TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 8, Pages 3095–3115
S 0002-9947(02)03009-X
Article electronically published on April 3, 2002

SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS WITH PATCHES OF ZERO GRADIENT: EXISTENCE, REGULARITY AND CONVEXITY OF LEVEL CURVES

L. CAFFARELLI AND J. SALAZAR

Abstract. In this paper, we first construct "viscosity" solutions (in the Crandall-Lions sense) of fully nonlinear elliptic equations of the form

$$F(D^2u, x) = g(x, u) \quad \text{on} \quad \{|\nabla u| \neq 0\}$$

In fact, viscosity solutions are surprisingly weak. Since candidates for solutions are just continuous, we only require that the "test" polynomials P (those tangent from above or below to the graph of u at a point x_0) satisfy the correct inequality only if $|\nabla P(x_0)| \neq 0$. That is, we simply disregard those test polynomials for which $|\nabla P(x_0)| = 0$.

Nevertheless, this is enough, by an appropriate use of the Alexandroff-Bakelman technique, to prove existence, regularity and, in two dimensions, for $F = \Delta$, $g = cu$ ($c > 0$) and constant boundary conditions on a convex domain, to prove that there is only one convex patch.

Introduction

We study some properties of viscosity solutions of fully nonlinear elliptic equations of the form

$$F(D^2u, x) = g(x, u) \quad \text{on} \quad \{|\nabla u| \neq 0\} \tag{1}$$

or, more precisely,

$$F(D^2u, x) = g(x, u)\chi_{\{|\nabla u| \neq 0\}}(x)\,.$$

Equations of this kind appear in several contexts. For example, the stationary equation for the mean field theory of superconducting vortices, derived formally by Chapman in [7], takes this form when the scalar stream function admits a functional dependence on the scalar magnetic potential. In general, solutions are expected to be $C^{1,1}$ or at least $W^{2,p}$ and satisfy the equation a.e. outside "patches" where the gradient vanishes.

The time dependent equations of Chapman's mean field model form a degenerate parabolic elliptic system. Viscosity solutions of this system were investigated in two dimensions by Elliott, Schätzle, and Stoth in [8]. They also found special solutions of the stationary problem.

Received by the editors May 15, 2000.
2000 *Mathematics Subject Classification.* Primary 35R35, 31B20.
Key words and phrases. Viscosity solutions, free boundary problems, regularity.
L. Caffarelli was supported by NSF grant DMS 9714758.
J. Salazar was partially supported by FCT Praxis/2/2.1/MAT/124/94, and also thanks the Mathematics Department of the University of Texas at Austin for its warm hospitality.

In two dimensions, for $F = \Delta$, $g = cu$ and constant boundary conditions on a convex domain, there is supposedly only one convex patch.

In this paper, we first construct "viscosity" solutions of (1), in the Crandall-Lions sense. In fact, viscosity solutions are surprisingly weak. Since candidates for solutions are just continuous, we only require that the "test" polynomials P (those tangent from above or below to the graph of u at a point x_0) satisfy the correct inequality only if $|\nabla P(x_0)| \neq 0$. That is, we simply disregard those test polynomials for which $|\nabla P(x_0)| = 0$.

Nevertheless, this is enough, by an appropriate use of the Alexandroff-Bakelman technique, to prove existence, regularity, and the "one convex patch" theorem.

Existence of solutions of Dirichlet's problem in an arbitrary domain Ω is established given a continuous subsolution dominated by a continuous supersolution. The continuity of the candidate for a solution is proved assuming that F does not depend on x. We believe this restriction has much to do with the method (Jensen's approximation), and it would be interesting to find a proof that releases it.

The first and main result leading to regularity is Proposition 5, which shows that the solutions actually satisfy uniform elliptic inequalities with bounded right hand side and no gradient restriction (see Corollary 6). This result allows us to apply the powerful machinery of nonlinear elliptic theory and obtain the Alexandroff-Bakelman-Pucci estimates, Harnack's inequality and C^α regularity (Corollary 7). We also discuss $W^{2,p}$ regularity, using the notion of L^p-viscosity solutions (introduced by Caffarelli, Crandall, Kocan and Śviech in [4]).

In the particular case of equation $\Delta u = cu$ on $\{|\nabla u| \neq 0\}$, where c is a positive constant, we prove that u is $C^{1,1}$. The main tool is the monotonicity lemma of Alt, Caffarelli and Friedman [1], and the proof is adapted from that of Theorem I in [5]. If the connected components of the set $\{|\nabla u| = 0\}$ are isolated, the mathematical problem becomes, locally, identical to an inverse problem treated by Caffarelli, Karp and Shahgholian in [5].

Section 3 is devoted to the finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary. A couple of tools are needed and previously proved: the strict positivity of nonnegative supersolutions and the quadratic growth of subsolutions.

We finish with an application to the equation $\Delta u = u$ on $\{|\nabla u| \neq 0\}$, on a bounded, convex, plane domain $\Omega \subset \mathbb{R}^2$, such that $u \equiv 1$ on $\partial\Omega$. We prove that the interior of the set $\{|\nabla u| = 0\}$ is convex (see Proposition 20). In particular, there is at most one connected component of $\{|\nabla u| = 0\}$ with nonempty interior; this answers a question posed by C. M. Elliott. B. Kawohl has kindly told us that an n-dimensional version of this result can be obtained using the methods in his book [12].

Before starting, let us give the precise meaning of (1): A subsolution is an upper semicontinuous (u.s.c.) function u, $u < \infty$, such that the inequality

$$F(D^2P, x) \geq g(x, u(x))$$

holds for any paraboloid P touching u from above at x, provided $|\nabla P(x)| \neq 0$. A supersolution of (1) is a lower semicontinuous (l.s.c.) function u, $u > -\infty$, such that the opposite inequality holds for paraboloids touching u from below with the same extra condition ($|\nabla P(x)| \neq 0$). A solution is simultaneously a sub- and supersolution.

Throughout the paper, Ω denotes the domain of u, which is a domain in n-dimensional Euclidean space. λ and Λ are the ellipticity constants of F, which is assumed to be continuous and uniformly elliptic. Also, g is assumed to be continuous. New hypotheses on F and g will be imposed as needed.

1. Existence of solutions

The Perron-Wiener method has been used extensively by Ishii [11] to solve the Dirichlet problem by the viscosity approach. Boundedness of the domain Ω is not needed, but we do assume that a continuous subsolution and a continuous supersolution dominating the subsolution already exist.

In order to apply the Perron-Wiener method, we shall verify the usual properties of the family of subsolutions (or supersolutions). The proposition below is a well known result for elliptic equations. Its proof follows the classical pattern, but special care shall be taken to deal with the condition on the gradient.

We denote by u^* the upper semicontinuous envelope of a given function u, i.e.,

$$u^*(x) = \limsup_{z\to x} u(z)\,.$$

In a similar way, we define the lower semicontinuous envelope

$$u_*(x) = \liminf_{z\to x} u(z)\,.$$

u^* and u_* are upper and lower semicontinuous respectively.

Proposition 1. *Let $\{u_\alpha\}$ be a nonempty family of subsolutions of* (1) *and put*

$$u = \sup_\alpha u_\alpha\,.$$

Then, u^* is a subsolution, provided $u^* < \infty$.

Proof. Let P be a paraboloid touching u^* from above at some point x_0. Assume that $|\nabla P(x_0)| \neq 0$. Fix $\varepsilon > 0$ and put

$$Q(x) = P(x) + \frac{\varepsilon}{2\Lambda}|x - x_0|^2\,.$$

By continuity of the functions involved, there is $\delta > 0$ such that for all $x \in B_\delta(x_0)$, the following assertions hold:

a) $|\nabla P(x)| \neq 0, \quad |\nabla Q(x)| \neq 0;$
b) $F(D^2P, x) \leq F(D^2P, x_0) + \varepsilon;$
c) $g(x, r) \geq g(x_0, u^*(x_0)) - \varepsilon$, for all r such that $|r - u^*(x_0)| \leq \frac{\varepsilon\delta^2}{\Lambda}$.

Now, choose $\eta < \delta/2$ such that

$$|P(x) - P(x_0)| \leq \frac{\varepsilon\delta^2}{16\Lambda}, \quad \forall x \in B_\eta(x_0)\,.$$

Note that there exist an index α and a point $x' \in B_\eta(x_0)$, such that

$$u_\alpha(x') \geq u^*(x_0) - \frac{\varepsilon\delta^2}{16\Lambda}\,.$$

Accordingly,

$$\begin{aligned} Q(x') - u_\alpha(x') &\leq Q(x') - u^*(x_0) + \frac{\varepsilon\delta^2}{16\Lambda} \\ &= P(x') - P(x_0) + \frac{\varepsilon}{2\Lambda}|x' - x_0|^2 + \frac{\varepsilon\delta^2}{16\Lambda} \\ &\leq \frac{\varepsilon\delta^2}{4\Lambda}\,. \end{aligned}$$

Since

$$Q(x) - u_\alpha(x) \geq \frac{\varepsilon}{2\Lambda}|x - x_0|^2 \,,$$

the infimum of $Q - u_\alpha$ is attained at a point $x_1 \in \overline{B}_{\delta/\sqrt{2}}$. At this point, we have

$$F(D^2Q, x_1) \geq g(x_1, u_\alpha(x_1))$$

and

$$\begin{aligned} |u^*(x_0) - u_\alpha(x_1)| &\leq |Q(x_0) - Q(x_1)| + Q(x_1) - u_\alpha(x_1) \\ &\leq \frac{\varepsilon\delta^2}{16\Lambda} + \frac{\varepsilon}{2\Lambda}|x_1 - x_0|^2 + \frac{\varepsilon\delta^2}{4\Lambda} \leq \frac{13}{16}\frac{\varepsilon\delta^2}{\Lambda} \,. \end{aligned}$$

By the ellipticity of F,

$$F(D^2Q, x) = F(D^2P + \frac{\varepsilon}{\Lambda}I, x) \leq F(D^2P, x) + \varepsilon \,.$$

Putting all these inequalities together, we obtain

$$\begin{aligned} F(D^2P, x_0) + 3\varepsilon &\geq F(D^2P, x_1) + 2\varepsilon \\ &\geq F(D^2Q, x_1) + \varepsilon \\ &\geq g(x_1, u_\alpha(x_1)) + \varepsilon \\ &\geq g(x_0, u^*(x_0)) \,. \end{aligned}$$

□

The following proposition is a first approach for solving the Dirichlet problem. It will be refined below under additional hypotheses on F and g.

Proposition 2. *For any given continuous subsolution $\underline{v}$ and a continuous supersolution $\overline{v}$ such that $\underline{v} \leq \overline{v}$, there exist a function u, $\underline{v} \leq u \leq \overline{v}$, such that u_* is a supersolution and u^* is a subsolution.*

Proof. Denote by u the supremum of all continuous subsolutions less than or equal to $\overline{v}$. By Proposition 1, u^* is a subsolution.

To prove that u is a supersolution ($u = u_*$, since u is the supremum of continuous functions), let P be a paraboloid touching u from below at a point x_0, such that $|\nabla P(x_0)| \neq 0$.

If $u(x_0) = \overline{v}(x_0)$, then P touches $\overline{v}$ from below at x_0. Consequently, $F(D^2P, x_0) \leq g(x_0, u(x_0))$.

Now, suppose

$$u(x_0) < \overline{v}(x_0) \quad \text{and} \quad F(D^2P, x_0) > g(x_0, u(x_0)) \,.$$

Let

$$a = F(D^2P, x_0) - g(x_0, u)$$

and choose $\delta_1 > 0$ and $\nu > 0$ such that for all $x \in B_{\delta_1}(x_0)$ and $|r - u^*(x_0)| < \nu$, we have

$$g(x, r) \leq g(x_0, u(x_0)) + \frac{a}{3}$$

and

$$F(D^2P, x) \geq F(D^2P, x_0) - a/3 \,.$$

Let $\delta_2 > 0$ be such that for all $x \in B_{\delta_2}(x_0)$

$$\left| P(x) - P(x_0) - \frac{a}{6\Lambda}|x - x_0|^2 \right| \le \frac{\nu}{2}\,.$$

Then, for $|\beta| < \nu/2$, the paraboloid

$$Q(x) = P(x) - \frac{a}{6\Lambda}|x - x_0|^2 + \beta$$

is a subsolution of (1) in $B_{\delta_1} \cap B_{\delta_2}$. In fact, by the ellipticity of F and the above inequalities, we get

$$\begin{aligned} F(D^2Q, x) &= F\Big(D^2P - \frac{a}{3\Lambda}I, x\Big) \\ &\ge F(D^2P, x) - \frac{a}{3} \\ &\ge F(D^2P, x_0) - \frac{2a}{3} \\ &\ge g(x_0, u(x_0)) + \frac{a}{3} \\ &\ge g(x, Q(x))\,. \end{aligned}$$

The last inequality holds because

$$|Q(x) - u(x_0)| = \left| P(x) - P(x_0) - \frac{a}{6\Lambda}|x - x_0|^2 \right| \le \nu\,.$$

To reach a contradiction, we shall construct a continuous subsolution less than or equal to $\overline{v}$ and strictly greater than u at x_0.

First we choose $\gamma > 0$ and $\delta_3 > 0$ such that

$$\overline{v} - P \ge \gamma \quad \text{on} \quad B_{\delta_3}(x_0)\,.$$

Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. By the axiom of choice and the compactness of ∂B_δ, there is a continuous subsolution $v \le \overline{v}$, such that

$$v - P \ge -\frac{a\delta^2}{12\Lambda} \quad \text{on} \quad \partial B_\delta\,.$$

Taking $\beta < \min\{\nu/2, \gamma, \frac{a\delta^2}{12\Lambda}\}$, $\beta > 0$, we see that the function

$$w(x) = \begin{cases} v(x) \vee Q(x)\,, & x \in B_\delta, \\ v(x)\,, & x \notin B_\delta, \end{cases}$$

is a continuous subsolution less than or equal to $\overline{v}$, and $w(x_0) > u(x_0)$. This is a contradiction. $\square$

A most natural question is whether the function u above is actually continuous. We answer this question under suitable additional hypotheses.

Proposition 3. *Let $\underline{v}$ and $\overline{v}$ be the two functions in Proposition 2. Suppose that*

$$\Omega_\alpha = \{x \in \Omega; \overline{v} - \underline{v} \ge \alpha\}$$

is compact for all $\alpha > 0$.

We also assume at this point that F does not depend on x and that for any given compact set $D \subset \Omega$, there is $c > 0$ such that for all $x \in D$, all $r \in \mathbb{R}$, and all $h > 0$,

$$g(x, r + h) \ge g(x, r) + ch\,.$$

Then, there is a viscosity solution u such that $\underline{v} \le u \le \overline{v}$.

Remark. By Proposition 2, taking u equal to the supremum of all continuous subsolutions less than or equal to $\overline{v}$, only the continuity of u remains to be proved.

Before going into the proof of this proposition, we need some notation and properties of Jensen's approximate solutions.

Suppose there is a point $x_0 \in \Omega$ such that

$$u^*(x_0) > u(x_0);$$

otherwise, u is continuous and there is nothing to prove.

Following Jensen's idea, define

$$u^\varepsilon(x) = \sup_{y \in \Omega_\alpha} \left\{ u^*(y) + \varepsilon - \frac{1}{\varepsilon}|y - x|^2 \right\}, \qquad x \in \Omega_\alpha$$

where α is a positive constant, whose precise value will be fixed later.

Jensen's approximation of a continuous solution enjoys many nice properties; a list of them can be found in the book by Caffarelli and Cabré [3], p. 43, Theorem 5.1. Suitable versions of those properties, adapted to our case, are listed below. We omit the proofs since those given in [3] work with minor changes.

a) u^ε is a decreasing family of continuous functions.

b) Let f be a continuous function such that $f \geq u^*$. For each $\beta > 0$ there is an $\varepsilon_0 > 0$ such that

$$u^\varepsilon \leq f + \beta \;\text{ on }\; \Omega_{2\alpha}, \qquad \forall \varepsilon < \varepsilon_0 .$$

c) There is a point $x' \in \Omega_\alpha$ such that

$$u^\varepsilon(x) = u^*(x') + \varepsilon - \frac{1}{\varepsilon}|x - x'|^2 .$$

d) The point x' in c) satisfies

$$|x - x'|^2 \leq \varepsilon \sup_{\Omega_\alpha}(s - t) .$$

Now, we can state the key fact in the proof of Proposition 3.

Lemma 4. *Under the hypothesis of Proposition 3, for each $\delta > 0$, there exists an $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$, the function $u^\varepsilon(x) - \delta$ is a viscosity subsolution of (1) in $\Omega_{2\alpha}$.*

Proof. Let P be a paraboloid touching $u^\varepsilon - \delta$ from above at a point $x_0 \in \Omega_{2\alpha}$. Assume $|\nabla P(x_0)| \neq 0$ and define

$$Q(x) = P(x + x_0 - x') + \delta + \frac{1}{\varepsilon}|x_0 - x'|^2 - \varepsilon .$$

Then, one readily verifies that

$$u^*(x) \leq u^\varepsilon(x + x_0 - x') + \frac{1}{\varepsilon}|x_0 - x'|^2 - \varepsilon \leq Q(x) ,$$

$$u^*(x') = Q(x') ,$$

and

$$\nabla Q(x') = \nabla P(x_0) \neq 0 .$$

Hence

$$
\begin{aligned}
F(D^2Q) &\geq g\big(x', u^*(x')\big) \\
&= g\big(x', u^\varepsilon(x_0) + \frac{1}{\varepsilon}|x_0 - x'|^2 - \varepsilon\big) \\
&\geq g\big(x', u^\varepsilon(x_0) - \delta\big) + c\big(\delta + \frac{1}{\varepsilon}|x_0 - x'|^2 - \varepsilon\big)\,,
\end{aligned}
$$

provided $\delta + \frac{1}{\varepsilon}|x_0 - x'|^2 - \varepsilon \geq 0$.

By d), since $\Omega_\alpha \times I$ (where $I = \{r \in \mathbb{R};\ \inf_{\Omega_\alpha} \underline{v} - \delta \leq r \leq \sup_{\Omega_\alpha} \overline{v} + 1\}$) is compact and g is continuous, we can find $\varepsilon_1 > 0$ such that for all $\varepsilon \leq \varepsilon_1$, $|x_0 - x'|$ is small enough and we have

$$
g\big(x', u^\varepsilon(x_0) - \delta\big) \geq g\big(x_0, u^\varepsilon(x_0) - \delta\big) - c\,\frac{\delta}{2}\,.
$$

Consequently, for $\varepsilon_1 \leq \delta/2$, we arrive at

$$
F(D^2P) \geq g\big(x_0, u^\varepsilon(x_0) - \delta\big)\,.
$$

□

Proof of Proposition 3. Let $\delta = u^*(x_0) - u(x_0)$ and fix $\varepsilon_0 > 0$ such that

$$
u^\varepsilon \leq \underline{v} + \delta/3 \ \text{ on } \ \Omega_{2\delta/3}\,, \qquad \forall \varepsilon < \varepsilon_0;
$$

see property b) above. In addition, by Lemma 4, let $\varepsilon_1 > 0$ be such that the function $u^\varepsilon - \delta$ is a continuous viscosity subsolution of (1) in $\Omega_{2\delta/3}$.

Then, for $\varepsilon \leq \varepsilon_0 \wedge \varepsilon_1$, we have

i) $u^\varepsilon(x_0) - \delta \geq u(x_0) + \varepsilon$,
ii) $u^\varepsilon - \delta \leq \overline{v}$ in $\Omega_{2\delta/3}$,
iii) $u^\varepsilon - \delta \leq\leq \overline{v} - 2\delta/3 = \underline{v}$ on $\partial\Omega_{2\delta/3}$.

This in particular implies that the function

$$
w(x) = \begin{cases} (u^\varepsilon(x) - \delta) \vee \underline{v}(x)\,, & x \in \Omega_{2\delta/3}, \\ \underline{v}(x)\,, & x \notin \Omega_{2\delta/3}, \end{cases}
$$

is a continuous subsolution less than or equal to $\overline{v}$ and $w(x_0) > u(x_0)$. This leads to a contradiction. □

2. Regularity of solutions

From now on, we assume that

$$
F(0, x) = 0 \quad \forall x \in \Omega\,.
$$

The following proposition shows that the solutions of (1) actually satisfy uniform elliptic inequalities with bounded right hand side and no gradient restriction. This result allows us to apply the powerful machinery of the nonlinear elliptic theory.

Proposition 5. *Let u be a continuous supersolution of* (1)*. Then,*

$$
F(D^2u, x) \leq g^+(x, u)
$$

in the viscosity sense.

Proof. Assume that there is a paraboloid P, touching u from below at a point x_0, such that

$$F(D^2P, x_0) > g^+(x_0, u(x_0))\,.$$

Since u is a supersolution of (1), we must have $|\nabla P(x_0)| = 0$. Now, put

$$a = F(D^2P, x_0) - g^+(x_0, u(x_0))\,,$$

and fix $\delta > 0$ such that for all $x \in B_\delta(x_0)$

$$F(D^2P, x) \geq g^+(x_0, u(x_0)) + \frac{3a}{4}\,,$$

and

$$g(x, u(x)) \leq g^+(x_0, u(x_0)) + \frac{a}{4}\,.$$

Define

$$P_1(x) = P(x) + \frac{a}{8\Lambda}(\delta^2 - |x - x_0|^2)\,.$$

If Q is a paraboloid touching $u - P_1$ from below at $x \in B_\delta(x_0)$ such that

$$\nabla Q(x) \neq -\nabla P_1(x)\,,$$

then

$$F(D^2Q + D^2P_1, x) \leq g(x, u(x))\,.$$

By the ellipticity of F,

$$F(D^2P, x) \leq F(D^2P_1, x) + \frac{a}{4}$$

and

$$F(D^2P_1, x) \leq F(D^2Q + D^2P_1, x) - \mathcal{M}^-\Big(D^2Q, \frac{\lambda}{n}, \Lambda\Big)\,.$$

Here, $\mathcal{M}^-$ denotes Pucci's minimal operator defined (for $0 < \lambda \leq \Lambda$ fixed) on the set of $n \times n$ real symmetric matrices by

$$\mathcal{M}^-\Big(M, \lambda, \Lambda\Big) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

where the e_i are the eigenvalues of M. We will also make use of Pucci's maximal operator, defined by

$$\mathcal{M}^+\Big(M, \lambda, \Lambda\Big) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i\,.$$

From this, we obtain

$$\begin{aligned}\mathcal{M}^-\Big(D^2Q, \frac{\lambda}{n}, \Lambda\Big) &\leq g(x, u(x)) - g^+(x_0, u(x_0)) - \frac{a}{2}\\ &\leq -\frac{a}{4}\,.\end{aligned}$$

In particular, if Q is affine and touches $u - P_1$ from below at a point $x \in B_\delta(x_0)$, then

$$\nabla Q(x) = -\nabla P_1(x)\,. \tag{2}$$

Now, consider the convex envelope of $u - P_1$ on $B_\delta(x_0)$, denoted by Γ. By (2), there is a unique supporting plane at each point of the set $\{x; u - P_1 = \Gamma\}$. In particular, Γ is differentiable at the contact points $\{x; u - P_1 = \Gamma\}$, and

$$\nabla\Gamma = -\nabla P_1 \quad \text{on} \quad \{x; u - P_1 = \Gamma\}\,.$$

Since Γ is convex, for all $x, x' \in \{u - P_1 = \Gamma\}$,

$$\langle \nabla\Gamma(x) - \nabla\Gamma(x'), x - x' \rangle \geq 0\,.$$

Then,

$$\langle \nabla P_1(x) - \nabla P_1(x'), x - x' \rangle \leq 0\,. \tag{3}$$

On the other hand, since $F(0, x) = 0$,

$$\mathcal{M}^+\Big(D^2 P_1, \frac{\lambda}{n}, \Lambda\Big) \geq g^+\big(x_0, u(x_0)\big) + \frac{3a}{4} > 0\,.$$

This implies the existence of a coordinate system in which P_1 is strictly convex with respect to the first variable. In particular, it is not possible to find two contact points $x, x' \in \{u - P_1 = \Gamma\}$ that differ only by the first component, since, by convexity, the scalar product in (3) would be strictly positive. We conclude that $\{x; u - P_1 = \Gamma\}$ is a closed graph in the x_1 direction and has Lebesgue measure equal to zero.

Since $u - P_1 \geq 0$ on ∂B_δ and $u - P_1(x_0) < 0$, there is an $\eta > 0$ such that

$$\nabla\Gamma\left(\{x; u - P_1 = \Gamma\}\right) \supset B_\eta(0)\,.$$

Then

$$|B_\eta(0)| \leq \int\limits_{\{u - P_1 = \Gamma\}} \det D^2\Gamma = 0\,,$$

which is a contradiction. $\square$

Using a similar argument, we can prove the subsolution version of Proposition 5. If u is a continuous subsolution of (1), then

$$F(D^2 u, x) \geq -g^-(x, u)$$

in the viscosity sense.

For future reference, let us state the following corollary,

Corollary 6. *If u is a solution of* (1)*, then*

$$-g^-(x, u) \leq F(D^2 u, x) \leq g^+(x, u) \tag{4}$$

in the viscosity sense.

Denote by $\overline{S}(\lambda, \Lambda, g)$ the set of supersolutions of the corresponding Pucci's minimal equation with right hand side equal to g, and by $\underline{S}(\lambda, \Lambda, g)$ the set of subsolutions of the Pucci's maximal equation. Put $S^* = \overline{S} \cap \underline{S}$. The following properties are straightforward applications of Pucci's extremal operator theory; see [3], chapter 2.2.

Corollary 7. a) *If u is a solution of* (1), *then*

$$u \in \overline{S}\Big(\frac{\lambda}{n}, \Lambda, g^+\Big) \cap \underline{S}\Big(\frac{\lambda}{n}, \Lambda, -g^-\Big) \subset S^*\Big(\frac{\lambda}{n}, \Lambda, |g|\Big).$$

b) Alexandroff-Bakelman-Pucci estimate:
Let u be a solution of (1) *in $\Omega = B_d$ (a ball of radius d). If $u \geq 0$ on ∂B_d, then*

$$\sup_{B_d} u^- \leq Cd\bigg(\int_{B_d \cap \{u=\Gamma_u\}} (g^+)^n\bigg)^{1/n},$$

where C is a universal constant and Γ_u denotes the convex envelope on B_{2d} of the function equal to $-u^-$ in B_d and zero on $B_{2d} \setminus B_d$.

c) Harnack Inequality:
Let $\Omega = Q_1$ be the cube $\{\max |x_i| < 1/2\}$. Denote by $Q_{1/2}$ the concentric cube with sides half as long. Suppose the function $\tilde{g} : x \to g(x, u(x))$ is bounded. Then, there is a universal constant C such that for all solutions u in Q_1, $u \geq 0$, we have

$$\sup_{Q_{1/2}} u \leq C\left(\inf_{Q_{1/2}} u + \|\tilde{g}\|_{L^n(Q_1)}\right).$$

d) C^α regularity:
If u is a solution of (1) *in $\Omega = Q_1$, we have:*
i) *For some universal constant $0 < \mu < 1$,*

$$\operatorname*{Osc}_{Q_{1/2}} u \leq \mu \operatorname*{Osc}_{Q_1} u + 2\|\tilde{g}\|_{L^n(Q_1)}.$$

ii) *There exist universal constants $0 < \alpha < 1$ and $c > 0$ such that $u \in C^\alpha(\overline{Q}_{1/2})$ and*

$$\|u\|_{C^\alpha(\overline{Q}_{1/2})} \leq C\Big(\|u\|_{L^\infty(Q_1)} + \|\tilde{g}\|_{L^n(Q_1)}\Big).$$

Let us briefly discuss $W^{2,p}$ regularity. To this end, we need the notion of L^p-viscosity solution, introduced by Caffarelli, Crandall, Kocan and Śviech in [4].

Remark. Thanks to a priori estimates found by Caffarelli [2] (extended later by Escauriaza [9]), leading in particular to a generalized maximum principle for strong solutions, (4) also holds in the sense of L^p-viscosity, for all $p \geq n$; see Proposition 2.9 in [4].

Corollary 8. *If F is concave and independent of x, and if there exists a constant K such that, for all symmetric matrices A and all $0 \leq \delta \leq 1$,*

$$|F(\delta A)| \leq K|F(A)|,$$

then a solution u of (1) *is in $W^{2,p}_{loc}$, and* (4) *holds for a.e. $x \in \Omega$.*

Proof. As remarked above, (4) is also verified in the sense of L^p-viscosity ($p \geq n$). By Theorem 3.6 in [4], pointwise a.e., u is twice sub- and superdifferentiable. By the generalized Rademacher-Stepanov theorem, u is twice differentiable a.e.; see [6] and [13]. By Proposition 3.4 in [4], (4) holds a.e. In particular, $F(D^2u) \in L^p$. By the existence and uniqueness result of Corollary 3.10 in [4], u is an L^p-strong solution and $u \in W^{2,p}_{\mathrm{loc}}$ for all $p \geq n$. ∎

By Corollary 8 and direct estimates of the Green function, viscosity solutions of

$$\Delta u = cu \quad \text{in} \quad \{|\nabla u| \neq 0\}\,, \tag{5}$$

where c is a positive constant, are in $C^{1,\alpha}$. To improve this result, namely, to prove $u \in C^{1,1}$, we use the monotonicity lemma of Alt, Caffarelli and Friedman [1] in a way already exploited in [5] (see the proof of Theorem I in that paper).

Lemma 9. *If u is a viscosity solution of* (5)*, then $u \in C^{1,1}$.*

Proof. We obtain a Lipschitz constant for ∇u if the second partial derivatives of u are uniformly bounded in $B_{r_0/4}(x_0) \cap \{|\nabla u| > 0\}$ for all $x_0 \in \partial\{|\nabla u| > 0\} \cap \Omega$ and $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega$. For this, it is enough to show the existence of a constant C such that

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \leq Cr^2 \qquad \forall r \leq r_0\,. \tag{6}$$

In fact, for all $x \in B_{r_0/4}(x_0) \cap \{|\nabla u| > 0\}$, putting $r_x = \operatorname{dist}(x, \partial\{|\nabla u| > 0\})$, we have then

$$|u(x) - u(x_0)| \leq C{r_x}^2\,.$$

Define

$$v(y) = \frac{u(x + r_x y) - u(x_0)}{{r_x}^2} \qquad \forall y \in B_1(0)\,,$$

Note that v is bounded on the unit ball and satisfies $\Delta v(y) = \Delta u(x + r_x y)$. By elliptic estimates, $D_{i,j}v(0) = D_{i,j}u(x)$ are uniformly bounded. $\square$

We will prove (6) for the sequence $r_i = 2^{-i}r_0$ (which is enough). The proof will be done in two steps.

1^{st} *Step.* Define

$$M_i = \sup_{x \in B_{r_i}(x_0)} |u(x) - u(x_0)|\,.$$

We can assume that there is a sequence $i_j \to \infty$ such that

$$4M_{i_j+1} \geq M_{i_j}$$

(if there is not such a sequence, (6) is proved). Suppose (6) already fails for the sequence i_j defined above. Taking a subsequence if necessary, we can always assume that

$$M_{i_j} \geq j2^{-2i_j}\,. \tag{7}$$

Define

$$u_j(x) = \frac{u(x_0 + 2^{-i_j}x) - u(x_0)}{M_{i_j+1}} \qquad \forall x \in B_1 = B_1(0)\,.$$

Then

i) $\|\Delta u_j\|_{\infty,B_1} \leq \frac{CM_0 2^{-2i_j}}{M_{i_j+1}} \leq \frac{CM_0 M_{i_j}}{jM_{i_j+1}} \leq \frac{4CM_0}{j} \to 0\,,$

ii) $\sup_{x\in B_{1/2}} |u_j(x)| = 1\,,$

iii) $\|u_j\|_{\infty,B_1} \leq \frac{M_{i_j}}{M_{i_j+1}} \leq 4\,,$ and

iv) $u_j(0) = |\nabla u_j(0)| = 0\,.$

Then, there is a subsequence of u_j converging in $C^{1,\alpha}(B_1)$ to a nonzero harmonic function u_0 satisfying $u_0(0) = |\nabla u_0(0)| = 0$. This follows from the compactness and regularity properties of harmonic functions (Gilbarg and Trudinger [10], Theorem 8.32), coupled with the uniform regularity of the u_j's, and the construction of the correcting term w (in (8.82) in [10]) that shows that the limit is harmonic.

Now, fix a unit vector $\nu \in S^{n-1}$ and denote by $u_{j,\nu}$ the directional derivative of u_j in the direction ν. By the monotonicity lemma ([1]), since ${u_\nu}^+$ and ${u_\nu}^-$ are subharmonic on $B_{r_0}(x_0)$ and $u_\nu(x_0) = 0$,

$$\frac{1}{r^{2n}} \int\limits_{B_r(x_0)} |\nabla {u_\nu}^+|^2 \int\limits_{B_r(x_0)} |\nabla {u_\nu}^-|^2 \leq C,$$

where C depends only on the $W^{2,2}$ norm of u on $B_{r_0}(x_0)$. By a change of variable (treating C below as a generic constant that may change from line to line),

$$\int\limits_{B_1} |\nabla {u_{j,\nu}}^+|^2 \int\limits_{B_1} |\nabla {u_{j,\nu}}^-|^2 \leq C \left(\frac{2^{-2i_j}}{M_{i_j+1}}\right)^4 \leq C \left(\frac{2^{-2i_j}}{M_{i_j}}\right)^4 .$$

By Poincaré's inequality and (7),

$$\int\limits_{B_1} |{u_{j,\nu}}^+ - m_j^+|^2 \int\limits_{B_1} |{u_{j,\nu}}^- - m_j^-|^2 \leq Cj^-4$$

where $m_j^\pm$ are the meanvalues of ${u_{j,\nu}}^\pm$ respectively. Letting $j \to \infty$, we get

$$\int\limits_{B_1} |{u_{0,\nu}}^+ - m_0^+|^2 \int\limits_{B_1} |{u_{0,\nu}}^- - m_0^-|^2 = 0$$

Then, either ${u_{0,\nu}}^+$ or ${u_{0,\nu}}^-$ vanishes identically. Since $u_{0,\nu}$ is harmonic, $u_{0,\nu}(0) = 0$ and $u_{0,\nu}$ does not change sign, $u_{0,\nu}$ vanishes identically. Since ν is arbitrary, $u_0 \equiv u_0(0) = 0$, which is a contradiction.

Then, there is a constant C such that

$$M_i \leq C2^{-2i} \quad \forall i \in I = \{i \in \mathbb{N}; 4M_{i+1} \geq M_i\}\,. \tag{8}$$

2^{nd} *Step.* Now, suppose that there exists an integer $i > \min I$ such that

$$M_i > 4C2^{-2i}\,.$$

For the first i with the above property, we must have

$$M_{i-1} \leq 4C2^{-2(i-1)} = 16C2^{-2i} \leq 4M_i\,.$$

Then, $i-1 \in I$ (I was defined in (8)). By (8),

$$M_i \leq M_{i-1} \leq C2^{-2(i-1)} = 4C2^{-2i}\,.$$

This contradicts our assumption. $\square$

Remark. One can show that C depends (linearly) only on $\sup_{B_{r_0}(x_0)} |u(x) - u(x_0)|$, but for simplicity and because in this paper we only need the regularity of u and not the estimates for the $C^{1,1}$ norm, we decided to prove (6) for a given u.

3. Hausdorff measure of the free boundary

We start by establishing two technical lemmas needed to prove the finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary (Proposition 13).

Lemma 10. *Let $u \geq 0$ be a viscosity supersolution of*

$$\mathcal{M}^-(D^2u, \lambda, \Lambda) = cu(x)\,.$$

Then $u > 0$ or $u \equiv 0$.

Proof. Fix a point x_0 such that $u(x_0) > 0$ and put

$$v(r) = \inf_{x \in B_r(x_0)} u(x)\,.$$

Since $\mathcal{M}^-(D^2v, \lambda, \Lambda) \leq cv$ and v_r is negative,

$$\lambda v_{rr} + (n-1)\Lambda \frac{1}{r} v_r \leq cv(r)$$

in the viscosity sense. In particular, since $v(0) > 0$, v cannot vanish for $r < r_0 = \mathrm{dist}(x_0, \partial\Omega)$. ☐

Lemma 11. *Let u be a viscosity subsolution of*

$$\mathcal{M}^+(D^2u) = c \text{ in } \{|\nabla u| \neq 0\}, \tag{9}$$

where $c > 0$ is a constant. Let $x_0 \in \Omega$ and assume there is a paraboloid P, touching u from above at x_0, such that $|\nabla P(x_0)| \neq 0$. Then,

$$\sup_{x \in B_r(x_0)} u(x) \geq u(x_0) + \frac{c}{2n\Lambda} r^2\,, \text{ for all } r < \mathrm{dist}(x_0, \Omega^c)\,. \tag{10}$$

Proof. Fix $a < c$ and put

$$v(x) = u(x) - u(x_0) - \frac{a}{2n\Lambda}|x - x_0|^2\,.$$

The supremum of v is attained on ∂B_r. In fact, if $x_1 \in B_r$ and

$$v(x_1) = \sup_{x \in B_r} v(x)\,,$$

then the paraboloid

$$Q(x) = v(x_1) + u(x_0) + \frac{a}{2n\Lambda}|x - x_0|^2$$

touches u from above at x_1.

If $x_1 \neq x_0$, then $|\nabla Q(x_1)| \neq 0$. By (9),

$$a = \mathcal{M}^+\Big(\frac{a}{n\Lambda} I, \lambda, \Lambda\Big) \geq c\,,$$

which, by the choice of a, is a contradiction.

If $x_1 = x_0$, Q touches u from above at x_0. We cannot conclude directly, because $|\nabla Q(x_0)| = 0$. But, by hypothesis, there is another paraboloid P touching u from above at x_0, such that $|\nabla P(x_0)| \neq 0$. Using these two paraboloids, it is easy to construct a third paraboloid P_1, touching u from above at x_0, satisfying $|\nabla P_1(x_0)| \neq 0$ and such that

$$\mathcal{M}^+(D^2 P_1, \lambda, \Lambda) \leq a\,,$$

which is a contradiction again.

In particular,

$$v(x_1) = u(x_1) - u(x_0) - \frac{ar^2}{2n\Lambda} \geq 0\,.$$

Since $a < c$ is arbitrary, the lemma is proved. □

Remark. By approximation, (10) is true for all x_0 in the closure of the set of points for which there is a paraboloid P, touching u from above at x_0, such that $|\nabla P(x_0)| \neq 0$.

Corollary 12. *Suppose $u \in C^{1,1}(\Omega)$ is a viscosity subsolution of* (9). *There exist two positive constants ε_0 and c_0, depending on c, Λ, n and the Lipschitz constant of ∇u (denoted c_1), such that for all $x_0 \in \partial\{|\nabla u| > 0\} \cap \Omega$, for all $r < \operatorname{dist}(x_0, \Omega^c)$,*

$$\left|\left\{|\nabla u| > \varepsilon_0 r\right\} \cap B_r(x_0)\right| \geq c_0 r^n\,.$$

Proof. Assume, without loss of generality, that $u(x_0) = 0$. For all $x, x' \in B_r(x_0)$,

$$|u(x) - u(x')| \leq c_1 r|x - x'|\,.$$

Let x_1 be some point in ∂B_r where $\sup_{B_r} u$ is attained. Then, for all $x \in B_r(x_0)$,

$$|u(x)| \geq \frac{c}{2n\Lambda} r^2 - c_1 r|x - x_1|\,.$$

Putting $\rho = cr/6n\Lambda c_1$, the above inequalities become:

i) $|u(x)| \leq cr^2/6n\Lambda$ in $B_\rho(x_0)$.

ii) $|u(x)| \geq cr^2/3n\Lambda$ in $B_\rho(x_1) \cap B_r(x_0)$.

Denote by u_ν the directional derivative of u in the direction

$$\nu = \frac{x_1 - x_0}{|x_1 - x_0|}\,.$$

Let $x, x' \in B_r(x_0)$ be such that $x' - x = \lambda\nu$ for some positive λ. Integrating u_ν along the segment $[x, x']$, we obtain

$$u(x) - u(x') = \int_{[x,x']} u_\nu\, dx \leq \varepsilon r|x - x'| + c_1 r\left|\left\{|\nabla u| \geq r\varepsilon\right\} \cap [x, x']\right|\,.$$

Take $\varepsilon_0 = c/12n\Lambda$. Denote by $H_\nu(x_0)$ the hyperplane $x_0 + \{(x - x_0)\cdot\nu = 0\}$. If $x \in H_\nu(x_0) \cap B_\rho(x_0)$ and $x' \in B_\rho(x_1) \cap \partial B_r$, then

$$\frac{cr}{12n\Lambda} \leq c_1\left|\left\{|\nabla u| \geq r\varepsilon\right\} \cap [x, x']\right|\,.$$

The result is obtained by integrating both sides of this inequality on the disc $H_\nu(x_0) \cap B_{\rho/2}(x_0)$. □

Notation. $\mathcal{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Proposition 13. *Let $u \in C^{1,1}(\Omega)$, smooth in $\{|\nabla u| > 0\}$. Assume that in the set $\{|\nabla u| > 0\}$, u satisfies:*

$$\begin{aligned} &\text{i)} \qquad \Delta u \geq c \ \text{ and} \\ &\text{ii)} \qquad |\nabla(\Delta u)| \leq M\,, \end{aligned} \tag{11}$$

where c and M are positive constants.

Then, there is a constant h, depending on c, n and the Lipschitz constant of ∇u (denoted c_1), such that for all balls $B_r \subset \Omega$

$$H_{n-1}(\partial\{|\nabla u| > 0\} \cap B_r) \leq h r^{n-1}\,. \tag{12}$$

Proof. Denote by u_j the partial derivatives of u in the j coordinate. For $\varepsilon > 0$, define

$$u_j^\varepsilon = (u_j \wedge c_1\varepsilon) \vee (-c_1\varepsilon)\,.$$

Since $\{u_j = 0\} \cap \{|\nabla u| > 0\}$ has null measure (unless u_j vanishes identically), we have

$$\int_{\{|\nabla u|>0\}\cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon = \lim_{\zeta\to 0} \int_{D_j^\zeta \cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon,$$

where $D_j^\zeta = \{|u_j| > \zeta\}$.

Applying Green's theorem, since $\partial D_j^\zeta \cap \partial B_r$ has zero $(n-1)$-dimensional measure, we obtain

$$\int_{D_j^\zeta\cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon = -\int_{D_j^\zeta\cap B_r} \Delta u_j u_j^\varepsilon \ + \int_{\partial B_r \cap D_j^\zeta} \frac{\partial u_j}{\partial \nu} u_j^\varepsilon \ + \int_{B_r\cap\partial D_j^\zeta} \frac{\partial u_j}{\partial\nu} u_j^\varepsilon\,.$$

The last integral is negative, since

$$\frac{\partial u_j}{\partial \nu} u_j^\varepsilon = -\zeta|\nabla u_j| \quad \text{on} \quad B_r \cap \partial D_j^\zeta\,.$$

Then

$$\int_{D_j^\zeta\cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon \leq M c_1 \varepsilon r^n |B_1| + {c_1}^2 \varepsilon r^{n-1} \sigma_n\,,$$

where σ_n denotes the measure of S^{n-1} and $|B_1|$ the measure of the unit ball.

By (11), for any (economic) cover of $\partial\{|\nabla u| > 0\} \cap B_r$ by ε-balls centered on $\partial\{|\nabla u| > 0\}$, with finite overlapping, we have

$$\varepsilon^n N \leq \sum_{i=1}^N \frac{1}{c_0}\Big|\big\{|\nabla u| > \varepsilon\varepsilon_0\big\} \cap B_i\Big| \leq \frac{m}{c_0}\Big|\big\{0 < |\nabla u| < c_1\varepsilon\big\} \cap B_r\Big|\,,$$

where B_i denotes the balls in the covering, N is the number of balls and m is the maximal number of overlapping for economic covers.

Since

$$\int_{\{|\nabla u|>0\}\cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon \geq \int_{\{0<|\nabla u|<c_1\varepsilon\}\cap B_r} |\nabla u_j|^2$$

and

$$\sum_j |\nabla u_j|^2 = \|D^2 u\|_2^2 \geq \frac{1}{n}(\Delta u) \geq \frac{c^2}{n}\,,$$

we get

$$\sum_j \int_{\{|\nabla u|>0\}\cap B_r} \nabla u_j \cdot \nabla u_j^\varepsilon \geq \frac{c^2}{n}\Big|\big\{0<|\nabla u|<c_1\varepsilon\big\}\cap B_r\Big|\,.$$

All these inequalities together give

$$\varepsilon^{n-1} N \leq \frac{mn^2 c_1}{c_0 c^2}\Big(M|B_1|r + c_1\sigma_n\Big)r^{n-1}\,.$$

□

Corollary 14. *Let $u \not\equiv 0$ be a nonnegative, viscosity solution of* (5). *Then the $(n-1)$-dimensional Hausdorff measure of the free boundary ($\partial\{|\nabla u|>0\}$) is locally finite and satisfies* (12) *(locally as well).*

Proof. By Lemma 9, $u \in C^{1,1}(\Omega)$. By Proposition 5 and Lemma 10, $u > 0$ in Ω. Since u is smooth on $\{|\nabla u| > 0\}$, we can apply Proposition 13 and get (12) (h depends locally on $\inf u$ and $\sup|\nabla u|$). □

4. Convexity of the free set in a plane convex domain

Before treating the two dimensional case, we need some technical tools that work in higher dimensions as well. We start by giving a definition.

Definition. Given a ball $B \subset \mathbb{R}^n$ and a cone $V \subset B^c$, with its vertex at a point $x_0 \in \partial B$, we say that V *is non-tangential to ∂B at x_0* if the hyperplane H tangent to ∂B at x_0 does not intersect $\overline{V}\setminus\{x_0\}$.

Lemma 15. *Let $v \geq 0$ be a Lipschitz-continuous, subharmonic function. Suppose there is a ball B such that $\overline{B} \subset \{v=0\}$, and fix a second ball $B' \subset \Omega$, concentric with B.*

Denote by w the harmonic function in $B'\setminus B$ equal to 1 *on $\partial B'$ and* 0 *on ∂B. Fix $x_0 \in \partial B$ and define, for any $\delta > 0$ such that $B_\delta(x_0) \subset B'$,*

$$\alpha_\delta = \inf\big\{\alpha > 0\,;\, v \leq \alpha w \text{ in } B_\delta(x_0)\setminus B\big\}\,.$$

If

$$\alpha_0 = \inf_\delta \alpha_\delta > 0\,, \tag{14}$$

then for every cone $V \subset B^c$, non-tangential to ∂B at x_0, there is $r > 0$ such that

$$v > 0 \text{ in } V \cap B_r(x_0)\,.$$

Proof. There is a constant $a > 0$ such that for all sufficiently small $\rho > 0$,

$$w \geq a\rho \text{ on } \partial B_\rho(x_0) \cap V\,.$$

If the conclusion fails, there is a sequence $\{x_n\} \subset V$, $x_n \to x_0$, such that $v(x_n) = 0$. Let

$$r_n = |x_n - x_0| \quad\text{and}\quad \rho_n = \frac{\alpha_0 a}{2c_1} r_n\,.$$

Denoting by c_1 the Lipschitz constant of v, we have

$$v \leq \frac{\alpha_0 a}{2} r_n \quad \text{in} \quad B_{\rho_n}(x_n) \setminus B\,.$$

For any $\alpha > \alpha_0$, there is an n such that

$$v \leq \alpha w \quad \text{in} \quad B_{r_n}(x_0) \setminus B\,.$$

By the maximum principle,

$$\alpha w - v \geq \frac{\alpha_0 a}{2} r_n v_n \quad \text{in} \quad B_{\rho_n}(x_n) \setminus B\,, \tag{15}$$

where v_n denotes the harmonic function on $B_{r_n}(x_0) \setminus B$ equal to 1 on $\partial B_{r_n}(x_0) \cap B_{\rho_n}(x_n)$ and zero elsewhere on the boundary.

Let w_n be the harmonic function in $B_{r_n}(x_0) \setminus B$ equal to 1 on $\partial B_{r_n}(x_0) \setminus B$ and 0 on $\partial B\ \cap B_{r_n}(x_0)$. Since there is a constant $b > 0$ such that

$$w(x) \leq b|x - x_0|\,, \tag{16}$$

then

$$w \leq b r_n w_n \quad \text{in} \quad B_{r_n}(x_0) \setminus B\,.$$

Dividing (15) by this inequality, we obtain

$$\frac{\alpha}{\alpha_0} - \frac{v}{\alpha_0 w} \geq \frac{a}{2b}\,\frac{v_n}{w_n} \quad \text{in} \quad B_{r_n}(x_0) \setminus B\,.$$

By Harnack estimates valid till the boundary, there is a universal constant $\beta_0 > 0$ such that

$$\frac{v_n}{w_n} \geq \beta_0 \quad \text{in} \quad B_{\frac{1}{2}r_n}(x_0) \setminus B\,.$$

Then

$$v \leq \left(\alpha - \frac{a\alpha_0\beta_0}{2b}\right) w \quad \text{in} \quad B_{\frac{1}{2}r_n}(x_0) \setminus B\,.$$

Choosing, for example,

$$\alpha = \left(1 + \frac{a\beta_0}{4b}\right)\alpha_0\,,$$

we get a contradiction. □

Corollary 16. *Let $u \not\equiv 0$ be a nonnegative, bounded, viscosity solution of* (5). *Denote by u_ν the directional derivative of u in the direction $\nu \in S^{n-1}$.*

Fix $x_0 \in \partial\{|\nabla u| > 0\} \cap \Omega$ and assume that there is a ball $B \subset \{|\nabla u| = 0\}$ such that $x_0 \in \partial B$. Denote by η the outward normal vector to ∂B at x_0.

If $\langle \nu, \eta \rangle > 0$, then for any cone $V \subset B^c$, non-tangential to ∂B at x_0, there is an $r > 0$ such that

$$u_\nu > 0 \quad \textit{in} \quad V \cap B_r(x_0)\,.$$

Proof. Since u_ν^+ is subharmonic, by Lemma 15, all we need to prove is (14). By (16), this reduces to showing that there is an $\varepsilon_0 > 0$ such that, for all r small enough,

$$\sup_{B_r(x_0)} u_\nu \geq \varepsilon_0 r\,.$$

Since $\langle \eta, \nu \rangle > 0$, there is a constant $\gamma > 0$ (depending on $\langle \eta, \nu \rangle$) such that for all small r, any point $x \in B_{\gamma r}(x_0)$ can be joined to B by a segment parallel to ν, contained in $B_r(x_0)$. The length of the segment is less than r.

By Lemma 10, $u > 0$. Then, locally, u is a subsolution of (9) with right hand side equal to $c \inf u$. By Lemma 11, there is a point $x_1 \in \partial B_{\gamma r}(x_0)$ such that

$$u(x_1) - u(x_0) \geq \frac{c}{2n\Lambda} \gamma^2 r^2 \times \inf_{B_{\gamma r}(x_0)} u \,.$$

Denote by I_{x_1} the segment, parallel to ν, joining x_1 to a point on ∂B. Since $u \equiv u(x_0)$ on B, then

$$\frac{c}{2n\Lambda} \gamma^2 r^2 \times \inf_{B_{\gamma r}(x_0)} u \leq \int_{I_{x_1}} u_\nu \, dx \leq r \times \sup_{B_r(x_0)} u_\nu \,.$$

□

Corollary 17. *Let u be a non-vanishing, nonnegative, bounded, viscosity solution of* (5). *For any $\nu \in S^{n-1}$, for any compact connected component K of $\{|\nabla u| = 0\}$ such that $K^\circ \neq \emptyset$ we have*

$$K \cap \overline{\{u_\nu < 0\}} \neq \emptyset \quad \text{and} \quad K \cap \overline{\{u_\nu > 0\}} \neq \emptyset \,.$$

Proposition 18. *Let Ω be a bounded, convex domain contained in $\mathbb{R}^2$. Let u be a viscosity solution of* (5) *such that $u \in C(\overline{\Omega})$ and $u \equiv 1$ on $\partial\Omega$.*

Then, there is at most one connected component of $\{|\nabla u| = 0\}$ with nonempty interior. Besides, this component is convex.

Proof. It will be done in two steps. We first prove that any connected component of $\Omega \setminus \overline{\{|\nabla u| > 0\}}$ is convex and then we prove that there is at most one component.

Note. By Hopf's lemma, $\{|\nabla u| = 0\}$ is compact. By the maximum principle, for all $\nu \in S^{n-1}$, the sets $\{u_\nu < 0\}$ and $\{u_\nu > 0\}$ are both connected.

1^{st} *Step.* Let C be a connected component of $\Omega \setminus \overline{\{|\nabla u| > 0\}}$. We shall prove that C is convex.

Fix two points z_0 and z_1 in C and put $r_0 = \min\{\operatorname{dist}(z_0, C^c), \operatorname{dist}(z_1, C^c)\}$. Then, for all $z \in [z_0, z_1]$

$$B_{r_0}(z) \subset C \,.$$

In fact, if this fails, there exist $r' < r_0$ and $z' \in [z_0, z_1]$ such that

$$B_{r'}(z') \cap C^c \neq \emptyset \,.$$

Let $r(z)$ be any affine function on the segment $z \in [z_0, z']$, satisfying

$$r' < r(z_0) < r_0 \quad \text{and} \quad r(z') = r' \,.$$

Then, there is a first point $z'' \in [z_0, z_1]$ (going from z_0 to z_1) such that

$$\overline{B}_r(z'') \cap C^c \neq \emptyset \,,$$

where $r = r(z'')$.

Define

$$\nu = \frac{z_1 - z_0}{|z_1 - z_0|}$$

and denote by η the outward normal vector on $\partial B_r(z'')$ at a point $z_0^1 \in \overline{B}_r(z'') \cap C^c$. Note that

$$\langle \eta, \nu \rangle > 0 \, .$$

By Corollary 16, there are a cone V_0^1, with vertex at z_0^1, and a $\rho > 0$ such that $V_0^1 \cap B_\rho(z_0^1) \subset \{u_\nu > 0\}$.

In a similar way, we can find a point z_1^1 and a cone V_1^1 with vertex at z_1^1 such that

$$[z_1, z_1^1[\subset C \quad \text{and} \quad V_1^1 \cap B_{\rho'}(z_1^1) \subset \{u_\nu > 0\} \, ,$$

for some $\rho' > 0$.

Let z_0^0 and z_1^0 two points satisfying

$$[z_i, z_i^0[\, \subset C \quad \text{and} \quad z_i^0 \in \overline{\{u_\nu < 0\}}, \quad i = 0, 1 \, .$$

Join z_0 to z_1 by a piecewise affine curve Γ_0 contained in C, disjoint from $[z_i, z_i^0]$ and from $[z_i, z_i^1]$ $(i = 0, 1)$. Join z_1^1 to z_0^1 by a piecewise affine curve Γ_+ in $\{u_\nu > 0\}$.

The curve

$$\Gamma = \Gamma_0 \cup [z_1, z_1^1] \cup \Gamma_+ \cup [z_0^1, z_0]$$

is a closed Jordan curve. By the minimum principle, the interior of Γ is contained in $\{u_\nu \geq 0\}$. Then, z_0^0 and z_1^0 lie in the exterior of Γ.

On the other hand, since z_0^0 and z_1^0 are linked to Γ by the segments $[z_0, z_0^0]$ and $[z_1, z_1^0]$ respectively, which lie on opposite sides of Γ, z_0^0 and z_1^0 are in different components. This contradiction proves the claim.

Remark. A similar argument shows that $\overline{C}$ is equal to the connected component of C in $\{|\nabla u| = 0\}$.

2^{nd} *Step.* Suppose there exist a connected component K of $\{|\nabla u| = 0\}$, different from $\overline{C}$, such that for all $\nu \in S^{n-1}$, both sets $K \cap \overline{\{u_\nu < 0\}}$ and $K \cap \overline{\{u_\nu > 0\}}$ are not empty (if $K^\circ \neq \emptyset$, then, by Corollary 17, K satisfies this property).

Notation. Denote by $\nu^\perp$ the $\pi/2$ anticlockwise rotation of ν. Define the north pole of C (with respect to ν) by

$$P_N(\nu) = \{\zeta \in \partial C; \; \forall \; z \in C, \; \langle z - \zeta, \nu^\perp \rangle \leq 0\}$$

and the south pole by

$$P_S(\nu) = \{\zeta \in \partial C; \; \forall \; z \in C, \; \langle z - \zeta, \nu^\perp \rangle \geq 0\} \, .$$

Let $\{|\nabla u| = 0\}^{\circ -}$ denote the closure of the interior of $\{|\nabla u| = 0\}$. The set

$$F(\nu) = \left(\overline{\{u_\nu > 0\}} \cap \overline{\{u_\nu < 0\}} \right) \cup \{|\nabla u| = 0\}^{\circ -}$$

is connected, and $F(\nu) \backslash \overline{C}$ has exactly two components, one adherent to P_N, denoted by $F_N(\nu)$, and the other one adherent to P_S, denoted by $F_S(\nu)$.

Obviously

$$F_N(\nu) = F_S(-\nu)\,. \tag{17}$$

Since K is a connected component of $F(\nu)$, then $K \subset F_N(\nu)$ or $K \subset F_S(\nu)$. This defines a partition of S^{n-1}

$$\Theta = \{\nu \in S^{n-1};\ K \subset F_N(\nu)\}\,.$$

By (17), this partition is symmetric:

$$\nu \in \Theta \Longleftrightarrow -\nu \in S^{n-1} \setminus \Theta\,.$$

In particular, both Θ and $S^{n-1} \setminus \Theta$ are not empty. To reach a contradiction, we shall show that Θ is open (then $S^{n-1} \setminus \Theta$ is also open).

Indeed, from Corollary 16, we conclude that for any $\xi \in \partial C$ with the interior ball property, and for any cone $V \subset C^c$, non-tangential to ∂C at ξ, $\nabla u/|\nabla u|$ tends to the outward normal unit vector to ∂C at ξ, along V.

Now, fix $\nu \in S^{n-1}$. Choose two points $\xi_0, \xi_1 \in \partial C \setminus (P_N \cup P_S)$, one on each component, both having the interior ball property. Since $\langle \nu, \xi_1 - \xi_0\rangle > 0$, by the above remark, we can extend the segment $[\xi_0, \xi_1]$ from both ends, so that the new segment, say $[\xi_0', \xi_1']$, satisfies $[\xi_0', \xi_0[\subset \{u_\vartheta < 0\}$ and $]\xi_1, \xi_1'] \subset \{u_\vartheta > 0\}$ for all ϑ in a neighborhood of ν.

Join ξ_0' to the boundary of Ω by a Jordan arc Γ^- such that $\Gamma^- \subset \{u_\vartheta < 0\}$ for all ϑ in a neighborhood of ν. Do the corresponding with ξ_1' by an arc $\Gamma^+ \subset \{u_\vartheta > 0\}$, ϑ in a neighborhood of ν.

$\Gamma^- \cup [\xi_0', \xi_1'] \cup \Gamma^+$ divides Ω into two components, and K lies in one of them. $F_N(\vartheta)$ and $F_S(\vartheta)$ lie in different components and $F_N(\vartheta)$ stays in the same, for all ϑ in a neighborhood of ν. This shows that Θ is open.

By Corollary 17, there is at most one component of $\Omega \setminus \overline{\{|\nabla u| > 0\}}$. This completes the proof of the proposition. □

References

[1] H.W. Alt, L.A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), 431–461. MR **85h:**49014

[2] L.A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. Math. **130** (1989), 189-213. MR **90i:**35046

[3] L.A. Caffarelli, X. Cabré, *Fully Nonlinear Elliptic Equations*, vol. 43, AMS Colloquium Publications, Rhode Island, 1995. MR **96h:**35046

[4] L.A. Caffarelli, M.G. Crandall, M. Kocan, A. Śviech, *On viscosity solutions of fully nonlinear equations with measurable ingredients*, Comm. Pure Appl. Math. **XLIX** (1996), 365–397. MR **97a:**35051

[5] L. Caffarelli, L. Karp, H. Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. of Math. 151 (2000), 269–292. MR **2001a:**35188

[6] A. Calderon, A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Stud. Math. **20** (1961), 171-225. MR **25:**310

[7] S.J. Chapman, *A mean-field model of superconducting vortices in three dimensions*, SIAM J. App. Math. **55** (1995), 1259–1274. MR **97d:**82085

[8] C.M. Elliott, R. Schätzle, B.E.E. Stoth, *Viscosity solutions of a degenerate parabolic-elliptic system arising in the mean-field theory of superconductivity*, Arch. Rat. Mech. Anal. **145** (1998), 99-127. MR **2000j:**35112

[9] L. Escauriaza, $W^{2,n}$ *a priori estimates for solutions to fully non-linear equations*, Indiana U. Math. J. **42** (1993), 413-423. MR **94h:**35022

[10] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2^{nd} edition, Springer-Verlag, Berlin, 1983. MR **86c:**35035

[11] H. Ishii, *On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic equations*, Comm. Pure Appl. Math. **XLII** (1989), 15–45. MR **89m:**35070

[12] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985. MR **87a:**35001

[13] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970. MR **44:**7280

Department of Mathematics, University of Texas at Austin, RLM 8.100, Austin, Texas 78712-1082

E-mail address: caffarel@math.utexas.edu

CMAF–University of Lisbon, Av. Gama Pinto 2, 1649-003 Lisbon, Portugal

E-mail address: salazar@alf1.cii.fc.ul.pt