARACTERS

We need the following.

(6.1) Lemma. *Suppose that G is p -solvable, and let $\varphi \in \text{IBr}(G)$.*

(a) Suppose that $G = MN$, where $M, N \triangleleft G$. If the irreducible constituents of φ_N and of φ_M have p' -degree, then φ has p' -degree.

(b) There exists a unique maximal normal subgroup E of G such that the irreducible constituents of φ_E have p' -degree. Furthermore, if $\varphi_E = e\theta$ for some $\theta \in \text{IBr}(E)$, then φ has p' -degree.

Proof. (a) Arguing as in Theorem (2.2), we may easily assume that G/N and G/M are simple groups. If, for instance, G/N is a p' -group, then φ has p' -degree by Theorem (8.30) of [8]. So we may assume that $G/N \cap M$ is a p -group. Now, let $\delta \in \text{IBr}(N \cap M)$ lie under φ . If $\theta \in \text{IBr}(N)$ lies under φ and over δ , we have that $\theta_{N \cap M} = \delta$. Hence, δ is N -invariant. By the same argument, δ is M -invariant. Hence, δ is G -invariant. Also, since θ has p' -degree, so does δ . By Green's theorem (Theorem (8.11) of [8]), $\varphi_{N \cap M} = \delta$, and the proof of the first part follows.

(b) It is clear that such a normal subgroup E exists, by part (a). Also, by Theorem (8.30) of [8], we have that $\mathbf{O}_{p'}(G/E) = 1$. So if $E < G$, we may find a normal subgroup K of G with K/E a p -group. By Green's theorem, there is a unique irreducible Brauer character τ of K which lies over θ . In fact, τ extends θ , and therefore also has p' -degree. By uniqueness, we have that $\varphi_K = e\tau$. This contradicts the maximality of E . \square

For our convenience, we will use the following characterization of the vertex of an irreducible Brauer character of a p -solvable group. If G is p -solvable and $\varphi \in \text{IBr}(G)$, by Huppert's theorem (Theorem (10.11) of [8]), we know that there is some $\alpha \in \text{IBr}(U)$ of p' -degree, such that $\alpha^G = \varphi$. In fact, if $\beta \in \text{IBr}(V)$ is some other irreducible Brauer character of p' -degree, it is a fact that the Sylow p -subgroups of U and V are G -conjugate. (Apply Theorem B of [5] with $\pi = p'$.) This conjugacy class of p -subgroups of G uniquely determined by φ up to G -conjugacy are the vertices of φ . Recall that we denote by $\text{IBr}(G|Q)$ the set of irreducible Brauer characters of G with vertex Q .

As a consequence of the definition, notice that if some Brauer character μ induces φ , then every vertex of μ is a vertex of φ .

Fix a p -subgroup Q of a p -solvable group G and suppose that $\varphi \in \text{IBr}(G|Q)$. Let $N \triangleleft G$, and let $\theta \in \text{IBr}(N)$ lie under φ . We say that θ is Q -good for φ if the Clifford correspondent of φ over θ has vertex Q .

(6.2) Lemma. *Suppose that G is p -solvable, let Q be a p -subgroup of G and let $\varphi \in \text{IBr}(G|Q)$. Suppose that $N \triangleleft G$. Then there is an irreducible constituent θ of φ_N which is Q -good, and any two of them are $\mathbf{N}_G(Q)$ -conjugate.*

Proof. Let $\eta \in \text{IBr}(N)$ be any irreducible constituent of φ_N . Let I be the inertia group of η in G and let $\mu \in \text{IBr}(I)$ be the Clifford correspondent of φ over η . Let P be a vertex of μ . Hence, P is a vertex of φ , and we deduce that $P^g = Q$ for some $g \in G$. Let $\theta = \eta^g$. Then $T = I^g$ is the stabilizer of θ in G , and $\tau = \mu^g$, which is the Clifford correspondent of φ over θ , has vertex $P^g = Q$. Suppose now that θ^x is also Q -good for φ for some $x \in G$. Since τ^x is the Clifford correspondent of φ over θ^x , we have that τ^x has vertex Q . However, since τ has vertex Q , it easily follows that τ^x has vertex Q^x . Hence, $Q^x = Q^{t^x}$ for some $t \in T$. Thus $t^{-1}x \in \mathbf{N}_G(Q)$ and $\theta^{t^{-1}x} = \theta^x$, as desired. □

The next result, which is Theorem A(a) of the Introduction, proves that the irreducible characters of G with vertex $(Q, 1_Q)$ provide a canonical lifting of the irreducible Brauer characters of a finite p -solvable group G with vertex Q . M. Isaacs constructed in [3] another (apparently) canonical set of liftings $B_{p'}(G) \subseteq \text{Irr}(G)$ of $\text{IBr}(G)$. We do know that for p odd, both liftings coincide. However, at the time of this writing, we do not know what happens in general. Perhaps surprisingly, we need the Isaacs $B_{p'}$ -lifting in order to prove ours. The only facts that we need to know about $B_{p'}$ -characters are that normal irreducible constituents of $B_{p'}$ -characters are $B_{p'}$ -characters (something that we do not know about our lifting) and that the $B_{p'}$ -characters of p' -degree are exactly the p' -special characters.

(6.3) Theorem. *Let Q be a p -subgroup of G . Then restriction to p -regular elements defines a natural bijection $\text{Irr}(G|Q, 1_Q) \rightarrow \text{IBr}(G|Q)$.*

Proof. We argue by induction on $|G|$. Let $\chi \in \text{Irr}(G|Q, 1_Q)$. First, we prove that $\chi^0 \in \text{IBr}(G|Q)$. By the definition of the vertex $(Q, 1_Q)$, we know that there exists a nucleus (W, γ) of χ such that $Q \in \text{Syl}_p(W)$ and $(\gamma_p)_Q = 1_Q$. Now, by definition of the nucleus, we know that there exists (N, θ) , a maximal p -factorable normal pair under χ , such that $N \subseteq W \subseteq T$, where $T = I_G(\theta)$, θ lies under γ , and (W, γ) is a nucleus for the Clifford correspondent ψ of χ over θ . Thus $\psi \in \text{Irr}(T|Q, 1_Q)$. Now, since $(\gamma_p)_Q = 1_Q$, by Theorem (2.1.c), we have that $\gamma_p = 1_W$, and we deduce that γ is p' -special. Since θ is under γ , we deduce that θ is p' -special. By Theorem (2.1.e), we have that $\varphi = \theta^0 \in \text{IBr}(N)$, and by the uniqueness in that result, we have that T is also the stabilizer of φ in G . If $T = G$, then $\theta = \chi$, $Q \in \text{Syl}_p(G)$ and $\chi^0 = \varphi \in \text{IBr}(G)$. Since φ has p' -degree, we also have that Q is a vertex of φ in this case. So we may assume that $T < G$. By induction, $\psi^0 \in \text{IBr}(T|Q)$. Also, ψ^0 lies over $\theta^0 = \varphi$, and by the Clifford correspondence for Brauer characters, we have that

$$\chi^0 = (\psi^G)^0 = (\psi^0)^G \in \text{IBr}(G|Q).$$

Write $\sigma = \chi^0 \in \text{IBr}(G)$. With the previous notation, we claim that N is the maximal normal subgroup of G such that the irreducible constituents of σ_N have

p' -degree. Suppose that E is such a normal subgroup. Since the irreducible constituents of σ_N (which are G -conjugates of φ) have p' -degree, we have that $N \subseteq E$. Let $\tau \in \text{IBr}(E)$ lie under σ and over φ . By Theorem (2.1.e), let $\kappa \in \text{Irr}(E)$ be p' -special such that $\kappa^0 = \tau$. By Theorem (2.1.f), we have that κ lies over θ . Since (N, θ) is maximal p -factorable, we have that $N = E$, as claimed.

Now, we prove that our map is one-to-one. Suppose that $\chi^0 = \eta^0 = \sigma$ for some $\eta \in \text{Irr}(G|Q, 1_Q)$. We want to prove that $\chi = \eta$. As before, we may find a maximal p -factorable pair (M, ν) for η such that the Clifford correspondent ξ of η over ν lies in $\text{Irr}(I|Q, 1_Q)$, where I is the inertia group of ν in G . Now, by the previous paragraph, we know that N is the maximal normal subgroup such that the irreducible constituents of σ_N have p' -degree. By the same argument applied to η , we deduce that $N = M$, and that θ^0 and ν^0 are G -conjugate. Hence, by Theorem (2.1.e), we have that θ and ν are G -conjugate. If $T = G$, then $N = M = G$, $\eta = \nu = \theta = \chi$, and in this case, we are done. So we may assume that T is proper in G . By induction, we have that ψ^0 and ξ^0 have vertex Q . Since both are Clifford correspondents of σ over N , by Lemma (6.2), we deduce that θ^0 and ν^0 (and therefore, θ and ν) are $\mathbf{N}_G(Q)$ -conjugate. Suppose that $\nu^n = \theta$ for $n \in \mathbf{N}_G(Q)$. Now, $I^n = T$, and $\xi^n \in \text{Irr}(T|Q, 1_Q)$. Now, ψ^0 and $(\xi^n)^0$ are the Clifford correspondent of σ over θ . By uniqueness, $\psi^0 = (\xi^n)^0$, and by induction we have that $\psi = \xi^n$. Hence

$$\chi = \psi^G = (\xi^n)^G = \eta,$$

and the injectivity is proven.

Finally, we prove that our map is surjective. Suppose that $\varphi \in \text{IBr}(G|Q)$. By Lemma (6.1), let N be maximal such that the irreducible constituents of φ_N have p' -degree. By Lemma (6.2), let $\theta \in \text{IBr}(N)$ be an irreducible constituent of φ_N such that, if T is the stabilizer of θ in G and $\mu \in \text{IBr}(T|\theta)$ is the Clifford correspondent of φ over θ , then μ has vertex Q . If $T = G$, by Lemma (6.1) we have that φ has p' -degree. In this case $Q \in \text{Syl}_p(G)$. By Theorem (2.1.e), there is some p' -special character ψ of G such that $\psi^0 = \varphi$. It is clear in this case that $\psi \in \text{Irr}(G|Q, 1_Q)$. So we may assume that T is proper in G . Now, let $\chi \in \text{B}_{p'}(G)$ be an Isaacs lifting for φ . The irreducible constituents of χ_N are in $\text{B}_{p'}(N)$, so they lift irreducible characters of N . Hence, there is some $\nu \in \text{B}_{p'}(N)$ such that $\nu^0 = \theta$. Since ν has p' -degree, ν is p' -special by Lemma (5.4) of [3]. Also, we know that $T = I_G(\nu)$ by uniqueness. We claim that (N, ν) is a maximal p -factorable normal pair below χ . Suppose that $N \subseteq M \triangleleft G$ is such that the irreducible constituents of χ_M are p -factorable. Let $\eta \in \text{Irr}(M)$ be one of them over ν . Since $\eta \in \text{B}_{p'}(M)$ is p -factorable, we have that η is p' -special by Lemma (5.4) of [3]. Now η^0 lies under φ and has p' -degree. By the maximality of N , we conclude that $N = M$. Thus (N, ν) is a maximal normal pair, as desired. By induction, there is some $\xi \in \text{Irr}(T|Q, 1_Q)$ such that $\xi^0 = \mu$. By definition, there is some nucleus (W, γ) for ξ such that γ is p' -special and $Q \in \text{Syl}_p(W)$. Suppose that (M, ρ) is a maximal normal p -factorable pair for ξ . By the second paragraph, we know that ρ is p' -special and that M is the maximal normal subgroup of T such that the irreducible constituents of μ_M have p' -degree. Since $\mu_N = e\theta$ and θ has p' -degree, we deduce that $N \subseteq M$ and that ρ^0 lies over θ . By Theorem (2.1.f), we deduce that ρ lies over θ . Hence, ξ lies over ν . Therefore, we conclude that (W, γ) is a nucleus of ξ^G . Therefore, $\xi^G \in \text{Irr}(G|Q, 1_Q)$ lifts φ , and the proof of the theorem is complete. \square

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