

## REPRESENTATIONS OF EXCEPTIONAL SIMPLE ALTERNATIVE SUPERALGEBRAS OF CHARACTERISTIC 3

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ABSTRACT. We study representations of simple alternative superalgebras  $B(1, 2)$  and  $B(2, 4)$ . The irreducible bimodules and bimodules with superinvolution over these superalgebras are classified, and some analogues of the Kronecker factorization theorem are proved for alternative superalgebras that contain  $B(1, 2)$  and  $B(4, 2)$ .

### 1. INTRODUCTION

The simple alternative superalgebras were classified in [6] and [5]. In particular, it was proved in [5] that a simple alternative superalgebra  $B = B_0 + B_1$ , which is not just a  $Z_2$ -graded alternative algebra, should necessarily have characteristic 3 and be isomorphic to one of the following superalgebras over a field  $F$  of characteristic 3.

1)  $B = B(1, 2)$ , where  $B_0 = F \cdot 1$ ,  $B_1 = F \cdot x + F \cdot y$ , with 1 being the unit of  $B$  and  $xy = -yx = 1$ ,  $x^2 = y^2 = 0$ .

2)  $B = B(4, 2)$ , where  $B_0 = M_2(F)$ ,  $B_1 = F \cdot m_1 + F \cdot m_2$  is the 2-dimensional irreducible Cayley bimodule over  $B_0$ ; that is,  $B_0$  acts on  $B_1$  by

$$(1) \quad e_{ij} \cdot m_k = \delta_{ik} m_j, \quad i, j, k \in \{1, 2\},$$

$$(2) \quad m \cdot a = \bar{a} \cdot m,$$

where  $a \in B_0$ ,  $m \in B_1$ ,  $a \rightarrow \bar{a}$  is the symplectic involution in  $B_0 = M_2(F)$ . The odd multiplication on  $B_1$  is defined by

$$m_1^2 = -e_{21}, \quad m_2^2 = e_{12}, \quad m_1 m_2 = e_{11}, \quad m_2 m_1 = -e_{22}.$$

3) **The twisted superalgebra of vector type**  $B = B(E, D, \gamma)$ . Let  $E$  be a commutative and associative algebra over  $F$ ,  $D$  be a nonzero derivation of  $E$  such that  $E$  is  $D$ -simple, and  $\gamma \in E$ . Denote by  $\bar{E}$  an isomorphic copy of the vector space  $E$ , with an isomorphism mapping  $a \rightarrow \bar{a}$ . Consider the vector space direct sum  $B(E, D, \gamma) = E + \bar{E}$  and define multiplication on it by the rules

$$a \cdot b = ab, \quad a \cdot \bar{b} = \bar{a} \cdot b = \overline{ab}, \quad \bar{a} \cdot \bar{b} = \gamma ab + 2D(a)b + aD(b),$$

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where  $a, b \in E$  and  $ab$  is the product in  $E$ . A  $Z_2$ -grading on  $B = B(E, D, \gamma)$  is defined by  $B_0 = E$  and  $B_1 = \overline{E}$ . In any characteristic,  $B$  is a simple right alternative superalgebra; and when  $\text{char } F = 3$ ,  $B$  is alternative.

In this work, we study birepresentations of  $B(1, 2)$  and of  $B(4, 2)$ . First, we classify the irreducible superbimodules over these superalgebras. It occurs that, besides a certain two-parametric series of bimodules  $V(\lambda, \mu)$  over  $B(1, 2)$ , all the other unital irreducible superbimodules for these superalgebras are regular or opposite to them. As a corollary, we prove that every unital  $B(4, 2)$ -superbimodule is completely reducible. Besides, every alternative superalgebra  $B$  that contains  $B(4, 2)$  as a unital subsuperalgebra admits a graded Kronecker factorization  $B = B(4, 2) \widetilde{\otimes} U$  for a certain associative commutative superalgebra  $U$ .

It was shown in [5] that both  $B(1, 2)$  and  $B(4, 2)$  admit  $J$ -admissible superinvolutions; that is, superinvolutions with symmetric elements in the nucleus. This was used in [5] for constructing new simple exceptional Jordan superalgebras of characteristic 3 as  $3 \times 3$  Hermitian matrices over  $B(1, 2)$  and  $B(4, 2)$ . Motivated by the future study of representations of these Jordan superalgebras, we classify the irreducible bimodules with  $J$ -admissible superinvolution over  $B(1, 2)$  and  $B(4, 2)$ . In the case of  $B(4, 2)$ , the list of irreducible bimodules with superinvolution coincides with that of irreducible bimodules, and for  $B(1, 2)$  this list contains only regular supermodules and their opposites, while the supermodules  $V(\lambda, \mu)$  do not enter in the list. As a corollary, every unital supermodule with  $J$ -admissible superinvolution over  $B(1, 2)$  is completely reducible; and every alternative superalgebra with  $J$ -admissible superinvolution that contains  $B(1, 2)$  as a unital subsuperalgebra admits a Kronecker factorization as above.

Now, let us recall some definitions and fix certain notation.

A superalgebra  $A = A_0 + A_1$  over a field  $F$  is called *alternative* if it satisfies the superidentities

$$(x, y, z) = -(-1)^{d(x)d(y)}(y, x, z) = -(-1)^{d(y)d(z)}(x, z, y),$$

where  $(x, y, z) = (xy)z - x(yz)$ ,  $x, y, z \in A_0 \cup A_1$ , and  $d(r)$  stands for the parity index of a homogeneous element  $r$ :  $d(r) = i$  if  $r \in A_i$ . In this case, it is easy to see that  $A_0$  is an alternative algebra and  $A_1$  is an alternative bimodule over  $A_0$ .

An  $A$ -superbimodule  $M = M_0 + M_1$  is called an *alternative superbimodule* if the corresponding split extension superalgebra  $E = A + M$  is alternative.

For an  $A$ -superbimodule  $M$ , the *opposite superbimodule*  $M^{op} = M_0^{op} + M_1^{op}$  is defined by the conditions  $M_0^{op} = M_1$ ,  $M_1^{op} = M_0$ , and the following action of  $A$ :  $a \cdot m = (-1)^{d(a)}am$ ,  $m \cdot a = ma$ , for any  $a \in A_0 \cup A_1$ ,  $m \in M^{op}$ . If  $M$  is an alternative  $A$ -superbimodule, then one can easily check that so is  $M^{op}$ .

A *regular superbimodule*,  $\text{Reg } A$ , for a superalgebra  $A$ , is defined on the vector superspace  $A$  with the action of  $A$  coinciding with the multiplication in  $A$ .

We will denote, for any homogeneous  $a$  and  $b$ ,

$$\begin{aligned} [a, b] &:= ab - (-1)^{d(a)d(b)}ba, \\ a \circ b &:= ab + (-1)^{d(a)d(b)}ba. \end{aligned}$$

If not stated otherwise, throughout the paper  $F$  will denote a field of characteristic 3. All the algebras and superalgebras will be considered over  $F$ .

2. REPRESENTATIONS OF  $B(1, 2)$

In this section, we classify irreducible superbimodules over the superalgebra  $B(1, 2)$ , defined in the Introduction.

We start with the following general result.

**Proposition 2.1.** *Let  $B$  be a simple commutative non-associative alternative superalgebra, and let  $V$  be an irreducible alternative  $B$ -superbimodule. Then  $V$  is commutative, that is, for any  $v \in V_0 \cup V_1$ ,  $a \in B_0 \cup B_1$ ,  $[v, a] = 0$  holds.*

*Proof.* Let us show first that for any homogeneous  $a \in B$  the set  $[V, a] := \{[v, a] \mid v \in V\}$  forms a subbimodule of  $V$ . Recall two identities that are valid in alternative superalgebras (see [5, 7]):

$$(3) \quad [xy, z] - x[y, z] - (-1)^{d(y)d(z)}[x, z]y - 3(x, y, z) = 0,$$

$$(4) \quad [[x, y], z] - (-1)^{d(y)d(z)}[[x, z], y] - [x, [y, z]] - 6(x, y, z) = 0.$$

Since  $B$  is commutative and  $char B = 3$ , we have by (3) for any homogeneous  $v \in V$ ,  $b \in B$

$$\begin{aligned} [v, a]b &= (-1)^{d(a)d(b)}[vb, a], \\ b[v, a] &= [bv, a], \end{aligned}$$

which proves that  $[V, a]$  is a subbimodule of  $V$ . Assume that there exists  $z \in B_1$  such that  $[V, z] \neq 0$ . Then, by irreducibility,  $V = [V, z] = [[V, z], z]$ . But it follows from (4) that  $[[v, z], z] = -[[v, z], z] = 0$ ; hence  $V = [[V, z], z] = 0$ , a contradiction. Therefore,

$$[V, B_1] = 0.$$

Now, the set  $B_1 + B_1^2$  is an ideal in  $B$ . If it were zero, then  $B = B_0$  would be a field; so we have  $B = B_1 + B_1^2$ . Let  $x, y \in B_1$ ,  $v \in V$ . Then we have by (3)

$$[xy, v] = x[y, v] + (-1)^{d(v)}[x, v]y = 0.$$

Thus,  $[B, V] = 0$ , proving the proposition. □

**Corollary 2.1.** *Every unital alternative superbimodule  $V$  over the superalgebra  $B = B(1, 2)$  satisfies the condition*

$$[[[V, B], B], B] = 0.$$

*Proof.* It was proved above that, for any  $v \in V$ ,  $z \in B_1$ , the equality  $[[v, z], z] = 0$  holds. Linearizing it, we have  $[[v, x], y] = -[[v, y], x]$ . In particular,  $[[V, B], B] = [[V, x], y] = [[V, y], x]$ . Therefore,

$$[[[V, B], B], x] = [[[V, y], x], x] = 0,$$

and similarly  $[[[V, B], B], y] = 0$ , proving the corollary. □

Denote by  $V(\lambda, \mu)$ , for  $\lambda, \mu \in F$ , the commutative superbimodule over  $B(1, 2) = F \cdot 1 + F \cdot x + F \cdot y$ , with the basis

$$v_0, v_1y, v_0y^2 \text{ for } V_0, \quad v_1, v_0y, v_1y^2 \text{ for } V_1,$$

and the action of  $x$  and  $y$  defined as follows. Let  $v$  stand for any of the elements  $v_0, v_1$  and  $v_i^s = v_{1-i}$ . Then

$$\begin{aligned} vy^j \cdot y &= vy^{j+1}, \quad j = 0, 1; \quad vy^2 \cdot y = \mu v^s; \\ vy^j \cdot x &= \lambda v^s y^j + jvy^{j-1}, \quad j = 0, 1, 2. \end{aligned}$$

**Proposition 2.2.** *The superbimodule  $V(\lambda, \mu)$  is alternative for any  $\lambda, \mu$  and irreducible if  $\lambda \neq 0$  or  $\mu \neq 0$ .*

*Proof.* It is easy to see that in any commutative superalgebra the equality

$$(a, b, c) = -(-1)^{d(a)d(b)+d(a)d(c)+d(b)d(c)}(c, b, a)$$

holds. This implies easily that every right alternative commutative superbimodule over a commutative superalgebra is also left alternative. Hence, it suffices to prove that  $V(\lambda, \mu)$  is right alternative. For this we need to check the following identities:

- (5)  $(u, x, y) - (u, y, x) = 0,$
- (6)  $(x, u, y) + (-1)^{d(u)}(x, y, u) = 0,$
- (7)  $(y, u, x) + (-1)^{d(u)}(y, x, u) = 0,$
- (8)  $(x, u, x) + (-1)^{d(u)}(x, x, u) = 0,$
- (9)  $(y, u, y) + (-1)^{d(u)}(y, y, u) = 0,$

where  $u$  is any element of the base. Let us start with (5). For  $u = vy^j, j = 0, 1,$  we have

$$\begin{aligned} (vy^j, x, y) &= \lambda v^s y^{j+1} + (j - 1)vy^j, \\ (vy^j, y, x) &= \lambda v^s y^{j+1} + (j + 1)vy^j + vy^j, \end{aligned}$$

which gives (5) since  $\text{char } F = 3$ . Similarly,

$$\begin{aligned} (vy^2, x, y) &= \lambda\mu v + 2vy^2 - vy^2, \\ (vy^2, y, x) &= \mu v^s \cdot x + vy^2 = \mu\lambda v + vy^2, \end{aligned}$$

which proves (5).

Furthermore, by commutativity,

$$\begin{aligned} (x, u, y) &= (-1)^{d(u)}(ux \cdot y + uy \cdot x), \\ (x, y, u) &= u \cdot xy + uy \cdot x; \end{aligned}$$

hence  $(x, u, y) + (-1)^{d(u)}(x, y, u) = (-1)^{d(u)}(ux \cdot y + uy \cdot x + u \cdot xy + uy \cdot x) = (-1)^{d(u)}(ux \cdot y - uy \cdot x - u \cdot xy + u \cdot yx) = (-1)^{d(u)}((u, x, y) - (u, y, x)) = 0$  by (5). Similarly, we have (7). Finally, we have

$$\begin{aligned} (x, u, x) &= (-1)^{d(u)}(ux \cdot x + ux \cdot x), \\ (x, x, u) &= ux \cdot x, \end{aligned}$$

which proves (8) and, similarly, (9). Hence, the module  $V(\lambda, \mu)$  is alternative. One can easily check that if  $\lambda \neq 0$  or  $\mu \neq 0$ , then this module is irreducible.  $\square$

Observe that the opposite bimodule  $(V(\lambda, \mu))^{op}$  is isomorphic to  $V(\lambda, \mu)$  under the isomorphism  $vy^j \mapsto v^s y^j$ . It is also easy to see that the modules  $V(\lambda, \mu)$  and  $V(\lambda', \mu')$  are isomorphic if and only if  $(\lambda, \mu) = \pm(\lambda', \mu')$ .

**Theorem 2.1.** *Every irreducible unital alternative superbimodule  $V$  over  $B(1, 2)$ , in the case where the ground field  $F$  (of characteristic 3) is algebraically closed, is isomorphic to one of the bimodules:  $\text{Reg } B(1, 2), (\text{Reg } B(1, 2))^{op}, V(\lambda, \mu)$ .<sup>1</sup>*

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<sup>1</sup>V.N.Zhelyabin informed the authors that a classification of irreducible alternative superbimodules over  $B(1, 2)$  was also obtained by M. Trushina.

*Proof.* According to Proposition 2.1, we can assume that  $V$  is commutative; so we may restrict ourselves to considering only the right actions  $\rho(x)$  and  $\rho(y)$  of  $x$  and  $y$  on  $V$ . Let us prove first that the elements  $\rho(x)^3$  and  $\rho(y)^3$  lie in the centralizer of  $V$  as a right  $B(1, 2)$ -module.

We will use in this proof non-graded (ordinary) commutators, which we will denote by

$$[a, b]_0 := ab - ba,$$

in order to distinguish them from the graded commutators, defined in the Introduction. By super-rightalternativity, we have for any  $v \in V$

$$(vx)y - (vy)x = v(xy - yx) = 2v = -v,$$

which gives

$$(10) \quad [\rho(x), \rho(y)]_0 = -id_V.$$

Now

$$\begin{aligned} [\rho(x)^3, \rho(y)]_0 &= \rho(x)^2[\rho(x), \rho(y)]_0 + [\rho(x), \rho(y)]_0\rho(x)^2 + \rho(x)[\rho(x), \rho(y)]_0\rho(x) \\ &= -3\rho(x)^2 = 0. \end{aligned}$$

Thus  $\rho(x)^3$  lies in the centralizer of  $V$ , and similarly so does  $\rho(y)^3$ .

Consider the two possible cases separately.

1°.  $\rho(x)^3 = \rho(y)^3 = 0$ .

Let us prove that in this case  $V$  is isomorphic to  $Reg B(1, 2)$  or to its opposite bimodule. Observe first that  $\rho(x)^2 \neq 0$ . In fact, we have by (10)

$$[\rho(x)^2, \rho(y)]_0 = \rho(x)[\rho(x), \rho(y)]_0 + [\rho(x), \rho(y)]_0\rho(x) = -2\rho(x);$$

so  $\rho(x)^2 = 0$  would imply  $\rho(x) = 0$ , which is impossible. Assume that  $\rho(x)^2|_{V_i} \neq 0$  for some  $i \in \{0, 1\}$ , that is, there exists  $v \in V_i$  such that  $u = (vx)x \neq 0$ ,  $u \in V_i$ . Then we have

$$ux = ((vx)x)x = 0.$$

Observe that, by (10),  $(ux)y - (uy)x = -u \neq 0$ ; hence  $uy \neq 0$ . Furthermore,

$$\begin{aligned} (uy)x &= (ux)y + u = u, \\ ((uy)y)y &= 0, \\ ((uy)y)x &= uy + ((uy)x)y = uy + uy = -uy. \end{aligned}$$

Therefore, the elements  $u, uy, (uy)y$  span a  $B(1, 2)$ -submodule of  $V$ , which, by irreducibility, coincides with  $V$ . It is easy to check that if  $i = 0$ , then  $V \cong (Reg B(1, 2))^{op}$ , and if  $i = 1$ , then  $V \cong Reg B(1, 2)$ .

2°.  $\rho(x)^3 \neq 0$ .

We claim that in this case  $V$  is isomorphic to a module of the type  $V(\lambda, \mu)$ . Let  $A = alg_F\langle \rho(x), \rho(y) \rangle$  be a subalgebra of  $End_F V$  generated by  $\rho(x), \rho(y)$ . Since  $V$  is irreducible, the center  $Z = Z(A)$  is a graded division algebra; besides,  $Z_1 \ni \rho(x)^3 \neq 0$ . It is easy to see that in this case  $Z = Z_0 + Z_0s$  for any fixed  $0 \neq s \in Z_1$ ; in particular,  $\rho(x)^3 = \alpha s$ ,  $\rho(y)^3 = \mu s$  for some  $\alpha, \mu \in Z_0$ . Let  $E = alg_F\langle \alpha, \mu, s^2 \rangle$ . Then  $E \subseteq Z_0$  and  $A$  is spanned over  $E$  by the elements  $\rho(x)^i \rho(y)^j, s\rho(x)^i \rho(y)^j, 0 \leq i, j \leq 2$ . In particular,  $V$  is finite dimensional over  $Z_0$ . Since  $V$  is a commutative supermodule, by [1, Proposition 4.2], it is irreducible as an ordinary (non-graded)  $A$ -module. This implies, by the density theorem, that  $A = End_{Z_0} V$ . Let us show that  $Z_0 = E$ . Consider some  $z \in Z_0, z = \alpha_0 + \alpha_1\rho(y) + \alpha_2\rho(y)^2$ , where  $\alpha_i$  depend

only on  $\rho(x)$  and  $s$ . We have  $0 = [z, \rho(x)]_0 = \alpha_1 + 2\alpha_2\rho(y)$ . Multiplying this by  $\rho(y)$  and subtracting from  $z$ , we get  $z = \alpha_0 - \alpha_2\rho(y)^2$ . Commuting  $z$  with  $\rho(x)$  again, we get  $\alpha_2\rho(y) = 0$  and so  $z = \alpha_0 = \beta_0 + \beta_1\rho(x) + \beta_2\rho(x)^2$ , where  $\beta_0, \beta_2 \in E$ ,  $\beta_1 \in Es$ . Commuting now  $z$  with  $\rho(y)$  and arguing as before, we obtain finally that  $z = \beta_0 \in E$ .

Thus, the field  $Z_0$  is a finitely generated algebra over  $F$ . Since  $F$  is algebraically closed, this implies that  $Z_0 = F$ . We can now choose  $s \in Z_1$  such that  $s^2 = 1$ . Let  $0 \neq \lambda \in F$  be a root of the polynomial  $X^3 - \alpha$  and  $v \in V$  such that  $s\rho(x)(v) = v^s \cdot x = \lambda v$ . We can assume, without loss of generality, that  $v = v_0 \in V_0$ . Denote  $v_1 := v^s$ ,  $\rho(y)^j(v_i) := v_i y^j$  for  $0 \leq j \leq 2$ . Then we have

$$\begin{aligned} v_0 \cdot x &= \lambda v_1, & v_1 \cdot x &= \lambda v_0; \\ v_i y^j \cdot y &= v_i y^{j+1}, & j < 2; & & v_i y^2 \cdot y &= v_i \rho(y)^3 = \mu v_{1-i}; \\ v_i y \cdot x &= v_i [\rho(y), \rho(x)]_0 + (v_i \cdot x) \cdot y &= v_i + \lambda v_{1-i} y; \\ v_i y^2 \cdot x &= v_i y [\rho(y), \rho(x)]_0 + (v_i y \cdot x) \cdot y &= v_i y + \lambda v_{1-i} y^2 + v_i y = \lambda v_{1-i} y^2 + 2v_i y. \end{aligned}$$

These relations show that  $V$  is a homomorphic image of the module  $V(\lambda, \mu)$ . In order to prove that  $V$  is isomorphic to  $V(\lambda, \mu)$ , it suffices to prove that the elements  $v_0, v_1 y, v_0 y^2$  are linearly independent over  $F$ . It is easy to see that they are nonzero. Assume that

$$(11) \quad \alpha v_0 + \beta v_1 y + \gamma v_0 y^2 = 0$$

for some  $\alpha, \beta, \gamma \in F$ . Applying  $s$  to this equality, we get

$$(12) \quad \alpha v_1 + \beta v_0 y + \gamma v_1 y^2 = 0.$$

On the other hand, multiplying (11) by  $x$ , we get

$$\alpha \lambda v_1 + \beta (\lambda v_0 y + v_1) + \gamma (\lambda v_1 y^2 + 2v_0 y) = 0,$$

which, by (12), gives

$$(13) \quad \beta v_1 + 2\gamma v_0 y = 0.$$

Applying  $s$  to (13), we get  $\beta v_0 + 2\gamma v_1 y = 0$ , and multiplying (13) by  $x$ , we obtain

$$0 = \beta \lambda v_0 + 2\gamma (\lambda v_1 y + v_0) = \lambda (\beta v_0 + 2\gamma v_1 y) + 2\gamma v_0 = 2\gamma v_0.$$

Thus  $\gamma = 0$ , which implies easily that  $\beta = \alpha = 0$  as well. This finishes the proof of the theorem. □

### 3. REPRESENTATIONS OF $B(4, 2)$

We will use in this section certain results about alternative bimodules over composition algebras that were proved in [5]. For the convenience of the reader, we state these results below.

Recall that a bimodule  $V$  over a composition algebra  $C$  is called a *Cayley bimodule* if it satisfies the relation

$$(14) \quad av = v\bar{a},$$

where  $a \in C$ ,  $v \in V$ , and  $a \rightarrow \bar{a}$  is the canonical involution in  $C$ .

**Proposition 3.1** ([5, Lemma 11 and its proof]). *Let  $B = B_0 + B_1$  be a unital alternative superalgebra over a field  $F$  which contains an even composition subalgebra  $C$  with the same unit. Assume that a subspace  $V$  of  $B$  is  $C$ -invariant and satisfies (14). Then, the following identities hold for any  $a, b \in C$ ,  $r \in B$ ,  $u, v \in V$ .*

$$\begin{aligned} (15) \quad & (ab)v = b(av), \quad v(ab) = (vb)a, \\ (16) \quad & a(ur) = u(\bar{a}r), \\ (17) \quad & a(uv) = u(va), \quad (uv)a = (au)v, \\ (18) \quad & (u, v, a) = [uv, a]. \end{aligned}$$

**Proposition 3.2** ([5, Lemma 12 and its proof]). *Let  $H$  be a generalized quaternion algebra. Then, any unital alternative  $H$ -bimodule  $V$  admits the decomposition  $V = V_a \oplus V_c$ , where  $V_a$  is an associative  $H$ -bimodule and  $V_c$  is a Cayley bimodule over  $H$ ; moreover, the subbimodule  $V_c$  coincides with the subspace  $(V, H, H)$ .*

In this section we are going to prove the following theorems which describe the alternative superbimodules over the superalgebra  $B(4, 2)$ .

**Theorem 3.1.** *Let  $V$  be a unital irreducible alternative superbimodule over  $B(4, 2)$ . Then  $V$  is isomorphic to  $\text{Reg}(B(4, 2))$  or to  $\text{Reg}(B(4, 2))^{op}$ .*

**Theorem 3.2.** *Every unital alternative superbimodule over  $B(4, 2)$  is completely reducible.*

We divide the proof into a sequence of lemmas.

Let  $B = B(4, 2) = H + M$ , with  $H = M_2(F)$ ,  $M = F \cdot m_1 + F \cdot m_2$ , the 2-dimensional Cayley  $H$ -bimodule defined by (1) and (2), and let  $V$  be a unital irreducible alternative superbimodule over  $B$ . By Proposition 3.2,  $V = V_a \oplus V_c$  where  $V_a$  is an associative  $H$ -bimodule and  $V_c$  is a Cayley  $H$ -bimodule.

**Lemma 3.1.** *Let  $V = V_a \oplus V_c$  be a unital alternative superbimodule over  $B(4, 2) = H + M$ . Then, for any  $v \in V_c$ ,  $m \in M$ ,  $a \in H$ ,*

$$\begin{aligned} (19) \quad & (vm)a = (av)m, \\ (20) \quad & (mv)a = (am)v, \end{aligned}$$

and for any  $u \in V_a$ ,  $m \in M$ ,  $a, b \in H$ ,

$$\begin{aligned} (21) \quad & (um)a = (u\bar{a})m, \\ (22) \quad & a(mu) = m(\bar{a}u), \\ (23) \quad & ((um)a)b = (um)(ba), \\ (24) \quad & b(a(mu)) = (ab)(mu), \\ (25) \quad & (um, a, b) = (um)[b, a], \\ (26) \quad & (b, a, mu) = [b, a](mu). \end{aligned}$$

*Proof.* First, consider  $v \in V_c$ ,  $m \in M$ ,  $a \in H$ . By (14),  $(vm)a - (av)m = (vm)a - (v\bar{a})m = (v, m, a) - (v, \bar{a}, m) + v(ma - \bar{a}m) = (v, m, a) + (v, a, m) = 0$ , and similarly  $(mv)a - (am)v = 0$ .

Now, let  $u \in V_a$ ,  $m \in M$ ,  $a, b \in H$ . Then  $(um)a - (u\bar{a})m = (u, m, a) - (u, \bar{a}, m) + u(ma - \bar{a}m) = 0$ , and similarly  $a(mu) - (a\bar{u})m = 0$ , which proves (21) and (22). Furthermore, by (21),  $(um)a \cdot b = (u\bar{a} \cdot m)b = (u\bar{a} \cdot \bar{b})m = (u \cdot \bar{b}\bar{a})m = (um)(ba)$ , which proves (23). Similarly, by (22), one gets (24). Finally, (25) and (26) follow easily from (23) and (24). □

**Lemma 3.2.** *Let  $V = V_a \oplus V_c$  be a unital alternative superbimodule over  $B(4, 2) = H + M$ . Then,  $V_aM, MV_a, V_cM$  and  $MV_c$  are  $H$ -invariant subspaces. Moreover  $V_aM + MV_a \subseteq V_c$  and  $V_cM + MV_c \subseteq V_a$ .*

*Proof.* Since  $V_a, V_c,$  and  $M$  are  $H$ -invariant, it suffices to prove, for the first part of the lemma, that the product of any  $H$ -invariant subspaces  $U$  and  $W$  in the split extension superalgebra  $E = B + V$  is again  $H$ -invariant.

We have  $(UW)H \subseteq U(WH) + (U, W, H) \subseteq UW + (U, H, W) \subseteq UW$ , and similarly  $H(UW) \subseteq UW$ .

Now, let us prove that  $V_aM + MV_a \subseteq V_c$ . Recall that, by Proposition 3.2,  $V_c = (V, H, H)$ . Choose  $a, b \in H$  such that  $[a, b]^2 \neq 0$ . Then  $0 \neq [a, b]^2 \in F$ , and, by (26),

$$MV_a = [a, b]^2(MV_a) \subseteq [a, b](MV_a) \subseteq (a, b, MV_a) \subseteq (H, H, V) = V_c,$$

and similarly  $V_aM \subseteq V_c$ .

Finally, for any  $v \in V_c, m \in M, a \in H$ , we have by (19) and (15)

$$((vm)a)b = ((av)m)b = (b(av))m = ((ab)v)m = (vm)(ab),$$

which proves that  $V_cM \subseteq V_a$ . Similarly, by (20) and (15),  $MV_c \subseteq V_a$ . □

**Corollary 3.1.** *In the notation of the lemma,  $V_a \neq 0$ .*

Really, if  $V_a = 0$ , then  $V = V_c$  and  $VM = MV = 0$ , which yields, for any  $v \in V$ ,

$$v = v \cdot (m_1m_2 - m_2m_1) = (vm_1)m_2 - (vm_2)m_1 = 0,$$

a contradiction. □

**Lemma 3.3.** *Let  $V$  be a unital alternative superbimodule over  $B = B(4, 2) = H + M$ , and let  $Z_a = Z_a(V) = \{v \in V_a \mid [v, H] = 0\}$ . Then,  $Z_a \neq 0$  and satisfies the following conditions:*

- i)  $[Z_a, B] = 0$ ,
- ii)  $(Z_a, B, B) = 0$ .

*Proof.* By Corollary 3.1,  $V_a$  is a nonzero unital bimodule over  $H$ . The category of unital  $H$ -bimodules is equivalent to the category of right unital  $H^\circ \otimes H$ -modules [4], where  $H^\circ$  is the algebra anti-isomorphic to  $H$ . Since  $H^\circ \otimes H \cong M_4(F)$ , this means that every unital  $H$ -bimodule is completely reducible and that any two unital irreducible  $H$ -bimodules are isomorphic. The regular  $H$ -bimodule  $Reg H$  is unital and irreducible; therefore, the bimodule  $V_a = \bigoplus_i W_i$ , where each  $W_i$  is isomorphic to  $Reg H$ . It is now clear that  $Z_a \neq 0$ .

Let us prove first that

$$(27) \quad (Z_a, H, M) = 0.$$

By Lemma 3.2, for any  $u \in Z_a, a \in H, m \in M$  we have

$$(a, u, m) = (au)m - a(um) \stackrel{(14)}{=} (au)m - (um)a \stackrel{(21)}{=} (au)m - (ua)m = [a, u]m = 0,$$

which proves (27). Furthermore, consider the identity

$$(28) \quad ([x, y], y, z) = [y, (x, y, z)],$$

which holds in any alternative algebra. Using its superized linearization, we have for any  $u \in Z_a, m \in M, a, b \in H$

$$([u, m], a, b) = -([u, a], m, b) + (-1)^{d(m)d(u)}[m, (u, a, b)] + [a, (u, m, b)] = 0,$$

since  $[u, a] = (u, a, b) = 0$  and  $(u, m, b) = 0$ . Therefore,  $([Z_a, M], H, H) = 0$ .

By (15),

$$0 = ([u, m], a, b) = ([u, m]a)b - [u, m](ab) = [u, m](ba) - [u, m](ab) = [u, m][b, a].$$

Therefore,  $[Z_a, M][H, H] = 0$ , which yields  $[Z_a, M] = 0$ , proving *i*).

Consider now the identity

$$(29) \quad 2[(x, y, z), t] = ([x, y], z, t) + ([y, z], x, t) + ([z, x], y, t),$$

which holds in every alternative algebra (see [7], Lemma 3.2). Using the corresponding superidentity, we have for any  $u \in Z_a$ ,  $m, n \in M$ ,  $a \in H$ ,

$$2[(u, m, n), a] = ([u, m], n, a) + ([m, n], u, a) - (-1)^{d(u)}([n, u], m, a) = 0,$$

by *i*) and (27). Therefore,  $[(Z_a, M, M), H] = 0$ , and by superized linearization of (28) we have

$$0 = [a, (u, m, n)] = -(-1)^{d(u)}[m, (u, a, n)] + (u, m, [n, a]) - (u, a, [n, m]).$$

By (27) and the fact that  $Z_a \subseteq V_a$ , this implies the equality  $(Z_a, M, [M, H]) = 0$ . But it is easy to see that  $[M, H] = M$ ; hence  $(Z_a, M, M) = 0$ , yielding *ii*).  $\square$

*Proof of Theorem 3.1.* Let  $V = V_a \oplus V_c$  be a unital irreducible alternative superbimodule over  $B = B(4, 2) = H + M$ . By Lemma 3.3,  $Z_a \neq 0$ ; so we can choose some homogeneous element  $0 \neq u \in Z_a$ . The conditions *i*) and *ii*) of Lemma 3.3 show that the subspace  $u \cdot B$  is a  $B$ -subbimodule of  $V$  and the mapping  $\varphi : a \mapsto u \cdot a$  is a  $B$ -bimodule homomorphism of  $Reg B$  onto  $uB$ , in the case where  $u$  is even, or of  $(Reg B)^{op}$  onto  $uB$ , in the case where  $u$  is odd. Since both  $Reg B$  and  $(Reg B)^{op}$  are irreducible, and  $\varphi(1) = u \neq 0$ , we have that  $uB = V$  is isomorphic to  $Reg B$  or to  $(Reg B)^{op}$ .  $\square$

*Proof of Theorem 3.2.* Let  $U = U_a + U_c$  be a unital superbimodule over  $B = B(4, 2) = H + M$ . It was shown in the proof of Lemma 3.3 that the bimodule  $U_a$  is isomorphic to a direct sum of regular  $H$ -bimodules:  $U_a = \bigoplus_i U_i$ , where, for every  $i$ ,  $U_i = u_i H$ , and  $u_i \in Z_a(U_i)$  is the image of the unit 1 under the isomorphism of  $Reg H$  onto  $U_i$ . In particular,  $[u_i, H] = 0$ ; hence, by Lemma 3.3,  $u_i \in Z_a(U)$ .

Consider  $W = \sum_i u_i B$ . Evidently,  $W$  is a  $B$ -subbimodule of  $U$  and  $U_a \subseteq W$ . Let  $v \in U_c$ . Then  $v = v(m_1 \circ m_2) = (vm_1)m_2 - (vm_2)m_1$ . By Lemma 3.2,  $vm_i \in U_a \subseteq W$ ; so  $v \in W$  as well, and  $U = W$ . Since every bimodule  $u_i \cdot B$  is irreducible,  $U = W$  is completely reducible.  $\square$

#### 4. BIMODULES WITH SUPERINVOLUTION

Recall that a linear even mapping  $* : A \rightarrow A$  is called a *superinvolution* of a superalgebra  $A$ , if it satisfies the conditions

$$(a^*)^* = a, \quad (ab)^* = (-1)^{d(a)d(b)}b^*a^*,$$

for any homogeneous elements  $a, b \in A$ .

Now, let  $V$  be a superbimodule over a superalgebra  $(A, *)$  with superinvolution. By analogy with the non-graded case (see [2]), we will call  $V$  an *A-bimodule with superinvolution*, if there exists a linear mapping  $\bar{-} : V \rightarrow V$  such that the mapping

$$a + v \mapsto a^* + \bar{v}$$

is a superinvolution of the split null extension superalgebra  $E = A + V$ . Evidently, for a superalgebra with superinvolution  $A$ , the bimodules  $Reg A$  and  $(Reg A)^{op}$  have the superinvolutions induced by that of  $A$ .

It was shown in [5] that the superalgebras  $B(1, 2)$  and  $B(4, 2)$  admit the following superinvolutions:

In  $B(1, 2)$ ,  $a_0 + a_1 \mapsto a_0 - a_1$ ; and in  $B(4, 2)$ ,  $a_0 + a_1 \mapsto \overline{a_0} - a_1$ , where the mapping  $a \mapsto \overline{a}$  is the symplectic involution of the matrix algebra  $M_2(F)$ .

Now, we will study the structure of superbimodules with superinvolution over  $B(1, 2)$  and  $B(4, 2)$ . Our first objective is to prove that every irreducible superbimodule with superinvolution over these superalgebras is of the type  $Reg B$  or  $(Reg B)^{op}$ .

In fact, we will consider the superbimodules with involution that satisfy the additional condition of so-called *J-admissibility* (see [2]). A superbimodule with superinvolution  $(V, -)$  over a superalgebra with superinvolution  $(A, *)$  is called *J-admissible* if all the symmetric elements of the superalgebra with superinvolution  $E = A + V$  lie in the associative center (the nucleus) of  $E$ . In fact, only *J-admissible* bimodules are needed for applications to Jordan algebras.

**Theorem 4.1.** *Every irreducible unital J-admissible superbimodule V with superinvolution over  $B = B(1, 2)$  is isomorphic to  $Reg B$  or to  $(Reg B)^{op}$ .*

*Proof.* Let  $V$  be a superbimodule under consideration, with a superinvolution  $v \mapsto \overline{v}$ . Observe first that for any  $a \in B$ ,  $v \in V$ , we have

$$\overline{[a, v]} = \overline{av} - (-1)^{d(v)d(a)}\overline{va} = (-1)^{d(a)d(v)}\overline{v}a - \overline{a}v = -[\overline{a}, \overline{v}].$$

This means that the subspace  $[V, a]$  is invariant with respect to the superinvolution and so is a subbimodule with superinvolution. Now, all the arguments of the proof of Proposition 2.1 are applied to our case, and we conclude that  $V$  is a commutative  $B$ -supermodule.

It is clear that  $V = Sym V \oplus Skew V$ , where, for any  $h \in Sym V$ ,  $k \in Skew V$ , we have  $\overline{h} = h$ ,  $\overline{k} = -k$ . Assume first that  $Sym V \neq 0$  and choose some  $0 \neq h \in Sym V$ . By *J-admissibility*,  $(h, B, B) = 0$ , and so we have

$$\begin{aligned} (hx)x &= (h, x, x) + h(xx) = 0, & (hy)y &= 0, \\ (hx)y &= (h, x, y) + h(xy) = h(xy) = h, & (hy)x &= -h, \\ \overline{hx} &= (-1)^{d(h)}\overline{x}h = -(-1)^{d(h)}xh = -hx, & \overline{hy} &= -hy. \end{aligned}$$

Therefore, the subspace  $U = Fh + F(hx) + F(hy)$  is a  $B$ -subbimodule with involution of  $V$ , and hence  $U = V$ . It is clear that  $U \cong Reg B$  for even  $h$ , and  $U \cong (Reg B)^{op}$  for odd  $h$ .

Now, assume that  $Sym V = 0$ , that is,  $\overline{v} = -v$  for any  $v \in V$ . Then we have

$$\overline{vx} = (-1)^{d(v)}\overline{x}v = (-1)^{d(v)}xv = vx;$$

hence  $vx \in Sym V = 0$ . Similarly,  $vy = 0$ , and finally  $v = v(xy - yx) = (vx)y - (vy)x = 0$ , a contradiction. □

**Theorem 4.2.** *Every unital J-admissible alternative superbimodule V with superinvolution over the superalgebra  $B = B(1, 2)$  is completely reducible.*

*Proof.* It suffices to prove that  $V$  is a sum of irreducible subbimodules with involution, or, equivalently, that every element  $v \in V$  lies in a sum of irreducible

subbimodules with involution. Assume first that  $v = h \in \text{Sym } V$ . We know that  $(h, B, B) = 0$ . Now let us show that also  $[h, B] = 0$ . Consider

$$(xhy)x = (xh \cdot y)x = (xh, y, x) + (xh)(yx) = (y, x, xh) - xh = -xh - xh = xh.$$

On the other hand,

$$\begin{aligned} (xhy)x &= (x \cdot hy)x = x(hy \cdot x) + (x, hy, x) = -xh + (-1)^{d(h)}(hy, x, x) \\ &= -xh - (-1)^{d(h)}hx. \end{aligned}$$

Hence,  $[x, h] = xh - (-1)^{d(h)}hx = 0$ . Similarly,  $[y, h] = 0$ , and so  $[B, h] = 0$ .

We can now apply the arguments from the proof of Theorem 4.1 which show that the elements  $h, hx, hy$  span an irreducible subbimodule with involution of  $V$ . So, in this case we are done.

Now, let  $v = k \in \text{Skew } V$ . By the previous arguments, the subbimodule  $(\text{Sym } V)B$  generated by symmetric elements of  $V$  is completely reducible; so it suffices to prove that  $k \in (\text{Sym } V)B$ . Below, for  $v \in V$  we will write  $v \equiv 0$  if  $v \in (\text{Sym } V)B$ .

It is easy to see that

$$(30) \quad \text{Skew } V \circ B_1 \subseteq \text{Sym } V, \quad [\text{Skew } V, B_1] \subseteq \text{Skew } V;$$

hence  $k \circ z \equiv 0$  for any  $z \in B_1$ . Moreover, we have

$$\begin{aligned} 0 &\equiv (k \circ z)z = kz \cdot z + (-1)^{d(k)}zk \cdot z = (k, z, z) + (-1)^{d(k)}zk \cdot z \\ &= -(-1)^{d(k)}(z, k, z) + (-1)^{d(k)}zk \cdot z = (-1)^{d(k)}z \cdot kz. \end{aligned}$$

Linearizing this relation on  $z$ , we have

$$(31) \quad x \cdot ky + y \cdot kx \equiv 0.$$

Now, consider the element  $(k \circ x)y \in (\text{Skew } V \circ B_1)B_1 \subseteq (\text{Sym } V)B_1 = (\text{Sym } V) \circ B_1 \subseteq \text{Skew } V$ . We have

$$(k \circ x)y = k + (k, x, y) + (-1)^{d(k)}xk \cdot y.$$

Since the elements  $k, (k, x, y), (k \circ x)y$  are skewsymmetric, so is  $xk \cdot y$ . We have

$$\overline{xk \cdot y} = (-1)^{d(x)d(k)+d(x)d(y)+d(y)d(k)}\overline{y} \cdot \overline{k} \cdot \overline{x} = y \cdot kx;$$

hence

$$xk \cdot y = -y \cdot kx.$$

Comparing this relation with (31), we get

$$xk \cdot y = -y \cdot kx \equiv x \cdot ky,$$

which yields  $(x, k, y) \equiv 0$ . Now, we have by (30),

$$k = k \cdot xy \equiv kx \cdot y \equiv \frac{1}{2}[k, x]y \equiv \frac{1}{4}[[k, x], y] = [[k, x], y].$$

By Corollary 2.1, for any  $B$ -superbimodule  $V$ , the equality  $[[[V, B], B], B] = 0$  holds. Therefore, we have

$$k \equiv [[k, x], y] \equiv [[[[k, x], y], x], y] = 0,$$

which proves the theorem. □

**Corollary 4.1.** *Every unital alternative  $J$ -admissible superbimodule with superinvolution over the superalgebra  $B(1, 2)$  is commutative.* □

Now, we turn to bimodules with superinvolution over  $B(4, 2)$ .

**Theorem 4.3.** *Every unital  $J$ -admissible superbimodule with superinvolution  $V$  over the superalgebra with superinvolution  $B = B(4, 2)$  is completely reducible and is a direct sum of irreducible bimodules with superinvolution isomorphic to  $\text{Reg } B$  or to  $(\text{Reg } B)^{op}$ .*

*Proof.* By Theorem 3.2,  $V = \bigoplus_i Bu_i$  for certain elements  $u_i \in Z_a = Z_a(V)$ . In particular, we always have  $Z_a \neq 0$ . Let us show that  $Z_a \subseteq \text{Sym } V$ . First, it is easy to see that  $Z_a$  is invariant under the superinvolution; so  $Z_a = (\text{Sym } V \cap Z_a) \oplus (\text{Skew } V \cap Z_a)$ . Assume that there exists  $0 \neq u \in Z_a$  such that  $\bar{u} = -u$ . Consider the element  $s = um_1 = \frac{1}{2}u \circ m_1$  (recall that  $[u, B] = 0$ ), where  $m_1$  is one of the two basic elements of  $M$ . It is easy to check that  $\bar{s} = s$ ; hence, by  $J$ -admissibility of  $V$ , we should have  $(s, B, B) = 0$ . But, by Lemma 3.3,  $(um_1, m_2, m_1) = -um_1$ . Hence  $s = 0$ , a contradiction.

Now, if  $V$  is irreducible then, for any homogeneous  $0 \neq u \in Z_a$  we have  $V = uB$ , which is isomorphic to  $\text{Reg } B$  or to its opposite, according to the parity of  $u$ , under the isomorphism  $b \mapsto ub$ .

In the general case, it suffices to notice that every  $u_i$  generates an irreducible subsuperbimodule which is invariant under the superinvolution and is isomorphic to  $\text{Reg } B$  or to its opposite. □

### 5. FACTORIZATION THEOREMS

In this section, we will prove for the superalgebras  $B(1, 2)$  and  $B(4, 2)$  some analogue of the Kronecker factorization theorem for Cayley algebras from [3].

**Theorem 5.1.** *Let  $B$  be an alternative superalgebra with  $J$ -admissible superinvolution (that is, every symmetric element lies in the nucleus of  $B$ ) such that  $B$  contains  $B(1, 2)$  as a unital subsuperalgebra with superinvolution. Then  $B \cong U \tilde{\otimes} B(1, 2)$  for a certain commutative associative superalgebra  $U$ , where  $\tilde{\otimes}$  denotes a graded tensor product, that is,*

$$(32) \quad (u \tilde{\otimes} a)(v \tilde{\otimes} b) = (-1)^{d(a)d(v)}(uv) \tilde{\otimes} (ab)$$

for any homogeneous  $u, v \in U$ ,  $a, b \in B(1, 2)$ . In particular, the superalgebra  $B$  is commutative.

*Proof.* Consider  $B$  as a  $B(1, 2)$ -superbimodule with superinvolution. By Theorem 4.2 and  $J$ -admissibility, we conclude that  $B = \sum_i u_i B(1, 2)$ , where  $\bar{u}_i = u_i$ ,  $(u_i, B, B) = 0$ . Moreover,  $[B, B(1, 2)] = 0$ , by Corollary 4.1. Let  $U = \text{Sym } B = \{u \in B \mid \bar{u} = u\}$ . Then  $B = UB(1, 2)$ , and we will show that this product is isomorphic to the tensor product we are looking for.

Consider the following identity, which is valid in any alternative algebra (see [7]):

$$(33) \quad [a, b](a, b, c) - (a, b, (a, b, c)) = 0.$$

Superlinearizing it, we have for any  $u, v \in U$ ,  $a, b, c \in B(1, 2)$

$$\begin{aligned} [u, v](a, b, c) &= \pm[a, v](u, b, c) \pm [u, b](a, v, c) \pm [a, b](u, v, c) \pm (u, v, (a, b, c)) \\ &\quad \pm (a, v, (u, b, c)) \pm (u, b, (a, v, c)) \pm (a, b, (u, v, c)) = 0. \end{aligned}$$

It is easy to see that  $(B(1, 2), B(1, 2), B(1, 2)) = (B(1, 2))_1 = Fx + Fy$ ; hence  $[u, v]x = [u, v]y = 0$  and  $[u, v] = -[u, v](xy - yx) = -([u, v]x)y + ([u, v]y)x = 0$ .

Therefore,  $[U, U] = 0$ . Since  $U \circ U \subseteq U$ , this proves that  $U$  is a commutative (and associative) subsuperalgebra of  $B$ .

Furthermore, we have for any  $u, v \in U$ ,  $a, b \in B(1, 2)$ ,

$$\begin{aligned} (ua)(vb) &= u(avb) = u([a, v]b + (-1)^{d(a)d(v)}vab) = (-1)^{d(a)d(v)}u(vab) \\ &= (-1)^{d(a)d(v)}(uv)(ab), \end{aligned}$$

which shows that  $B$  is a homomorphic image of  $U \widetilde{\otimes} B(1, 2)$ . Assume that  $u + vx + wy = 0$  for some  $u, v, w \in U$ . Then  $u \in \text{Sym } B$ ,  $vx + wy \in \text{Skew } B$ ; hence  $u = vx + wy = 0$ . Moreover, we have  $0 = (vx + wy)x = -w$  and  $0 = (vx + wy)y = v$ . Therefore,  $B \cong U \widetilde{\otimes} B(1, 2)$ .

One can easily see that, since  $U$  and  $B(1, 2)$  are commutative superalgebras, so is  $B$ . □

**Theorem 5.2.** *Let  $B$  be an alternative superalgebra such that  $B$  contains  $B(4, 2)$  as a unital subsuperalgebra. Then  $B \cong U \widetilde{\otimes} B(4, 2)$  for a certain commutative associative superalgebra  $U$ .*

*Proof.* As before, consider  $B$  as a  $B(4, 2)$ -superbimodule. By Theorem 4.3,  $B = \sum_i u_i B(4, 2)$ , where  $u_i \in Z_a(B) = \{u \in B \mid [u, B(4, 2)] = 0\}$ . Set  $U = Z_a$ . Then  $B = UB(4, 2)$ , and we will show that  $U$  is the desired superalgebra.

Let us see first that  $U$  is a subsuperalgebra of  $B$ . Fix arbitrary  $u, v, w \in U$ ,  $a, b, c \in B(4, 2)$ . Then, by (3),

$$[uv, a] = u[v, a] + (-1)^{d(v)d(a)}[u, a]v = 0;$$

hence  $UU \subseteq U$ . Furthermore, by Lemma 3.3,  $(U, B(4, 2), B(4, 2)) = 0$ , and so, by superization of (29),

$$([a, b], u, v) = \pm([b, u], a, v) \pm ([u, a], b, v) \pm [(a, b, u), v] = 0.$$

Since  $B(4, 2) = F1 + [B(4, 2), B(4, 2)]$ , this yields that  $(U, U, B(4, 2)) = 0$ .

Furthermore, by superized linearization of (33), we have

$$\begin{aligned} [a, b](u, v, w) &= \pm[a, v](u, b, w) \pm [u, b](a, v, w) \pm [u, v](a, b, w) \pm (a, b, (u, v, w)) \\ &\quad \pm (a, v, (u, b, w)) \pm (u, b, (a, v, w)) \pm (u, v, (a, b, w)) = 0. \end{aligned}$$

Choose  $a, b \in B(4, 2)_0 = M_2(F)$  such that  $[a, b]^2 = \alpha \in F$ ,  $\alpha \neq 0$ . Then  $\alpha(u, v, w) = [a, b]^2(u, v, w) = [a, b]([a, b](u, v, w)) = 0$  and  $(u, v, w) = 0$ . Thus,  $U$  is associative.

Applying again the superized linearization of (33), we get

$$\begin{aligned} [u, v](a, b, c) &= \pm[a, v](u, b, c) \pm [u, b](a, v, c) \pm [a, b](u, v, c) \pm (u, v, (a, b, c)) \\ &\quad \pm (a, v, (u, b, c)) \pm (u, b, (a, v, c)) \pm (a, b, (u, v, c)) = 0. \end{aligned}$$

Since  $m_i = -(e_{ii}, e_{ji}, m_j)$ ,  $i, j = 1, 2$ ,  $i \neq j$ , this implies  $[u, v]m_i = 0$ ,  $i = 1, 2$ , and finally

$$[u, v] = [u, v](m_1 m_2 - m_2 m_1) = ([u, v]m_1)m_2 - ([u, v]m_2)m_1 = 0.$$

Therefore,  $U$  is a commutative and associative subsuperalgebra of  $B$ .

It is clear that  $B$  is a homomorphic image of  $U \widetilde{\otimes} B(4, 2)$ . Assume that  $w = \sum_{ij} u_{ij} e_{ij} + u_1 m_1 + u_2 m_2 = 0$  for some  $u_i, u_{ij} \in U$ . Then we have

$$\begin{aligned} 0 &= (e_{11}, e_{21}, w) = -u_2 m_1, \\ 0 &= (e_{22}, e_{12}, w) = -u_1 m_2, \end{aligned}$$

which implies easily that  $u_1 = u_2 = 0$ . Furthermore,

$$0 = (e_{ii}w)e_{jj} = u_{ij}e_{ij},$$

which yields easily  $u_{ij} = 0$  for all  $i, j$ . □

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