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THE BERGMAN METRIC ON A STEIN MANIFOLD WITH A BOUNDED PLURISUBHARMONIC FUNCTION

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Abstract. In this article, we use the pluricomplex Green function to give a sufficient condition for the existence and the completeness of the Bergman metric. As a consequence, we proved that a simply connected complete Kähler manifold possesses a complete Bergman metric provided that the Riemann sectional curvature $\le -A/\rho^2$, which implies a conjecture of Greene and Wu. Moreover, we obtain a sharp estimate for the Bergman distance on such manifolds.

1. Introduction

Let M be a complex n-dimensional manifold. Let $\mathcal{H}$ be the space of holomorphic n-forms on M such that $|\int_M f \wedge \bar{f}| < \infty$. This space is a separable complex Hilbert space with an inner product $(f_1, f_2) = i^{n^2} \int_M f_1 \wedge \bar{f}_2$. Let $h_0, h_1, \cdots$ be a complete orthonormal basis for $\mathcal{H}$. Then the $2n$-form defined on $M \times M$ given by $K = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j$ is called the Bergman kernel form of M. Let $z = (z_1, \cdots, z_n)$ be a local coordinate system in M and let $K(z) = K^*(z) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, where K^* is a locally defined function. Then $\beta := \partial\bar{\partial} \log K^*$ is a well-defined Hermitian form of bidegree (1,1), whenever K^* is nonzero. We say that M possesses a Bergman metric iff β is everywhere positive definite. In 1959, Kobayashi [11] began to investigate the completeness of the Bergman metric. After that, there are a lot of papers concerning the Bergman completeness for bounded pseudoconvex domains in $\mathbf{C}^n$ (see [3], [18] for a review). There are two general results: One says that any bounded hyperconvex domain is Bergman complete (cf. [1], [7]); the other states that any bounded pseudoconvex domain whose boundary can be locally described as the graph of a continuous function is also Bergman complete (cf. [4]). However, little is known for the Bergman metric of manifolds except the early work of Greene and Wu [6]. They proved that a simply connected complete Kähler manifold possesses a complete Bergman metric if the sectional curvature is pinched between two negative constants, or the curvature is nonpositive and the

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following estimate holds outside a compact subset of M:

$$-\frac{B}{\rho^2} \leq \text{curvature} \leq -\frac{A}{\rho^2}$$

for some positive constants A, B (in this paper, curvature will mean sectional curvature). Here ρ denotes the distance function relative to some fixed point o in M. They conjectured that the lower estimate $-\frac{B}{\rho^2}$ is unnecessary. We will solve this conjecture in the present paper.

To formulate our results precisely, we need some notions. Let M be any complex manifold. We denote by $PSH(M)$ the set of all plurisubharmonic (psh) functions on M. According to Klimek [10], we define the pluricomplex Green function with a logarithmic pole y on M by

$$g_M(x,y) = \sup\{u(x)\},$$

where the supremum is taken over all negative functions $u \in PSH(M)$ satisfying the property that the function $u - \log|z|$ is bounded from above in a deleted neighborhood of y for some holomorphic local coordinates z centered at y, that is, $z(y) = 0$. It is known from [10] that for any $y \in M$ the function $g_M(\cdot, y)$ belongs to the above class, and it coincides with the classical (negative) Green function on hyperbolic Riemann surfaces (a Riemann surface is called hyperbolic if there is a negative nonconstant subharmonic function).

Definition. A complex manifold M is said to satisfy the property $(B1)$ if for any $y \in M$ there is a positive number $a > 0$ such that the sublevel set $A(y,a) := \{x \in M : g_M(x,y) < -a\}$ is relatively compact in M.

It is easy to see that any bounded domain D in $\mathbf{C}^n$ has the property $(B1)$ because of the trivial estimate $g_D(x,y) \geq \log\frac{|x-y|}{R_D}$, where R_D is the diameter of D. We will also show in section 2 that any hyperbolic Riemann surface, any complex manifold carrying a bounded continuous strictly psh function, and any hyperconvex Stein manifold have the property $(B1)$. Following Stehlé [17], we called a complex manifold M hyperconvex if there exists a negative psh function u such that the sublevel set $\{u < -c\}$ is relatively compact in D for every $c > 0$.

Definition. A complex manifold M is said to satisfy the property $(B2)$ if for any sequence of points $\{y_k\}, k = 1, 2, \cdots$, which has no adherent point in M there exist a subsequence $\{y_{k_j}\}, j = 1, 2, \cdots$, and a number $a > 0$ such that for any compact subset K one has $A(y_{k_j}, a) \subset M \backslash K$ for all sufficiently large j.

Theorem 1. *If M is a Stein mainfold which satisfies the property $(B1)$, then it possesses a Bergman metric. If furthermore, M satisfies the property $(B2)$, then the Bergman metric is complete.*

With an application of Theorem 1, we solve the conjecture of Greene and Wu in the sequel.

Theorem 2. *Let M be a simply-connected complete Kähler manifold of dimension n with nonpositive sectional curvature such that the inequality*

$$\textit{curvature} \leq -\frac{A}{\rho^2} \tag{1}$$

holds outside a compact subset of M for a suitable positive constant A. Then M possesses a complete Bergman metric.

We also have the following consequences of Theorem 1:

Corollary 3. *Any hyperconvex Riemann surface is Bergman complete.*

Corollary 4. *Let D be a domain in $\mathbf{C}^n$, not necessarily bounded. Suppose that there exists a negative C^2 psh exhaustion function ψ on D, such that*

$$\partial\bar{\partial}\psi \geq \partial\bar{\partial}|z|^2.$$

Then D is Bergman complete.

This domain was introduced by Sibony in [15], where he obtained an estimate of the Kobayashi metric for this domain.

In fact, this paper is a continuation of the paper [2], where the first-named author proved the Bergman completeness under the assumption of curvature $\leq -c$ for some positive constant. Greene and Wu used the geometry method of Siu and Yau [16] to get a comparison of the Bergman metric and the Kähler metric of the manifold which implies the completeness of the Bergman metric; hence the hypothesis that the curvature is bounded from below is essential. In this paper we just verify Kobayashi's criterion for Theorem 1 with help of the L^2 estimates of Hörmander type. Under the curvature condition of Theorem 2, Greene and Wu [6] constructed some special bounded psh exhaustion functions on the manifold. These functions enable us to show that the manifold satisfies the properties $(B1)$ and $(B2)$.

Using a recent result of Jost and Zuo [9] together with Theorem 1, we obtain a vanishing theorem for the L^2-cohomology groups with respect to the Bergman metric. Let (M, ds^2) be a complete Kähler manifold of dimension n and let $\mathcal{H}^{p,q}(M)$ denote the space of square-integrable harmonic (p,q) forms on M. The result of Jost and Zuo says that if the sectional curvature is nonpositive, then $\mathcal{H}^{p,q}(M) = \{0\}$ for $p+q \neq n$. This implies the following

Corollary 5. *Let M be as in Theorem 1. Suppose that the sectional curvature of the Bergman metric is nonpositive. Then one has $\mathcal{H}^{p,q}_\beta(M) = \{0\}$ for $p+q \neq n$, where $\mathcal{H}^{p,q}_\beta(M)$ denotes the space of harmonic (p,q) forms on M which are square-integrable with respect to the Bergman metric β.*

In 1995, Diederich and Ohsawa [5] introduced a method of estimating the Bergman distance, which is based on Kobayashi's alternative definition of the Bergman metric. Inspired by their work, we are able to improve Theorem 2 as follows:

Theorem 6. *Let M be a simply-connected complete Kähler manifold of dimension n with nonpositive sectional curvature.*

1) *If the inequality* (1) *holds outside a compact subset of M for suitable positive constant A, then there exists a positive constant C' such that*

$$\operatorname{dist}_\beta(o,x) \geq C' \log \rho(x),$$

where $\operatorname{dist}_\beta(o,x)$ *denotes the distance between o and x with respect to the Bergman metric.*

2) *If the curvature is bounded from above by a negative constant $-A$, then*

$$\operatorname{dist}_\beta(o,x) \geq C'' \rho(x)$$

for a suitable constant $C'' > 0$.

2. Proof of Theorem 1

We assume first that M satisfies the property $(B1)$, that is, for any $y \in M$ there is a number $a > 0$ so that $A(y,a) \subset\subset M$. To prove the existence of the Bergman metric, it suffices to show, according to [11], the following two statements:

(i) Given any point y of M, there exists a form $f \in \mathcal{H}$ such that $f(y) \neq 0$.

(ii) For any holomorphic vector X at y, there exists a form $f \in \mathcal{H}$ such that $f^*(0) = 0$ and $Xf^*(0) \neq 0$, where $f(z) = f^*(z)dz_1 \wedge \cdots \wedge dz_n$ in a local coordinate system centered at y.

Since M is Stein, there exist n holomorphic functions $\zeta_1, \cdots, \zeta_n$ on M which form a local coordinate system centered at y. Without loss of generality, we assume $X = \partial/\partial\zeta_1$. We take a cut-off function $\chi : \mathbf{R} \to [0,1]$ such that $\chi \equiv 1$ on $(-\infty, -1]$ and $\chi \equiv 0$ on $[0, \infty)$. We set

$$\eta = \begin{cases} \chi\left(-\log(-g_M(\cdot,y)+a) + \log(2a)\right) & \text{for case } (i), \\ \chi\left(-\log(-g_M(\cdot,y)+a) + \log(2a)\right)\zeta_1 & \text{for case } (ii), \end{cases}$$

and

$$\varphi = 2(n+1)g_M(\cdot,y) - \log(-g_M(\cdot,y)+a).$$

Clearly $\varphi \in PSH(M)$. We will show that there exists a constant $C = C(y,a)$ so that the equation $\overline{\partial}u = \overline{\partial}\eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$ has a solution in the distribution sense such that the following inequality holds:

$$\left| \int_M u \wedge \overline{u} e^{-\varphi} \right| \leq C.$$

If we have proved the above fact under the assumption that the function $g_M(\cdot,y)$ is C^∞, then for the general case, we can exhaust M by an increasing sequence of relatively compact Stein domains $M_j, j = 1, 2, \cdots$, and for each j the psh function $g_M(\cdot,y)$ can be approximated uniformly on $\overline{M}_j$ by negative strictly psh functions $\psi_{j,k}, k = 1, 2, \cdots$. We replace $g_M(\cdot,y)$ by $\psi_{j,k}$. It follows that there is a solution to the equation $\overline{\partial}u_{j,k} = \overline{\partial}\eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$ on M_j together with the estimate

$$\left| \int_{M_j} u_{j,k} \wedge \overline{u}_{j,k} e^{-\varphi} \right| \leq C$$

for a suitable constant $C > 0$ depending only on y and a. To obtain the desired solution, we only need to take a weak limit as $k, j \to \infty$. This limiting procedure is standard (cf. [4]). Hence we may assume $g_M(\cdot,y)$ is C^∞. Next, we need an L^2 estimate of Hörmander type for the $\bar{\partial}$-equation on complete Kähler manifolds.

Proposition 7 (cf. [13])**.** *Let M be a complete Kähler manifold and let φ be a C^∞ strictly psh function on M. Then for any $\bar{\partial}$-closed $(n,1)$ form with $\int_M |v|^2_{\partial\bar{\partial}\varphi} e^{-\varphi} dV_\varphi < \infty$, there is an n-form u on M such that $\bar{\partial}u = v$ and*

$$\left| \int_M u \wedge \bar{u} e^{-\varphi} \right| \leq \int_M |v|^2_{\partial\bar{\partial}\varphi} e^{-\varphi} dV_\varphi,$$

where dV_φ denotes the volume with respect to $\partial\bar{\partial}\varphi$.

This proposition gives us a solution to the equation $\overline{\partial}u = \overline{\partial}\eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$ with the following inequality:

$$\begin{aligned} \left| \int_M u \wedge \bar{u} e^{-\varphi} \right| &\leq \int_M |\overline{\partial}\eta|^2_{\partial\bar{\partial}\varphi} e^{-\varphi} dV_\varphi \\ &\leq C_1, \end{aligned}$$

noting that $\left|\overline{\partial}\chi(\,\cdot\,)\right|_{\partial\overline{\partial}\varphi} \leq \sup|\chi'|$ because

$$\partial\overline{\partial}\varphi \geq -\partial\overline{\partial}\log(-g_M(\cdot,y)+a) \geq \partial\log(-g_M(\cdot,y)+a)\overline{\partial}\log(-g_M(\cdot,y)+a).$$

Here C_1 is a positive constant depending only on y, a and the choice of χ; $|\cdot|_{\partial\overline{\partial}\varphi}$ denotes the point norm w.r.t. the metric $\partial\overline{\partial}\varphi$. This implies that the form $f := \eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n - u \in \mathcal{H}$, because φ is bounded from above. Moreover, the singularity of φ shows that $f^*(0) = 1$ for case (i), while $f^*(0) = 0,\ \partial f^*/\partial\zeta_1(0) = 1$ for case (ii). This completes the first part of the proof.

To prove the second part, we need the criterion of Kobayashi for Bergman completeness:

Proposition 8 (cf. [12])**.** *Let M be a complex manifold which possesses a Bergman metric. Assume that there exists a dense subspace $\mathcal{H}'$ of $\mathcal{H}$ such that for any $f \in \mathcal{H}'$ and for any sequence of points $\{y_k\}_{k=1}^\infty$ of M which has no adherent point in M, there is a subsequence $\{y_{k_j}\}_{j=1}^\infty$ such that*

$$\lim_{j\to\infty} \frac{f(y_{k_j}) \wedge \bar{f}(y_{k_j})}{K(y_{k_j})} = 0.$$

Then M is Bergman complete.

Let $f \in \mathcal{H}$, and let $\{y_k\}_{k=1}^\infty$ be a sequence of points which has no adherent point in M. For any $\epsilon > 0$ one can find a compact subset M_ϵ of M such that

$$|\int_{M\setminus M_\epsilon} f \wedge \overline{f}| < \epsilon.$$

By the hypothesis $(B2)$, one can find a subsequence $\{y_{k_j}\}$ with the following property: there exists a positive number a (independent of ϵ and j) such that $A(y_{k_j}, a) \subset M\setminus M_\epsilon$ for all sufficiently large j. Let χ be as before, and set

$$\begin{aligned} \eta_j &= \chi\left(-\log(-g_M(\cdot,y_{k_j})+a) + \log(2a)\right) f, \\ \varphi_j &= 2ng_M(\cdot,y_{k_j}) - \log(-g_M(\cdot,y_{k_j})+a). \end{aligned}$$

We can solve the equation $\overline{\partial}u_j = \overline{\partial}\eta_j$ essentially as above together with the following estimate:

$$\begin{aligned} \left| \int_M u_j \wedge \bar{u}_j e^{-\varphi_j} \right| &\leq \int_M |\overline{\partial}\eta_j|^2_{\partial\bar{\partial}\varphi_j} e^{-\varphi_j} dV_{\varphi_j} \\ &\leq C_2 \left| \int_{\operatorname{supp}\overline{\partial}\eta_j} f \wedge \overline{f} \right| \\ &\leq C_2\epsilon, \end{aligned}$$

because $\operatorname{supp}\overline{\partial}\eta_j \subset A(y_{k_j}, a) \subset M\setminus M_\epsilon$ for j sufficiently large. Here C_2 is a constant depending only on a and the choice of χ. We set $f_j = \eta_j - u_j$. Then $f_j(y_{k_j}) = f(y_{k_j})$

and $\left|\int_M f_j \wedge \overline{f}_j\right| \leq C_3\epsilon$. It follows that

$$\frac{f(y_{k_j}) \wedge \bar{f}(y_{k_j})}{K(y_{k_j})} \leq \left|\int_M f_j \wedge \overline{f}_j\right| \leq C_3\epsilon.$$

Hence the assertion follows immediately from Proposition 8.

Let us see that there are various complex manifolds satisfying $(B1)$:

1. D is a hyperbolic Riemann surface, that is, M carries a bounded non-constant subharmonic function. It is well known that this condition is equivalent to the fact that M carries a (negative) Green function. Since $g_M(x, y)$ is harmonic in $M\backslash\{y\}$ and $g_M(x, y) - \log|z|$ is harmonic in a local coordinate chart at y, we see that M satisfies the property $(B1)$.

2. Let M be a complex manifold carrying a bounded continuous strictly psh function ψ. By the well-known theorem of Richberg [14], we may assume that ψ is C^∞. For any $y \in M$, we take a function κ which is compactly supported in a coordinate chart at y and identically equal to 1 in a neighborhood of y. One can find a constant a_y such that $\kappa \log|z| + a_y\psi$ is a psh function on M wth a logarithmic pole at $z(y) = 0$ which is bounded above by a constant depending only on y. It follows from the definition of the pluricomplex Green function that M satisfies $(B1)$.

3. Let M be a hyperconvex Stein manifold. We will show that M satisfies $(B1)$. We first prove the following fact.

Claim. *Let M be a Stein manifold and let $y \neq y'$ be two points of M. Then there is a holomorphic function f on M such that $f(y) = 0, df(y) = 0$ and $f(y') = 1$.*

Proof. The proof is standard. Let ψ be a C^∞ strictly psh exhaustion function on M. Similarly as before, one can find psh functions ψ_y, $\psi_{y'}$ on M with a logarithmic pole at y, y' respectively. We choose a cutoff function τ which is compactly supported in M and is such that $\tau \equiv 0$ in a neighborhood of y and $\tau \equiv 1$ in a neighborhood of y' . Now we take a convex, rapidly increasing function γ such that there is, according to Theorem 5.2.4 in [8], a solution to the equation $\overline{\partial} u = \overline{\partial}\tau$ satisfying the following estimate:

$$\int_M |u|^2 e^{-\varphi} dV \leq \int_M |\overline{\partial}\tau|^2 e^{-\varphi} dV \leq \widetilde{C},$$

where $\varphi = \gamma \circ \psi + 2(n+1)\psi_y + 2n\psi_{y'}$; the point norm and the volume are associated to some fixed Kähler metric on M. Then the function defined by $f = \tau - u$ is holomorphic on M and satisfies $f(y) = 0$, $df(y) = 0$ and $f(y') = 1$. □

Now let μ be a negative psh exhaustion function on M. Again we take n globally defined holomorphic functions $\zeta_1, \cdots, \zeta_n$ which form a local coordinate system centered at y, and denote by U the coordinate neighborhood of y. We set

$$K = \{x \in M : \mu(x) \leq \mu(y)/8\}\backslash U.$$

Since it is compact in M, we obtain from the above claim finite holomorphic functions $\zeta_{n+1}, \cdots, \zeta_{n+m}$ on M such that $\zeta_{n+j}(y) = 0, d\zeta_{n+j}(y) = 0$ for all $1 \leq j \leq m$, and the function $\sum_{j=1}^m |\zeta_{n+j}|^2$ is nowhere vanishing on K. We denote $\zeta = (\zeta_1, \cdots, \zeta_{n+m})$ and set

$$\lambda = \inf_{\{\mu(x)=\mu(y)/2\}} \log|\zeta(x)|/R_y,$$

$$\widetilde{\mu}(x) = \lambda \frac{\log(-\mu(x) - \mu(y)/4) - \log(-\mu(y)/2)}{\log 3/2},$$

where

$$R_y = \sup_{\{\mu(x)=\mu(y)/2\}} |\zeta(x)| + 1.$$

It follows that $\widetilde{\mu}$ is a psh function on M satisfying

$$\begin{aligned} \widetilde{\mu}(x) &= \lambda \leq \log|\zeta(x)|/R_y \quad \text{if } \mu(x) = \mu(y)/2, \\ \widetilde{\mu}(x) &= 0 \geq \log|\zeta(z)|/R_y \quad \text{if } \mu(x) = \mu(y)/4. \end{aligned}$$

Hence the function defined by

$$v(x) = \begin{cases} \log|\zeta(x)|/R_y & \text{if } \mu(x) < \mu(y)/2, \\ \max\{\log|\zeta(x)|/R_z, \widetilde{\mu}(x)\} & \text{if } \mu(y)/2 \leq \mu(x) \leq \mu(y)/4, \\ \widetilde{\mu}(x) & \text{if } \mu(x) > \mu(y)/4 \end{cases}$$

is a psh function on M with a logarithmic pole at y which is bounded from above by a constant depending only on y. Then, similarly as above, M satisfies $(B1)$.

Proof of Corollary 3. It suffices to show that M satisfies $(B2)$. Let μ be a negative psh exhaustion function on M. We denote

$$M_c = \{x \in M : \mu(x) < -c\}$$

for any $c > 0$. Now let c be fixed and let $y \in M_{2c}$ be arbitrary. We set

$$\psi_y(x) = \begin{cases} \max\{C\mu(x), g_M(x,y) - 1\} & \text{if } x \in M\backslash M_c, \\ g_M(x,y) - 1 & \text{if } x \in M_c, \end{cases}$$

where

$$C = -c^{-1} \min_{\{\mu(x)=-c\}} (g_M(x,y) - 1)$$

is a constant depending only on c because g_M is a continuous function off the diagonal on $M \times M$. Clearly, ψ_y is a negative psh function with a logarithnic pole at y, and furthermore, there is a positive constant $c' << c$ such that $\psi_y(x) \geq -1$ on $M\backslash M_{c'}$. It follows from the extremal property of the Green function that the inequality $g_M(x,y) \geq -1$ holds there. Since g_M is symmetric, we have

$$g_M(x,y) \geq -1, \ \forall x \in M_{2c}, \ y \in M\backslash M_{c'}.$$

It follows that $A(y,-1) \subset M\backslash M_{2c}$ for any $y \in M\backslash M_{c'}$, which implies the property $(B2)$. The proof is complete. □

Proof of Corollary 4. Since ψ is a negative C^2 strictly psh function on D, according to the above facts, D carries a Bergman metric. Moreover, it is the standard Bergman metric since D is a domain in $\mathbf{C}^n$. Let $\{y_k\}$ be an arbitrary sequence of points which has no adherent point in D. We distinguish two cases:

(a) There is a subsequence $\{y_{k_j}\}$ such that $|y_{k_j}| \to \infty$ as $j \to +\infty$. We take a cutoff finction $\chi : \mathbf{R} \to [0,1]$ such that $\chi|_{(-\infty,1/2]} = 1$ and $\chi|_{[1,+\infty)} = 0$. Since $\partial\bar{\partial}\psi \geq \partial\bar{\partial}|z|^2$, there is a constant $C' > 0$, depending only on the choice of χ, such that the function $\varphi_j := C'\psi + \chi(|z - y_{k_j}|)\log|z - y_{k_j}|$ is a negative psh function on D with a logarithmic pole at y_{k_j}. Hence for any compact subset K of D, one has $A(y_{k_j}, -1) \subset D\backslash K$ for all sufficiently large j. Similarly as in the proof of Theorem 1, the criterion of Kobayashi holds for $\{y_k\}$.

(b) Otherwise, there is a subsequence $\{y_{k_j}\}$ such that y_{k_j} converges to a boundary point y_0. Take a ball $B(y_0,1)$ and set $D' = D \cap B(y_0,1)$. Clearly D' is a bounded hyperconvex domain. Without loss of generality, we may assume $y_{k_j} \in B(y_0, 1/4)$. If we have proved that

$$K_D(y_{k_j}) \geq C'' K_{D'}(y_{k_j})$$

for some constant C'' independent of j, then for any $f \in \mathcal{H}(D)$

$$\lim_{k\to\infty} \frac{|f(y_{k_j})|^2}{K_D(y_{k_j})} \leq \frac{1}{C''} \lim_{k\to\infty} \frac{|f(y_{k_j})|^2}{K_{D'}(y_{k_j})} = 0,$$

where the last equality was shown in [1], [7]. Hence Kobayashi's criterion holds for $\{y_k\}$. The proof is reduced to showing the localization property of the Bergman kernel. Let φ_j be as above. We solve the equation

$$\bar{\partial} u_j = \bar{\partial}\chi(|z-y_0|) K_{D'}(z, y_{k_j})/K_{D'}^{1/2}(y_{k_j})$$

together with the following estimate:

$$\int_D |u_j|^2 e^{-2n\varphi_j - \psi} dV \leq \int_D |\bar{\partial}\chi(|z-y_0|)|^2_{\partial\bar{\partial}\psi} e^{-2n\varphi_j-\psi} dV \leq C''',$$

where C''' is a constant independent of j. Set again

$$f_j = \chi(|z-y_0|) K_{D'}(z, y_{k_j})/K_{D'}^{1/2}(y_{k_j}) - u_j.$$

We obtain

$$K_D(y_{k_j}) \geq \frac{|f(y_{k_j})|^2}{\int_D |f|^2 dV} \geq C'' K_{D'}(y_{k_j})$$

for a suitable constant C'' independent of j.

The proof follows immediately from Proposition 8. □

3. Proof of Theorem 2

We will follow the argument of Greene and Wu [6] throughout this section. Let M be as in Theorem 1. Suppose that (1) holds in $M \backslash B(o, c)$ for some positive constant c, where $B(x, \delta)$ denotes the geodesic ball with radius δ around x. Let x_0 be any point in $M \backslash B(o, 2c)$. Let ρ_0 denote the distance function relative to x_0. Let G be the complete Kähler metric of M and let $\mathrm{K}_G(x)$ denotes the maximum of the sectional curvatures at x. It is not difficult to see that the inequality

$$\mathrm{K}_G(x) \leq -\frac{A}{4\rho_0(x)^2} \tag{2}$$

holds for all $x \in M \backslash B(x_0, 2\rho(x_0))$. Consider the new Kähler metric $H = \frac{G}{\rho(x_0)^2}$. Let γ_0 denote the distance function of H relative to x_0. Then $\gamma_0 = \frac{\rho_0}{\rho(x_0)}$ and $\mathrm{K}_H = \rho(x_0)^2 \mathrm{K}_G$. Hence inequality (3) is equivalent to

$$\mathrm{K}_H(x) \leq -\frac{A}{4\gamma_0(x)^2}$$

for all $x \in M$ with $\gamma_0(x) \geq 2$. Notice also that $\mathrm{K}_H \leq 0$ everywhere. By Lemma 5.15 in [6], there is a complete Hermitian metric h on the unit disc D which is rotationally symmetric, and its Gaussian curvature K_h satisfies (a) $\mathrm{K}_h \leq 0$ and (b) if $\widetilde{\rho}$ denotes the distance function of h relative to the origin, then

$$\mathrm{K}_h = \begin{cases} 0 & \text{on } \{\widetilde{\rho} \leq 2\}, \\ -A/(4\widetilde{\rho}^2) & \text{on } \{\widetilde{\rho} \geq 3\}, \end{cases}$$

and in the annulus $\{2 < \widetilde{\rho} < 3\}$, K_h is rotationally symmetric. Write $h = d\widetilde{\rho}^2 + f(\widetilde{\rho})^2 d\theta^2$ in terms of geodesic polar coordinates. Since $f'' = -\mathrm{K}_h f$, it follows that $f'' \equiv 0$ on $[0,2]$; hence $f(\widetilde{\rho}) = \widetilde{\rho}$ there. Next we write h as follows:

$$h = \eta(r) dz d\bar{z} = \eta(r)(dr^2 + r^2 d\theta^2),$$

where $r : D \to [0,1)$ is the ordinary radial function on D. Clearly, one has

$$\eta(r) = [\widetilde{\rho}'(r)]^2, \tag{3}$$

$$r^2\eta(r) = f(\widetilde{\rho}(r))^2. \tag{4}$$

We will regard r as a function $\widetilde{\rho}$ so that $r : [0,\infty) \to [0,1)$. By (3), (4), one has

$$\frac{1}{r} = \frac{\widetilde{\rho}'(r)}{f(\widetilde{\rho}(r))}.$$

Integrating both sides relative to dr from r to 1, we obtain

$$\begin{aligned} r(\widetilde{\rho}) &= \exp\left\{-\int_r^1 \frac{\widetilde{\rho}'(r)}{f(\widetilde{\rho}(r))} dr\right\} \\ &= \exp\left\{-\int_{\widetilde{\rho}}^{\infty} \frac{1}{f}\right\}. \end{aligned}$$

Set $\phi_{x_0} = r(\gamma_0)^2$. Using a Hessian comparison theorem, Greene and Wu proved that ϕ_{x_0} is a bounded exhaustion function on M which is C^∞ strictly psh, and satisfies $0 \le \phi_{x_0} < 1, \phi_{x_0}^{-1}(0) = x_0, \phi_{x_0} = O\left(\gamma_0^2\right)$ near x_0, and $\log \phi_{x_0}$ is also psh. Observe that

$$\begin{aligned} \log \phi_{x_0}(x) &= 2\log r(\gamma_0(x)) = -2\int_{\gamma_0(x)}^{\infty} \frac{1}{f} \\ &= -2\left(\int_{\gamma_0(x)}^{1} \frac{1}{f} + \int_1^{\infty} \frac{1}{f}\right) = 2\log\frac{\gamma_0(x)}{b} \end{aligned}$$

for any $x \in M$ with $\gamma_0(x) \le 1$, since $f(t) = t$ for $t \le 2$. Here $b = \exp\left(\int_1^\infty \frac{1}{f}\right) > 1$, which is a constant depending only on A. On the other hand, one has

$$\log \phi_{x_0}(x) \ge -2\int_1^\infty \frac{1}{f} = -2\log b$$

whenever $\gamma_0(x) > 1$. If we set

$$\widetilde{A}(x_0, c) := \{x \in M : \log \phi_{x_0}(x) < -c\}$$

for any $c > 0$, then we immediately obtain the following fact.

Lemma 9. *Under the condition of Theorem 2, one has*

$$\begin{aligned} \widetilde{A}(x_0, 2\log(2b)) &\subset \left\{x \in M : \gamma_0(x) < \frac{1}{2}\right\} \\ &= \left\{x \in M : \rho_0(x) < \frac{1}{2}\rho(x_0)\right\} \end{aligned}$$

for any $x_0 \in M\backslash B(o, 2c)$.

Proof of Theorem 2. For any $x_0 \in M \backslash B(o, 2c)$, $\phi_{x_0} - 1$ is a negative C^∞ strictly psh exhaustion function of M. It follows from the previous section that M satisfies the property $(B1)$. By Lemma 9 we claim that, for any sequence of points $y_k, k = 1, 2, \cdots$, which has no adherent point in M,

$$\begin{aligned} & A(y_k, 2\log(2b)) \subset \widetilde{A}(y_k, 2\log(2b)) \\ \subset \; & \left\{x \in M : \rho_k(x) < \frac{1}{2}\rho(y_k)\right\} \subset \left\{x \in M : \rho(x) > \frac{1}{2}\rho(y_k)\right\} \end{aligned}$$

provided k is sufficiently large. Here ρ_k denotes the distance associated to y_k. This implies that the property $(B2)$ is also satisfied. Thus the assertion follows immediately from Theorem 1. □

4. Proof of Theorem 6

We first prove 1). Let x_1, x_2 be two arbitrary points which satisfy $\rho(x_2) \geq 2c$ and $\rho(x_1) = 4\rho(x_2)$. Take a complete orthonormal basis $\{h_j\}_{j=0}^\infty$ for $\mathcal{H}$ such that $h_j(x_2) = 0$ for all $j \geq 1$. We claim that the following holds:

Lemma 10. *There is a constant $C_4 > 0$ such that*

$$\begin{aligned} & C_4 h_0(x_1) \wedge \bar{h}_0(x_1) \\ \leq \; & \sup\left\{f(x_1) \wedge \bar{f}(x_1) : f \in \mathcal{H},\ f(x_2) = 0,\ \left|\int_M f \wedge \bar{f}\right| \leq 1\right\}, \end{aligned} \tag{5}$$

where for any two forms $f(z) = f^*(z) dz_1 \wedge \cdots \wedge dz_n$, $g(z) = g^*(z) dz_1 \wedge \cdots \wedge dz_n$, $f(z) \wedge \bar{f}(z) \leq g(z) \wedge \bar{g}(z)$ iff $|f^*(z)| \leq |g^*(z)|$.

Proof. We will use Lemma 9 with $x_0 = x_1, x_2$ respectively. Set

$$\varphi = n\left(\log \phi_{x_1} + \log \phi_{x_2}\right) + \phi_{x_1} - \log\left(-\log \frac{\phi_{x_1}}{2}\right).$$

Clearly, it is a C^∞ strictly psh function on $M \backslash \{x_1, x_2\}$ which satisfies the following estimate:

$$\partial\bar{\partial}\varphi \geq \partial\bar{\partial}\left(-\log\left(-\log\frac{\phi_{x_1}}{2}\right)\right) \geq \frac{\partial \log \phi_{x_1} \bar{\partial} \log \phi_{x_1}}{\left(\log \frac{\phi_{x_1}}{2}\right)^2}. \tag{6}$$

Choose a C^∞ cutoff function $\chi : \mathbf{R} \to [0,1]$ such that $\chi|_{(-\infty,-2)} = 1, \chi|_{(-1,\infty)} = 0$. Set

$$\eta = \chi\left(\frac{\log \phi_{x_1}}{2\log(2b)}\right) h_0.$$

Clearly, $\eta(x_1) = h_0(x_1)$, and it follows from Lemma 9 that

$$\begin{aligned} \operatorname{supp} \eta \;\; & \subset \;\; \{x \in M : \log \phi_{x_1}(x) < -2\log(2b)\} \\ & \subset \;\; \left\{x \in M : \rho_1(x) \leq \frac{1}{2}\rho(x_1)\right\}, \end{aligned} \tag{7}$$

where $\rho_1(x)$ denotes the distance function relative to x_1. It follows that $\eta(x_2) = 0$ and

$$\log \phi_{x_2}(x) \geq -2\log(2b), \quad \forall x \in \operatorname{supp} \eta, \tag{8}$$

because $\rho(x_1) = 4\rho(x_2)$. By (7), one has

$$\left|\bar{\partial}\chi\left(\frac{\log \phi_{x_1}}{2\log(2b)}\right)\right|_{\partial\bar{\partial}\varphi} \leq C_5, \tag{9}$$

where $|\cdot|_{\partial\bar\partial\varphi}$ denotes the point norm with respect to the metric $\partial\bar\partial\varphi$ and $C_6 > 0$ is a constant that only depends on b and the choice of χ.

Observe that $M\backslash\{x_1, x_2\}$ still carries a complete Kähler metric, defined as follows:

$$\partial\bar\partial\left(-\log\left(-\log\phi_{x_1}\right) - \log\left(-\log\phi_{x_2}\right) + \phi_{x_1}\right).$$

By Proposition 7, there is a solution to the equation $\bar\partial u = \bar\partial\eta$ on $M\backslash\{x_1, x_2\}$ which satisfies

$$\begin{aligned}\left|\int_{M\backslash\{x_1,x_2\}} u\wedge\bar u e^{-\varphi}\right| &\leq \int_{M\backslash\{x_1,x_2\}} |\bar\partial\eta|^2_{\partial\bar\partial\varphi} e^{-\varphi} dV_\varphi \\ &\leq \left|\int_{M\backslash\{x_1,x_2\}} \left|\bar\partial\chi\left(\frac{\log\phi_{x_1}}{2\log(2b)}\right)\right|^2_{\partial\bar\partial\varphi} h_0\wedge\bar{h_0} e^{-\varphi}\right| \\ &\leq C_6 \qquad (10)\end{aligned}$$

because of (8)–(10). Set $f = \eta - u$. It is a holomorphic n-form on $M\backslash\{x_1, x_2\}$, and by (11), f can be extended holomorphically across x_1, x_2; moreover, $f(x_1) = h_0(x_1), f(x_2) = 0$ because of the singularity of φ at x_1, x_2. Since φ is bounded from above on M, one has

$$\begin{aligned}\left|\int_M f\wedge\bar f\right| &\leq 2\left|\int_M \eta\wedge\bar\eta\right| + 2\left|\int_M u\wedge\bar u\right| \\ &\leq 2\left|\int_M h_0\wedge\bar h_0\right| + C_7\left|\int_M u\wedge\bar u e^{-\varphi}\right| \\ &\leq C_8,\end{aligned}$$

from which the assertion immediately follows with the constant $C_4 = C_8^{-1}$. □

We proceed to prove the theorem. According to Kobayashi's alternative definition of the Bergman metric, β is nothing but the pullback of the Fubini-Study metric of the infinite-dimensional complex projective space $\mathbf{CP}(\mathcal{H})$ (cf. [11]). It follows that the Bergman distance $\mathrm{dist}_\beta(x_1, x_2)$ is no less than the Fubini-Study distance between the points $p_1 = (a_0 : a_1 : \cdots)$ and $p_2 = (1 : 0 : \cdots)$, where the a_j are given by

$$|a_j|^2 = \frac{h_j(x_1)\wedge\bar h_j(x_1)}{\sum_{j=0}^\infty h_j(x_1)\wedge\bar h_j(x_1)} = \frac{h_j(x_1)\wedge\bar h_j(x_1)}{K(x_1)}.$$

This implies that

$$\mathrm{dist}_\beta(x_1, x_2) \geq \sqrt{|1-a_0|^2 + \sum_{j=1}^\infty |a_j|^2}.$$

Assume that the supremum on the right side of (5) is realized by a certain n-form f. Then, without loss of generality, we can take $h_1 = f$. If $|a_0| \geq 1/2$, we have

$$\begin{aligned}\mathrm{dist}_\beta(x_1, x_2) &\geq |a_1| = \sqrt{\frac{f(x_1)\wedge\bar f(x_1)}{K(x_1)}} \\ &\geq \sqrt{C_4}\sqrt{\frac{h_0(x_1)\wedge\bar h_0(x_1)}{K(x_1)}} = \sqrt{C_4}|a_0| \geq \frac{1}{2}\sqrt{C_4}.\end{aligned}$$

Otherwise, it is clear that $\text{dist}_{\beta}(x_1, x_2) \geq 1 - |a_0| \geq \frac{1}{2}$. Therefore, there is a positive constant $C_9 > 0$ such that

$$\text{dist}_\beta(x_1, x_2) \geq C_9$$

holds for any $x_1, x_2 \in M$ satisfying $\rho(x_1) = 4\rho(x_2)$. From this the inequality (2) immediately follows.

Next we prove 2). The idea is similar, but simpler. It is known from page 109 of [6] that the bounded psh exhaustion function has an explicit form:

$$\phi_{x_0} = \left(\tanh \frac{\sqrt{A}\rho_0}{2}\right)^2$$

for any $x_0 \in M$. Hence there exists a constant $b_1 > 0$ such that

$$\widetilde{A}(x_0, b_1) \subset \{x \in M : \rho_0(x) < 1\}.$$

Repeating the argument as above, one can find a positive constant C_{10} such that for any two points $x_1, x_2 \in M$ with $\rho(x_1) = \rho(x_2) + 3$, we have

$$\text{dist}_\beta(x_1, x_2) \geq C_{10},$$

from which the assertion immediately follows.

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