

## TRANSITION PROBABILITIES FOR SYMMETRIC JUMP PROCESSES

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ABSTRACT. We consider symmetric Markov chains on the integer lattice in  $d$  dimensions, where  $\alpha \in (0, 2)$  and the conductance between  $x$  and  $y$  is comparable to  $|x - y|^{-(d+\alpha)}$ . We establish upper and lower bounds for the transition probabilities that are sharp up to constants.

### 1. INTRODUCTION

There is a huge literature on the subject of transition probabilities of random walks on graphs. For a recent and comprehensive account, see the book [Wo]. The vast majority of the work, however, has been for nearest neighbor Markov chains. The purpose of this paper is to obtain good transition probability estimates for Markov chains on the integer lattice  $\mathbb{Z}^d$  in  $d$  dimensions in the case when the probability of a jump from a point  $x$  to a point  $y$  is comparable to that of a symmetric stable process of index  $\alpha \in (0, 2)$ .

To be more precise, for  $x, y \in \mathbb{Z}^d$  with  $x \neq y$ , let  $C_{xy}$  be positive finite numbers such that  $C_{xy} = C_{yx}$  for all  $x, y$  and  $\sum_z C_{xz} < \infty$  for all  $x$ . Set  $C_{xx} = 0$  for all  $x$ . We call  $C_{xy}$  the *conductance* between  $x$  and  $y$ . Define a symmetric Markov chain by

$$(1.1) \quad \mathbb{P}(X_1 = y \mid X_0 = x) = \frac{C_{xy}}{\sum_z C_{xz}}, \quad x, y \in \mathbb{Z}^d.$$

In this paper we will assume that  $\alpha \in (0, 2)$  and there exists  $\kappa > 1$  such that for all  $x \neq y$

$$(1.2) \quad \frac{\kappa^{-1}}{|x - y|^{d+\alpha}} \leq C_{xy} \leq \frac{\kappa}{|x - y|^{d+\alpha}}.$$

Write  $p(n, x, y)$  for  $\mathbb{P}^x(X_n = y)$ . The main result of this paper is

**Theorem 1.1.** *There exist positive finite constants  $c_1$  and  $c_2$  such that*

$$(1.3) \quad p(n, x, y) \leq c_1 \left( n^{-d/\alpha} \wedge \frac{n}{|x - y|^{d+\alpha}} \right),$$

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and for  $n \geq 2$

$$(1.4) \quad p(n, x, y) \geq c_2 \left( n^{-d/\alpha} \wedge \frac{n}{|x - y|^{d+\alpha}} \right).$$

If  $n = 1$  and  $x \neq y$ , (1.4) also holds.

The Markov chain  $X_n$  is discrete in time and in space. Closely related to  $X_n$  is the continuous time process  $Y_t$ , which is the process that waits at a point in  $\mathbb{Z}^d$  a length of time that is exponential with parameter 1, jumps according to the jump probabilities of  $X$ , then waits at the new point a length of time that is exponential with parameter 1 and independent of what has gone before, and so on. A continuous-time continuous state space process related to both  $X_t$  and  $Y_t$  is the process  $U_t$  on  $\mathbb{R}^d$  whose Dirichlet form is

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 C(x, y) dx dy,$$

where  $C(x, y)$  is a measurable function with

$$\frac{\kappa^{-1}}{|x - y|^{d+\alpha}} \leq C(x, y) \leq \frac{\kappa}{|x - y|^{d+\alpha}}.$$

The process  $U_t$  stands in the same relationship to  $X_n$  as the diffusion process corresponding to a uniformly elliptic operator in divergence form does to a nearest neighbor Markov chain.

The methods of this paper allow one to obtain bounds for the transition probabilities of  $Y_t$  and the transition densities of  $U_t$ . In fact, these are considerably easier than the bounds for  $X_n$ , so we concentrate in this paper only on the estimates for  $X_n$ . Some results for  $Y_t$  are needed, however, along the way.

Our methods are quite different from those used for diffusions or nearest neighbor chains. Recall that for a nearest neighbor Markov chain on  $\mathbb{Z}^d$ , the transition probabilities are bounded above and below by expressions of the form

$$c_1 n^{-d/2} \exp(-c_2 |x - y|^2 / n)$$

as long as  $|x - y|$  is not larger than  $n$ ; see [SZ]. One way of obtaining these results is to use a method of Davies as developed in [CKS]. The lack of a suitably fast decay in the conductances in (1.2) makes the powerful theorem of [CKS] only partially successful. We use that theorem to handle the small jumps and use a perturbation argument to handle the large jumps. Another difficulty that shows up is that, unlike the diffusion case,  $\mathbb{P}^x(|X_n - y| < 1)$  is not comparable to  $\mathbb{P}^x(\max_{k \leq n} |X_k - x| > |x - y|)$  when  $|x - y|$  is relatively large. We circumvent this by proving a parabolic Harnack inequality and using another perturbation argument.

Previous work related to this paper includes [Kl] and [Km]. In both these works partial results were obtained for estimates for the process  $U_t$  mentioned above. [SZ] studies nearest neighbor chains on  $\mathbb{Z}^d$ . In [HS-C] upper bounds of Gaussian type were obtained for Markov chains whose jumps had bounded range or where the conductances decayed at a Gaussian rate.

After some preliminaries, we obtain in Section 2 some probability estimates on the time for our Markov chain  $X_n$  to leave a ball. This is followed in Section 3 by a parabolic Harnack inequality. In Section 4 we obtain the upper bound in Theorem 1.1, and in Section 5 we prove the lower bound.

2. TIME TO LEAVE A BALL

We denote the ball of radius  $r$  centered at  $x$  by  $B(x, r)$ ; throughout we use the Euclidean metric.  $T_A$  will denote the first hit of a set  $A$  by whichever process is under consideration, while  $\tau_A$  will denote the first exit. The letter  $c$  with subscripts will denote positive finite constants whose exact value is unimportant and may change from occurrence to occurrence.

We assume we are given reals  $C_{xy}$  satisfying (1.2), and we define the transition probabilities for the Markov chain  $X_n$  by

$$(2.1) \quad p(1, x, y) = \mathbb{P}^x(X_1 = y) = \frac{C_{xy}}{C_x}, \quad x \neq y,$$

where  $C_x = \sum_z C_{xz}$ , and  $p(1, x, x) = 0$  for every  $x$ . The process  $X_n$  is symmetric (or reversible):  $C_x$  is an invariant measure for which the kernel  $C_x p(1, x, y)$  is symmetric in  $x, y$ . Note that  $c_1^{-1} \leq C_x/C_y \leq c_1$  for some positive and finite constant  $c_1$ .

Our main goal in this section is to get estimates for the time for  $X_n$  to leave a ball. See Theorem 2.8 for the exact statement.

We will need to introduce several processes to obtain our estimates. For the convenience of the reader, we give a brief summary.

- $X_n$  is a discrete time, discrete space Markov chain,
- $Y_t$  is a continuous time, discrete Markov chain,
- $U_t$  is a continuous time continuous space pure jump process,
- $Z_t$  is a pure jump Lévy process,
- $V_t$  is a rescaled version of  $Y_t$ , with generator  $\mathcal{A} + \mathcal{B}$ ,
- $W_t$  is the process  $V_t$  without large jumps, with generator  $\mathcal{A}$ .

We start with  $Y_t$ , the continuous time version of  $X_n$ , which we construct as follows: Let  $U_1, U_2, \dots$  be an i.i.d. sequence of exponential random variables with parameter 1 that is independent of the chain  $X_n$ . Let  $T_0 = 0$  and  $T_k = \sum_{i=1}^k U_i$ . Define  $Y_t = X_n$  if  $T_n \leq t < T_{n+1}$ . If we define  $A(x, y) = |x - y|^{d+\alpha} C_{xy}/C_x$ , then by (1.2), there is a finite positive constant  $\kappa_1$  so that  $\kappa_1^{-1} \leq A(x, y) \leq \kappa_1$ , and the infinitesimal generator of  $Y_t$  is

$$\sum_{y \neq x} [f(y) - f(x)] \frac{A(x, y)}{|x - y|^{d+\alpha}}.$$

Now we introduce several processes related to  $Y_t$ , needed in what follows. The rescaled process  $V_t = D^{-1}Y_{D^\alpha t}$  takes values in  $\mathcal{S} = D^{-1}\mathbb{Z}^d$  and has infinitesimal generator

$$\sum_{y \in \mathcal{S}, y \neq x} [f(y) - f(x)] \frac{A^D(x, y)}{D^d |x - y|^{d+\alpha}},$$

where  $A^D(x, y) = A(Dx, Dy)$  for  $x, y \in \mathcal{S}$ . If the large jumps of  $V_t$  are removed, we obtain the process  $W_t$  with infinitesimal generator

$$(2.2) \quad \mathcal{A}f(x) = \sum_{\substack{y \in \mathcal{S}, y \neq x \\ |x - y| \leq 1}} [f(y) - f(x)] \frac{A^D(x, y)}{D^d |x - y|^{d+\alpha}}.$$

To analyze  $W_t$ , we compare it to a Lévy process with a comparable transition kernel. Let  $Z_t$  be the Lévy process which has no drift and no Gaussian component and whose Lévy measure is

$$n_Z(dh) = \sum_{\substack{y \neq 0, |y| \leq 1 \\ y \in \mathcal{S}}} \frac{1}{D^d |y|^{d+\alpha}} \delta_y(dh).$$

Write  $q_Z(t, x, y)$  for the transition density for  $Z_t$ .

**Proposition 2.1.** *There exist  $c_1, c_2$  such that the transition density  $q_Z(t, x, y)$  satisfies*

$$q_Z(t, x, y) \leq \begin{cases} c_1 D^{-d} t^{-d/\alpha}, & t \leq 1, \\ c_2 D^{-d} t^{-d/2}, & t > 1. \end{cases}$$

*Proof.* The characteristic function  $\varphi_t(u)$  of  $Z_t$  is periodic with period  $2\pi D$  since  $Z_t$  is supported on  $\mathcal{S} = D^{-1}\mathbb{Z}^d$ . By the Lévy-Khintchine formula and the symmetry of  $n_Z$ ,

$$(2.3) \quad \varphi_t(u) = \exp \left( -2t \sum_{x \in \mathcal{S}, |x| \leq 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \right).$$

Let

$$(2.4) \quad Q(a) = \{(u_1, \dots, u_d) : -a < u_i \leq a, i = 1, \dots, d\}.$$

We estimate  $\varphi_t$  as follows.

*Case 1:*  $|u| \leq \frac{1}{2}$ .

Since  $|x| \leq 1$ , we have  $1 - \cos u \cdot x \geq c_3(u \cdot x)^2 = c_3|u|^2|x|^2 h_u(x)$ , where  $h_u(x) = (u \cdot x)^2/|u|^2|x|^2$ . Thus

$$\begin{aligned} \sum_{|x| \leq 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} &\geq c_3|u|^2 \sum_{|x| \leq 1} h_u(x) |x|^{2-d-\alpha} D^{-d} \\ &\geq c_4 D^{\alpha-2} |u|^2 \int_{B(0,D)} |x|^{2-d-\alpha} h_u(x) dx \\ &= c_4 D^{\alpha-2} |u|^2 \int_0^D r^{1-\alpha} \left[ \int_{S(r)} h_u(s) \sigma_r(ds) \right] dr, \end{aligned}$$

where  $S(r)$  is the  $(d-1)$ -dimensional sphere of radius  $r$  centered at 0, and  $\sigma_r(ds)$  is normalized surface measure on  $S(r)$ . Since  $h_u(x)$  depends on  $x$  only through  $x/|x|$ , the inner integral does not depend on  $r$ . Furthermore, by rotational invariance, it does not depend on  $u$ . Thus,

$$\sum_{|x| \leq 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \geq c_5 |u|^2.$$

*Case 2:*  $\frac{1}{2} \leq |u| \leq D/32$ .

Let  $A = \{x \in \mathcal{S} : \frac{1}{4|u|} \leq |x| \leq \frac{4}{|u|} \wedge 1, 1 \geq u \cdot x \geq \frac{1}{16}\}$ . If  $x \in A$ , then  $[1 - \cos u \cdot x] \geq c_6$ , the minimum value of  $|x|^{-d-\alpha}$  is  $c_7|u|^{d+\alpha}$ , and a bit of geometry shows that there are at least  $c_8|u|^{-d}D^d$  points in  $A$ . (Notice that  $|u| < D/32$  is required to prevent  $A$  from being empty.) We then have

$$\begin{aligned} \sum_{|x| \leq 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} &\geq \sum_A [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \\ &\geq c_6 c_7 |u|^{d+\alpha} c_8 |u|^{-d} = c_9 |u|^\alpha. \end{aligned}$$

Case 3:  $D/32 < |u|, u \in Q(\pi D)$ .

At least one component of  $u$  must be larger than  $c_{10}D$ , where  $c_{10} = 1/(32\sqrt{d})$ ; without loss of generality we may assume it is the first component. Let  $y_0 = (D^{-1}, 0, \dots, 0)$ . Since  $|u_1| \leq \pi D$  and  $u \cdot y_0 \geq c_{10}$ , then  $1 - \cos u \cdot y_0 \geq c_{11}$ . Hence

$$\sum_{|x| \leq 1} [1 - \cos u \cdot x] \frac{1}{D^d |x|^{d+\alpha}} \geq c_{11} D^{-d} |y_0|^{-d-\alpha} \geq c_{12} D^\alpha \geq c_{13} |u|^\alpha,$$

since  $u \in Q(\pi D)$ .

For  $u \in Q(\pi D)$ , we then have that  $\varphi_t(u)$  is real and

$$0 < \varphi_t(u) \leq e^{-c_{14}t|u|^2} + e^{-c_{15}t|u|^\alpha}.$$

Since  $Z_t$  is supported on  $\mathcal{S}$ ,

$$\begin{aligned} q_Z(t, x, y) &= \frac{1}{|Q(\pi D)|} \int_{Q(\pi D)} e^{iu \cdot (x-y)} \varphi_t(u) du \\ &\leq \frac{1}{|Q(\pi D)|} \int_{Q(\pi D)} \varphi_t(u) du \\ &\leq \frac{c_{16}}{D^d} \int_{\mathbb{R}^d} (e^{-c_{14}t|u|^2} + e^{-c_{15}t|u|^\alpha}) du, \end{aligned}$$

where  $|Q(\pi D)|$  denotes the Lebesgue measure of  $Q(\pi D)$ . Our result follows from applying a change of variables to each of the integrals on the right hand side.  $\square$

We now obtain bounds for the transition probabilities of  $W_t$ :

**Proposition 2.2.** *If  $q_W(t, x, y)$  is the transition density for  $W$ , then*

$$q_W(t, x, y) \leq \begin{cases} c_1 D^{-d} t^{-d/\alpha}, & t \leq 1, \\ c_2 D^{-d} t^{-d/2}, & t > 1. \end{cases}$$

The proof of Proposition 2.2 is almost identical with that of Theorem 1.2 in [BBG], and is omitted here.

To obtain off-diagonal bounds for  $q_W$  we again proceed as in [BBG]. Let

$$\begin{aligned} \Gamma(f, f)(x) &= \sum_{\substack{y \in \mathcal{S} \\ 0 < |x-y| \leq 1}} (f(x) - f(y))^2 \frac{A^D(x, y)}{D^d |x-y|^{d+\alpha}}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi} \Gamma(e^\psi, e^\psi)\|_\infty \vee \|e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi})\|_\infty, \\ E(t, x, y) &= \sup\{|\psi(x) - \psi(y)| - t\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}. \end{aligned}$$

**Proposition 2.3.** *For  $t \leq 1$  and  $x, y \in \mathcal{S}$ ,*

$$q_W(t, x, y) \leq c_1 D^{-d} t^{-d/\alpha} e^{-E(2t, x, y)}.$$

*Proof.* Allowing for slight differences in notation, the proof is very similar to the proof of Lemma 1.4 in [BBG]. The principal difference is the following. Let  $K$  be an integer larger than  $\frac{1}{2} + \frac{1}{\alpha}$ . Let  $M$  be a sufficiently regular manifold with volume growth given by  $V(x, r) \approx r^{2Kd}$ ,  $r > 1$ , and  $V(x, r) \approx r^d$ ,  $r < 1$ , where  $V(x, r)$  is the volume of the ball in  $M$  of radius  $r$  centered at  $x$ . We can then find a symmetric

Markov process  $\tilde{V}_t$  on  $M$ , independent of  $W$ , whose transition density with respect to a measure  $m$  on  $M$  satisfies

$$\begin{aligned} q_{\tilde{V}}(t, x, y) &\leq c_2 t^{-d/2}, & 0 < t \leq 1, \\ q_{\tilde{V}}(t, x, y) &\leq c_2 t^{-dK}, & 1 < t < \infty, \\ q_{\tilde{V}}(t, x, x) &\geq c_3 t^{-d/2}, & 0 < t \leq 1, \\ q_{\tilde{V}}(t, x, x) &\geq c_3 t^{-dK}, & 1 < t < \infty. \end{aligned}$$

Then  $q_W(t, x, y)q_{\tilde{V}}(t, x', y') \leq c_4 D^{-dt} t^{-d(\frac{1}{2} + \frac{1}{\alpha})}$  for all  $t$  while  $q_W(t, x, y)q_{\tilde{V}}(t, 0, 0) \geq c_5 D^{-dt} t^{-d(\frac{1}{2} + \frac{1}{\alpha})}$  for  $t \leq 1$ . With these changes, the proof is now as in [BBG].  $\square$

The next step is to estimate  $E(t, x, y)$  and use this in Proposition 2.3.

**Proposition 2.4.** *Suppose  $t \leq 1$ . Then*

$$q_W(t, x, y) \leq c_1 D^{-d} t^{-d/\alpha} e^{-|x-y|}.$$

In particular, for  $\frac{1}{4} \leq t \leq 1$ ,

$$q_W(t, x, y) \leq c_1 D^{-d} e^{-|x-y|}.$$

*Proof.* Let  $\psi(\xi) = B \cdot \xi$ , where  $B = (y - x)/|y - x|$ . Note that if  $|\xi - \zeta| \leq 1$ , then  $(e^{\psi(\zeta) - \psi(\xi)} - 1)^2 = (e^{B \cdot (\zeta - \xi)} - 1)^2$  is bounded by  $c_2 |B|^2 |\zeta - \xi|^2 = c_2 |\zeta - \xi|^2$ . Hence

$$e^{-2\psi(\xi)} \Gamma(e^\psi, e^\psi)(\xi) = \sum_{\substack{\zeta \in \mathcal{S} \\ 0 < |\xi - \zeta| \leq 1}} (e^{\psi(\zeta) - \psi(\xi)} - 1)^2 \frac{A^D(\xi, \zeta)}{D^d |\xi - \zeta|^{d+\alpha}}$$

is bounded by

$$c_3 \sum_{\substack{\zeta \in \mathcal{S} \\ 0 < |\xi - \zeta| \leq 1}} D^{-d} |\xi - \zeta|^{2-d-\alpha}.$$

Since the sum is over  $\zeta \in \mathcal{S}$  that are within a distance 1 from  $\xi$ , this in turn is bounded by  $c_4$ . We have the same bound when  $\psi$  is replaced by  $-\psi$ , so  $\Lambda(\psi)^2 \leq c_4^2$ . Moreover, the bound does not depend on  $x$  or  $y$ . On the other hand,

$$\psi(y) - \psi(x) = (y - x) \cdot (y - x)/|y - x| = |y - x|.$$

Using this in Proposition 2.3 and recalling that  $t \leq 1$ , we have our result.  $\square$

From the above estimate we can obtain an estimate for the time for  $W_t$  to leave a ball.

**Proposition 2.5.** *There exists  $c_1$  such that if  $t \leq 1$  and  $\lambda > 0$ , then*

$$\mathbb{P}^x(\sup_{s \leq t} |W_s - x| > \lambda) \leq c_1 e^{-\lambda/8}.$$

*Proof.* From Proposition 2.4 and summing, if  $t \in [\frac{1}{4}, 1]$  and  $\lambda > 0$ ,

$$(2.5) \quad \mathbb{P}^x(|W_t - x| \geq \lambda) \leq \sum_{\substack{y \in \mathcal{S} \\ |y-x| \geq \lambda}} c_2 t^{-d/\alpha} D^{-d} e^{-|y-x|} \leq c_3 e^{-\lambda/2}.$$

Let  $S_\lambda = \inf\{t : |W_t - W_0| \geq \lambda\}$ . Then, using (2.5),

$$\begin{aligned} \mathbb{P}^x(\sup_{s \leq 1/2} |W_s - x| \geq \lambda) &= \mathbb{P}^x(S_\lambda \leq 1/2) \\ &= \mathbb{P}^x(|W_1 - x| > \lambda/2) + \mathbb{P}^x(S_\lambda \leq 1/2, |W_1 - x| \leq \lambda/2) \\ &\leq c_3 e^{-\lambda/4} + \int_0^{1/2} \mathbb{P}^x(|W_1 - W_s| > \lambda/2, S_\lambda \in ds). \end{aligned}$$

By the Markov property, the last term on the right is bounded by

$$\begin{aligned} &\int_0^{1/2} \mathbb{E}^x[\mathbb{P}^{W_s}(|W_{1-s} - W_0| > \lambda/2); S_\lambda \in ds] \\ &\leq c_3 e^{-\lambda/4} \int_0^{1/2} \mathbb{P}^x(S_\lambda \in ds) \leq c_3 e^{-\lambda/4}, \end{aligned}$$

using (2.5) again. Adding gives

$$(2.6) \quad \mathbb{P}^x(\sup_{s \leq t} |W_s - x| > \lambda) \leq c_4 e^{-\lambda/4}$$

as long as  $t \leq \frac{1}{2}$ . For  $t \in (\frac{1}{2}, 1]$ , I've made the observation that if  $\sup_{s \leq t} |W_s - x| > \lambda$ , then  $\sup_{s \leq \frac{1}{2}} |W_s - x| > \lambda/2$  or  $\sup_{\frac{1}{2} < s \leq 1} |W_s - W_{1/2}| > \lambda/2$ . The probability of the first event is bounded using (2.6), while the probability of the second event is bounded using the Markov property at time  $\frac{1}{2}$  and (2.6).  $\square$

Define  $\mathcal{B}$  to be the infinitesimal generator of  $V_t$  without small jumps:

$$(2.7) \quad \mathcal{B}f(x) = \sum_{\substack{y \in \mathcal{S} \\ |y-x| > 1}} [f(y) - f(x)] \frac{A^D(x, y)}{D^d |x - y|^{d+\alpha}}.$$

Our next goal is to obtain estimates for the process  $V_t = D^{-1}Y_{D^\alpha t}$ , whose generator is  $\mathcal{A} + \mathcal{B}$ .

**Proposition 2.6.** *Let  $V_t$  be the process whose generator is  $\mathcal{A} + \mathcal{B}$ . There exist  $c_1, c_2$  and  $\delta_0$  such that if  $\delta \leq \delta_0$  and  $\lambda \geq 1$ , then*

$$\mathbb{P}^x(\sup_{s \leq \delta} |V_s - x| > \lambda) \leq c_1 e^{-c_2 \lambda} + c_2 \delta.$$

*Proof.* Since summing  $D^d |y - x|^{-d-\alpha}$  over  $|y - x| \geq 1$  is a constant,

$$(2.8) \quad |\mathcal{B}f(x)| \leq c_3 \|f\|_\infty,$$

and hence  $\mathcal{B}$  is a bounded operator on  $L^\infty$ .

Define  $Q_t^W f(x) = \sum q_W(t, x, y) f(y)$ . Let  $Q_t^V$  be the corresponding transition semigroup for  $V_t$ . Let  $S_0(t) = Q_t^W$  and for  $n \geq 1$ , let  $S_n(t) = \int_0^t S_{n-1} \mathcal{B} Q_{t-s}^W ds$ . Then

$$Q_t^V = \sum_{n=0}^\infty S_n(t);$$

see [Le], Theorem 2.2, for example. Obviously  $Q_t^W$  is a bounded operator on  $L^\infty$  of norm 1, so for  $t < \delta_0 = 1/(2c_3)$  the sum converges by (2.8). In particular, for  $t \leq \delta \leq \delta_0$ , we have

$$|Q_t^W f(x) - Q_t^V f(x)| \leq c_4 \delta \|f\|_\infty.$$

Fix  $x$  and apply this to  $f(y) = 1_{B(x,\lambda)}(y)$ . We obtain

$$\begin{aligned} \mathbb{P}^x(|V_t - x| > \lambda) &= Q_t^V f(x) \leq Q_t^W f(x) + c_4\delta \\ &= \mathbb{P}^x(|W_t - x| > \lambda) + c_4\delta \leq c_5e^{-\lambda/8} + c_4\delta. \end{aligned}$$

We now obtain our result by applying the method of proof of Proposition 2.5.  $\square$

Now notice that  $Y_t = DV_{t/D^\alpha}$ . Translating Proposition 2.6 in terms of  $Y_t$ , we have

**Corollary 2.7.** *If  $\lambda \geq 1$  and  $\delta \leq \delta_0$ , then*

$$\mathbb{P}^x\left(\sup_{s \leq \delta D^\alpha} |Y_s - x| > \lambda D\right) \leq c_1e^{-c_2\lambda} + c_1\delta$$

for every  $D$ .

We can now obtain the result for the time for  $X_n$  to leave a ball.

**Theorem 2.8.** *Given  $C > 1$  and  $\beta \in (0, 1)$ , there exists  $\gamma$  such that*

$$(2.9) \quad \mathbb{P}^x\left(\max_{k \leq \lceil \gamma S^\alpha \rceil} |X_k - x| > CS\right) \leq \beta$$

for all  $S > 0$ .

*Proof.* Let  $\beta \in (0, 1)$ . By Corollary 2.7 we may choose  $\lambda$  and  $\delta \leq \delta_0/2$  so that

$$\mathbb{P}^x\left(\sup_{s \leq 2\delta D^\alpha} |Y_s - x| > \lambda D\right) \leq \beta/2$$

for every  $D$ . Define  $D = CS/\lambda$ . We may suppose  $Y$  is constructed as in Section 2. Then

$$\begin{aligned} \mathbb{P}^x\left(\max_{k \leq \lceil \delta D^\alpha \rceil} |X_k - x| > CS\right) &\leq \mathbb{P}^x\left(\sup_{s \leq 2\delta D^\alpha} |Y_s - x| > CS\right) \\ &\quad + \mathbb{P}^x(|T_{\lceil \delta D^\alpha \rceil} - \lceil \delta D^\alpha \rceil| > \lceil \delta D^\alpha \rceil) \\ &\leq \frac{\beta}{2} + \frac{c_3}{\delta D^\alpha}. \end{aligned}$$

We used Chebyshev’s inequality and the fact that  $T_{\lceil \delta D^\alpha \rceil}$  is the sum of i.i.d. exponentials to bound the second probability on the right hand side. Choose  $S_0$  large enough so that  $c_3/\delta D^\alpha < \beta/2$  if  $S \geq S_0$ . We thus have the desired result for  $S \geq S_0$ .

Finally, choose  $\gamma$  smaller if necessary so that  $\gamma S_0^\alpha < 1$ . If  $S < S_0$ , then  $\gamma S^\alpha < 1$ . But  $X_k$  needs at least one unit of time to make a step; hence the left hand side of (2.9) is 0 if  $S < S_0$ .  $\square$

*Remark 2.9.* Given the above probability estimate, one could formulate a central limit theorem. Under a suitable normalization, a sequence of Markov chains whose jump structure is similar to that of a symmetric stable process should converge weakly to a process such as the  $U_t$  described in Section 1.

*Remark 2.10.* We expect that our techniques could also be used to prove tightness for Markov chains where the conductances decay more rapidly than the rates given in this paper. In this case one might have a central limit theorem where the limiting distributions are those of processes corresponding to elliptic operators in divergence form. It would be quite interesting to formulate a central limit theorem for Markov chains where the limit processes are diffusions but the Markov chains do not have bounded range.

3. HARNACK INEQUALITY

It is fairly straightforward at this point to follow the argument of [BL] and obtain a Harnack inequality of Moser type for functions that are harmonic with respect to  $X_n$ . In this paper, however, we are primarily interested in transition probability estimates. As a tool for obtaining these, we turn to a parabolic Harnack inequality.

Let  $\mathcal{T} = \{0, 1, 2, \dots\} \times \mathbb{Z}^d$ . We will study here the  $\mathcal{T}$ -valued Markov chain  $(V_k, X_k)$ , where  $V_k = V_0 + k$ . We write  $\mathbb{P}^{(j,x)}$  for the law of  $(V_k, X_k)$  started at  $(j, x)$ . Let  $\mathcal{F}_j = \sigma((V_k, X_k) : k \leq j)$ . A bounded function  $q(k, x)$  on  $\mathcal{T}$  will be said to be *parabolic* on  $D \subset \mathcal{T}$  if  $q(V_{k \wedge \tau_D}, X_{k \wedge \tau_D})$  is a martingale.

Define

$$(3.0) \quad Q(k, x, r) = \{k, k + 1, \dots, k + \lceil \gamma r^\alpha \rceil\} \times B(x, r).$$

Our goal in this section is the following result:

**Theorem 3.1.** *There exists  $c_1$  such that if  $q$  is bounded and nonnegative on  $\mathcal{T}$  and parabolic on  $Q(0, z, R)$ , then*

$$\max_{(k,y) \in Q(\lceil \gamma R^\alpha \rceil, z, R/3)} q(k, y) \leq c_1 \min_{y \in B(z, R/3)} q(0, y).$$

We prove this after first establishing a few intermediate results.

From Theorem 2.8 there exists  $\gamma$  such that for all  $r > 0$

$$(3.1) \quad \mathbb{P}^x \left( \max_{k \leq \lceil \gamma r^\alpha \rceil} |X_k - x| > r/2 \right) \leq \frac{1}{4}.$$

Without loss of generality we may assume  $\gamma \in (0, \frac{1}{3})$ .

We will often write  $\tau_r$  for  $\tau_{Q(0,x,r)}$ . For  $A \subset Q(0, x, r)$  set  $A(k) = \{y : (k, y) \in A\}$ . Define  $N(k, x)$  to be  $\mathbb{P}^{(k,x)}(X_1 \in A(k+1))$  if  $(k, x) \notin A$ , and 0 otherwise.

**Lemma 3.2.** *Let*

$$J_n = 1_A(V_n, X_n) - 1_A(V_0, X_0) - \sum_{k=0}^{n-1} N(V_k, X_k).$$

*Then  $J_{n \wedge T_A}$  is a martingale.*

*Proof.* We have

$$\mathbb{E} [J_{(k+1) \wedge T_A} - J_{k \wedge T_A} \mid \mathcal{F}_k] = \mathbb{E} [1_A(V_{(k+1) \wedge T_A}, X_{(k+1) \wedge T_A}) - 1_A(V_{k \wedge T_A}, X_{k \wedge T_A}) - N(V_{k \wedge T_A}, X_{k \wedge T_A}) \mid \mathcal{F}_k].$$

On the event  $\{T_A \leq k\}$ , this is 0. If  $T_A > k$ , this is equal to

$$\mathbb{P}^{(V_k, X_k)}((V_1, X_1) \in A) - N(V_k, X_k) = \mathbb{P}^{X_k}(X_1 \in A(V_k + 1)) - N(V_k, X_k) = 0.$$

□

Given a set  $A \subset \mathcal{T}$ , we let  $|A|$  denote the cardinality of  $A$ .

**Proposition 3.3.** *There exists  $\theta_1$  such that if  $A \subset Q(0, x, r/2)$  and  $A(0) = \emptyset$ , then*

$$\mathbb{P}^{(0,x)}(T_A < \tau_r) \geq \theta_1 \frac{|A|}{r^{d+\alpha}}.$$

*Proof.* Observe that  $T_A$  cannot equal  $\tau_r$ . If  $\mathbb{P}^{(0,x)}(T_A \leq \tau_r) \geq \frac{1}{4}$  we are done, so assume without loss of generality that  $\mathbb{P}^{(0,x)}(T_A \leq \tau_r) < \frac{1}{4}$ . Let  $S = T_A \wedge \tau_r$ . From Lemma 3.2 and optional stopping we have

$$\mathbb{E}^{(0,x)} 1_A(S, X_S) \geq \mathbb{E}^{(0,x)} \sum_{k=0}^{S-1} N(k, X_k).$$

Note that if  $(k, x) \in Q(0, x, r)$ , then

$$N(k, x) = \mathbb{P}^{(k,x)}(X_1 \in A(k+1)) \geq \sum_{y \in A(k+1)} \frac{c_1}{|x-y|^{d+\alpha}} \geq \frac{c_2}{r^{d+\alpha}} |A(k+1)|.$$

So on the set  $(S \geq \lceil \gamma r^\alpha \rceil)$  we have  $\sum_{k=0}^{S-1} N(k, X_k) \geq c_3 |A|/r^{d+\alpha}$ . Therefore, since  $\tau_r \leq \lceil \gamma r^\alpha \rceil$ ,

$$\begin{aligned} \mathbb{E}^{(0,x)} 1_A(S, X_S) &\geq c_4 \frac{|A|}{r^{d+\alpha}} \mathbb{P}^x(S \geq \lceil \gamma r^\alpha \rceil) \\ &\geq c_4 \frac{|A|}{r^{d+\alpha}} [1 - \mathbb{P}^x(T_A \leq \tau_r) - \mathbb{P}^x(\tau_r < \lceil \gamma r^\alpha \rceil)]. \end{aligned}$$

Now  $\mathbb{P}^x(\tau_r < \lceil \gamma r^\alpha \rceil) \leq \frac{1}{4}$  by (3.1). Therefore  $\mathbb{E}^{(0,x)} 1_A(S, X_S) \geq c_5 |A|/r^{d+\alpha}$ . Since  $A \subset Q(0, x, r/2)$ , the proposition follows.  $\square$

With  $Q(k, x, r)$  defined as in (3.0), let  $U(k, x, r) = \{k\} \times B(x, r)$ .

**Lemma 3.4.** *There exists  $\theta_2$  such that if  $(k, x) \in Q(0, z, R/2)$ ,  $r \leq R/4$ , and  $k \geq \lceil \gamma r^\alpha \rceil + 2$ , then*

$$\mathbb{P}^{(0,z)}(T_{U(k,x,r)} < \tau_{Q(0,z,R)}) \geq \theta_2 r^{d+\alpha} / R^{d+\alpha}.$$

*Proof.* Let  $Q' = \{k, k-1, \dots, k - \lceil \gamma r^\alpha \rceil\} \times B(x, r/2)$ . By Proposition 3.3,

$$\mathbb{P}^{(0,z)}(T_{Q'} < \tau_{Q(0,z,R)}) \geq c_1 r^{d+\alpha} / R^{d+\alpha}.$$

Starting at a point in  $Q'$ , by (3.1) there is probability at least  $\frac{3}{4}$  that the chain stays in  $B(x, r)$  for at least time  $\gamma r^\alpha$ . So by the strong Markov property, there is probability at least  $\frac{3}{4} c_1 r^{d+\alpha} / R^{d+\alpha}$  that the chain hits  $Q'$  before exiting  $Q(0, z, R)$  and stays within  $B(x, r)$  for an additional time  $c_2 r^\alpha$ , hence hits  $U(k, x, r)$  before exiting  $Q(0, z, R)$ .  $\square$

**Lemma 3.5.** *Suppose  $H(k, w)$  is nonnegative and 0 if  $w \in B(x, 2r)$ . There exists  $\theta_3$  (not depending on  $x, r$ , or  $H$ ) such that*

$$\mathbb{E}^{(0,x)} [H(V_{\tau_r}, X_{\tau_r})] \leq \theta_3 \mathbb{E}^{(0,y)} [H(V_{\tau_r}, X_{\tau_r})], \quad y \in B(x, r/3).$$

*Proof.* Fix  $x$  and  $r$ , and suppose  $k \leq \lceil \gamma r^\alpha \rceil$  and  $w \notin B(x, 2r)$ . Assume for now that  $\lceil \gamma r^\alpha \rceil \geq 4$ . We claim there exists  $c_1$  such that

$$M_j = 1_{(k,w)}(V_{j \wedge \tau_r}, X_{j \wedge \tau_r}) - \sum_{i=0}^{j-1} \frac{c_1}{|w-x|^{d+\alpha}} 1_{(i < \tau_r)} 1_{k-1}(V_i)$$

is a submartingale. To see this we observe that the quantity

$$\mathbb{E} [1_{(k,w)}(V_{(i+1) \wedge \tau_r}, X_{(i+1) \wedge \tau_r}) - 1_{(k,w)}(V_{i \wedge \tau_r}, X_{i \wedge \tau_r}) \mid \mathcal{F}_i]$$

is 0 if  $i \geq \tau_r$ , and otherwise it equals

$$\mathbb{E}^{(V_i, X_i)} 1_{(k,w)}(V_{1 \wedge \tau_r}, X_{1 \wedge \tau_r}).$$

This is 0 unless  $k = V_i + 1$ . When  $k = V_i + 1$  and  $i < \tau_r$  this quantity is equal to

$$\mathbb{P}^{X_i}(X_1 = w) \geq \frac{c_2}{|X_i - w|^{d+\alpha}} \geq \frac{c_3}{|x - w|^{d+\alpha}}.$$

Thus  $\mathbb{E}[M_{i+1} - M_i \mid \mathcal{F}_i]$  is 0 if  $i \geq \tau_r$  or  $k \neq V_i + 1$ , and greater than or equal to 0 otherwise if  $c_1$  is less than  $c_3$ , which proves the claim.

Since  $\mathbb{P}^y(\max_{i \leq \lceil \gamma r^\alpha \rceil} |X_i - X_0| > r/2) \leq \frac{1}{4}$ , then

$$(3.3) \quad \mathbb{E}^{(0,y)} \tau_r \geq \lceil \gamma r^\alpha \rceil \mathbb{P}^{(0,x)}(\tau_r \geq \lceil \gamma r^\alpha \rceil) \geq \lceil \gamma r^\alpha \rceil / 2.$$

The random variable  $\tau_r$  is obviously bounded by  $\lceil \gamma r^\alpha \rceil$ , so by optional stopping,

$$\mathbb{P}^{(0,y)}((V_{\tau_r}, X_{\tau_r}) = (k, w)) \geq \left( \mathbb{E}^{(0,y)} \tau_r - 1 \right) \frac{c_4}{|x - w|^{d+\alpha}} \geq \frac{c_4 r^\alpha}{|x - w|^{d+\alpha}}.$$

Similarly, there exists  $c_5$  such that

$$1_{(k,w)}(V_{j \wedge \tau_r}, X_{j \wedge \tau_r}) - \sum_{i=1}^{j-1} \frac{c_5}{|w - x|^{d+\alpha}} 1_{(i < \tau_r)} 1_{k-1}(V_i)$$

is a supermartingale, and so

$$\mathbb{P}^{(0,x)}((V_{\tau_r}, X_{\tau_r}) = (k, w)) \leq \left( \mathbb{E}^{(0,x)} \tau_r \right) \frac{c_6}{|x - w|^{d+\alpha}} \leq \frac{c_6 r^\alpha}{|x - w|^{d+\alpha}}.$$

Letting  $\theta_3 = c_6/c_4$ , we have

$$\mathbb{E}^{(0,x)}[1_{(k,w)}(V_{\tau_r}, X_{\tau_r})] \leq \theta_3 \mathbb{E}^{(0,y)}[1_{(k,w)}(V_{\tau_r}, X_{\tau_r})].$$

It is easy to check that  $\theta_3$  can be chosen so that this inequality also holds when  $\lceil \gamma r^\alpha \rceil < 4$ . Multiplying by  $H(k, w)$  and summing over  $k$  and  $w$  proves our lemma.  $\square$

**Proposition 3.6.** *For each  $n_0$  and  $x_0$ , the function  $q(k, x) = p(n_0 - k, x, x_0)$  is parabolic on  $\{0, 1, \dots, n_0\} \times \mathbb{Z}^d$ .*

*Proof.* We have

$$\begin{aligned} \mathbb{E}[q(V_{k+1}, X_{k+1}) \mid \mathcal{F}_k] &= \mathbb{E}[p(n_0 - V_{k+1}, X_{k+1}, x_0) \mid \mathcal{F}_k] \\ &= \mathbb{E}^{(V_k, X_k)}[p(n_0 - V_1, X_1, x_0)] \\ &= \sum_z p(1, X_k, z) p(n_0 - V_k - 1, z, x_0). \end{aligned}$$

By the semigroup property this is

$$p(n_0 - V_k, X_k, x_0) = q(V_k, X_k).$$

$\square$

*Proof of Theorem 3.1.* By multiplying by a constant, we may suppose that

$$\min_{y \in B(z, R/3)} q(0, y) = 1.$$

Let  $v$  be a point in  $B(z, R/3)$  where  $q(0, v)$  takes the value one. Suppose  $(k, x) \in Q(\lceil \gamma R^\alpha \rceil, z, R/3)$  with  $q(k, x) = K$ . By Proposition 3.3 there exists  $c_2 \leq 1$  such that if  $r < R/3$ ,  $C \subset Q(k + 1, x, r/3)$ , and  $|C|/|Q(k + 1, x, r/3)| \geq \frac{1}{3}$ , then

$$(3.4) \quad \mathbb{P}^{(k,x)}(T_C < \tau_r) \geq c_2.$$

Set

$$(3.5) \quad \eta = \frac{c_2}{3}, \quad \zeta = \frac{1}{3} \wedge (\theta_3 \eta).$$

Define  $r$  to be the smallest number such that

$$(3.6) \quad \frac{|Q(0, x, r/3)|}{R^{d+\alpha}} \geq \frac{3}{\theta_1 \zeta K}$$

and

$$(3.7) \quad \frac{r^{d+\alpha}}{R^{d+\alpha}} \geq \frac{2}{\zeta K \theta_2}.$$

This implies

$$(3.8) \quad r/R = c_3 K^{-1/(d+\alpha)}.$$

Let

$$A = \{(i, y) \in Q(k + 1, x, r/3) : q(i, y) \geq \zeta K\}.$$

Let  $U = \{k\} \times B(x, r/3)$ . If  $q \geq \zeta K$  on  $U$ , we would then have by Lemma 3.4 that

$$\begin{aligned} 1 = q(0, v) &= \mathbb{E}^{(0,v)}[q(V_{T_U \wedge \tau_{Q(0,z,R)}}, X_{T_U \wedge \tau_{Q(0,z,R)}})] \\ &\geq \zeta K \mathbb{P}^{(0,v)}(T_U < \tau_{Q(0,z,R)}) \geq \frac{\theta_2 r^{d+\alpha} \zeta K}{R^{d+\alpha}}, \end{aligned}$$

a contradiction to our choice of  $r$ . So there must exist at least one point in  $U$  for which  $q$  takes a value less than  $\zeta K$ .

If  $\mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x, 2r)] \geq \eta K$ , then by Lemma 3.5 we would have

$$\begin{aligned} q(k, y) &\geq \mathbb{E}^{(k,y)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x, 2r)] \\ &\geq \theta_3 \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x, 2r)] \geq \theta_3 \eta K \geq \zeta K \end{aligned}$$

for  $y \in B(x, r/3)$ , a contradiction to the preceding paragraph. Therefore

$$(3.9) \quad \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); X_{\tau_r} \notin B(x, 2r)] \leq \eta K.$$

By Proposition 3.3,

$$\begin{aligned} 1 = q(0, v) &\geq \mathbb{E}^{(0,v)}[q(V_{T_A}, X_{T_A}); T_A < \tau_{Q(0,z,R)}] \\ &\geq \zeta K \mathbb{P}^{(0,v)}(T_A < \tau_{Q(0,z,R)}) \geq \frac{\theta_1 |A| \zeta K}{R^{d+\alpha}}; \end{aligned}$$

hence

$$\frac{|A|}{|Q(k + 1, x, r/3)|} \leq \frac{R^{d+\alpha}}{\theta_1 |Q(k + 1, x, r/3)| \zeta K} \leq \frac{1}{3}.$$

Let  $C = Q(k + 1, x, r/3) - A$ . Let  $M = \max_{Q(k+1,x,2r)} q$ . We write

$$\begin{aligned} q(k, x) &= \mathbb{E}^{(k,x)}[q(V_{T_C}, X_{T_C}); T_C < \tau_r] \\ &\quad + \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \notin B(x, 2r)] \\ &\quad + \mathbb{E}^{(k,x)}[q(V_{\tau_r}, X_{\tau_r}); \tau_r < T_C, X_{\tau_r} \in B(x, 2r)]. \end{aligned}$$

The first term on the right is bounded by  $\zeta K \mathbb{P}^{(k,x)}(T_C < \tau_r)$ . The second term on the right is bounded by  $\eta K$ . The third term is bounded by  $M \mathbb{P}^{(k,x)}(\tau_r < T_C)$ . Therefore

$$K \leq \zeta K \mathbb{P}^{(k,x)}(T_C < \tau_r) + \eta K + M(1 - \mathbb{P}^{(k,x)}(T_C < \tau_r)).$$

It follows that

$$M/K \geq 1 + \beta$$

for some  $\beta$  not depending on  $x$  or  $r$ , and therefore it follows that there exists a point  $(k', x') \in Q(k + 1, x, 2r)$  such that  $q(k', x') \geq (1 + \beta)K$ .

We use this to construct a sequence of points: suppose there exists a point  $(k_1, x_1)$  in  $Q([\gamma R^\alpha], z, R/6)$  such that  $q(k_1, x_1) = K$ . We let  $x = x_1, k = k_1$  in the above and construct  $r_1 = r, x_2 = x',$  and  $k_2 = k'$ . We define  $r_2$  by the analogues of (3.6) and (3.7). We then use the above (with  $(k, x)$  replaced by  $(k_2, x_2)$  and  $(k', x')$  replaced by  $(k_3, x_3)$ ) to construct  $k_3, x_3,$  and so on. We thus have a sequence of points  $(k_i, x_i)$  for which  $k_{i+1} - k_i \leq (2r_i)^\alpha, |x_{i+1} - x_i| \leq 2r_i,$  and  $q(k_i, x_i) \geq (1 + \beta)^{i-1}K$ . By (3.8) there exists  $K'$  such that if  $K \geq K',$  then  $(k_i, x_i) \in Q([\gamma R^\alpha], z, R/3)$  for all  $i$ . We show this leads to a contradiction. One possibility is that for large  $i$  we have  $r_i < 1,$  which means that  $B(x_i, r_i)$  is a single point, contradicting the fact that there is at least one point in  $B(x_i, r_i)$  for which  $q(k_i, \cdot)$  is less than  $\eta(1 + \beta)^{i-1}K$ . The other possibility is that  $q(k_i, x_i) \geq (1 + \beta)^{i-1}K' > \|q\|_\infty$  for large  $i,$  again a contradiction. We conclude that  $q$  is bounded by  $K'$  in  $Q([\gamma R^\alpha], z, R/3)$ .  $\square$

#### 4. UPPER BOUNDS

In this section our goal is to obtain upper bounds on the transition probabilities for our chain  $X_n$ . Again we need some auxiliary processes.

$\tilde{V}_t$  is a rescaled version of  $Y_t,$  with generator  $\tilde{A} + \tilde{B},$

$\tilde{W}_t$  is  $\tilde{V}_t$  without large jumps, with generator  $\tilde{A},$

$\zeta_t$  is a pure jump Lévy process.

We start by proving a uniform upper bound. Let us begin by considering the Lévy process  $\zeta_t$  whose Lévy measure is

$$n(dx) = \sum_{y \in \mathbb{Z}^d, y \neq 0} |y|^{-(d+\alpha)} \delta_y(dx).$$

**Proposition 4.1.** *The transition density for  $\zeta_t$  satisfies  $q_\zeta(t, x, y) \leq c_1 t^{-d/\alpha}.$*

*Proof.* The proof is similar to Proposition 2.1 (with  $D = 1$ ). The characteristic function  $\varphi_t(u)$  is given by

$$\varphi_t(u) = \exp\left(-2t \sum_{x \in \mathbb{Z}^d} [1 - \cos(u \cdot x)] \frac{1}{|x|^{d+\alpha}}\right).$$

For  $|u| \leq 1/32,$  we proceed similarly to Case 2 of the proof of Proposition 2.1: we set  $D = 1$  and  $A = \{x \in \mathbb{Z}^d : \frac{1}{4|u|} \leq |x| \leq \frac{4}{|u|}, 1 \geq u \cdot x \geq \frac{1}{16}\},$  and obtain

$$\sum [1 - \cos u \cdot x] \frac{1}{|x|^{d+\alpha}} \geq c_2 |u|^\alpha.$$

Let  $Q(a)$  be defined by (2.4). For  $|u| > 1/32$  with  $u \in Q(\pi),$  we proceed as in Case 3 of the proof of Proposition 2.1 and obtain the same estimate. We then proceed as in the remainder of the proof of Proposition 2.1 to obtain our desired result.  $\square$

**Proposition 4.2.** *The transition densities for  $Y_t$  satisfy*

$$q_Y(t, x, y) \leq c_1 t^{-d/\alpha}.$$

*Proof.* This is similar to the proof of Proposition 2.2, but considerably simpler, as we do not have to distinguish between  $t \leq 1$  and  $t > 1$ .  $\square$

Now we can obtain global bounds for the transition probabilities for  $X_n$ .

**Theorem 4.3.** *There exists  $c_1$  such that the transition probabilities for  $X_n$  satisfy*

$$p(n, x, y) \leq c_1 n^{-d/\alpha}, \quad x, y \in \mathbb{Z}^d.$$

*Proof.* Recall the construction of  $Y_t$  in Section 1. First, by the law of large numbers,  $T_n/n \rightarrow 1$  a.s. Thus there exists  $c_2$  such that  $\mathbb{P}(T_{[n/2]} \leq \frac{3}{4}n < T_n) \geq c_2$  for all  $n$ .

Let  $C_x = \sum_z C_{xz}$ , and set  $r(n, x, y) = C_x p(2n, x, y)$ . Since  $C_x p(1, x, y)$  is symmetric, it can be seen by induction that  $C_x p(n, x, y)$  is symmetric. The kernel  $r(n, x, y)$  is nonnegative definite because

$$\begin{aligned} \sum_x \sum_y f(x)r(n, x, y)f(y) &= \sum_x \sum_y \sum_z f(x)C_x p(n, x, z)p(n, z, y)f(y) \\ &= \sum_x \sum_y \sum_z f(x)f(y)C_z p(n, z, x)p(n, z, y) \\ &= \sum_z C_z \left( \sum_x f(x)p(n, z, x) \right)^2 \geq 0. \end{aligned}$$

If we set  $r_M(n, x, y) = r(n, x, y)$  if  $|x|, |y| \leq M$  and 0 otherwise, we have an eigenfunction expansion for  $r_M$ :

$$(4.1) \quad r_M(n, x, y) = \sum_i \lambda_i^n \varphi_i(x)\varphi_i(y),$$

where each  $\lambda_i \in [0, 1]$ . By Cauchy-Schwarz,

$$\begin{aligned} r_M(n, x, y) &\leq \left( \sum_i \lambda_i^n \varphi_i(x)^2 \right)^{1/2} \left( \sum_i \lambda_i^n \varphi_i(y)^2 \right)^{1/2} \\ &= r_M(n, x, x)^{1/2} r_M(n, y, y)^{1/2}. \end{aligned}$$

Also, by (4.1)  $r_M(n, x, x)$  is decreasing in  $n$ . Letting  $M \rightarrow \infty$ , we see that  $p(2n, x, x)$  is decreasing in  $n$  and

$$p(2n, x, y) \leq p(2n, x, x)^{1/2} p(2n, y, y)^{1/2}.$$

Suppose now that  $n$  is even and  $n \geq 8$ . It is clear from (1.2) and (2.1) that there exists  $c_3$  such that  $p(3, z, z) \geq c_3$  for all  $z \in \mathbb{Z}^d$ . If  $k$  is even and  $k \leq n$ , then  $\mathbb{P}^x(X_k = x) \geq \mathbb{P}^x(X_n = x)$ . If  $k$  is odd and  $k \leq n$ , then

$$\begin{aligned} \mathbb{P}^x(X_k = x) &= p(k, x, x) \geq p(k-3, x, x)p(3, x, x) \geq c_3 \mathbb{P}^x(X_{k-3} = x) \\ &\geq c_3 \mathbb{P}^x(X_n = x). \end{aligned}$$

Setting  $t = \frac{3}{4}n$ , using Proposition 4.2, and the independence of the  $T_i$  from the  $X_k$ , we have

$$\begin{aligned} c_4 t^{-d/\alpha} &\geq \mathbb{P}^x(Y_t = x) = \sum_{k=0}^{\infty} \mathbb{P}^x(X_k = x, T_k \leq t < T_{k+1}) \\ &\geq \sum_{[n/2] \leq k \leq n} \mathbb{P}^x(X_k = x) \mathbb{P}(T_k \leq t < T_{k+1}) \\ &\geq c_3 \mathbb{P}^x(X_n = x) \mathbb{P}^x(T_{[n/2]} \leq t < T_n) \geq c_2 c_3 \mathbb{P}^x(X_n = x). \end{aligned}$$

We thus have an upper bound for  $p(n, x, x)$  when  $n$  is even, and by the paragraph above, for  $p(n, x, y)$  when  $n \geq 8$  is even.

Now suppose  $n$  is odd and  $n \geq 5$ . Then

$$c_5(n + 3)^{-d/\alpha} \geq p(n + 3, x, y) \geq p(n, x, y)p(3, y, y) \geq c_3p(n, x, y),$$

which implies the desired bound when  $n$  is odd and  $n \geq 5$ .

Finally, since  $p(n, x, y) = \mathbb{P}^x(X_n = y) \leq 1$ , we have our bound for  $n \leq 8$  by taking  $c_1$  larger if necessary.  $\square$

We now turn to the off-diagonal bounds, that is, when  $|x - y|/n^{1/\alpha}$  is large. We begin by bounding  $\mathbb{P}^x(Y_{t_0} \in B(y, rt_0^{1/\alpha}))$ . To do this, it is more convenient to look at  $\tilde{V}_t = t_0^{-1/\alpha}Y_{t_0t}$  and to obtain a bound on  $\mathbb{P}^x(\tilde{V}_1 \in B(y, r))$  for  $x, y \in \mathcal{S} = t_0^{-1/\alpha}\mathbb{Z}^d$ . The infinitesimal generator for  $\tilde{V}_t$  is

$$\sum_{y \in \mathcal{S}} [f(y) - f(x)] \frac{A_0^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha}|x - y|^{d+\alpha}}.$$

Fix  $D$  and let  $E = D^{1/2}$ . Let  $Q_t$  be the transition operator for the process  $\tilde{W}_t$  corresponding to the generator

$$\tilde{A}f(x) = \sum_{\substack{y \in \mathcal{S} \\ |y-x| \leq E}} [f(y) - f(x)] \frac{A_0^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha}|x - y|^{d+\alpha}}.$$

(Compare with the process defined by (2.2).) Define

$$\tilde{B}f(x) = \sum_{\substack{y \in \mathcal{S} \\ |y-x| > E}} [f(y) - f(x)] \frac{A_0^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha}|x - y|^{d+\alpha}}$$

and  $\|f\|_1 = \sum_{\mathcal{S}} |f(y)|$ . (Compare with the process defined in (2.7).)

**Proposition 4.4.** *There exists  $c_1$  such that*

$$(4.2) \quad \|Q_t f\|_1 \leq c_1 \|f\|_1, \quad \|Q_t f\|_\infty \leq \|f\|_\infty.$$

Also

$$(4.3) \quad \|\tilde{B}f\|_1 \leq \frac{c_1}{E^\alpha} \|f\|_1, \quad \|\tilde{B}f\|_\infty \leq \frac{c_1}{E^\alpha} \|f\|_\infty.$$

*Proof.* The second inequality in (4.2) follows because  $Q_t$  is a Markovian semigroup. Notice that  $C_x Q_t(x, y)$  is symmetric in  $x, y$ . Then

$$\|Q_t f\|_1 \leq \sum_x \sum_y Q_t(x, y) |f(y)| = \sum_y |f(y)| \sum_x Q_t(x, y) \leq c_2 \sum_y |f(y)|$$

because  $\sum_x Q_t(x, y) = \sum_x \frac{C_y}{C_x} Q_t(y, x) \leq c_2 \sum_x Q_t(y, x) = c_2$ . This establishes the first inequality.

Note that

$$(4.4) \quad \sum_{\substack{y \in \mathcal{S} \\ |y-x| > E}} \frac{A_0^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha}|x - y|^{d+\alpha}} \leq c_3 E^{-\alpha}.$$

Then

$$|\tilde{B}f(x)| \leq 2\|f\|_\infty \sum_{\substack{y \in \mathcal{S} \\ |y-x| > E}} \frac{A_0^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha}|x - y|^{d+\alpha}} \leq 2c_3 E^{-\alpha} \|f\|_\infty.$$

To get the first inequality in (4.3),

$$\begin{aligned} \sum_x |\tilde{\mathcal{B}}f(x)| &\leq \sum_x \sum_{|y-x|>E} |f(y)| \frac{A^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha} |x-y|^{d+\alpha}} + \sum_x |f(x)| \sum_{|y-x|>E} \frac{A^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha} |x-y|^{d+\alpha}} \\ &\leq \sum_y |f(y)| \sum_{|x-y|>E} \frac{A^{t_0^{1/\alpha}}(x, y)}{t_0^{d/\alpha} |x-y|^{d+\alpha}} + c_3 E^{-\alpha} \sum_x |f(x)|. \end{aligned}$$

Applying (4.4) completes the proof. □

Let  $K$  be the smallest integer larger than  $2(d + \alpha)/\alpha$  and let

$$A_n = D^{(1/2)+(n/4K)}.$$

Let us say that a function  $g$  is in  $\mathcal{L}(n, \eta)$  if

$$|g(z)| \leq \eta \left[ \frac{1}{D^{d+\alpha}} + \frac{1}{|z-y|^{d+\alpha}} 1_{B(y, A_n)^c}(z) + H(z) \right]$$

for all  $z$ , where  $H$  is a nonnegative function supported in  $B(y, A_n)$  with  $\|H\|_1 + \|H\|_\infty \leq 1$ .

**Lemma 4.5.** *Suppose  $D^{1/(4K)} \geq 4$  and  $n \leq K$ . There exists  $c_1$  such that if  $g \in \mathcal{L}(n, \eta)$ , then*

- (a)  $\tilde{\mathcal{B}}g \in \mathcal{L}(n + 1, c_1\eta)$ ;
- (b) for each  $s \leq 1$ ,  $Q_s g \in \mathcal{L}(n + 1, c_1\eta)$ .

*Proof.* In view of (4.2) and (4.3),  $\|\tilde{\mathcal{B}}(D^{-d+\alpha})\|_\infty \leq c_2 D^{-d+\alpha}$  and the same bound holds when  $\tilde{\mathcal{B}}$  is replaced by  $Q_t$ .

Next, set

$$v(z) = \frac{1}{|z-y|^{d+\alpha}} 1_{B(y, A_n)^c}(z).$$

Note that  $\|v\|_1 + \|v\|_\infty \leq c_3$ , where  $c_3$  does not depend on  $n$  or  $D$ . Let

$$J_0(z) = |\tilde{\mathcal{B}}(v + H)(z)| 1_{B(y, A_{n+1})}(z)$$

and  $J(z) = J_0(z)/(\|J_0\|_1 + \|J_0\|_\infty)$ . Because of (4.2), we see that  $J_0$  has  $L^1$  and  $L^\infty$  norms bounded by a constant, so  $J$  is a nonnegative function supported on  $B(y, A_{n+1})$  with  $\|J\|_1 + \|J\|_\infty \leq 1$ . The same argument serves for  $Q_t$  in place of  $\tilde{\mathcal{B}}$ .

It remains to get suitable bounds on  $|\tilde{\mathcal{B}}v|$  and  $Q_t v$  when  $|z - y| \geq A_{n+1}$ . We have

$$(4.5) \quad |\tilde{\mathcal{B}}v(z)| \leq \sum_{|w-z|>E} v(w) \frac{c_4}{|w-z|^{d+\alpha}} + \sum_{|w-z|>E} v(z) \frac{c_4}{|w-z|^{d+\alpha}}.$$

Clearly the second sum is bounded by  $c_5 v(z)$ , as required. We now consider the first sum. Let  $C = \{w : |w - z| \geq |w - y|\}$ . If  $w \in C$ , then  $|w - z| \geq |y - z|/2$ . Hence

$$\begin{aligned} \sum_{w \in C, |z-w|>E} \frac{1}{|w-z|^{d+\alpha}} \frac{1}{|w-y|^{d+\alpha}} &\leq c_6 \frac{1}{|y-z|^{d+\alpha}} \sum_{|w-y|>1} \frac{1}{|w-y|^{d+\alpha}} \\ &\leq \frac{c_7}{|y-z|^{d+\alpha}}. \end{aligned}$$

If  $w \in C^c$ , then  $|w - y| \geq |y - z|/2$ , and we get a similar bound. Combining gives the desired bound for (4.5).

Finally, we examine  $Q_t v(z)$  when  $z \in B(y, A_{n+1})^c$ . We write

$$(4.6) \quad Q_t v(z) = \sum_{|z-w| \leq A_{n+1}/2} Q_t(z, w)v(w) + \sum_{|z-w| > A_{n+1}/2} Q_t(z, w)v(w).$$

If  $|z - y| \geq A_{n+1}$  and  $|z - w| \leq A_{n+1}/2$ , then  $|w - y| \geq |z - y|/2$ . For such  $w$ ,  $v(w) \leq c_8/|z - y|^{d+\alpha}$ , and hence the first sum in (4.6) is bounded by

$$\frac{c_8}{|z - y|^{d+\alpha}} \sum_w Q_t(z, w) = \frac{c_8}{|z - y|^{d+\alpha}}.$$

For  $|z - w| > A_{n+1}/2$ ,  $v$  is bounded, and the second sum in (4.6) is less than or equal to

$$\sum_{|z-w| > A_{n+1}/2} Q_t(z, w) \leq \mathbb{P}^z(|\tilde{W}_t - z| \geq A_{n+1}/2) \leq c_9 e^{-c_{10}(A_{n+1}/2A_n)}$$

using Proposition 2.5. This is less than

$$c_{11} \left( \frac{A_n}{A_{n+1}} \right)^{8K^2} \leq c_{12} D^{-d-\alpha}.$$

Combining the estimates proves the lemma. □

**Proposition 4.6.** *There exists  $c_1$  such that  $\mathbb{P}^x(Y_t \in B(y, 1)) \leq c_1/|x - y|^{d+\alpha}$ .*

*Proof.* Let  $D = |x - y|$ . Assume first that  $D \geq D_0$ , where  $D_0 = 4^{4K}$ . Let  $f = 1_{B(y, 1)}$ . Clearly there exists  $\eta$  such that  $f \in \mathcal{L}(1, \eta)$ . Then  $Q_t f \in \mathcal{L}(2, c_2\eta)$  for all  $t \leq 1$  by Lemma 4.5. Set  $S_0(t) = Q_t$  and  $S_1(t) = \int_0^t Q_s \tilde{\mathcal{B}} Q_{t-s} ds$ . Since  $Q_1 f \in \mathcal{L}(2, c_2\eta)$  and  $|x - y| = D > A_2$ , we have

$$|S_0(1)f(x)| \leq c_3|x - y|^{-d-\alpha}.$$

By Lemma 4.5, for each  $s \leq t \leq 1$ ,  $Q_s \tilde{\mathcal{B}} Q_{t-s} f \in \mathcal{L}(4, c_2^3\eta)$ . Hence  $|Q_s \tilde{\mathcal{B}} Q_{t-s} f(x)| \leq c_4 D^{-d-\alpha}$ . Integrating over  $s \leq t$ , we have

$$|S_1(t)f(x)| \leq c_4 D^{-d-\alpha}.$$

Set  $S_2(t) = \int_0^t S_1(s) \tilde{\mathcal{B}} Q_{t-s} ds = \int_0^t \int_0^s Q_r \tilde{\mathcal{B}} Q_{s-r} \tilde{\mathcal{B}} Q_{t-s} dr ds$ . By Lemma 4.5 we see that  $Q_r \tilde{\mathcal{B}} Q_{s-r} \tilde{\mathcal{B}} Q_{t-s} f \in \mathcal{L}(6, c_2^5\eta)$  and therefore  $|Q_r \tilde{\mathcal{B}} Q_{s-r} \tilde{\mathcal{B}} Q_{t-s} f(x)| \leq c_6 D^{-d-\alpha}$ . Integrating over  $r$  and  $s$ , we have

$$|S_2(t)f(x)| \leq c_6 D^{-d-\alpha}.$$

We continue in this fashion and find that for all  $n \leq K$  we have

$$|S_n(1)f(x)| \leq c_7(n) D^{-d-\alpha}.$$

On the other hand, by Proposition 4.4

$$\|\tilde{\mathcal{B}}\|_\infty \leq c_8/E^\alpha.$$

Take  $D_0$  larger if necessary, so that  $c_8 D_0^{-\alpha/2} < \frac{1}{2}$ . If  $D \geq D_0$ , we have by the argument of Proposition 2.6 that

$$\|S_n(1)f\|_\infty \leq (c_8/E^\alpha)^n.$$

Consequently,

$$\sum_{n=K}^{\infty} |S_n f(x)| \leq c_9/E^{\alpha K} \leq c_{10}D^{-d-\alpha}.$$

If we set  $P_t = \sum_{n=0}^{\infty} S_n(t)$ , we then have

$$|P_1 f(x)| \leq \left( \sum_{n=0}^K c_7(n) + c_{10} \right) D^{-d-\alpha} = c_{11}D^{-d-\alpha}.$$

This is precisely what we wanted to show, because, by [Le],  $P_t$  is the semigroup corresponding to  $\tilde{V}_t$ .

This proves the result for  $D \geq D_0$ . For  $D < D_0$  we have our result by taking  $c_1$  larger if necessary. □

From the probabilities of being in a set for  $Y_t$  we can obtain hitting probabilities.

**Proposition 4.7.** *There exist  $c_1$  and  $c_2$  such that*

$$\mathbb{P}^x(Y_t \text{ hits } B(y, c_1 t_0^{1/\alpha}) \text{ before time } t_0) \leq c_2 \left( \frac{t_0^{1/\alpha}}{|x - y|} \right)^{d+\alpha}.$$

*Proof.* There is nothing to prove unless  $|x - y|/t_0^{1/\alpha}$  is large. Let  $D = |x - y|$  and let  $A$  be the event that  $Y_t$  hits  $B(y, t_0^{1/\alpha})$  before time  $t_0$ . Let  $C$  be the event that  $\sup_{s \leq t_0} |Y_s - Y_0| \leq c_3 t_0^{1/\alpha}$ . From Theorem 2.8,  $\mathbb{P}^z(C) \geq \frac{1}{2}$  if  $c_3$  is large enough. By the strong Markov property,

$$\mathbb{P}^x(Y_{t_0} \in B(y, (1 + c_3)t_0^{1/\alpha})) \geq \mathbb{E}^x[\mathbb{P}^{Y_{t_0}}(C); A] \geq \frac{1}{2}\mathbb{P}^x(A),$$

where  $S = \inf\{t : Y_t \in B(y, t_0^{1/\alpha})\}$ . We can cover  $B(y, (1 + c_3)t_0^{1/\alpha})$  by a finite number of balls of the form  $B(z, t_0^{1/\alpha})$ , where the number  $M$  of balls depends only on  $c_3$  and the dimension  $d$ . Then by Proposition 4.6, the left hand side is bounded by  $c_4 M (t_0^{1/\alpha}/D)^{d+\alpha}$ . □

We now get the corresponding result for  $X_n$ . We suppose that  $Y_t$  is constructed in terms of  $X_n$  and stopping times  $T_n$  as in Section 2.

**Proposition 4.8.** *There exist  $c_1$  and  $c_2$  such that*

$$\mathbb{P}^x(X_n \text{ hits } B(y, c_1 n_0^{1/\alpha}) \text{ before time } n_0) \leq c_2 \left( \frac{n_0^{1/\alpha}}{|x - y|} \right)^{d+\alpha}.$$

*Proof.* Let  $A$  be the event that  $X_n$  hits  $B(y, n_0^{1/\alpha})$  before time  $n_0$ ,  $C$  the event that  $Y_t$  hits  $B(y, n_0^{1/\alpha})$  before time  $2n_0$ , and  $D$  the event that  $T_{n_0} \leq 2n_0$ . By the independence of  $A$  and  $D$ , we have

$$\mathbb{P}^x(A)\mathbb{P}(D) = \mathbb{P}^x(A \cap D) \leq \mathbb{P}^x(C).$$

Using the bound on  $\mathbb{P}^x(C)$  from Proposition 4.7 and the fact that  $\mathbb{P}(D) > c_2$ , where  $c_2$  does not depend on  $n_0$ , proves the proposition. □

We now come to the main result of this section.

**Theorem 4.9.** *There exists  $c_1$  such that*

$$p(n, x, y) \leq c_1 \left( n^{-d/\alpha} \wedge \frac{n}{|x - y|^{d+\alpha}} \right).$$

*Proof.* Let  $D = |x - y|$ . Fix  $c_2$  sufficiently large. If  $D \leq c_2 n^{1/\alpha}$ , the result follows from Theorem 4.3. So suppose  $D > c_2 n^{1/\alpha}$ . Let  $m = n + \lceil \gamma n \rceil$ , where  $\gamma$  is the constant in Theorem 2.8. By Proposition 4.8,

$$\mathbb{P}^x(X_m \in B(y, m^{1/\alpha})) \leq c_3 \frac{m^{1+d/\alpha}}{D^{d+\alpha}}.$$

On the other hand, the left hand side is  $\sum_{z \in B(y, m^{1/\alpha})} p(m, x, z)$ . So for at least one  $z \in B(y, m^{1/\alpha})$ , we have  $p(m, x, z) \leq c_4 m / D^{d+\alpha} \leq c_5 n / D^{d+\alpha}$ . Let

$$q(k, w) = p(n + \lceil \gamma n \rceil - k, w, x).$$

By Proposition 3.6,  $q$  is parabolic in  $\{0, 1, \dots, \lceil \gamma n \rceil\} \times \mathbb{Z}^d$ , and we have shown that

$$\min_{w \in B(z, n^{1/\alpha})} q(0, w) \leq c_5 n / D^{d+\alpha}.$$

Thus by Theorem 3.1 we have

$$p(n, x, y) = \frac{C_y}{C_x} p(n, y, x) = \frac{C_y}{C_x} q(\lceil \gamma n \rceil, y) \leq c_6 n / D^{d+\alpha}.$$

□

### 5. LOWER BOUNDS

Lower bounds are considerably easier to prove.

**Proposition 5.1.** *There exist  $c_1$  and  $c_2$  such that if  $|x - y| \leq c_1 n^{1/\alpha}$  and  $n \geq 2$ , then*

$$p(n, x, y) \geq c_2 n^{-d/\alpha}.$$

*Proof.* Let  $m = n - \lceil \gamma n \rceil$ . By Theorem 2.8 there exists  $c_3$ , not depending on  $x$  or  $m$ , such that

$$\mathbb{P}^x(\max_{k \leq m} |X_k - x| > c_3 m^{1/\alpha}) \leq \frac{1}{2}.$$

By Theorem 4.9, provided  $m$  is sufficiently large, there exists  $c_4 < c_3/2$ , not depending on  $x$  or  $m$ , such that

$$\mathbb{P}^x(X_m \in B(x, c_4 m^{1/\alpha})) \leq \frac{1}{4}.$$

Let  $E = B(x, c_3 m^{1/\alpha}) - B(x, c_4 m^{1/\alpha})$ . Therefore

$$\mathbb{P}^x(X_m \in E) \geq \frac{1}{4}.$$

This implies, since  $\mathbb{P}^x(X_m \in E) = \sum_{z \in E} p(m, x, z)$ , that for some  $z \in E$  we have  $p(m, x, z) \geq c_5 m^{-d-\alpha} \geq c_6 n^{-d-\alpha}$ . If  $w \in E$ , then by Theorem 3.1 with  $q(k, \cdot) = p(n - k, x, \cdot)$ , we have

$$p(n, x, w) \geq c_7 n^{-d/\alpha}.$$

This proves our proposition when  $n$  is greater than some  $n_1$ .

By (1.2) and (2.1) it is easy to see that there exists  $c_8$  such that

$$p(2, x, x) \geq c_8, \quad p(3, x, x) \geq c_8.$$

If  $n \leq n_1$  and  $n = 2\ell + 1$  is odd, then

$$p(n, x, y) \geq p(2, x, x)^\ell p(1, x, y) \geq c_8^\ell n^{-d/\alpha}.$$

The case when  $n \leq b_1$  and  $n$  is even is done similarly. □

**Theorem 5.2.** *There exists  $c_1$  such that if  $n \geq 2$ , then*

$$p(n, x, y) \geq c_1 \left( n^{-d/\alpha} \wedge \frac{n}{|x - y|^{d+\alpha}} \right).$$

*Proof.* Again, our result follows for small  $n$  as a consequence of (1.2) and (2.1), so we may suppose  $n$  is larger than some  $n_1$ . In view of Proposition 5.1 we may suppose  $|x - y| \geq c_2 n^{1/\alpha}$ . Let  $A = B(y, n^{1/\alpha})$ . Let  $N(z) = \mathbb{P}^z(X_1 \in A)$  if  $z \notin A$ , and 0 otherwise. For  $z \in B(x, n^{1/\alpha})$ , note that  $N(z) \geq c_3 n^{d/\alpha} / D^{d+\alpha}$ . As in the proof of Lemma 3.2,

$$1_A(X_{j \wedge T_A}) - 1_A(X_0) - \sum_{i=1}^{j \wedge T_A} N(X_i)$$

is a martingale. By optional stopping at the time  $S = n \wedge \tau_{B(x, n^{1/\alpha})}$  we have

$$\mathbb{P}^x(X_S \in A) = \mathbb{E}^x \sum_{i=1}^S N(X_i) \geq \frac{c_3 n^{d/\alpha}}{D^{d+\alpha}} \mathbb{E}^x[S - 1].$$

Arguing as in (3.3),  $\mathbb{E}^x S \geq c_4 n$ . We conclude that

$$\mathbb{P}^x(X_n \text{ hits } B(y, n^{1/\alpha}) \text{ before time } n) \geq c_5 n^{1+d/\alpha} / D^{d+\alpha}.$$

By Theorem 2.8, starting at  $z \in B(y, n^{1/\alpha})$ , there is positive probability, not depending on  $z$  or  $n$ , that the chain does not move more than  $c_6 n^{1/\alpha}$  in time  $n$ . Hence by the strong Markov property, there is probability at least  $c_7 n^{1+d/\alpha} / D^{d+\alpha}$  that  $X_n \in B(y, c_8 n^{1/\alpha})$ . Let  $m = n - \lceil \gamma n \rceil$ . Applying the above with  $m$  in place of  $n$ , we get

$$\mathbb{P}^x(X_m \in B(y, c_9 n^{1/\alpha})) \geq c_{10} \frac{m^{1+d/\alpha}}{D^{d+\alpha}} \geq c_{11} \frac{n^{1+d/\alpha}}{D^{d+\alpha}}.$$

But this is also  $\sum_{w \in B(y, c_9 n^{1/\alpha})} p(m, x, w)$ . So there must exist  $w \in B(y, c_9 n^{1/\alpha})$  such that  $p(m, x, w) \geq c_{12} n / D^{d+\alpha}$ . A use of Theorem 3.1 as in the proof of Theorem 5.1 finishes the current proof. □

*Proof of Theorem 1.1.* This is a combination of Theorems 4.9 and 5.2. If  $n = 1$  and  $x \neq y$ , the result follows from (1.2) and (2.1). □

*Remark 5.3.* Similar (but a bit easier) arguments show that the transition probabilities for  $Y_t$  satisfy

$$(5.1) \quad c_1 \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \leq q_Y(t, x, y) \leq c_2 \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).$$

One can also show that the transition densities of the process  $U_t$  described in Section 1 also satisfy bounds of the form (5.1). One can either modify the proofs suitably, or else approximate  $U_t$  by a sequence of processes of the form  $Y_t$  but with state space  $\varepsilon \mathbb{Z}^d$ , and then let  $\varepsilon \rightarrow 0$ .

## REFERENCES

- [BBG] M.T. Barlow, R.F. Bass, and C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. **53** (2000), 1007–1038. MR **2001m**:35095
- [BL] R.F. Bass and D.A. Levin, *Harnack inequalities for jump processes*, Potential Anal., to appear.
- [CKS] E.A. Carlen, S. Kusuoka, and D.W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré Probab. Statist. **23** (1987), no. 2, suppl., 245–287. MR **88i**:35066
- [HS-C] W. Hebisch and L. Saloff-Coste, *Gaussian estimates for Markov chains and random walks on groups*, Ann. Probab. **21** (1993), 673–709. MR **94m**:60144
- [KI] V. Kolokoltsov, *Symmetric stable laws and stable-like jump-diffusions*, Proc. London Math. Soc. **80** (2000), 725–768. MR **2001f**:60016
- [Km] T. Komatsu, *Uniform estimates for fundamental solutions associated with non-local Dirichlet forms*, Osaka J. Math. **32** (1995), 833–860. MR **97d**:31010
- [Le] T. Leviatan, *Perturbations of Markov processes*, J. Functional Analysis **10** (1972), 309–325. MR **53**:4243
- [SZ] D.W. Stroock and W. Zheng, *Markov chain approximations to symmetric diffusions*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), 619–649. MR **98k**:60125
- [Wo] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge Univ. Press (2000). MR **2001k**:60006

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