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ON A CLASS OF JOINTLY HYPONORMAL TOEPLITZ OPERATORS

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ABSTRACT. We characterize when a pair of Toeplitz operators $\mathbf{T} = (T_\phi, T_\psi)$ is jointly hyponormal under various assumptions—for example, ϕ is analytic or ϕ is a trigonometric polynomial or $\phi - \psi$ is analytic. A typical characterization states that $\mathbf{T} = (T_\phi, T_\psi)$ is jointly hyponormal if and only if an algebraic relation of ϕ and ψ holds and the single Toeplitz operator T_ω is hyponormal, where ω is a combination of ϕ and ψ. More general results for an n-tuple of Toeplitz operators are also obtained.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For $T, S \in B(\mathcal{H})$, the commutator of T and S is

$$[T, S] := TS - ST.$$

An operator T is hyponormal if the self-commutator $[T^*, T] = T^*T - TT^*$ is positive. Let $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. The tuple $\mathbf{A}$ is jointly hyponormal (or simply, hyponormal) if the self-commutator $[\mathbf{A}^*, \mathbf{A}]$ of $\mathbf{A}$, defined by $[\mathbf{A}^*, \mathbf{A}] = ([A_j{}^*, A_i])$, is a positive operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. This notion of joint hyponormality was first introduced by Athavale [3] in 1988. We say $\mathbf{A}$ is weakly hyponormal if $\alpha_1 A_1 + \alpha_2 A_2 + \cdots \alpha_n A_n$ is hyponormal for all constants $\alpha_1, \cdots, \alpha_n$. It was observed in [3] that if $\mathbf{A}$ is jointly hyponormal then $\mathbf{A}$ is weakly hyponormal. These two notions are in general different even for commuting tuples, see [7], [15]. Joint hyponormality has been studied by several authors [3], [7], [11], [12], [14], [21].

In this paper we will study the joint hyponormality of Toeplitz operators. Let L^2 be the space of Lebesgue square integrable functions on the unit circle, and L^∞ the space of essentially bounded functions on the unit circle. The Hardy space H^2 is the closed linear span of analytic polynomials in L^2. Let P be the projection of L^2 onto H^2. For $\phi \in L^\infty$, the Toeplitz operator T_ϕ with symbol ϕ on the Hardy space H^2 is defined by the rule

$$T_\phi h = P(\phi h).$$

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The unilateral shift U is the Toeplitz operator T_z. The normality, subnormality and hyponormality of a single Toeplitz operator have been discussed in [1], [2], [6], [8], [9], [10] [16], [19], [22]. In particular, Cowen [8] gave an elegant characterization of hyponormal Toeplitz operators.

The study of jointly hyponormal Toeplitz operators started with the interesting observation, made by Farenick and MaEachin [17], that the joint hyponormality of (U, T) implies that T is necessarily a Toeplitz operator; see Proposition 2.3 below. They went on to characterize when (U, T) is hyponormal. Let $\psi \in L^\infty$. Recently Curto and Lee [13] studied the joint hyponormality of the Toeplitz pair $\mathbf{T} = (T_\phi, T_\psi)$ when both symbols ψ and ϕ are trigonometric polynomials. A complete characterization of hyponormal Toeplitz pairs in this case was given. Moreover, they showed that the weak hyponormality and the hyponormality of $\mathbf{T}$ are equivalent properties when both symbols are trigonometric polynomials.

In this paper we give an example of $\mathbf{T} = (T_\phi, T_\psi)$ where ϕ and ψ are rational functions such that $\mathbf{T}$ is weakly hyponormal but not hyponormal. This will be done in Section 3 after some preliminary results in Section 2. This is achieved by noting that the weak hyponormality of $\mathbf{T} = (T_\phi, T_\psi)$ for a certain class of rational symbols is equivalent to the classical Hermite-Fejér interpolation problem via Cowen's characterization. For a single Toeplitz operator, this connection was observed by the author [19].

In Section 4, we characterize when $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal for the case $\phi_i \in H^\infty$ for $i = 1, \cdots, n$ and $\psi \in L^\infty$. This extends the result of [17], where it was assumed that $n = 1$ and $\phi_1 = z$, and the result of [13], where it was assumed that $n = 1$ and ϕ_1 is an analytic polynomial. This also allows us to construct a tuple of three Toeplitz operators $\mathbf{T} = (T_{\phi_1}, T_{\phi_2}, T_{\phi_3})$ for which each subpair is hyponormal but $\mathbf{T}$ is not hyponormal, which gives a negative answer to a conjecture of Curto and Lee [13]. Indeed, it was shown in [13] that $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ is hyponormal if and only if each subpair is hyponormal when all the ϕ_i are assumed to be trigonometric polynomials. The examples above indicate that when we deal with symbols more general than trigonometric polynomials, new phenomena occur. In particular, the technique of using weak hyponormality to study hyponormality as in [13] is not available, and the extension of the results for a pair of Toeplitz operators to a tuple of Toeplitz operators requires much more work.

In Section 5 we study the hyponormality of $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ when all the ϕ_i have the same anti-analytic parts. We make some observations in Section 6 and give various sufficient and necessary conditions for the hyponormality of $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$. In particular, we give a complete characterization of the hyponormality of $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ when only one of them is assumed to have a trigonometric polynomial symbol. A typical characterization states that $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if a certain algebraic relation of ϕ and ψ holds and a single Toeplitz operator whose symbol is a variation of ψ or ϕ is hyponormal.

The approach we take is quite different from the ones in [17] and [13], where the proofs often rely on intricate and explicit computations using the coefficients of trigonometric polynomials involved. For more general symbols, explicit computation is very difficult, if not impossible. Indeed, here most computations are done at the operator and function levels in a more abstract way. Besides yielding more general results, our proofs seem to be shorter and more insightful. This is made possible by systematic use of Hankel operators and by avoiding direct computation of the inverses of certain operators.

2. Some preliminary results

We first establish some results for positive 2 by 2 block matrices. Let $\mathcal{H}$ and $\mathcal{G}$ be two separable complex Hilbert spaces. Let

$$M(A,B,C) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad \text{on} \quad \mathcal{H} \oplus \mathcal{G},$$

where A, B and C are operators acting between appropriate spaces. Let $Ker(A)$, $Range(A)$ and $Rank(A)$ be the kernel, range and rank of A. For an operator A, the Moore-Penrose inverse $A^{\#}$ of A is an operator satisfying

$$AA^{\#}A = A, \quad A^{\#}AA^{\#} = A^{\#}, \quad (AA^{\#})^* = AA^{\#}, \quad (A^{\#}A)^* = A^{\#}A.$$

An operator on a Hilbert space has a Moore-Penrose inverse if and only if it has closed range, and the Moore-Penrose inverse is unique whenever it exists. In particular, if A is positive and has closed range, and we write

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : Range(A) \oplus Ker(A) \to Range(A) \oplus Ker(A),$$

then the Moore-Penrose inverse $A^{\#}$ of A is defined by

$$A^{\#} = \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} : Range(A) \oplus Ker(A) \to Range(A) \oplus Ker(A).$$

Lemma 2.1. *a) $M(A,B,C)$ is positive if and only if both A and C are positive and*

$$|(h, Bg)|^2 \leq (Ah, h)(Cg, g) \quad \textit{for all } h \in \mathcal{H}, g \in \mathcal{G}.$$

b) If $M(A,B,C)$ is positive, then

$$Ker(B^*) \supset Ker(A) \quad \textit{and} \quad Ker(B) \supset Ker(C).$$

In particular, if $A = 0$, then $B = 0$.

c) If $Range(A)$ is closed, let $A^{\#}$ be the Moore-Penrose inverse of A. Then $M(A,B,C)$ is positive if and only if both A and C are positive,

$$Ker(B^*) \supset Ker(A) \quad \textit{and} \quad C \geq B^* A^{\#} B.$$

d) If $M(A,B,C)$ is positive and both A and C are of finite rank, then $M(A,B,C)$ is of finite rank and

$$Rank[M(A,B,C)] = Rank(A) + Rank(C - B^* A^{\#} B).$$

Proof. The proof is essentially from [14]. $M(A,B,C)$ is positive if and only if

$$\langle M(A,B,C)(th \oplus g), (th \oplus g) \rangle \geq 0 \quad \text{for all } h \in \mathcal{H}, g \in \mathcal{G}$$

and any real number t. Equivalently,

$$t^2(Ah,h) + 2tRe(h, Bg) + (Cg, g) \geq 0.$$

Analyzing the above quadratic function of t yields the result in part a).

Part b) clearly follows from part a). Part c) is well known in the case when A is invertible. Part c) in this general form also appeared in [13]. Here we give a slightly more direct proof, which is an adaptation of the proof for the invertible case and also yields the rank formula in part d).

We next prove the sufficiency in part c). Let $\mathcal{H}_0$ be the range of A. It is easy to see that $AA^\# = A^\# A = P_{\mathcal{H}_0}$ The relation $Ker(B^*) \supset Ker(A) = Ker(P_{\mathcal{H}_0})$ implies that

$$-B^*A^\# A + B^* = B^*(I - P_{\mathcal{H}_0}) = 0.$$

Therefore,

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -B^*A^\# & I \end{pmatrix} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} I & -A^\# B \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A & B - AA^\# B \\ -B^*A^\# A + B^* & C - B^*A^\# B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - B^*A^\# B \end{pmatrix}. \end{aligned}$$

The sufficiency in part c) follows from the above matrix identity. The necessity is also clear by part b) and the above matrix identity. □

The following lemma provides a contrast between the weak hyponormality and hyponormality of $\mathbf{A} = (A_1, A_2)$. The first part is Lemma 1.4 in [14] and the second part is Proposition 2.5 in [7].

Lemma 2.2. *a)* $\mathbf{A} = (A_1, A_2)$ *is hyponormal if and only if*

$$|([A_1^*, A_2]\, x, y)|^2 \le ([A_1^*, A_1]\, x, x)([A_2^*, A_2]\, y, y) \quad \textit{for all } x, y \in \mathcal{H}.$$

b) $\mathbf{A} = (A_1, A_2)$ *is weakly hyponormal if and only if*

$$|([A_1^*, A_2]\, x, x)|^2 \le ([A_1^*, A_1]\, x, x)([A_2^*, A_2]\, x, x) \quad \textit{for all } x \in \mathcal{H}.$$

Let U be the unilateral shift on H^2.

Proposition 2.3 (Farenick and McEachin, 1995)**.** *If (U^n, T) is hyponormal, then $U^{*n}TU^n = T$. Therefore T is a block Toeplitz operator with block size n. In particular, if (U, T) is hyponormal, then T is a Toeplitz operator.*

Proof. If (U^n, T) is hyponormal, then

$$\begin{aligned} &\begin{pmatrix} U^{*n} & 0 \\ 0 & U^n \end{pmatrix} \begin{pmatrix} [U^{*n}, U^n] & [T^*, U^n] \\ [U^{*n}, T] & [T^*, T] \end{pmatrix} \begin{pmatrix} U^n & 0 \\ 0 & U^{*n} \end{pmatrix} \\ &= \begin{pmatrix} U^{*n}\,[U^{*n}, U^n]\,U^n & U^{*n}\,[T^*, U^n]\,U^{*n} \\ U^n\,[U^{*n}, T]\,U^n & U^n\,[T^*, T]\,U^{*n} \end{pmatrix} \\ &= \begin{pmatrix} 0 & U^{*n}\,[T^*, U^n]\,U^{*n} \\ U^n\,[U^{*n}, T]\,U^n & U^n\,[T^*, T]\,U^{*n} \end{pmatrix} \end{aligned}$$

is positive. Thus $U^n\,[U^{*n}, T]\,U^n = 0$. Therefore

$$U^{*n}U^n\,[U^{*n}, T]\,U^n = U^{*n}TU^n - T = 0.$$

This completes the proof. □

The above proposition is still valid if we replace U by any isometry. The following proposition shows that if the other operator T is also an isometry, conditions of this type are indeed sufficient for hyponormality.

Proposition 2.4. *Let S_1 and S_2 be two isometries on H^2. Then $\mathbf{S} = (S_1, S_2)$ is hyponormal if and only if*

$$S_1 = S_2^* S_1 S_2, \quad S_2 = S_1^* S_2 S_1. \tag{1}$$

Proof. We need to show the sufficiency. Let H_1 and H_2 be the ranges of S_1 and S_2 respectively. Let P_{H_1} and P_{H_2} be the projections onto H_1 and H_2. First note that

$$Ker(S_1^*S_2 - S_2S_1^*) \supset Ker(I - P_{H_1}) = H_1.$$

Since, by (1), for any $h \in H^2$

$$(S_1^*S_2 - S_2S_1^*)S_1h = S_1^*S_2S_1h - S_2h = 0,$$

thus by Lemma 2.1,

$$[\mathbf{S}^*, \mathbf{S}] = \begin{pmatrix} I - P_{H_1} & S_2^*S_1 - S_1S_2^* \\ S_1^*S_2 - S_2S_1^* & I - P_{H_2} \end{pmatrix}$$

is positive if

$$I - P_{H_2} \geq (S_1^*S_2 - S_2S_1^*)(I - P_{H_1})(S_2^*S_1 - S_1S_2^*).$$

Since $S_1^*(I - P_{H_1}) = 0$ and $(I - P_{H_1})S_1 = 0$, the above inequality is the same as

$$I - P_{H_2} \geq S_1^*S_2(I - P_{H_1})S_2^*S_1.$$

Since S_1^* and S_2^* are contractions, the above inequality holds if and only if

$$(I - P_{H_1})S_2^*S_1P_{H_2} = (I - S_1S_1^*)S_2^*S_1S_2S_2^* = 0.$$

Equivalently,

$$S_2^*S_1S_2S_2^* = S_1S_1^*S_2^*S_1S_2S_2^*.$$

By (1), we see that the left side in the above equation is $S_1S_2^*$ and the right side is

$$S_1S_1^*S_2^*S_1S_2S_2^* = S_1S_2^*S_2S_2^* = S_1S_2^*.$$

This completes the proof. □

3. Weakly hyponormal Toeplitz operators

In this section we recall Cowen's characterization of hyponormal Toeplitz operators and prove a corollary. We give a necessary and sufficient condition for weak hyponormality for a class of Toeplitz pairs with rational symbols. We then give an example of a weakly hyponormal but not hyponormal Toeplitz pair.

The Hankel operator H_ψ with symbol $\psi \in L^\infty$ is the operator on H^2 defined by

$$H_\psi h = J(I - P)(\psi h),$$

where J is the unitary operator from $L^2 \ominus H^2$ onto H^2 defined by $J(e^{-in\theta}) = e^{i(n-1)\theta}$ for $n \geq 1$. Toeplitz operators and Hankel operators are connected by the following important relation:

$$T_{\phi\psi} - T_\phi T_\psi = H_{\overline{\phi}}^* H_\psi. \tag{2}$$

For $f \in L^\infty$, let $f_+ = P(f)$ and $f_- = \overline{(I - P)(f)}$. Set

$$\psi = P(\psi) + (I - P)(\psi) = \psi_+ + \overline{\psi_-},$$

$$\phi = P(\phi) + (I - P)(\phi) = \phi_+ + \overline{\phi_-}.$$

The following formula, which expresses the self-commutator of $\mathbf{T} = (T_\phi, T_\psi)$ by Hankel operators, will play a crucial role in this paper:

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} \left[T_\phi^*, T_\phi\right] & \left[T_\psi^*, T_\phi\right] \\ \left[T_\phi^*, T_\psi\right] & \left[T_\psi^*, T_\psi\right] \end{pmatrix}$$

$$= \begin{pmatrix} H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}} H_{\overline{\phi_-}} & H^*_{\overline{\phi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\phi_-}} \\ H^*_{\overline{\psi_+}} H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}} H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \end{pmatrix}. \tag{3}$$

For $f \in L^\infty$, let $\tilde{f} = \overline{f(\bar{z})}$. We note that $H^*_f = H_{\tilde{f}}$.

Lemma 3.1. *Let $\phi \in L^\infty$, $h \in H^\infty$ and $\theta \in H^\infty$ (an inner function). Then*

$$H_\phi U = U^* H_\phi, H_\phi T_h = H_{\phi h} = T^*_{\tilde{h}} H_\phi,$$

$$H^*_\phi H_\phi - H^*_{\theta\phi} H_{\theta\phi} = H^*_\phi H_{\overline{\theta}} H^*_{\overline{\theta}} H_\phi.$$

Proof. The first two equalities can be verified by the definition of a Hankel operator. By using the second equality and (2), we have

$$H^*_\phi H_\phi - H^*_{\theta\phi} H_{\theta\phi} = H^*_\phi (I - T_{\tilde{\theta}} T^*_{\tilde{\theta}}) H_\phi = H^*_\phi H^*_{\overline{\tilde{\theta}}} H_{\overline{\tilde{\theta}}} H_\phi = H^*_\phi H_{\overline{\theta}} H^*_{\overline{\theta}} H_\phi.$$

This is the desired third equality in the lemma. □

For an inner function $\theta \in H^\infty$, let $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$. The following lemma is well known.

Lemma 3.2. *$Ker(H_f) \neq \{0\}$ if and only if f is of the form $\overline{\theta} b$, where θ is some inner function and $b \in H^\infty$ has the property that the inner part of b and θ are coprime. Furthermore, we have*

$$Ker(H_{\overline{\theta} b}) = \theta H^2, \quad Closure\{Range(H_{\overline{\theta} b})\} = \{Ker(H^*_{\overline{\theta} b})\}^\perp = \mathcal{H}(\tilde{\theta}).$$

The classical result of Kronecker on finite rank Hankel matrices is that H_f is of finite rank if and only if $(I - P)f$ is a rational function, and in this case the rank is equal to the degree of $(I - P)f$. This can also be seen from the above lemma. The following result gives a way to compute the rank of a product of two Hankel operators.

Lemma 3.3 (Axler, Chang and Sarason, 1978)**.** *Let $f, g \in L^\infty$. Then*

$$Rank(H_f H_g) = \min\{Rank(H_f), Rank(H_g)\}.$$

In particular, $H_f H_g$ is of finite rank if and only if at least one of them is of finite rank.

We recall the following characterization of hyponormal Toeplitz operators.

Theorem 3.4 (Cowen, 1988)**.** *T_ϕ is hyponormal if and only if there exists a function $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that*

$$\overline{\phi_-} - k\overline{\phi_+} \in H^2.$$

A slight variation of the above condition in [22] is that

$$\phi - k\overline{\phi} \in H^\infty,$$

which only deals with bounded functions. A function f is of bounded type if f can be written as a quotient of two bounded analytic functions. A function of bounded type is of the form $\overline{q_1} q_2 f_0$ for some inner functions q_1, q_2 and an outer function f_0. By Lemma 3.2, it is interesting to note that a bounded function f is of bounded type if and only if $Ker(H_f) \neq \{0\}$. It was observed in Lemma 6 [1] that if T_ϕ is hyponormal and if ϕ is not in H^∞, then ϕ is of bounded type if and only if $\overline{\phi}$ is of bounded type. Assume this is the case. The positivity of

$$\left[T^*_\phi, T_\phi\right] = H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}} H_{\overline{\phi_-}}$$

implies that

$$Ker(H^*_{\overline{\phi_+}}H_{\overline{\phi_+}}) \subset Ker(H^*_{\overline{\phi_-}}H_{\overline{\phi_-}}).$$

By Lemma 3.2, we can write

$$\phi_+ = \theta_1\theta_0\overline{a}, \quad \phi_- = \theta_1\overline{b}$$

for some inner functions θ_1, θ_0 and $a \in \mathcal{H}(\theta_1\theta_0)$ and $b \in \mathcal{H}(\theta_1)$, where the inner part of a and $\theta_1\theta_0$ are relatively prime and the inner part of b and θ_1 are relatively prime.

Corollary 3.5. *Let $\phi = \phi_+ + \overline{\phi_-} \in L^\infty$. Assume*

$$\phi_+ = \theta_1\theta_0\overline{a}, \quad \phi_- = \theta_1\overline{b},$$

where θ_1, θ_0 are inner functions, and $a \in \mathcal{H}(\theta_1\theta_0)$ and $b \in \mathcal{H}(\theta_1)$. Let θ be a factor of θ_0. Then T_ϕ is hyponormal if and only if T_{ϕ_θ} is hyponormal, where

$$\phi_\theta = P(\overline{\theta}\phi_+) + \overline{\phi_-}.$$

Furthermore, if $\left[T^*_\phi, T_\phi\right]$ is of finite rank, then

$$Rank(\left[T^*_\phi, T_\phi\right]) = Degree(\theta) + Rank(\left[T^*_{\phi_\theta}, T_{\phi_\theta}\right]).$$

Proof. Assume T_ϕ is hyponormal. By Cowen's theorem, there exists a function $k \in H^\infty$ such that $\|k\|_\infty \leq 1$ and

$$\overline{\theta_1}b - k\overline{\theta_1\theta_0}a = h$$

for some $h \in H^2$. Equivalently,

$$ka = \theta_0(b - \theta_1 h).$$

Since the inner part of a and $\theta_1\theta_0$ are coprime, θ_0 is a factor of k. Thus $k = k_1\theta_0$. Let θ be a factor of θ_0 and write $\theta_0 = \theta\theta_2$. The above equation implies that

$$\overline{\theta_1}b - k_2\overline{\theta_1\theta_2}a = h,$$

where $\|k_2\|_\infty = \|k_1\theta_2\|_\infty \leq 1$. By Cowen's theorem, this proves that T_{ϕ_θ} is hyponormal. The other implication is also clear by the above argument.

We next prove the rank formula. By Theorem 10 in Nakazi and Takahashi [22], $\left[T^*_\phi, T_\phi\right]$ is of finite rank if and only if there exist a finite Blaschke product k and $h \in H^2$ satisfying

$$\overline{\phi_-} - k\overline{\phi_+} = h$$

and k can be chosen such that the degree of k is $Rank(\left[T^*_\phi, T_\phi\right])$. As before, we have $k = k_1\theta_0$, $\theta_0 = \theta\theta_2$, $k_2 = k_1\theta_2 = k/\theta$ and

$$\overline{\theta_1}b - k_2\overline{\theta_1\theta_2}a = \overline{\phi_-} - k_2\overline{\overline{\theta}\phi_+} = h.$$

We need to show that

$$Degree(k_2) = Rank(\left[T^*_{\phi_\theta}, T_{\phi_\theta}\right]).$$

Although the degree of k is chosen to be $Rank(\left[T^*_\phi, T_\phi\right])$, the degree of k_2 is not chosen a priori. Nevertheless, by (3) and Lemma 3.1,

$$\left[T^*_\phi, T_\phi\right] = H^*_{\overline{\phi_+}}H_{\overline{\psi_+}} - H^*_{\overline{k\phi_+}}H_{\overline{k\phi_+}} = H^*_{\overline{\phi_+}}H_{\overline{k}}H^*_{\overline{k}}H_{\overline{\phi_+}}.$$

It follows that

$$Degree(k) = Rank(\left[T_\phi^*, T_\phi\right]) = Rank(H_{\overline{\phi_+}}^* H_{\overline{k}}) \leq Rank(H_{\overline{\phi_+}}) = Degree(\theta_1\theta_0).$$

Therefore $Degree(\theta_1\theta_2) \geq Degree(k_2)$. Similarly,

$$\left[T_{\phi_\theta}^*, T_{\phi_\theta}\right] = H_{\overline{\theta\phi_+}}^* H_{\overline{k_2}} H_{\overline{k_2}}^* H_{\overline{\theta\phi_+}}.$$

By Lemma 3.3, we have

$$Rank(\left[T_{\phi_\theta}^*, T_{\phi_\theta}\right]) = \min\{Degree(\theta_1\theta_2), Degree(k_2)\} = Degree(k_2).$$

This completes the proof. □

It was observed by the author [19] that the hyponormality of T_ϕ with rational symbol ϕ can be reduced to a tangential Hermite-Fejér interpolation problem; see Foias and Frazho [18], in particular pp. 294-304, for an extensive treatment of such a problem. We next show how to reduce the weak hyponormality of $\mathbf{T} = (T_\phi, T_\psi)$ for a certain class of rational symbols ϕ and ψ to a Hermite-Fejér interpolation problem.

We first recall the classical Hermite-Fejér interpolation problem.

Problem. Let $\{\alpha_i, 1 \leq i \leq n\}$ be n distinct complex numbers inside the unit disk. Let $\{b_{ij} : 1 \leq i \leq n \text{ and } 0 \leq j \leq n_i\}$ be a given set of complex numbers. Find necessary and sufficient conditions for the existence of an analytic function $f \in H^\infty$ with $\|f\|_\infty \leq 1$ satisfying

$$f^{(j)}(\alpha_i) = b_{ij}, \quad 1 \leq i \leq n, 0 \leq j \leq n_i,$$

where $f^{(j)}(\alpha_i)$ is the j-th derivative evaluated at α_i.

If $n = 1$, this is the so-called Carathéodory interpolation problem. If $n_i = 1$ for all i, this is the so-called Nevanlinna-Pick interpolation problem.

For $|\alpha| < 1$, set

$$m_\alpha = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

Lemma 3.6. *Let α_i for $i = 1, \cdots, n$ and β_j for $j = 1, \cdots, l$ be distinct complex numbers inside the unit disk. Let*

$$\theta_1 = \prod_{i=1}^n m_{\alpha_i}^{m_i}, \quad \theta_2 = \prod_{j=1}^l m_{\beta_j}^{k_j}$$

$$\phi_+ = \theta_1 \overline{a}, \quad \phi_- = \theta_1 \overline{b}; \quad \psi_+ = \theta_2 \overline{c}, \quad \psi_- = \theta_2 \overline{d},$$

where $a, b \in \mathcal{H}(\theta_1), c, d \in \mathcal{H}(\theta_2)$. Let k_1, k_2 be bounded analytic functions satisfying

$$a - k_1 b = \theta_1 h_1, \quad c - k_2 d = \theta_2 h_2 \tag{4}$$

for some $h_1, h_2 \in H^\infty$. Then $\mathbf{T} = (T_\phi, T_\psi)$ is weakly hyponormal if and only if for all constant $|\delta| = 1$, there exists $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that

$$k^{(m)}(\alpha_i) = k_1^{(m)}(\alpha_i), \quad 0 \leq m \leq m_i, i = 1, \cdots, n,$$

$$k^{(k)}(\beta_j) = \delta k_2^{(k)}(\beta_j), \quad 0 \leq k \leq k_j, j = 1, \cdots, l.$$

Proof. $\mathbf{T} = (T_\phi, T_\psi)$ is weakly hyponormal if $T_{\alpha\phi+\beta\psi}$ is hyponormal for all α and β. By Cowen's theorem, this is equivalent to the existence of an analytic function k with $\|k\|_\infty \leq 1$ satisfying

$$\alpha \overline{\theta_1} b + \beta \overline{\theta_2} d - k(\overline{\alpha} \overline{\theta_1} a + \overline{\beta \theta_2} c) = h$$

for some $h \in H^2$. Equivalently,

$$\alpha\theta_2 b + \beta\theta_1 d - k(\overline{\alpha}\theta_2 a + \overline{\beta}\theta_1 c) = \theta_1\theta_2 h.$$

By taking an appropriate number of derivatives in the above equation and evaluating the resulting equation at α_i and β_j, we see that k satisfies the above equation if and only if

$$k^{(m)}(\alpha_i) = \frac{\alpha}{\overline{\alpha}} k_1^{(m)}(\alpha_i), \quad 0 \le m \le m_i, i = 1, \cdots, n,$$
$$k^{(m)}(\beta_j) = \frac{\beta}{\overline{\beta}} k_2^{(k)}(\beta_j), \quad 0 \le k \le k_j, j = 1, \cdots, l,$$

where k_1 are k_2 are defined by (4). Now setting $\delta = \overline{\alpha}\beta/\alpha\overline{\beta}$, we see that $|\delta| = 1$ and $\overline{\alpha}k/\alpha$ satisfies the interpolation conditions as in the lemma. □

We now give an example of $\mathbf{T} = (T_\phi, T_\psi)$ which is weakly hyponormal but not hyponormal. Farenick and MaEachin [17] gave an example of (U, W) for a weighted shift W which is weakly hyponormal but not hyponormal. See also [14] for an example of a commuting pair (T_1, T_2) which is weakly hyponormal but not hyponormal.

Example 1. Let a, b, c and d be constants such that $|a| > |b|, |c| > |d|$, $0 < |\alpha| < 1$ and

$$\phi = z\overline{a} + \bar{z}b, \quad \psi = \frac{z\overline{c}}{1 - \bar{\alpha}z} + \frac{\bar{z}d}{1 - \alpha\bar{z}}.$$

We claim that $\mathbf{T} = (T_\phi, T_\psi)$ is weakly hyponormal if and only if

$$|\alpha|(|ac| + |bd|) \ge (|ad| + |bc|). \tag{5}$$

To see this, by the previous lemma, for all $|\delta| = 1$, we need to find a bounded analytic function k with $\|k\|_\infty \le 1$ satisfying

$$k(0) = \frac{b}{a}, \quad k(\alpha) = \delta\frac{d}{c}.$$

This is a Nevanlinna-Pick interpolation problem with two interpolation conditions. It is solvable if and only if

$$\begin{pmatrix} 1 - |k(0)|^2 & 1 - \overline{k(0)}k(\alpha) \\ 1 - k(0)\overline{k(\alpha)} & \dfrac{1 - |k(\alpha)|^2}{1 - |\alpha|^2} \end{pmatrix}$$

is positive. A straightforward computation of the determinant of the above matrix shows that this is equivalent to (5). For example, if $\alpha = 1/2, a = c = 1, b = d = 1/4$, then $\mathbf{T} = (T_\phi, T_\psi)$ is weakly hyponormal. But (T_ϕ, T_ψ) is not hyponormal as long as none of a, b, c, d is zero. This follows from some observations as in Proposition 6.4 below.

4. One coordinate analytic Toeplitz operator

Theorem 4.1. *Assume $\phi = \phi_+ \in H^\infty$ and $\psi = \psi_+ + \overline{\psi_-} \in L^\infty$, where $\psi_+ \in H^2$ and $\psi_- \in zH^2$.*

a) If $Ker(H_{\overline{\phi}}) = \{0\}$, then (T_ϕ, T_ψ) is hyponormal if and only if $\psi \in H^\infty$.

b) If $Ker(H_{\overline{\phi}}) = \theta H^2$ for some inner function θ, then $\phi = \theta\overline{f}$ for some $f \in H^\infty$, and (T_ϕ, T_ψ) is hyponormal if and only if T_{ψ_θ} is hyponormal, where

$$\psi_\theta = P(\overline{\theta}\psi_+) + \overline{\psi_-}.$$

Furthermore, if θ is a finite Blaschke product, then

$$Rank([\mathbf{T}^*,\mathbf{T}]) = Degree(\theta) + Rank(\left[T^*_{\psi_\theta}, T_{\psi_\theta}\right]).$$

Proof. By the assumption $\phi_- = 0$ and (3), we have

$$[\mathbf{T}^*,\mathbf{T}] = \begin{pmatrix} \left[T^*_\phi, T_\phi\right] & \left[T^*_\psi, T_\phi\right] \\ \left[T^*_\phi, T_\psi\right] & \left[T^*_\psi, T_\psi\right] \end{pmatrix} = \begin{pmatrix} H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} & H^*_{\overline{\phi_+}} H_{\overline{\psi_+}} \\ H^*_{\overline{\psi_+}} H_{\overline{\phi_+}} & \left[T^*_\psi, T_\psi\right] \end{pmatrix}. \tag{6}$$

Therefore by Lemma 2.1, $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if

$$|(H^*_{\overline{\psi_+}} H_{\overline{\phi_+}} x, y)|^2 \leq (H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} x, x)(\left[T^*_\psi, T_\psi\right] y, y)$$

for all $x, y \in H^2$. Equivalently,

$$|(h, H_{\overline{\psi_+}} y)|^2 \leq (h,h)(\left[T^*_\psi, T_\psi\right] y, y) \tag{7}$$

for all $h \in Range(H_{\overline{\phi_+}})$ and $y \in H^2$, and hence for all h in the closure of $Range(H_{\overline{\phi_+}})$ and $y \in H^2$. There are two cases.

a) If $Ker(H_{\overline{\phi}}) = \{0\}$, then the closure of $Range(H_{\overline{\phi_+}})$ is H^2. Thus

$$|(h, H_{\overline{\psi_+}} y)|^2 \leq (h,h)(\left[T^*_\psi, T_\psi\right] y, y)$$

for all $h \in H^2$. Let $h = H_{\overline{\psi_+}} y$; we have

$$(H_{\overline{\psi_+}} y, H_{\overline{\psi_+}} y) \leq (H_{\overline{\psi_+}} y, H_{\overline{\psi_+}} y) - (H_{\overline{\psi_-}}, H_{\overline{\psi_-}} y)$$

for all $y \in H^2$. Therefore $H_{\overline{\psi_-}} = 0$, and so $\psi_- = 0$.

b) If $Ker(H_{\overline{\phi}}) = \theta H^2$ for some inner function θ, the closure of $Range(H_{\overline{\phi_+}})$ is $\mathcal{H}(\tilde{\theta})$, which is the same as $Range(H_{\overline{\theta}})$. Equation (7) is the same as

$$|(H_{\overline{\theta}} x, H_{\overline{\psi_+}} y)|^2 \leq (H_{\overline{\theta}} x, H_{\overline{\theta}} x)(\left[T^*_\psi, T_\psi\right] y, y)$$

for all $x, y \in H^2$. Note that $H_{\overline{\theta}} H^*_{\overline{\theta}}$ is the projection onto $\mathcal{H}(\tilde{\theta})$. By part c) of Lemma 2.1, the above equation is the same as

$$\left[T^*_\psi, T_\psi\right] - H^*_{\overline{\psi_+}} H_{\overline{\theta}} H^*_{\overline{\theta}} H_{\overline{\psi_+}} \geq 0.$$

A straightforward computation shows that

$$\begin{aligned}\left[T^*_{\psi_\theta}, T_{\psi_\theta}\right] &= H^*_{\theta\overline{\psi_+}} H_{\theta\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \\ &= H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} - (H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\theta\overline{\psi_+}} H_{\theta\overline{\psi_+}}) \\ &= \left[T^*_\psi, T_\psi\right] - H^*_{\overline{\psi_+}} H_{\overline{\theta}} H^*_{\overline{\theta}} H_{\overline{\psi_+}},\end{aligned}$$

where the last equality follows from Lemma 3.1.

If θ is a finite Blaschke product, by a classical result of Kronecker, $H_{\overline{\theta} f}$ is of finite rank and the rank is equal to the degree of θ. We note that in this case $H_{\overline{\phi}}(H^*_{\overline{\phi}} H_{\overline{\phi}})^{\#} H^*_{\overline{\phi}}$ is the projection $H_{\overline{\theta}} H^*_{\overline{\theta}}$ on $\mathcal{H}(\tilde{\theta})$, and

$$\left[T^*_\psi, T_\psi\right] - H^*_{\overline{\psi_+}} H_{\overline{\phi}} (H^*_{\overline{\phi}} H_{\overline{\phi}})^{\#} H^*_{\overline{\phi}} H_{\overline{\psi_+}} = \left[T^*_\psi, T_\psi\right] - H^*_{\overline{\psi_+}} H_{\overline{\theta}} H^*_{\overline{\theta}} H_{\overline{\psi_+}} = \left[T^*_{\psi_\theta}, T_{\psi_\theta}\right].$$

By part d) of Lemma 2.1 and (6), we have

$$Rank([\mathbf{T}^*,\mathbf{T}]) = Degree(\theta) + Rank(\left[T^*_{\psi_\theta}, T_{\psi_\theta}\right]).$$

This completes the proof. ∎

Remark 1. When $\phi = z$, the above theorem without the rank formula is due to Farenick and McEachin [17]. When ϕ is an analytic polynomial, it is due to Curto and Lee [13], but the rank formula was only obtained there when ψ is assumed to be a trigonometric polynomial.

Remark 2. The above theorem extends to block Toeplitz operators, and one consequence is a result conjectured by Curto and Lee [13]. The hyponormality of block Toeplitz operators will be discussed in a future work.

Corollary 4.2. *Assume $\phi = \phi_+ = \theta\overline{a(z)} \in H^\infty$, where θ is inner and $a \in \mathcal{H}(\theta)$. Assume also that $\psi = \psi_+ + \overline{\psi_-} \in L^\infty$ and*

$$\psi_+ = \theta_1\theta_0\overline{c(z)}, \quad \psi_- = \theta_1\overline{d(z)},$$

where θ_1 and θ_0 are inner, and $c(z) \in \mathcal{H}(\theta_0\theta_1)$ and $d(z) \in \mathcal{H}(\theta_1)$. If (T_ϕ, T_ψ) is hyponormal, then there exist inner functions Δ_1 and Δ_2 such that

$$\overline{\theta}\theta_1\theta_0 = \theta_1\Delta_1\overline{\Delta_2},$$

where $\theta_1\Delta_1$ and Δ_2 are coprime. In other words, the greatest common divisor of θ and $\theta_1\theta_0$ is a factor of θ_0.

Proof. By the previous theorem, the hyponormality of (T_ϕ, T_ψ) implies the hyponormality of T_{ψ_θ}, where

$$\psi_\theta = P(\overline{\theta}\psi_+) + \overline{\psi_-}.$$

Thus $\theta_1 H^2 = Ker(H_{\overline{\psi_-}}) \supset Ker(H_{\theta\overline{\psi_+}})$. Set

$$\overline{\theta}\theta_1\theta_0 = \Delta\overline{\Delta_2}, \tag{8}$$

where the inner functions Δ and Δ_2 are coprime. It is clear that $Ker(H_{\theta\overline{\psi_+}}) = \Delta H^2$. Therefore $\Delta = \theta_1\Delta_1$. Now equation (8) becomes

$$\overline{\theta}\theta_1\theta_0 = \theta_1\Delta_1\overline{\Delta_2}.$$

This completes the proof. □

Theorem 4.3. *Assume $\phi_i \in H^\infty$ for $i = 1, 2, \cdots, n$, and $\psi = \psi_+ + \overline{\psi_-} \in L^\infty$, where $\psi_+ \in H^2$ and $\psi_- \in zH^2$. Let θ_i $(i = 1, 2, \cdots, n)$ be inner functions. Assume $Ker(H_{\overline{\phi_i}}) = \theta_i H^2$. Then $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal if and only if T_{ψ_θ} is hyponormal, where θ is the least common multiple of $\theta_1, \theta_2, \cdots, \theta_n$ and*

$$\psi_\theta = P(\overline{\theta}\psi_+) + \overline{\psi_-}.$$

Moreover, if θ is a finite Blaschke product, then

$$Rank([\mathbf{T}^*, \mathbf{T}]) = Degree(\theta) + Rank(\left[T^*_{\psi_\theta}, T_{\psi_\theta}\right]).$$

Proof. The proof is similar to the proof of Theorem 4.1, with suitable notational changes. Let

$$\bigoplus_{i=1}^{n} H^2 = H^2 \oplus \cdots \oplus H^2$$

be the direct sum of n copies of H^2, and let

$$\mathbf{H}_{\overline{\phi}} = (H_{\overline{\phi_1}}, \cdots, H_{\overline{\phi_n}})$$

be an operator from $\bigoplus_{i=1}^{n} H^2$ into H^2. By the assumption $\phi_i \in H^\infty$ and (3), we have

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} \mathbf{H}^*_{\overline{\phi}} \mathbf{H}_{\overline{\phi}} & \mathbf{H}^*_{\overline{\phi}} H_{\overline{\psi_+}} \\ H^*_{\overline{\psi_+}} \mathbf{H}_{\overline{\phi}} & \left[T^*_\psi, T_\psi\right] \end{pmatrix}.$$

By Lemma 2.2, $\mathbf{T}$ is hyponormal if and only if

$$|(H^*_{\overline{\psi_+}} \mathbf{H}_{\overline{\phi}} x, y)|^2 \leq (\mathbf{H}^*_{\overline{\phi}} \mathbf{H}_{\overline{\phi}} x, x)(\left[T^*_\psi, T_\psi\right] y, y)$$

for all $x \in \bigoplus_{i=1}^{n} H^2$ and $y \in H^2$. Equivalently,

$$|(h, H_{\overline{\psi_+}} y)|^2 \leq (h, h)(\left[T^*_\psi, T_\psi\right] y, y) \tag{9}$$

for all h in the closure of $Range(\mathbf{H}_{\overline{\phi}})$ and $y \in H^2$. Note that

$$Range(\mathbf{H}_{\overline{\phi}}) = \{ \sum_{i=1}^{n} h_i : h_i \in Range(H_{\overline{\phi_i}})\},$$

and the closure of $Range(H_{\overline{\phi_i}})$ is $\mathcal{H}(\tilde{\theta}_i)$. Therefore the closure of $Range(\mathbf{H}_{\overline{\phi}})$ is $\mathcal{H}(\tilde{\theta})$, where θ is the least common multiple of $\theta_1, \theta_2, \cdots, \theta_n$. Again note that $\mathcal{H}(\tilde{\theta})$ is the same as $Range(H_{\overline{\theta}})$. Equation (9) now becomes

$$|(H_{\overline{\theta}} x, H_{\overline{\psi_+}} y)|^2 \leq (H_{\overline{\theta}} x, H_{\overline{\theta}} x)(\left[T^*_\psi, T_\psi\right] y, y)$$

for all $x, y \in H^2$. The exact same argument as in the proof of Theorem 4.1 shows that the above inequality is equivalent to the hyponormality of T_{ψ_θ}.

If θ is a finite Blaschke product, we note that $\mathbf{H}_{\overline{\phi}}(\mathbf{H}^*_{\overline{\phi}}\mathbf{H}_{\overline{\phi}})^{\#}\mathbf{H}_{\overline{\phi}}^*$ is the projection $H_{\overline{\theta}} H^*_{\overline{\theta}}$ on $\mathcal{H}(\tilde{\theta})$. The rank formula follows as in the proof of Theorem 4.1. □

Corollary 4.4. *Assume $\phi_i \in H^\infty$ and $Ker(H_{\overline{\phi_i}}) = \theta_i H^2$, where θ_i $(i = 1, 2, \cdots, n)$ are inner. Assume also*

$$\psi_+ = \Delta\Delta_0 \overline{c}, \quad \psi_- = \Delta\overline{d},$$

where Δ and Δ_0 are inner, $c \in \mathcal{H}(\Delta\Delta_0)$ and $d \in \mathcal{H}(\Delta)$. If θ_i is a factor of $\Delta\Delta_0$, then $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal if and only if θ_i is a factor of Δ_0. Moreover, if $\left[T^*_\psi, T_\psi\right]$ is of finite rank, then

$$Rank([\mathbf{T}^*, \mathbf{T}]) = Rank(\left[T^*_\psi, T_\psi\right]).$$

Proof. By the above theorem, the hyponormality of $\mathbf{T}$ is equivalent to the hyponormality of T_{ψ_θ}, where θ is the least common multiple of $\theta_1, \cdots, \theta_n$. If $\mathbf{T}$ is hyponormal, then (T_{ϕ_i}, T_ψ) is hyponormal. By Corollary 4.2, θ_i is a factor of Δ_0, since by assumption θ_i is a factor of $\Delta\Delta_0$. Hence θ is a factor of Δ_0. By Corollary 3.5, the hyponormality of T_{ψ_θ} is automatic by the hyponormality of T_ψ. The rank formula follows from the rank formulas in the above theorem and Corollary 3.5. This completes the proof. □

Curto and Lee [13] conjectured that $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ is hyponormal if and only if each subpair of $\mathbf{T}$ is hyponormal. The following example gives a negative answer to this conjecture.

Example 2. Let

$$m_\alpha = \frac{z - \alpha}{1 - \overline{\alpha} z}, \quad m_\beta = \frac{z - \beta}{1 - \overline{\beta} z},$$

where α and β are two distinct nonzero complex numbers inside the unit disk. Let $\psi = z + \gamma\overline{z}$, where γ is a complex number such that

$$|\alpha\beta| < |\gamma| < \min\{|\alpha|, |\beta|\}.$$

We claim that $(T_{m_\alpha}, T_{m_\beta}, T_\psi)$ is not hyponormal while both (T_{m_α}, T_ψ) and (T_{m_β}, T_ψ) are hyponormal. Equivalently, by the above theorem, $T_{\psi_{m_\alpha m_\beta}}$ is not hyponormal while both $T_{\psi_{m_\alpha}}$ and $T_{\psi_{m_\beta}}$ are hyponormal.

To see this, note that

$$\psi_{m_\alpha} = P(\overline{m_\alpha} z) + \gamma\overline{z} = 1 - |\alpha|^2 - \overline{\alpha} z + \gamma \overline{z}.$$

By the assumption that $|\alpha| > |\gamma|$, $T_{\psi_{m_\alpha}}$ is hyponormal. Similarly, by the assumption $|\beta| > |\gamma|$, $T_{\psi_{m_\beta}}$ is hyponormal. A simple computation shows that

$$\psi_{m_\alpha m_\beta} = P(\overline{m_\alpha m_\beta} z) + \gamma\overline{z} = -(\overline{\alpha} + \overline{\beta}) + (\alpha + \beta)\overline{\alpha\beta} + \overline{\alpha\beta} z + \gamma\overline{z}.$$

By the assumption $|\alpha\beta| < |\gamma|$, $T_{\psi_{m_\alpha m_\beta}}$ is not hyponormal.

5. Toeplitz operators with equal anti-analytic parts

The following simple lemma turns out to be very useful for studying the hyponormality of Toeplitz operators.

Lemma 5.1. *(1) Let* $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ *and* $\mathbf{S} = (T_{\phi_1 - \beta_1\psi}, \cdots, T_{\phi_n - \beta_n \psi}, T_\psi)$, *where* $\beta_1, \cdots, \beta_n$ *are constants.* $\mathbf{T}$ *is hyponormal if and only if* $\mathbf{S}$ *is hyponormal. Furthermore,*

$$Rank([\mathbf{S}^*, \mathbf{S}]) = Rank([\mathbf{T}^*, \mathbf{T}]).$$

(2) Let $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ *and* $\mathbf{S} = (T_{\beta_1\phi_1}, \cdots, T_{\beta_n\phi_1})$, *where* $\beta_1, \cdots, \beta_n$ *are constants and no* β_i *is zero.* $\mathbf{T}$ *is hyponormal if and only if* $\mathbf{S}$ *is hyponormal. Furthermore,*

$$Rank([\mathbf{S}^*, \mathbf{S}]) = Rank([\mathbf{T}^*, \mathbf{T}]).$$

Proof. To prove (1), without loss of generality, assume $n = 1$. Let $\mathbf{T} = (T_\phi, T_\psi)$ and $\mathbf{S} = (T_{\phi - \beta\psi}, T_\psi)$. A direct computation shows that

$$\begin{aligned}
[\mathbf{S}^*, \mathbf{S}] &= \begin{pmatrix} \left[T^*_{\phi-\beta\psi}, T_{\phi-\beta\psi}\right] & \left[T^*_{\psi}, T_{\phi-\beta\psi}\right] \\ \left[T^*_{\phi-\beta\psi}, T_{\psi}\right] & \left[T^*_{\psi}, T_{\psi}\right] \end{pmatrix} \\
&= \begin{pmatrix} I & -\beta I \\ 0 & I \end{pmatrix} \begin{pmatrix} \left[T^*_{\phi}, T_{\phi}\right] & \left[T^*_{\psi}, T_{\phi}\right] \\ \left[T^*_{\phi}, T_{\psi}\right] & \left[T^*_{\psi}, T_{\psi}\right] \end{pmatrix} \begin{pmatrix} I & 0 \\ -\overline{\beta} I & I \end{pmatrix}.
\end{aligned}$$

The result follows from the above relation.

Statement (2) follows immediately from the fact that

$$\begin{aligned}
[\mathbf{S}^*, \mathbf{S}] &= \left(\left[\overline{\beta_j} T^*_{\phi_j}, \beta_i T_{\phi_i}\right]\right) = \left(\overline{\beta_j}\beta_i \left[T^*_{\phi_j}, T_{\phi_i}\right]\right) \\
&= D\left(\left[T^*_{\phi_j}, T_{\phi_i}\right]\right) D^* = D\left[\mathbf{T}^*, \mathbf{T}\right] D^*,
\end{aligned}$$

where D is the block diagonal operator with diagonal $(\beta_1 I, \cdots, \beta_n I)$. $\square$

Proposition 5.2. *Assume $\phi_- = \alpha\psi_- \neq 0$ for some constant α.*

1. If ϕ or ψ is not of bounded type, then $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if $\overline{\phi} - \overline{\alpha}\overline{\psi}$ is of bounded type.

2. If both ϕ and ψ are of bounded type, we write

$$\phi_+ = \theta\theta_0\theta_1\overline{a}, \quad \phi_- = \alpha\theta\overline{b},$$

$$\psi_+ = \theta\theta_0\theta_2\overline{c}, \quad \psi_- = \theta\overline{b},$$

where θ_1 and θ_2 are coprime, $a \in \mathcal{H}(\theta\theta_0\theta_1)$, $c \in \mathcal{H}(\theta\theta_0\theta_2)$ and $b \in \mathcal{H}(\theta)$. Then $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if

$$\alpha P_{\mathcal{H}(\theta)}(\theta_1 c) = P_{\mathcal{H}(\theta)}(\theta_2 a), \tag{10}$$

θ_1 and θ are coprime, θ_2 and θ are coprime, and both $T_{\psi_{\theta_0\theta_1\theta_2}}$ and $T_{\phi_{\theta_0\theta_1\theta_2}}$ are hyponormal, where

$$\psi_{\theta_0\theta_1\theta_2} = P(\overline{\theta_0\theta_1\theta_2}\psi_+) + \overline{\psi_-} = P(\overline{\theta_1}\theta\overline{c}) + \overline{\theta}b,$$
$$\phi_{\theta_0\theta_1\theta_2} = P(\overline{\theta_0\theta_1\theta_2}\phi_+) + \overline{\phi_-} = P(\overline{\theta_2}\theta\overline{a}) + \overline{\alpha}\overline{\theta}b.$$

Proof. By the previous lemma, we can assume $\alpha = 1$. We first prove the necessity part of (1). Assume (T_ϕ, T_ψ) is hyponormal. By Lemma 5.1, $(T_{\phi-\psi}, T_\psi)$ is hyponormal. By the assumption $\phi_- = \psi_-$, $\phi - \psi = \phi_+ - \psi_+ \in H^\infty$. Thus by Theorem 4.1, $KerH_{\overline{\phi}-\overline{\psi}} = \theta H^2 \neq \{0\}$ for some inner function θ. Therefore $\overline{\phi} - \overline{\psi} = \overline{\theta}h$ for some $h \in \mathcal{H}(\theta)$. Hence $\overline{\phi} - \overline{\psi}$ is of bounded type.

Next we prove the sufficiency part of (1). Without loss of generality assume ϕ is not of bounded type. Assume that

$$\overline{\phi} - \overline{\psi} = \overline{\phi_+} - \overline{\psi_+} = \overline{\theta}h, \tag{11}$$

where θ is an inner function and $h \in \mathcal{H}(\theta) \cap H^\infty$. By Cowen's theorem, there exist $k_1, k_2, h_1, h_2 \in H^\infty$ such that $\|k_1\|_\infty \leq 1$, $\|k_2\|_\infty \leq 1$ and

$$\phi - k_1\overline{\phi} = h_1, \tag{12}$$

$$\psi - k_2\overline{\psi} = h_2. \tag{13}$$

Substituting (11) into (13), subtracting (12) from the resulting equation and multiplying by θ, we have

$$\theta(k_2 - k_1)\overline{\phi} = \theta(h_1 - h_2 - \theta\overline{h}) + k_2 h.$$

If $k_1 \neq k_2$, then $\overline{\phi}$ is of bounded type, which is a contradiction. Therefore $k_1 = k_2$. It follows that $k_2 = \theta k_3$ for some $k_3 \in H^\infty$. Now equation (13) reads

$$\psi - k_3\theta\overline{\psi} = h_2.$$

By Cowen's theorem, this implies that T_{ψ_θ} is hyponormal, where

$$\psi_\theta = P(\overline{\theta}\psi_+) + \overline{\psi_-}.$$

By Theorem 4.1, $(T_{\phi-\psi}, T_\psi)$ is hyponormal. This concludes the proof of part (1).

Now we prove the necessity part of (2). Assume (T_ϕ, T_ψ) is hyponormal. By Lemma 5.1, $(T_{\phi-\psi}, T_\psi)$ is hyponormal. Note that by our assumption $\phi - \psi \in H^\infty$. Therefore we can use Theorem 4.1 to study the hyponormality of $(T_{\phi-\psi}, T_\psi)$. We write

$$\phi - \psi = \phi_+ - \psi_+ = \theta\theta_0\theta_1\overline{a} - \theta\theta_0\theta_2\overline{c}$$
$$= \theta\theta_0\theta_1\theta_2[\overline{\theta_2}\overline{a} - \overline{\theta_1}\overline{c}]. \tag{14}$$

Let

$$\theta_2 a - \theta_1 c = \Delta_0\Delta_1 e \tag{15}$$

be the inner-outer factorization, where e is outer and $\Delta_0\Delta_1$ is inner such that Δ_0 is a factor of $\theta\theta_0\theta_1\theta_2$, and Δ_1 and $\theta\theta_0\theta_1\theta_2$ are coprime. We claim that Δ_0 and $\theta_1\theta_2$ are coprime. Equation (15) shows that a common inner factor of $\Delta_0\Delta_1$ and θ_1 will be an inner factor of $\theta_2 a$, which is impossible, since θ_2 and θ_1 are coprime and the inner part of a and θ_1 are also coprime. Similarly, θ_2 and Δ_0 are coprime. So in fact Δ_0 is a factor of $\theta\theta_0$.

We next show that θ is a factor of Δ_0. Set

$$\Delta = \theta\theta_0\theta_1\theta_2\overline{\Delta_0}. \tag{16}$$

With the above notation, equation (14) becomes

$$\phi_+ - \psi_+ = \Delta\overline{\Delta_1}\overline{e}.$$

Therefore by Theorem 4.1, T_{ψ_Δ} is hyponormal, where

$$\psi_\Delta = P(\overline{\Delta}\psi_+) + \overline{\psi_-}.$$

In particular, by Corollary 4.2, this implies that

$$\overline{\Delta}\theta\theta_0\theta_2 = \theta\omega_1\overline{\omega_2} \tag{17}$$

for some inner functions ω_1 and ω_2 such that $\theta\omega_1$ and ω_2 are coprime. Substituting (16) into (17) and multiplying both sides of the equation by appropriate inner functions, we have

$$\Delta_0\omega_2 = \theta\omega_1\theta_1.$$

Since ω_2 and $\theta\omega_1$ are coprime and Δ_0, and θ_1 are coprime, we have $\Delta_0 = \theta\omega_1$ and $\omega_2 = \theta_1$. Therefore θ is a factor of Δ_0, and θ_1 and θ are coprime.

Now equation (15) becomes

$$\theta_2 a - \theta_1 c = \theta\omega_1\Delta_1 e.$$

Therefore condition (10) holds. Note that ω_1 is a factor of θ_0, since $\Delta_0 = \theta\omega_1$ is a factor of $\theta\theta_0$. We next show that the hyponormality of T_{ψ_Δ} implies the hyponormality of $T_{\psi_{\theta_0\theta_1\theta_2}}$. By Cowen's theorem, the hyponormality of T_{ψ_Δ} is equivalent to the existence of $k \in H^\infty$ and $h \in H^2$ such that $\|k\|_\infty \leq 1$ and

$$\overline{\theta}b - k\Delta\overline{\theta\theta_0\theta_2}c = \overline{\theta}b - k\theta_1\overline{\theta\omega_1}c = h,$$

where in the second equality we use (17) and $\omega_2 = \theta_1$. Equivalently,

$$\omega_1 b - k\theta_1 c = \omega_1\theta h.$$

Note that the inner part of c and ω_1 is coprime, since ω_1 is a factor of θ_0. Also, as noted before, ω_1 and θ_1 are coprime. Therefore $k = k_1\omega_1$, where $k_1 \in H^\infty$. The above equation becomes

$$\overline{\theta}b - k_1\overline{\theta}\theta_1 c = h.$$

By Cowen's theorem again, this implies that $T_{\psi_{\theta_0\theta_1\theta_2}}$ is hyponormal. Similarly, θ_2 and θ are coprime, and $T_{\phi_{\theta_0\theta_1\theta_2}}$ is hyponormal.

The proof of the sufficiency part of (2) is obtained by reversing the above argument and a direct application of Theorem 4.1. $\square$

Remark. It follows from the above proof that

$$Ker(H_{\overline{\phi_+}-\overline{\psi_+}}) = \Delta H^2 = \theta\theta_0\theta_1\theta_2\overline{\Delta_0}H^2.$$

Therefore by Lemma 5.1 and Theorem 4.1, we have

$$Rank([\mathbf{T}^*, \mathbf{T}]) = Degree(\Delta) + Rank(\left[T^*_{\psi_\Delta},\ T_{\psi_\Delta}\right]).$$

We also note that if condition (10) holds, then the hyponormality of $T_{\psi_{\theta_0\theta_1\theta_2}}$ implies the hyponormality of $T_{\phi_{\theta_0\theta_1\theta_2}}$, and vice versa.

Let $\phi_i \in L^\infty$ for $i = 1, 2, \cdots, n$ and $\psi \in L^\infty$.

Theorem 5.3. *Assume $(\phi_i)_- = \psi_-$ for $i = 1, 2, \cdots, n$.*

1. If one of $\phi_1, \cdots, \phi_n, \psi$ is not of bounded type, then $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal if and only if $\overline{\phi_i} - \overline{\psi}$ is of bounded type for $i = 1, \cdots, n$.

2. If all $\phi_1, \cdots, \phi_n, \psi$ are of bounded type, we write

$$(\phi_i)_+ = \theta\Delta_i\theta_i\overline{a_i}, \quad (\phi_i)_- = \theta\overline{b}, \quad i = 1, 2, \cdots, n,$$

$$\psi_+ = \theta\Delta_0\overline{c} = \theta\Delta_i\delta_i\overline{c}, \quad \psi_- = \theta\overline{b},$$

where θ_i and δ_i are coprime, $a_i \in \mathcal{H}(\theta\Delta_i\theta_i)$, $c \in \mathcal{H}(\theta\Delta_0)$ and $b \in \mathcal{H}(\theta)$. Then $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal if and only if

$$P_{\mathcal{H}(\theta)}(\theta_i c) = P_{\mathcal{H}(\theta)}(\delta_i a_i), \quad i = 1, 2, \cdots, n, \tag{18}$$

and $T_{\psi_{\theta_0\Delta_0}}$ is hyponormal, where θ_0 is the least common multiple of $\theta_1, \theta_2, \cdots, \theta_n$ and

$$\psi_{\theta_0\Delta_0} = P(\overline{\theta_0\Delta_0}\psi_+) + \overline{\psi_-}.$$

Proof. We first prove part (1). The necessity clearly follows from the previous proposition. For the sufficiency, we write

$$\overline{\phi_i} - \overline{\psi} = \overline{\theta_i}h_i, \quad i = 1, \cdots, n,$$

where θ_i is inner and $h_i \in \mathcal{H}(\theta_i)$. Let $k \in H^\infty, h \in H^2$ be such that $\|k\|_\infty \le 1$ and

$$\overline{\psi_-} - k\overline{\psi_+} = h. \tag{19}$$

As in the proof of the previous proposition, θ_i is an inner factor of k. Therefore θ_0, the least common multiple of $\theta_1, \cdots, \theta_n$, is an inner factor of k; that is, $k = \theta_0 k_1$. By Cowen's theorem, equation (19) implies the hyponormality of $T_{\psi_{\theta_0}}$. By Theorem 4.3, this implies the hyponormality of $(T_{\phi_1-\psi}, \cdots, T_{\phi_n-\psi}, T_\psi)$, hence the hyponormality of $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$.

We now show the sufficiency of part (2). Equation (18) implies that there exists some $h_i \in H^2$ such that

$$\delta_i a_i - \theta_i c = \theta h_i.$$

Note that the inner part of h_i and $\delta_i\theta_i$ are coprime, since δ_i and θ_i are coprime. Thus

$$\begin{aligned}\phi_i - \psi = (\phi_i)_+ - \psi_+ &= \theta\Delta_i\theta_i\delta_i(\overline{\delta_i a_i} - \overline{\theta_i c}) \\ &= \theta\Delta_i\theta_i\delta_i(\overline{\theta h_i}) = \Delta_i'\delta_i\theta_i\overline{h_i'},\end{aligned}$$

where Δ_i' is a factor of Δ_i and the inner part of h_i' and $\Delta_i'\delta_i\theta_i$ are coprime. Let Δ be the least common multiple of the $\Delta_i'\delta_i\theta_i$, $i = 1, \cdots, n$. Note that $\Delta = \delta\theta_0$, where θ_0 is the least common multiple of $\theta_1, \ldots, \theta_n$ and δ is a factor of $\Delta_i\delta_i = \Delta_0$. Similarly to the proof of the previous proposition, we can use Cowen's theorem to show that the hyponormality of $T_{\psi_{\theta_0\Delta_0}}$ implies the hyponormality of $T_{\psi_\Delta}(=T_{\psi_{\theta_0\delta}})$. By Theorem 4.3, we have that $(T_{\phi_1-\psi},\cdots,T_{\phi_n-\psi},T_\psi)$ is hyponormal. Thus $\mathbf{T}=(T_{\phi_1},\cdots,T_{\phi_n},T_\psi)$ is hyponormal.

Next we prove the necessity of part (2). The hyponormality of $(T_{\phi_1},\cdots,T_{\phi_n},T_\psi)$ implies the hyponormality of (T_{ϕ_i},T_ψ) for each i. By the previous proposition, this shows that equation (18) holds. Now the arguments in the proof of sufficiency are reversible. This completes the proof. □

Remark. It follows from the above proof that for $i=1,\cdots,n$,

$$Ker(H_{\overline{\phi_i-\psi}})=\Delta_i'\delta_i\theta_i H^2.$$

Therefore by Lemma 5.1 and Theorem 4.3, we have

$$Rank([\mathbf{T}^*,\mathbf{T}])=Degree(\Delta)+Rank(\left[T^*_{\psi_\Delta},T_{\psi_\Delta}\right]),$$

where Δ is the least common multiple of the $\Delta_i'\delta_i\theta_i$, $i=1,\cdots,n$.

A similar result for the hyponormality of $(T_{\phi_1},\cdots,T_{\phi_l},T_{\phi_{l+1}},\cdots,T_{\phi_n},T_\psi)$ holds in the more general case in which $\phi_i\in H^\infty$ for $1\le i\le l$ and $(\phi_i)_-=\psi_-$ for $l+1\le i\le n$. A special case of this result is the following corollary.

Corollary 5.4. *Assume that*

$$\begin{gathered}\phi_i\in H^\infty,\quad Ker(H_{\overline{\phi_i}})=\Delta_i H^2,\quad i=1,\cdots,l,\\ (\phi_i)_+=\theta\Delta_i\overline{a_i},\quad (\phi_i)_-=\theta\overline{b},\quad i=l+1,\cdots,n,\\ \psi_+=\theta\Delta_0\overline{c}=\theta\Delta_i\delta_i\overline{c},\quad \psi_-=\theta\overline{b},\end{gathered}$$

where Δ_i $(i=1,\cdots,n)$ are factors of Δ_0, $a_i\in\mathcal{H}(\theta\Delta_i)$, $c\in\mathcal{H}(\theta\Delta_0)$ and $b\in\mathcal{H}(\theta)$. Then $\mathbf{T}=(T_{\phi_1},\cdots,T_{\phi_n},T_\psi)$ is hyponormal if and only if

$$P_{\mathcal{H}(\theta)}(c)=P_{\mathcal{H}(\theta)}(\delta_i a_i),\quad i=l+1,\cdots,n. \tag{20}$$

If this is the case, then

$$Rank([\mathbf{T}^*,\mathbf{T}])=Rank(\left[T_\psi^*,T_\psi\right]).$$

Proof. By the proof of the previous theorem, $\mathbf{T}$ is hyponormal if (20) holds and $T_{\phi_{\theta_0\Delta_0}}$ is hyponormal for a certain inner function θ_0. It follows from the assumption that $\theta_0=1$. Therefore by Corollary 3.5, the hyponormality of $T_{\phi_{\Delta_0}}$ is guaranteed by that of T_ϕ. The rank formula follows from the remark above and Corollary 3.5. □

6. Miscellaneous cases

We begin with a general fact about the kernels of Hankel operators.

Lemma 6.1. *Let θ be an inner function and $\psi\notin H^2$. If $Ker(H_\psi^*H_\phi)\supset\theta H^2$, then $Ker(H_\phi)\supset\theta H^2$. In particular, if $Ker(H_\psi^*H_\phi)\supset Ker(H_\delta)$, then $Ker(H_\phi)\supset Ker(H_\delta)$.*

Proof. The assumption implies that

$$H_\psi^*H_\phi T_\theta=H_\psi^*H_{\phi\theta}=0.$$

By a result of Brown and Halmos [6], $H_{\phi\theta}$=0. Therefore $Ker(H_\phi)\supset\theta H^2$. □

Lemma 6.2. *Assume neither ϕ nor ψ is analytic. If (T_ϕ,T_ψ) is hyponormal, then $Ker(H_{\overline{\phi_+}})\subset Ker(H_{\overline{\psi_-}})$ and $Ker(H_{\overline{\psi_+}})\subset Ker(H_{\overline{\phi_-}})$.*

Proof. First, note that the hyponormality of T_ϕ implies that

$$Ker(H^*_{\overline{\phi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\phi_-}}) \supset Ker(H_{\overline{\phi_+}}).$$

By equation (3) and Lemma 2.1, we have

$$Ker(H^*_{\overline{\psi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\psi_-}}) \supset Ker(H^*_{\overline{\phi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\phi_-}}) \supset Ker(H_{\overline{\phi_+}}).$$

Therefore $Ker(H^*_{\overline{\phi_-}}H_{\overline{\psi_-}}) \supset Ker(H_{\overline{\phi_+}})$. By the previous lemma, $Ker(H_{\overline{\psi_-}}) \supset Ker(H_{\overline{\phi_+}})$. Similarly we can show that $Ker(H_{\overline{\phi_-}}) \supset Ker(H_{\overline{\psi_+}})$. □

Remark. The above observation shows that if (T_ϕ, T_ψ) is hyponormal, then ϕ is not of bounded type implies that ψ is not of bounded type and vice versa.

The following two results follow from the above observation.

Proposition 6.3. *Let* $m_\alpha = (z-\alpha)/(1-\overline{\alpha}z)$,

$$\phi_+ = m_\alpha \theta_0 \theta_1 \overline{a}, \quad \phi_- = m_\alpha \theta_0 \overline{b},$$

$$\psi_+ = m_\alpha \theta_2 \theta_3 \overline{c}, \quad \psi_- = m_\alpha \theta_2 \overline{d},$$

where $a \in \mathcal{H}(m_\alpha\theta_0\theta_1)$, $b \in \mathcal{H}(m_\alpha\theta_0)$, $c \in \mathcal{H}(m_\alpha\theta_2\theta_3)$ *and* $d \in \mathcal{H}(m_\alpha\theta_2)$. *Assume* m_α *and* $\theta_0\theta_1$ *are coprime, and* m_α *and* $\theta_2\theta_3$ *are coprime. Then* (T_ϕ, T_ψ) *is hyponormal if and only if* $\phi = \beta\psi$, *where*

$$\overline{\beta} = \frac{\theta_2(\alpha)\theta_3(\alpha)a(\alpha)}{\theta_0(\alpha)\theta_1(\alpha)c(\alpha)}. \tag{21}$$

Proof. Let β be defined as in (21). Note that

$$\begin{aligned} \phi_+ - \beta\psi_+ &= m_\alpha\theta_0\theta_1 - \beta m_\alpha \theta_2\theta_3 \\ &= \theta_0\theta_1\theta_2\theta_3 m_\alpha(\overline{\theta_2}\,\overline{\theta_3}\,\overline{a} - \beta\overline{\theta_0}\,\overline{\theta_1}\,\overline{c}). \end{aligned}$$

By the definition of β, we have $m_\alpha(\overline{\theta_2}\,\overline{\theta_3}\,\overline{a} - \beta\overline{\theta_0}\,\overline{\theta_1}\,\overline{c}) \in \overline{H^2}$. Thus

$$Ker(H_{\overline{\phi_+}-\beta\overline{\psi_+}}) \supset \theta_0\theta_1\theta_2\theta_3 H^2.$$

On the other hand, by Lemma 5.1, the hyponormality of (T_ϕ, T_ψ) implies the hyponormality of $(T_{\phi-\beta\psi}, T_\psi)$. Now by Lemma 6.2, we must have

$$\theta_0\theta_1\theta_2\theta_3 H^2 \subset Ker(H_{\overline{\phi_+}-\beta\overline{\psi_+}}) \subset Ker(H_{\overline{\psi_-}}) = m_\alpha\theta_2 H^2.$$

This is a contradiction, unless $\phi = \beta\psi$. □

Proposition 6.4. *Assume that*

$$\phi_+ = \theta_0\overline{a}, \quad \phi_- = \theta_0\overline{b},$$

$$\psi_+ = \theta_1\overline{c}, \quad \psi_- = \theta_1\overline{d},$$

where θ_0 *and* θ_1 *are inner,* $a, b \in \mathcal{H}(\theta_0)$ *and* $c, d \in \mathcal{H}(\theta_1)$. *Then* (T_ϕ, T_ψ) *hyponormal implies that* $\theta_0 = \theta_1$. *Furthermore, if* θ_0 *is not a singular inner function, then* $\phi = \beta\psi$ *for some constant* β.

Proof. By Lemma 6.2

$$Ker(H_{\overline{\psi_-}}) = \theta_1 H^2 \supset Ker(H_{\overline{\phi_+}}) = \theta_0 H^2,$$
$$Ker(H_{\overline{\phi_-}}) = \theta_0 H^2 \supset Ker(H_{\overline{\psi_+}}) = \theta_1 H^2.$$

Therefore $\theta_1 = \theta_0$. If θ_0 is not a singular inner function, say

$$\theta_0 = \theta_1 = m_\alpha m_\alpha^n \Delta,$$

where $n \geq 0$, m_α and Δ are coprime and $m_\alpha = (z-\alpha)/(1-\overline{\alpha}z)$. Let $\beta = \bar{a}(\alpha)/\bar{c}(\alpha)$. By Lemma 5.1, the hyponormality of (T_ϕ, T_ψ) implies the hyponormality of $(T_{\phi-\beta\psi}, T_\psi)$. By Lemma 6.2

$$Ker(H_{\overline{\psi_-}}) = m_\alpha m_\alpha^n H^2 \supset Ker(H_{\overline{(\phi-\beta\psi)_+}}).$$

But $Ker(H_{\overline{(\phi-\beta\psi)_+}}) \supset m_\alpha^n \Delta H^2$, since

$$(\phi - \beta\psi)_+ = m_\alpha^n \Delta m_\alpha (\overline{a(z)} - \beta \overline{c(z)})$$

and $\overline{m_\alpha}(a(z) - \bar{\beta} c(z)) \in H^2$. This is a contradiction, unless $\phi = \beta\psi$. □

Recall that, for an inner function θ, $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$. In particular, we have

$$\mathcal{H}(z^n) = \{\text{all analytic polynomials of degress less than n}\}.$$

Theorem 6.5. *Assume that*

$$\phi_+ = z^N \theta_0 \theta_1 \overline{a(z)}, \quad \phi_- = z^n \overline{b(z)},$$
$$\psi_+ = z^L \theta_0 \theta_2 \overline{c(z)}, \quad \psi_- = z^l \overline{d(z)},$$

where $a \in \mathcal{H}(z^N \theta_0 \theta_1)$, $b \in \mathcal{H}(z^n)$ $c \in \mathcal{H}(z^L \theta_0 \theta_2)$, $d \in \mathcal{H}(z^l)$, $\theta_0(0)\theta_1(0)\theta_2(0) \neq 0$ *and* θ_1 *and* θ_2 *are coprime. Then* $\mathbf{T} = (T_\phi, T_\psi)$ *is hyponormal if and only if*

$$N = L, n = l, b(z) = \alpha d(z), \quad P_{\mathcal{H}(z^n)}[\theta_2 a(z)] = \alpha P_{\mathcal{H}(z^n)}[\theta_1 c(z)]$$

for some constant α *and* $T_{\psi_{z^{N-n}\theta_0\theta_1\theta_2}}$ *is hyponormal, where*

$$\psi_{z^{N-n}\theta_0\theta_1\theta_2} = P(\overline{z^{N-n}\theta_0\theta_1\theta_2}\psi_+) + \overline{\psi_-} = P(z^l \theta_1 \bar{c}(z)) + \overline{z}^l d(z).$$

Proof. The sufficiency follows from Proposition 5.2. We prove the necessity. Without loss of generality, assume $l \geq n \geq 1$. We prove the theorem by induction on n. Assume $n = 1$. If $l = 1$, then $b(z)$ and $d(z)$ are nonzero constants b and d. Therefore $b(z) = \alpha d(z)$ for $\alpha = b/d$. Now the remaining part of the theorem for $n = l = 1$ follows from Proposition 5.2. For example, assume $L \geq N$; by (10) in Proposition 5.2, we have

$$\alpha P_{\mathcal{H}(z)}(\theta_1 c) = P_{\mathcal{H}(z)}(z^{L-N}\theta_2 a).$$

Thus $L - N = 0$, since otherwise $\alpha P_{\mathcal{H}(z)}(\theta_1 c) = \alpha\theta_1(0)c(0) = 0$, which is a contradiction.

We now prove that $l = 1$. If $l > 1$, we first show that $L - N \geq l - 1$. Note that the positivity of

$$\begin{aligned}
&\begin{pmatrix} T_z^* & 0 \\ 0 & T_z^* \end{pmatrix} [\mathbf{T}, \mathbf{T}] \begin{pmatrix} T_z & 0 \\ 0 & T_z \end{pmatrix} \\
&= \begin{pmatrix} H^*_{z\overline{\phi_+}} H_{z\overline{\phi_+}} - H^*_{z\overline{\phi_-}} H_{z\overline{\phi_-}} & H^*_{z\overline{\phi_+}} H_{z\overline{\psi_+}} - H^*_{z\overline{\psi_-}} H_{z\overline{\phi_-}} \\ H^*_{z\overline{\psi_+}} H_{z\overline{\phi_+}} - H^*_{z\overline{\phi_-}} H_{z\overline{\psi_-}} & H^*_{z\overline{\psi_+}} H_{z\overline{\psi_+}} - H^*_{z\overline{\psi_-}} H_{z\overline{\psi_-}} \end{pmatrix} \\
&= \begin{pmatrix} H^*_{z\overline{\phi_+}} H_{z\overline{\phi_+}} & H^*_{z\overline{\phi_+}} H_{z\overline{\psi_+}} \\ H^*_{z\overline{\psi_+}} H_{z\overline{\phi_+}} & H^*_{z\overline{\psi_+}} H_{z\overline{\psi_+}} - H^*_{z\overline{\psi_-}} H_{z\overline{\psi_-}} \end{pmatrix} = [\mathbf{T_1}, \mathbf{T_1}]
\end{aligned}$$

implies the hyponormality of $\mathbf{T_1} = (T_{\phi_1}, T_{\psi_1})$, where

$$\phi_1 = P(\bar{z}\phi_+), \quad \psi_1 = P(\bar{z}\psi_+) + (I - P)(z\overline{\psi_-}).$$

Note that

$$\begin{aligned}
\bar{z}\psi_- &= \bar{z}z^l\overline{d(z)} = z^{l-1}\overline{d_1(z)} + \beta, \\
\bar{z}\psi_+ &= \bar{z}z^L\theta_0\theta_2\overline{c(z)} = z^{L-1}\theta_0\theta_2\overline{c_1(z)} + \gamma, \\
\bar{z}\phi_+ &= \bar{z}z^N\theta_0\theta_1\overline{a(z)} = z^{N-1}\theta_0\theta_1\overline{a_1(z)} + \delta,
\end{aligned}$$

where $d_1(z) \in \mathcal{H}(z^{l-1}), c_1(z) \in \mathcal{H}(z^{L-1}\theta_0\theta_2), a_1(z) \in \mathcal{H}(z^{N-1}\theta_0\theta_1)$, and β, γ, δ are constants such that

$$\begin{aligned}
d(z) &= d_1(z) + z^{l-1}\overline{\beta}, \\
c(z) &= c_1(z) + z^{L-1}\theta_0\theta_2\overline{\gamma}, \\
a(z) &= a_1(z) + z^{N-1}\theta_0\theta_1\overline{\delta}.
\end{aligned}$$

In the above we have used the fact that

$$\mathcal{H}(z^N\theta_0\theta_1) = \mathcal{H}(z^{N-1}\theta_0\theta_1) \oplus z^{N-1}\theta_0\theta_1\mathcal{H}(z).$$

Since by the (implicit) assumption, $z^L\theta_0\theta_2$ and the inner part of $c(z)$ are relatively prime, it follows that $z^{L-1}\theta_0\theta_2$ and the inner part of $c_1(z)$ are also relatively prime. Set

$$\begin{aligned}
\psi_2 &= z^{L-1}\theta_0\theta_2\overline{c_1(z)} + \overline{z^{l-1}\overline{d_1(z)}} = \psi_1 - \gamma, \\
\phi_2 &= z^{N-1}\theta_0\theta_1\overline{a_1(z)} = \phi_1 - \delta.
\end{aligned}$$

Then $\mathbf{T_1} = (T_{\phi_1}, T_{\psi_1})$ is hyponormal if and only if $\mathbf{T_2} = (T_{\phi_2}, T_{\psi_2})$ is hyponormal. Since ϕ_2 is analytic and ψ_2 is not analytic, by Corollary 4.2,

$$\overline{z^{N-1}\theta_0\theta_1} z^{L-1}\theta_0\theta_2 = z^{l-1}\Delta_1\overline{\Delta_2}$$

for some inner functions Δ_1 and Δ_2, and $z^{l-1}\Delta_1$ and Δ_2 are coprime. In particular, we have $L - 1 - (N - 1) \geq l - 1$.

Now $l > 1$ implies that $L - 1 \geq N - 1 + l - 1 \geq N$. Note that

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ 0 & T_z^* \end{pmatrix} [\mathbf{T}, \mathbf{T}] \begin{pmatrix} I & 0 \\ 0 & T_z \end{pmatrix} \\
&= \begin{pmatrix} H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}} H_{\overline{\phi_-}} & H^*_{\overline{\phi_+}} H_{z\overline{\psi_+}} \\ H^*_{z\overline{\psi_+}} H_{\overline{\phi_+}} & H^*_{z\overline{\psi_+}} H_{z\overline{\psi_+}} - H^*_{z\overline{\psi_-}} H_{z\overline{\psi_-}} \end{pmatrix}
\end{aligned}$$

is positive. Therefore by Lemma 2.1

$$\begin{aligned}&|(H^*_{z\overline{\psi_+}}H_{\overline{\phi_+}}x,y)|^2\\ &\le \left((H^*_{\overline{\phi_+}}H_{\overline{\phi_+}}-H^*_{\overline{\phi_-}}H_{\overline{\phi_-}})x,x\right)\left((H^*_{z\overline{\psi_+}}H_{z\overline{\psi_+}}-H^*_{z\overline{\psi_-}}H_{z\overline{\psi_-}})x,x\right)\end{aligned}$$

for all $x,y\in H^2$. In particular,

$$|(H_{\overline{\phi_+}}x,H_{z\overline{\psi_+}}y)|^2\le\left((H^*_{\overline{\phi_+}}H_{\overline{\phi_+}}-H^*_{\overline{\phi_-}}H_{\overline{\phi_-}})x,x)\right)\left(H_{z\overline{\psi_+}}y,H_{z\overline{\psi_+}}y\right).$$

It is easy to see that

$$Range(H_{z\overline{\psi_+}})\supset\mathcal{H}(z^{L-1})\supset\mathcal{H}(z^N)$$

and $H_{\overline{\phi_+}}$ maps $\theta_0\theta_1H^2$ onto $\mathcal{H}(z^N)$. Therefore

$$|\langle H_{\overline{\phi_+}}x,h\rangle|^2\le[(H_{\overline{\phi_+}}x,H_{\overline{\phi_+}}x)-(H_{\overline{\phi_-}}x,H_{\overline{\phi_-}}x)](h,h)$$

for all $x\in\theta_0\theta_1H^2, h\in\mathcal{H}(z^N)$. This implies that $H_{\overline{\phi_-}}x=0$ for all $x\in\theta_0\theta_1H^2$, which is a contradiction. Therefore $l=1$. This completes the proof of the theorem for $n=1$.

Assume the theorem is true for $n-1$ ($n\ge 2$). Let

$$b(z)=b_0+b_1z+\cdots+b_{n-1}z^{n-1},\quad d(z)=d_0+d_1z+\cdots+d_{l-1}z^{l-1}.$$

Note that

$$\begin{aligned}&\begin{pmatrix}T_z^* & 0\\ 0 & T_z^*\end{pmatrix}[\mathbf{T},\mathbf{T}]\begin{pmatrix}T_z & 0\\ 0 & T_z\end{pmatrix}\\ &=\begin{pmatrix}H^*_{z\overline{\phi_+}}H_{z\overline{\phi_+}}-H^*_{z\overline{\phi_-}}H_{z\overline{\phi_-}} & H^*_{z\overline{\phi_+}}H_{z\overline{\psi_+}}-H^*_{z\overline{\psi_-}}H_{z\overline{\phi_-}}\\ H^*_{z\overline{\psi_+}}H_{z\overline{\phi_+}}-H^*_{z\overline{\phi_-}}H_{z\overline{\psi_-}} & H^*_{z\overline{\psi_+}}H_{z\overline{\psi_+}}-H^*_{z\overline{\psi_-}}H_{z\overline{\psi_-}}\end{pmatrix}=[\mathbf{T_1},\mathbf{T_1}],\end{aligned}$$

where $\mathbf{T_1}=(T_{\phi_1},T_{\psi_1})$ and

$$\phi_1=P(\bar z\phi_+)+(I-P)(z\overline{\phi_-}),\quad \psi_1=P(\bar z\psi_+)+(I-P)(z\overline{\psi_-}).$$

It is easy to see that

$$\begin{aligned}(\phi_1)_+&=z^{N-1}\theta_0\theta_1\overline{a_{11}(z)},\quad (\phi_1)_-=z^{n-1}\overline{b_{11}(z)},\\ (\psi_1)_+&=z^{L-1}\theta_0\theta_2\overline{c_{11}(z)},\quad (\psi_1)_-=z^{l-1}\overline{d_{11}(z)},\end{aligned}$$

where $a_{11}(z)\in\mathcal{H}(z^{N-1}\theta_0\theta_1)$, $c_{11}(z)\in\mathcal{H}(z^{N-1}\theta_0\theta_2)$ and

$$b_{11}(z)=b_0+b_1z+\cdots+b_{n-2}z^{n-2},\quad d_{11}(z)=d_0+d_1z+\cdots+d_{l-2}z^{l-2}.$$

By the induction assumption for $n-1$, we have $N-1=L-1, n-1=l-1$, and $b_{11}(z)=\alpha d_{11}(z)$ for some constant α. By Lemma 5.1, $(T_{\phi-\alpha\psi},T_\psi)$ is hyponormal. Note that

$$\begin{aligned}b(z)-\alpha d(z)&=(b_{11}(z)+b_{n-1}z^{n-1})-\alpha(d_{11}(z)+d_{n-1}z^{n-1})\\ &=b_{n-1}z^{n-1}-\alpha d_{n-1}z^{n-1}.\end{aligned}$$

If $b_{n-1}\ne\alpha d_{n-1}$, then $(\phi-\alpha\psi)_-=z(\overline{b_{n-1}}-\alpha\overline{d_{n-1}})$. By the previous argument, we have $\psi_-=z\bar\beta$ for some constant β. But $\psi_-=z^l\overline{d(z)}$ and $l=n\ge 2$. This is a contradiction. Therefore $b(z)=\alpha d(z)$. Now by Proposition 5.2, we conclude that

$$P_{\mathcal{H}(z^n)}[\theta_2a(z)]=\alpha P_{\mathcal{H}(z^n)}[\theta_1c(z)]$$

and $T_{\psi_{z^{N-n}\theta_0\theta_1\theta_2}}$ is hyponormal. This completes the proof. $\square$

Let $\phi_i \in L^\infty$ for $i = 1, 2, \cdots, n$ and let $\psi \in L^\infty$. The next result follows easily from the previous theorem and Theorem 5.3.

Theorem 6.6. *Assume that the $(\phi_i)_-, i = 1, \cdots, n$, are all analytic polynomials. Write*

$$(\phi_i)_+ = z^{N_i}\Delta_i\theta_i\overline{a_i(z)}, \quad (\phi_i)_- = z^{n_i}\overline{b_i(z)}, \quad i = 1, \cdots, n,$$
$$\psi_+ = z^L\Delta_0\overline{c(z)} = z^L\Delta_i\delta_i\overline{c(z)}, \quad \psi_- = z^l\overline{d(z)},$$

where $a_i \in \mathcal{H}(z^{N_i}\Delta_i\theta_i)$, $b_i \in \mathcal{H}(z^{n_i})$ $c \in \mathcal{H}(z^L\Delta_0)$, $d \in \mathcal{H}(z^l)$, $\Delta_i(0)\theta_i(0)\delta_i(0) \neq 0$, and θ_i and δ_i are coprime. Then $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ is hyponormal if and only if

$$N_i = L, n_i = l, \quad b_i(z) = \alpha_i d(z),$$

$$P_{\mathcal{H}(z^l)}[\delta_i a_i(z)] = \alpha_i P_{\mathcal{H}(z^l)}[\theta_i c(z)],$$

$i = 1, \cdots, n$, for some constants α_i, and $T_{\psi_{z^{L-l}\theta\Delta_0}}$ is hyponormal, where θ is the least common multiple of $\theta_1, \cdots, \theta_n$ and

$$\psi_{z^{L-l}\theta\Delta_0} = P(\overline{z^{L-l}\theta\Delta_0}\psi_+) + \overline{\psi_-}.$$

Remark. The above theorem gives a complete characterization for the hyponormality of $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n}, T_\psi)$ when only one of the symbols is a trigonometric polynomial. We note that if ψ is a trigonometric polynomial, say

$$\psi_+ = z^L\overline{c(z)}, \quad \psi_- = z^l\overline{d(z)},$$

where $c(z) \in \mathcal{H}(z^L), d(z) \in \mathcal{H}(z^l)$, the hyponormality of (T_ϕ, T_ψ) will force ϕ to take a certain form. Namely, by Lemma 6.2

$$Ker(H_{\overline{\psi_-}}) = z^lH^2 \supset Ker(H_{\overline{\phi_+}}), \quad Ker(H_{\overline{\phi_-}}) \supset z^LH^2 = Ker(H_{\overline{\psi_+}}).$$

Therefore

$$\phi_+ = z^N\theta\overline{a(z)}, \quad \phi_- = z^n\overline{b(z)},$$

where $N \geq l$, $n \leq L$, $c \in \mathcal{H}(z^N\theta)$ and $d \in \mathcal{H}(z^n)$.

We conclude the paper with the following corollary.

Corollary 6.7 (Curto and Lee, 2001)**.** *Let $\mathbf{T} = (T_{\phi_1}, \cdots, T_{\phi_n})$ be an n-tuple of trigonometric Toeplitz operators. Then the following three statements are equivalent.*

i) The tuple T is hyponormal.

ii) Every subpair of T is hyponormal.

iii) The symbols ϕ_i possess the following properties:

a) All non-analytic trigonometric polynomials ϕ_i are of the form

$$\phi_i(z) = \sum_{k=-m}^{N} a_k z^k,$$

where a_{-m} and a_N are nonzero, every T_{ϕ_i} is hyponormal, and for every pair $\{\phi_i, \phi_j\}$ $(i \neq j)$ we have $\phi_i - c\phi_j = \sum_{k=0}^{N-m} d_k z^k$ for some constants c and $d_0, \cdots, d_m$.

b) $\max\{$*Degree*(ϕ_i)*: ϕ_i is an analytic polynomial*$\} \leq N - m$.

Proof. i) implies ii) is evident. We next show ii) implies iii). Without loss of generality, assume

$$\phi_n(z) = \sum_{k=-m}^{N} a_k z^k = (\phi_n)_+ + \overline{(\phi_n)_-},$$

where a_{-m} and a_N are nonzero. Since the commutator of T_{ϕ_n} does not depend on the constant term a_0, we assume $a_0 = 0$. Set

$$a(z) = \overline{a_N} + \overline{a_{N-1}}z + \cdots + \overline{a_1}z^{N-1}, \quad b(z) = \overline{a_{-m}} + \overline{a_{-m+1}}z + \cdots + \overline{a_{-1}}z^{m-1}.$$

It is easy to see that

$$(\phi_n)_+ = z^N\overline{a(z)}, \quad (\phi_n)_- = z^m\overline{b(z)}.$$

Let $1 \le i \le n-1$. Assume (T_{ϕ_i}, T_{ϕ_n}) is hyponormal. If ϕ_i is an analytic polynomial of degree l, then $Ker(H_{\overline{\phi_i}}) = z^l H^2$. Since the greatest common divisor of z^l and z^N is z^k, where k is the minimum of l and N, by Corollary 4.2, z^k is a factor of z^{N-m}. Therefore $l \le N-m$. If ϕ_i is a non-analytic trigonometric polynomial, write

$$(\phi_i)_+ = z^L\overline{c(z)}, \quad (\phi_i)_- = z^l\overline{d(z)},$$

where $c \in \mathcal{H}(z^L)$ and $d \in \mathcal{H}(z^l)$. By Theorem 6.5, we have $N = L, m = l$, and

$$b(z) = \alpha d(z), \quad P_{\mathcal{H}(z^m)}[a(z)] = \alpha P_{\mathcal{H}(z^m)}[c(z)].$$

This is exactly the same as

$$\phi_n - \alpha\phi_i = \sum_{k=0}^{N-m} d_k z^k$$

for some constants α and $d_0, \cdots, d_{N-m}$. The implication of iii) to i) follows from Corollary 5.4. This completes the proof. $\square$

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