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ON CERTAIN CO–H SPACES RELATED TO MOORE SPACES

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ABSTRACT. We show that certain co–H spaces, constructed by Anick and Gray, carry a homotopy co–associative and co–commutative co–H structure.

1. INTRODUCTION AND STATEMENT OF RESULTS

In a series of papers, Anick and Gray constructed and studied a family of co–H spaces $G_k^{2n}(p^r)$, $p \geq 5$, with some remarkable properties [1], [2], [7], [8]. The limit space, or rather its loop space $\Omega G_\infty^{2n}(p)$, shows up in a secondary version of the EHP sequence. Furthermore, Gray used the $G_k^{2n}(p^r)$ to construct v_2–periodic families in the homotopy groups of Moore spaces [8], and Anick used them to decompose loop spaces of finite complexes [1]. It is the aim of this paper to prove the following theorem, which was conjectured in part in [2]:

Theorem 1. *The spaces $G_k^{2n}(p^r)$ carry a co–associative and co–commutative co–H structure.*

In section 2 we define the notions of co–A and co–C deviation. Some properties are established, which are used in section 3 to prove Theorem 1.

After the results in this paper were obtained, I learned of Stephen Theriault's thesis, in which he proved the homotopy co–associtivity of G_k (and much more) by using Ganea's characterisation of co–associtivity. But the argument given there has a gap. It is assumed during the proof that, given a homotopy commutative square of co–H spaces and co–H maps, the map induced on the cofibers is also a co–H map. But this is clearly false (see for example the discussion of Zabrodsky's formula in [10], p. 228).

2. THE CO–A AND CO–C DEVIATION

The co–A deviation was first defined by Harper [10]. It is an obstruction for the existence of a co–associative co–H structure on the mapping cone of a co–H map between co–associative co–H spaces. The dual A–deviation and C–deviation were studied by Zabrodsky in [13].

Let $f : X, \tau \to Y, \sigma$ be a co–H map between the co–H spaces X, Y with structure maps τ, σ. So there are based homotopies

$$\begin{array}{llcll} M_x & : & j \circ \tau & \to & \triangle_x \ , \\ M_y & : & j \circ \sigma & \to & \triangle_y \ , \\ F & : & (f \vee f) \circ \tau & \to & \sigma \circ f \ . \end{array}$$

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Here $\triangle$ is the diagonal, and j the inclusion of the wedge into the product.

Suppose X, τ and Y, σ are co–associative resp. co–commutative up to homotopy. In this case we have based homotopies:

$$L_x : (1 \vee \tau) \circ \tau \ \longrightarrow\ (\tau \vee 1) \circ \tau\ , \qquad L_y : (1 \vee \sigma) \circ \sigma \ \longrightarrow\ (\sigma \vee 1) \circ \sigma$$

and

$$G_x : \tau \ \longrightarrow\ T \circ \tau\ , \qquad G_y : \sigma \ \longrightarrow\ T \circ \sigma\ ,$$

where T denotes the switch map.

We will use the following notation for stringing together a list of homotopies. Given $H_1, \dots, H_n : Z \times I \to W$ such that $H_i(Z,1) = H_{i+1}(Z,0)$, define

$$\{H_1, \dots, H_n\}(z,t) = H_i(z, nt - i + 1), \quad \text{if } (i-1) \leq nt \leq i.$$

We also write H^{-1} for the homotopy with $H^{-1}(z,t) = H(z, 1-t)$.

Suppose X, τ and Y, σ are homotopy co–associative, and fix L_x, L_y. Following Harper we define the co–A deviation with respect to F as follows: let

$$A(f,F) = \{(f \vee F) \circ \tau,\ (1 \vee \sigma) \circ F,\ L_y \circ f,\ (\sigma \vee 1) \circ F^{-1},$$
$$(F^{-1} \vee f) \circ \tau,\ f_{(3)} \circ L_x^{-1}\}\ .$$

This defines a map

$$A(f,F) : X, * \ \longrightarrow\ \Lambda Y_{(3)}, c\ ,$$

where Λ denotes the free loop space, c is the map sending the whole circle to the base point, and $X_{(k)}$ and $f_{(k)}$ denote the k–fold wedge of the space and the map, respectively.

Let $u : X, * \to \Lambda Y_{(3)},\ c$ be the map which sends x to the constant map

$$f_{(3)} \circ (1 \vee \tau) \circ \tau(x).$$

Then $A(f,F) - u$ factors uniquely over a map from $X, *$ to $\Omega Y_{(3)}, *$, and the homotopy class of the adjoint of this map

$$A_*(f,F) \in [\Sigma X, Y_{(3)}]$$

is the co–A deviation.

The homotopy F is called primitive, if

$$j \circ F = \{(f \times f) \circ M_x,\ M_y^{-1} \circ f\}$$

as tracks, i.e., as homotopy classes relative to $X \times \partial I$. Berstein and Harper proved in [5] that one can choose F to be primitive.

The map f is called a co–A map, if there is a primitive homotopy F such that $A_*(f;F)$ is the class of the constant map.

The definition of the co–C deviation is similar. Suppose X, τ and Y, σ are homotopy co–commutative, and fix G_x, G_y. Define

$$C(f,F) = \{F, G_y \circ f, T \circ F^{-1}, (f \vee f) \circ G_x^{-1}\}.$$

This gives us a map $C(f,F) : X, * \to \Lambda Y_{(2)}, c$.

Again there is a map unique up to homotopy from X to $\Omega Y_{(2)}$ such that the composite with the inclusion $\Omega Y_{(2)} \hookrightarrow \Lambda Y_{(2)}$ is $C(f,F) - v$ with $v(x)$ constant of value $(f \vee f) \circ \tau(x)$. The homotopy class of the adjoint $C_*(f,F) \in [\Sigma X, Y_{(2)}]$ is called the co–C deviation of f, F.

The map f is called a co–C map, if there is a primitive homotopy F such that $C_*(f,F)$ is the class of the constant map.

A result from [13], which we wish to dualize, is that the A–deviation, which is a map from $X \times X \times X$ to ΩY, can be chosen to be constant on the fat wedge in $X \times X \times X$. The proof used the existence of a strict unit for H–spaces. Since there is never a strict counit for a co–H space unless the space is a point, we have not been able to prove the strictly dual theorem, i.e., that $A_*(f, F)$ lifts to the homotopy fiber of $Y_{(3)} \longrightarrow R$, where R is the homotopy limit of the diagram

$$\begin{tikzcd}
Y_1 \vee Y_2 \arrow[r] \arrow[d] & Y_1 & Y_1 \vee Y_3 \arrow[l] \arrow[d] \\
Y_2 & Y_2 \vee Y_3 \arrow[l] \arrow[r] & Y_3
\end{tikzcd}$$

with the obvious maps. However, the following weak version will suffice for the applications which we have in mind.

Theorem 2.1. *The homotopies L_x, L_y and G_x, G_y can be chosen such that*

a) $j \circ A_*(f, F) \simeq *$*, where* $j : Y_{(3)} \to Y \times Y \times Y$*, and*

b) $j \circ C_*(f, F) \simeq *$*, where* $j : Y_{(2)} \to Y \times Y$*.*

For a map $g : X \to Y_{(k)}$ we denote the projection onto the i–th component by g_i.

The proof of Theorem 2.1 will need

Lemma 2.2. *One can choose L_x and L_y such that*

$$\begin{aligned}
L_{y,1} &= \sigma_1 \circ M_{y,1}^{-1}, & L_{x,1} &= M_{x,1}^{-1} \circ \tau_1, \\
L_{y,2} &= \{M_{y,1} \circ \sigma_2,\, \sigma_2 \circ M_{y,1}^{-1}\}, & L_{x,2} &= \{M_{x,1} \circ \tau_2, \tau_2 \circ M_{x,1}^{-1}\}, \\
L_{y,3} &= \sigma_2 \circ M_{y,2}, & L_{x,3} &= M_{x,2} \circ \tau_2
\end{aligned}$$

as tracks, i.e., as homotopy classes relative to $Y \times \partial I$ and $X \times \partial I$.

Proof. Since the loop of the homotopy fiber sequence $F \to Y_{(3)} \to Y \times Y \times Y$ splits, the lemma follows from [5, 1.8]. □

Lemma 2.3. *One can choose G_x and G_y such that*

$$G_{x,1} = \{M_{x,1}, M_{x,2}^{-1}\},\ G_{y,1} = \{M_{y,1}, M_{y,2}^{-1}\},$$
$$G_{x,2} = \{M_{x,2}, M_{x,1}^{-1}\},\ G_{y,2} = \{M_{y,2}, M_{y,1}^{-1}\}.$$

Proof. Same argument as above. □

Proof of Theorem 2.1. a) We choose F primitive and L_x, L_y as in Lemma 2.2. We show that in this case $A(f, F)_i = u_i$, $i \in \{1, 2, 3\}$. (By a slight abuse of notation we also denote by g_i the composition of a map to $\Lambda(Y_{(k)})$ with the i-th projection to $\Lambda(Y)$.)

We first have a hard look at $A(f, F)_1$:

$$\begin{aligned}
A(f, F)_1 &= \{f \circ M_{x,1},\, M_{y,1}^{-1} \circ f,\, \sigma_1 \circ M_{y,1}^{-1} \circ f,\, \sigma_1 \circ M_{y,1} \circ f,\, \sigma_1 \circ f \circ M_{x,1}^{-1}, \\
&\qquad M_{y,1} \circ f \circ \tau_1,\, f \circ M_{x,1}^{-1} \circ \tau_1,\, f \circ M_{x,1} \circ \tau_1\} \\
&= \{f \circ M_{x,1},\, M_{y,1}^{-1} \circ f,\, \sigma_1 \circ f \circ M_{x,1}^{-1},\, M_{y,1} \circ f \circ \tau_1\}.
\end{aligned}$$

Claim.

$$\begin{aligned}
&\{f\circ M_{x,1},\, M_{y,1}^{-1}\circ f\} = M_{y,1}^{-1}\circ f\circ M_{x,1},\\
&\{\sigma_1\circ f\circ M_{x,1}^{-1},\, M_{y,1}\circ f\circ\tau_1\} = M_{y,1}\circ f\circ M_{x,1}^{-1}.
\end{aligned}$$

Here $M_{y,1}^{-1}\circ f\circ M_{x,1}$ is shorthand for $M_{y,1}^{-1}(f\circ M_{x,1},\ \mathrm{id}_I)$, and similarly for $M_{y,1}\circ f\circ M_{x,1}^{-1}$.

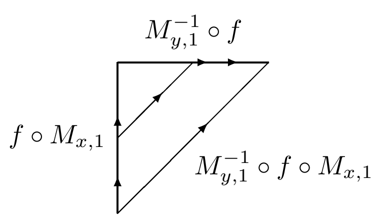
Moreover, for compositions, the first argument may be replaced by – if the expression is complicated and forced by contest.

To see this, consider the homotopies of tracks

$$K_1(x,t,s) := \begin{cases} f\circ M_{x,1}(x,2t) \quad \text{if } 0\le t\le \frac{s}{2},\\ M_{y,1}^{-1}(f\circ M_{x,1}(x,t+\frac{s}{2}),\, t-\frac{s}{2}) \quad \text{if } \frac{s}{2}\le t\le 1-\frac{s}{2},\\ M_{y,1}^{-1}(f(x),2t-1) \quad \text{if } 1-\frac{s}{2}\le t\le 1\,, \end{cases}$$

$$K_2(x,t,s) := \begin{cases} \sigma_1\circ f\circ M_{x,1}^{-1}(x,2t) \quad \text{if } 0\le t\le \frac{s}{2},\\ M_{y,1}(f\circ M_{x,1}^{-1}(x,t+\frac{s}{2}),\, t-\frac{s}{2}) \quad \text{if } \frac{s}{2}\le t\le 1-\frac{s}{2},\\ M_{y,1}(f(x),\, 2t-1) \quad \text{if } 1-\frac{s}{2}\le t\le 1\,. \end{cases}$$

The picture is:



where the path in the middle is $K_1(\dots,s)$. In conclusion, we have shown that $A(f,F)_1\simeq A(f,F)_1(0)=u_1$. $A(f,F)_2$ is a little more complex:

$$\begin{aligned}
A(f,F)_2 \quad &= \quad \{F_1\circ\tau_2,\, \sigma_1\circ F_2,\, L_{y,2}\circ f,\, \sigma_2\circ F_1^{-1},\, F_2^{-1}\circ\tau_1,\, f\circ L_{x,2}^{-1}\}\\
&= \quad \{f\circ M_{x,1}\circ\tau_2,\, M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f,\\
&\qquad M_{y,1}\circ\sigma_2\circ f,\ \sigma_2\circ M_{y,1}^{-1}\circ f,\, \sigma_2\circ M_{y,1}\circ f,\, \sigma_2\circ f\circ M_{x,1}^{-1},\\
&\qquad M_{y,2}\circ f\circ\tau_1,\ f\circ M_{x,2}^{-1}\circ\tau_1, f\circ\tau_2\circ M_{x,1},\ f\circ M_{x,1}^{-1}\circ\tau_2\}\\
&= \quad \{\{M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f,\, M_{y,1}\circ\sigma_2\circ f\}\,,\\
&\qquad \{\sigma_2\circ f\circ M_{x,1}^{-1},\, M_{y,2}\circ f\circ\tau_1,\, f\circ M_{x,2}^{-1}\circ\tau_1,\, f\circ\tau_2\circ M_{x,1}\}\}\,.
\end{aligned}$$

As in the case above, one sees that

$$\begin{aligned}
\{\sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f\} \quad &= \quad \sigma_1\circ M_{y,2}^{-1}\circ f\circ M_{x,2},\\
\{M_{y,2}\circ f\circ\tau_1,\, f\circ M_{x,2}^{-1}\circ\tau_1\} \quad &= \quad M_{y,2}\circ f\circ M_{x,2}^{-1}\circ\tau_1.
\end{aligned}$$

So it remains to show that

$$\begin{aligned}
&\{M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ M_{y,2}^{-1}\circ f\circ M_{x,2},\, M_{y,1}\circ\sigma_2\circ f\}\\
&\qquad = \{\sigma_2\circ f\circ M_{x,1}^{-1},\, M_{y,2}\circ f\circ M_{x,2}^{-1}\circ\tau_1,\, f\circ\tau_2\circ M_{x,1}\}^{-1}\,.
\end{aligned}$$

To see this, consider the homotopy between tracks:

$$H_1(x,t,s) := \begin{cases} M_{y,1}^{-1}(f\circ\tau_2(x), 3t) \quad \text{if } 0\le t\le \frac{s}{3}, \\ M_{y,1}^{-1}(-,s)\circ M_{y,2}^{-1}\left(-, \dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ f\circ M_{x,2}\left(-, \dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right) \\ \qquad\qquad \text{if } \frac{s}{3}\le t\le 1-\frac{s}{3}, \\ \sigma_2\circ M_{y,1}(f(x), 3t-2) \quad \text{if } 1-\frac{s}{3}\le t\le 1, \end{cases}$$

$$H_2(x,t,s) := \begin{cases} \sigma_2\circ f\circ M_{x,1}^{-1}(x, 3t) \quad \text{if } 0\le t\le \frac{s}{3}, \\ M_{y,2}\left(-, \dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ f\circ M_{x,2}^{-1}\left(-, \dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ M_{x,1}^{-1}(x,s) \\ \qquad \text{if } \frac{s}{3}\le t\le 1-\frac{s}{3}, \\ f\circ\tau_2\circ M_{x,1}(x, 3t-2) \quad \text{if } 1-\frac{s}{3}\le t\le 1. \end{cases}$$

Since $H_1(x,t,0) = H_2(x,1-t,0) = M_{y,2}^{-1}(-,t)\circ f\circ M_{x,2}(x,t)$, the identity above is proved.

The case $A(f,F)_3$ is like the first one.

Next we prove b), which is easier. Using the primitivity of F and Lemma 2.3, we find that

$$\begin{aligned}&\{F, G_y\circ f, T\circ F^{-1}, f\vee f\circ G_x^{-1}\}_{,1} \\ &\quad = \{f\circ M_{x,1}, M_{y,1}^{-1}\circ f, M_{y,1}\circ f, M_{y,2}^{-1}\circ f, M_{y,2}\circ f, f\circ M_{x,2}^{-1}, f\circ M_{x,2}, f\circ M_{x,1}^{-1}\},\end{aligned}$$

which is the constant track.

A similar argument works for the second coordinate. □

The following two lemmas were proved in [5].

Lemma 2.4. *Let $(f,F):\Sigma X,\tau\to Y,\sigma$ be a co–H map with primitive homotopy F. Suppose Y,σ is homotopy co–associative and $\Sigma X,\tau$ is the suspension co–H structure. Then for each suspension $\Sigma g:\Sigma Z\to\Sigma X$ the co–A deviation satisfies*

$$A_*(f\circ\Sigma g, F\circ\Sigma g) = A_*(f,F)\circ\Sigma^2 g\ .$$

□

Recall from [5] that for a co–H map $f: X,\tau\to Y,\sigma$ with primitive homotopy F, the mapping cone C_f carries a co–H structure $\overline{\sigma}$ defined by

$$\begin{aligned}\overline{\sigma}(x,t) \quad &= \quad \begin{cases} F^{-1}(x,2t) & \text{for} \quad 0\le 2t\le 1, \\ \tau(x), 2t-1 & \text{for} \quad 0\le 2t\le 1, \end{cases} \\ \overline{\sigma}(y) \quad &= \quad \sigma(y)\ .\end{aligned}$$

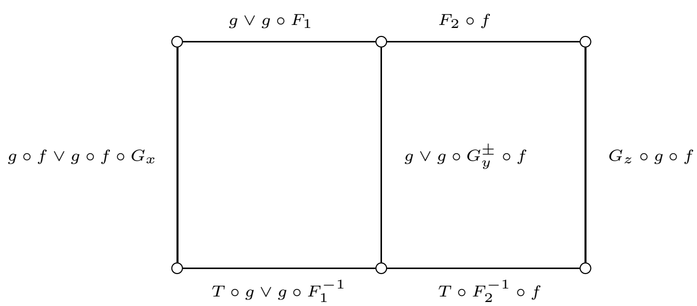
Lemma 2.5. *Let (f,F) be a co–A map. Then $\overline{\sigma}$ is homotopy co–associative.* □

There are similar results for the co–C deviations:

Lemma 2.6. *Let $(f,F_1): X,\tau\longrightarrow Y,\sigma$ and $(g,F_2): Y,\sigma\longrightarrow Z,\eta$ be co–H maps and primitive homotopies, and suppose all spaces are homotopy co–commutative. Then*

$$C_*(f\circ g,\ \{g\vee g\circ F_1, F_2\circ f\}) = (g\vee g)\circ C_*(f,F_1) + C_*(g,F_2)\circ\Sigma f.$$

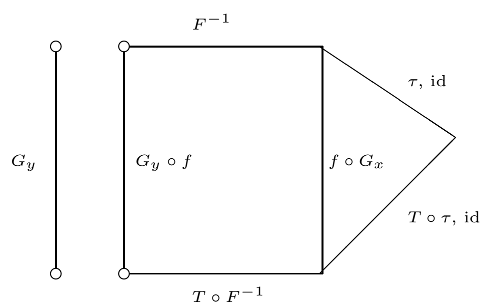
Proof. The proof is given by looking at the diagram



□

Lemma 2.7. *Let (f, F) be a co–C map and a primitive homotopy. Then the co–H structure on C_f defined by F is homotopy co–commutative.*

Proof. Consider the diagram



The square in the middle can be filled since f is a co–C map, and the triangle is extendable by the cone on G_*. □

3. Review on $G_k^{2n}(p^r)$ and proof of theorem 1

Recall from [2] that the space $G_k^{2n}(p^r)$ has a CW structure as follows:

$$G_k^{2n}(p^r) = S^{2n} \cup_{p^r} e^{2n+1} \cup_{g_1} \ \ldots \ \cup_{g_k} S^{2np^k} \cup_{p^{r+k}} e^{2np^k+1};$$

the attaching maps g_i are divisible by p^{r+i-1} in homotopy.

One also has $G_{k-1}^{2n}(p^r) \cup_{g_k} S^{2np^k} \cup e^{2np^k+1} = G_k^{2n}(p^r)$, and there is a compatible co–H structure on $G_k^{2n}(p^r)$. Denote by W_k^{r+k} the class of spaces that are of finite type and of the homotopy type of bouquets of Moore spaces of type p^{r+s} with $s \in \{0, \ldots, k\}$.

Lemma 3.1. *The following statements hold:*

a) $\Sigma^2 \Omega G_k^{2n}(p^r) \in W_r^{r+k}$.
b) $\Sigma \Omega G_k^{2n}(p^r) \wedge \Omega G_k^{2n}(p^r) \in W_r^{r+k}$.

c) *Let F be the homotopy fiber of*

$$j : \bigvee_{i=1}^{3} G_k^{2n}(p^r) \longrightarrow \prod_{i=1}^{3} G_k^{2n}(p^r) .$$

Then ΩF is homotopy equivalent to a weak product of spaces of the form ΩW_j, $W_j \in W_r^{r+k}$.

Proof. Parts a) and b) are in [2]. For c), recall the two homotopy equivalences

i) $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$,

ii) $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y)$.

Apply i) to $\Omega(Y \vee Y \vee Y)$ to find that

$$\Omega F \simeq \Omega\Sigma(\Omega Y)^{\wedge 2} \times \Omega[\Sigma\Omega Y \wedge (\Omega Y \times \Omega Y \times \Omega\Sigma(\Omega Y)^{\wedge 2}] ,$$

and ii) to see that

$$\begin{aligned}&\Sigma(\Omega Y \times \Omega Y \times \Omega\Sigma(\Omega Y)^{\wedge 2})\\ &\quad \simeq \Sigma\Omega Y \vee \Sigma\Omega Y \vee \ \Sigma(\Omega Y)^{\wedge 2} \ \vee \ \Sigma\Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma\Omega Y\\ &\qquad \wedge \ \Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma\Omega Y \wedge \Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma(\Omega Y)^{\wedge 2} \wedge \Omega\Sigma(\Omega Y)^{\wedge 2}.\end{aligned}$$

By Hilton–Milnor, and the fact that W_r^{r+k} is closed under smash products [12], it is enough to show that each wedge factor is in W_r^{r+k} for $Y = G_k^{2n}(p^r)$. By b) $\Sigma(\Omega G_k^{2n}(p^r))^{\wedge 2} \in W_r^{r+k}$, and since this space is 2–connected it is a double suspension. By a) the smash with $\Omega G_k^{2n}(p^r)$ is in W_r^{r+k}. The splitting of $\Sigma\Omega\Sigma X$ shows that also $\Sigma\Omega\Sigma\Omega G_k^{2n}(p^r)^{\wedge 2} \in W_r^{r+k}$. □

Proof of Theorem 1. The proof is by induction on k. For $k = 0$, $G_k^{2n}(p^r)$ is a simply connected Moore space of type p^r, p odd. Hence it is a suspension, and so homotopy co–associative. If $n \geq 2$ it is a double suspension, and the homotopy co–commutativity for $n = 1$ was proved in [4]. So suppose the assertion holds for $k - 1, k \geq 1$. We show that the co–A and the co–C deviations of g_k vanish. This suffices by Lemmas 2.5 and 2.7. Since g_k is divisible by p^{r+k-1}, it follows from Lemmas 2.4 and 2.6 that $A_*(g_k, F)$ and $C_*(g_k, F)$ are also divisible by p^{r+k-1}.

By Theorem 2.1 the maps $A_*(g_k, F)$ and $C_*(g_k, F)$ lift, uniquely up to homotopy, to the homotopy fibers F_1 and F_2, respectively, of

$$j : \bigvee_{i=1}^{s} G_k^{2n}(p^r) \longrightarrow \prod_{i=1}^{s} G_k^{2n}(p^r), \qquad s \in \{2, 3\} .$$

Moreover, the lift is also divisible by p^{r+k-1}, since the loop of this fibration splits.

We claim that in fact

$$p^{r+k-1}[P^{2np^k}(p^r), \Omega F_i] = 0.$$

The homotopy type of F_2 is well known to be $\Sigma\Omega G_{k-1}^{2n}(p^r) \wedge \Omega G_{k-1}^{2n}(p^r)$, and the type of ΩF_1 was determined during the proof of Lemma 3.1.

By [11] the homotopy exponent of $P^\ell(p^s)$ is p^{s+1} for p odd. It follows that the only factors in the product decomposition of ΩF_i which could contribute a class of order p^{r+k} are of the form $\Omega P^\ell(p^{r+k-1})$. So we have to determine the least ℓ for which such a factor occurs. The first class in $H_*(\Omega G_k^{2n}(p^r); \mathbb{Z})$ of order p^{r+k-1} shows up in dimension $2np^{k-1} - 1$ by [2, p. 864].

Consequently the first class of order p^{r+k-1} in $H_*(\Omega\Sigma\Omega G_k^{2n}(p^r) \wedge \Omega G_k^{2n}(p^r); \mathbb{Z})$ shows up in dimension $4np^{k-1} - 2$. Inspection of the proof of Lemma 3.1 shows that in the splitting of ΩF_1 each factor is of the form loop of $\Sigma(\Omega G_k^{2n}(p^r))^{\wedge 2} \wedge Z$. So also in $H_*(\Omega F_1, \mathbb{Z})$ and $H_*(\Omega F_2, \mathbb{Z})$ the first class of order p^{r+k-1} is in dimension $\geq 4np^{k-1} - 2$. This class comes from a factor $\Omega P^{4np^{k-1}\geq}(p^{r+k-1})$.

By [6], the first element of order p^{r+k} in $\pi_*(P^{2m+1}(p^{r+k-1}))$, respectively $\pi_*(P^{2m}(p^{r+k-1}))$ is in dimension $2mp-1$, resp. $4mp-2p-1$.

The universal coefficient sequence for homotopy groups splits at an odd prime. So it is enough to show that $\pi_*(P^\ell(p^{r+k-1}))$ does not contain a class of order p^{r+k} for $* \leq 2np^k+1$ and $\ell \geq 4np^{k-1} = 2m$. But this follows from what was said above and the trivial estimates

$$\begin{aligned} 2np^k+1 &< 4mp-2p-1 = 8np^k-2p-1, \\ 2np^k+1 &< 2mp-1 = 4np^k-1 \ . \end{aligned}$$

□

The next two corollaries are just mild strengthenings of two results from [2].

Corollary 3.2. *Suppose X is an H space, and $\varphi_{k-1} : G_{k-1}^{2n}(p^r) \to \Sigma X$ is a map such that $\varphi_{k-1}\big|_K$, where K is a skeleton, is a co–H map. Suppose also that $p^{r+k-1}\pi_{2np^k-1}(X; \mathbb{Z}/p^{r+k}) = 0$. Then φ_{k-1} has an extension $\varphi_k : G_k^{2n}(p^r) \to \Sigma X$.*

Proof. It was shown in [2, 4.1] that the corollary would follow if one could choose the coretraction $\sigma : G_k^{2n}(p^r) \to \Sigma\Omega G_k^{2n}(p^r)$ corresponding to the co–H structure to be a co–H map. Since every 1–connected co–associative co–H space is a cogroup [3], this follows from [9, 4.2] and Theorem 1. □

Corollary 3.3. *Let $0 \leq s \leq m \leq \infty$, and suppose that X is an H–space such that $p^{r+k-1}\pi_{2np^k-1}(X; \mathbb{Z}/p^{r+k}) = 0$ for $s < k \leq m$. Let $\varphi : P^{2mp^s}(p^{r+s}) \to X$ be a map. Then there is a map $\varphi_m : G_m^{2n}(p^s) \to \Sigma X$ which extends*

$$G_s^{2n}(p^r) \xrightarrow{pinch} G_s^{2n}(p^r)/G_{s-1}^{2n}(p^r) = P^{2np^s+1}(p^{r+s}) \xrightarrow{\Sigma\varphi} \Sigma X \ .$$

Proof. Since $G_{s-1}^{2n}(p^r)$ is a sub-co–H space of $G_s^{2n}(p^r)$ the pinch map is a co–H map for the induced co–H structure on the quotient space. This co–H structure is unique [4]. Thus the composite with $\Sigma\varphi$ is a co–H map, and the assertion follows from Corollary 3.2. □

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