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ON THE GLAUBERMAN AND WATANABE CORRESPONDENCES FOR BLOCKS OF FINITE p-SOLVABLE GROUPS

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Abstract. If G is a finite p-solvable group for some prime p, A a solvable subgroup of the automorphism group of G of order prime to $|G|$ such that A stabilises a p-block b of G and acts trivially on a defect group P of b, then there is a Morita equivalence between the block b and its Watanabe correspondent $w(b)$ of $C_G(A)$, given by a bimodule M with vertex ΔP and an endo-permutation module as source, which on the character level induces the Glauberman correspondence (and which is an isotypy by Watanabe's results).

1. Introduction

The *Glauberman correspondence* [8] for a finite group G and a solvable group A of automorphisms of G of order prime to $|G|$ is a bijection between the set of $A-$stable ordinary irreducible characters of G and the set of ordinary irreducible characters of the centraliser $C = C_G(A)$ of A in G.

Watanabe showed in [24], that under suitable hypotheses, the Glauberman correspondence gives rise to *isotypies* (a concept due to Broué [2]) between $A-$stable blocks of G and C in characteristic prime to the order of A. What we want to show here is, that if G is $p-$solvable, the perfect isometries arising in this way are induced by Morita equivalences. In order to explain this more precisely, we introduce the following notation that we maintain throughout the paper.

We denote by $\mathcal{O}$ a complete discrete valuation ring having an algebraically closed residue field k of prime characteristic p and a quotient field $\mathcal{K}$ of characteristic zero which will always be assumed to be large enough for any of the finite groups that we consider in the sequel.

If p does not divide the order of A and b is an $A-$stable block of $\mathcal{O}G$ having a defect group P contained in C, then by [24, Prop. 1], every ordinary irreducible character of G belonging to the block b is $A-$stable, and the main result of [24] shows then that the Glauberman correspondence maps the set $\mathrm{Irr}_{\mathcal{K}}(G, b)$ onto the set $\mathrm{Irr}_{\mathcal{K}}(C, w(b))$ for a uniquely determined block $w(b)$ of $\mathcal{O}C$, which is then shown still to have P as defect group and a $p-$local structure equivalent to that of b. Together with the signs occurring in the Glauberman correspondence in [24], this yields an isotypy between the blocks b and $w(b)$.

In the light of Broué's theorem [2, 3.1] asserting that any derived equivalence between two block algebras induces a perfect isometry, and more precisely, that every splendid derived equivalence induces an isotypy, it is natural to ask whether

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there is a splendid derived equivalence between the blocks b and $w(b)$ behind this isotypy. What we show in this paper is that if G is $p-$solvable, this isotypy is actually induced by a Morita equivalence between the block algebras of b and $w(b)$ with an endo-permutation source. Following [10], [20], such a Morita equivalence induces a splendid derived equivalence whenever the occurring endo-permutation source has a fusion-stable $p-$permutation resolution. Our main result is a consequence of Puig's structure theorem [19] on blocks of $p-$solvable groups, and has independently been observed by Puig himself. In the case where the defect groups are abelian, the existence of a Morita equivalence between the block algebras of b and $w(b)$ over the residue field k has also been shown by Koshitani and Michler [13].

Theorem 1.1. *Let G be a finite $p-$solvable group, A a solvable group of automorphisms of G of order prime to $|G|p$. Set $C = C_G(A)$. Let b be an $A-$stable block of $\mathcal{O}G$ with a defect group P contained in C. Denote by $w(b)$ the block of $\mathcal{O}C$ corresponding to b through the Glauberman correspondence.*

There is a Morita equivalence between the block algebras $\mathcal{O}Gb$ and $\mathcal{O}Cw(b)$ given by an indecomposable $\mathcal{O}Gb$-$\mathcal{O}Cw(b)$-bimodule M with the following properties:

(i) When viewed as an $\mathcal{O}(G \times C)-$module, M has $\Delta P = \{(u,u)\}_{u\in P}$ as vertex and a fusion-stable endo-permutation $\mathcal{O}\Delta P-$module W as source.

(ii) The bijection between the sets of ordinary irreducible characters $\mathrm{Irr}_{\mathcal{K}}(G,b)$ and $\mathrm{Irr}_{\mathcal{K}}(C,w(b))$ induced by the Morita equivalence given by M is precisely the Glauberman correspondence.

(iii) If b is the principal block of G, then $w(b)$ is the principal block of C and $\mathcal{O}Gb = \mathcal{O}Cw(b)$.

"Fusion-stable" means that for a suitable choice of a maximal $b-$Brauer pair (P, e_P) we have $\mathrm{Res}^P_R(W) \cong \mathrm{Res}_\varphi(W)$ for any subgroup R of P and any group homomorphism $\varphi : R \to P$ for which there is $x \in G$ satisfying $\varphi(r) = xrx^{-1}$ for all $r \in R$ and ${}^x(R, e_R) \subset (P, e_P)$, where e_R is the unique block of $kC_G(R)$ such that $(R, e_R) \subset (P, e_P)$. We freely use some standard results and concepts such as the Brauer correspondence for blocks of finite groups, relative traces (introduced by J. A. Green), the Brauer homomorphism, induction of interior algbras (due to L. Puig) and the inclusion of Brauer pairs (due to Alperin and Broué) which can, for instance, be found in J. Thévenaz' book [23]. Recall that an interior $P-$algebra, where P is a finite group, is an $\mathcal{O}-$algebra A endowed with a group homomorphism $\sigma : P \longrightarrow A^\times$. The opposite algebra A^0 of A is then considered as an interior $P-$algebra via the group homomorphism sending $u \in P$ to $\sigma(u^{-1})$.

Remark 1.2. It will be apparent from the proof of Theorem 1.1 that it suffices to require k to be perfect and large enough.

2. Quoted results on character correspondences

We recall in this section the definition and main properties of the Glauberman correspondence, as well as Watanabe's results in [24].

Theorem 2.1 (Glauberman [8], [11])**.** *For every pair (G, A) consisting of a finite group G and a finite solvable group A acting on G such that G and A have coprime orders, there is a bijection $\pi(G, A) : \mathrm{Irr}_{\mathcal{K}}(G)^A \longrightarrow \mathrm{Irr}_{\mathcal{K}}(C_G(A))$ satisfying, for any such pair (G, A), the following properties:*

(i) For any normal subgroup B of A, the bijection $\pi(G,B)$ maps $\mathrm{Irr}_{\mathcal{K}}(G)^A$ to $\mathrm{Irr}_{\mathcal{K}}(C_G(B))^A$, and we have $\pi(G,A) = \pi(C_G(B), A/B) \circ \pi(G,B)$ on $\mathrm{Irr}_{\mathcal{K}}(G)^A$.

(ii) If A is a q-group for some prime q, for any $\chi \in \mathrm{Irr}_{\mathcal{K}}(G)^A$ the corresponding irreducible character $\pi(G,A)(\chi)$ of $C_G(A)$ is the unique irreducible constituent of $\mathrm{Res}^G_{C_G(A)}(\chi)$ occurring with a multiplicity prime to q.

Since A is assumed to be solvable, the composition factors of A are cyclic of prime order. The strategy of the proof of Theorem 2.1 is to first show (ii) and then proceed inductively along a composition series of A (cf. [11, 13.3]). For any $\chi \in \mathrm{Irr}_{\mathcal{K}}(G)^A$, there is a unique character $\hat{\chi}$ to $G \rtimes A$ extending χ such that $A \subset \ker(\det(\hat{\chi}))$, called the *canonical extension of* χ. If A is cyclic, its value $\hat{\chi}(a)$ at any generator a of A does not depend on the choice of a, and this value is plus or minus the degree of the Glauberman correspondent of χ. In fact, more precisely, we have $\hat{\chi}(ac) = \epsilon_\chi \pi(G,A)(\chi)(c)$ for any generator a of A, any $c \in C_G(A)$ and a sign ϵ_χ depending only on χ (cf. [11, 13.6]).

If in the situation of Theorem 2.1, $C_G(A)$ contains a defect group of an $A-$stable block b of $\mathcal{O}G$, Watanabe showed in [24], that the Glauberman correspondence induces an isotypy between b and some block $w(b)$ of $\mathcal{O}C_G(A)$:

Theorem 2.2 (Watanabe [24, Theorem 1, Proposition 5])**.** *Let G be a finite group, A a solvable subgroup of $\mathrm{Aut}(G)$ of order prime to $|G|$ and set $C = C_G(A)$. Let b be an $A-$stable block of $\mathcal{O}G$ having a defect group P contained in C.*

(i) The group A acts trivially on $\mathrm{Irr}_{\mathcal{K}}(G,b)$, and there is a unique block $w(b)$ of $\mathcal{O}C$ such that the Glauberman correspondence $\pi(G,A)$ maps $\mathrm{Irr}_{\mathcal{K}}(G,b)$ onto $\mathrm{Irr}_{\mathcal{K}}(C,w(b))$.

(ii) The block $w(b)$ has again P as defect group, and the Brauer categories of b, $w(b)$ are equivalent.

(iii) There is an isotypy $\mathbb{Z}\mathrm{Irr}_{\mathcal{K}}(G,b) \cong \mathbb{Z}\mathrm{Irr}_{\mathcal{K}}(C,w(b))$ mapping $\chi \in \mathrm{Irr}_{\mathcal{K}}(G,b)$ to $\epsilon_\chi \pi(G,A)(\chi)$.

Our approach will yield an alternative proof of Watanabe's theorem in the case where G is $p-$solvable. We require the following induction and restriction theorem for the Glauberman correspondence, due to Isaacs and Navarro [12]. Since the results in [12] are in fact more general than what we need here, we include proofs for the convenience of the reader.

Theorem 2.3 ([12, Theorem A(a), Theorem (3.1)])**.** *Let G be a finite group, A a solvable subgroup of $\mathrm{Aut}(G)$ of order prime to $|G|$, and let H be an $A-$stable subgroup of G. Set $C = C_G(A)$. Let $\psi \in \mathrm{Irr}_{\mathcal{K}}(H)^A$ and $\chi \in \mathrm{Irr}_{\mathcal{K}}(G)^A$.*

(i) If $\mathrm{Res}^G_H(\chi)$ is irreducible, then $\pi(H,A)(\mathrm{Res}^G_H(\chi)) = \mathrm{Res}^C_{C\cap H}(\pi(G,A)(\chi))$.

(ii) If $\mathrm{Ind}^G_H(\psi)$ is irreducible, then $\pi(G,A)(\mathrm{Ind}^G_H(\psi)) = \mathrm{Ind}^C_{C\cap H}(\pi(H,A)(\psi))$.

Proof. If ψ is $A-$stable, so is $\mathrm{Ind}^G_H(\psi)$; similarly, if χ is $A-$stable, so is $\mathrm{Res}^G_H(\chi)$. We proceed for both statements by induction on the order of A.

(i) Assume that A is cyclic of prime order q. Set $\beta = \mathrm{Res}^G_H(\chi)$ and assume that β is irreducible. Denote by $\hat{\chi}$ and $\hat{\beta}$ the canonical extensions of χ and β to $G \rtimes A$ and $H \rtimes A$, respectively. Then $\mathrm{Res}^{G\rtimes A}_{H\rtimes A}(\hat{\chi}) = \hat{\beta}$. By [11, 13.6] there are signs $\epsilon, \epsilon' \in \{+1,-1\}$ such that, for any generator $a \in A$, we have $\hat{\chi}(ac) = \epsilon\pi(G,A)(\chi)(c)$ for all $c \in C$ and $\hat{\beta}(ad) = \epsilon'\pi(H,A)(\beta)(d)$ for all $d \in C \cap H$. For $d = c = 1$ we get $\epsilon = \epsilon'$, which implies (i) in this case. In general, since A is solvable, there is a normal subgroup B of A such that A/B is cyclic of prime order. Set $D = C_G(B)$. If χ is

$A-$stable, it is also $B-$stable. By induction, $\pi(H,B)(\beta) = \mathrm{Res}^D_{D\cap H}(\pi(G,B)(\chi))$. Moreover, $\pi(H,B)(\beta)$ and $\pi(G,B)(\chi)$ are $A/B-$stable. Applying $\pi(D\cap H, A/B)$ to the previous equality yields the statement.

(ii) Assume again that A is cyclic of prime order q. Set $\alpha = \mathrm{Ind}^G_H(\psi)$ and assume that α is irreducible. Denote by $\hat{\alpha}$ and $\hat{\psi}$ the canonical extensions of α and ψ to $G \rtimes A$ and $H \rtimes A$, respectively. By [11, 13.6] there are signs $\delta, \epsilon \in \{+1, -1\}$ such that, for any generator a of A, we have $\hat{\alpha}(ca) = \delta\pi(G,A)(\alpha)(c)$ for all $c \in C$, and $\hat{\psi}(da) = \epsilon\pi(H,A)(\psi)(d)$ for all $d \in C \cap H$. Moreover, since $\mathrm{Res}^{H\rtimes A}_H(\hat{\psi}) = \psi$, applying Mackey's formula yields

$$\mathrm{Res}^{G\rtimes A}_G(\mathrm{Ind}^{G\rtimes A}_{H\rtimes A}(\hat{\psi})) = \mathrm{Ind}^G_H(\mathrm{Res}^{H\rtimes A}_H(\hat{\psi})) = \alpha \ ,$$

which shows that $\mathrm{Ind}^{G\rtimes A}_{H\rtimes A}(\hat{\psi})$ is also an irreducible character extending α. Whence $\hat{\alpha} = \lambda\mathrm{Ind}^{G\rtimes A}_{H\rtimes A}(\hat{\psi})$ for some linear character λ of A inflated to $G \rtimes A$.

Fix a generator a of A, and let $c \in C$. Let R be a set of representatives of the different cosets $y(C\cap H)$ of $C\cap H$ in C for which we have $c^y \in C\cap H$. The formula [22, 3.3, Theorem 12] to compute induced characters yields

$$\mathrm{Ind}^C_{C\cap H}(\pi(H,A)(\psi))(c) = \sum_{t\in R}\pi(H,A)(\psi)(c^t) \ .$$

The key observation at this point is that R is also a system of representatives of the different cosets $y(H \rtimes A)$ of $H \rtimes A$ in $G \rtimes A$ for which we have $(ca)^y \in H \rtimes A$. Indeed, on one hand, different elements in R are easily seen to represent different cosets of $H \rtimes A$. On the other hand, if $(ca)^y \in H \rtimes A$, then actually $(ca)^y \in Ha$, and whence, by Glauberman's Lemma [11, 13.8], there is an $h \in H$ such that $(ca)^{yh} = da$ for some $d \in C\cap H$. As the order q of a is prime to the orders of c and d, we have $c^{yh} = d$ and $a^{yh} = a$, thus $yh \in C$. This shows that indeed R represents the cosets $y(H \rtimes A)$ such that $(ca)^y \in H \rtimes A$. Therefore, again by the formula [22, 3.3, Theorem 12], we have

$$\mathrm{Ind}^{G\rtimes A}_{H\rtimes A}(\hat{\psi})(ca) = \sum_{t\in R}\hat{\psi}(c^t a) = \epsilon\mathrm{Ind}^C_{C\cap H}(\pi(H,A)(\psi))(c) \ .$$

Whence $\pi(G,A)(\alpha)(c) = \delta\lambda(a)\epsilon\mathrm{Ind}^C_{C\cap H}(\pi(H,A)(\psi))(c)$. This holds for any $c \in C$, and thus, by the linear independence of irreducible characters, we have $\delta\lambda(a)\epsilon = 1$. This proves the theorem in the case where A is cyclic of prime order.

In the general case, as A is solvable, there is a normal subgroup B of A of index q for some prime q. Set $D = C_G(B)$. Then $D\cap H$ is $A-$invariant. By induction, we have $\pi(G,B)(\alpha) = \mathrm{Ind}^D_{D\cap H}(\pi(H,B)(\psi))$. Moreover, $\pi(G,B)(\alpha)$ and $\pi(H,B)(\psi)$ are $A/B-$stable. Applying $\pi(D, A/B)$ to the previous equality concludes the proof. □

Alperin showed in [1] that the Glauberman correspondence can be interpreted in terms of the Brauer correspondence, if the group A is a $q-$group for some prime q. We restate this in the following form, adapted to our needs in Section 4 below:

Proposition 2.4 (cf. [1])**.** *Let G be a finite group and A a group of prime order q acting on G. Suppose that q does not divide $|G|$. Let $\mathcal{O}'$ be a subring of $\mathcal{K}$ containing a primitive $|G|$-th root of unity in which $|G|$ is invertible and let $\mathcal{O}' \to k'$*

be a surjective ring homomorphism from $\mathcal{O}'$ onto a field k' of characteristic q. Let χ be an $A-$stable irreducible character of G with values in $\mathcal{O}'$. Denote by e and f the images in $k'G$ and $k'C_G(A)$ of the central primitive idempotents $e(\chi)$ and $e(\pi(G,A)(\chi))$. We have $Br_A(e) = f$.

Proof. Since $|G|$ is prime to q and χ is absolutely irreducible, the algebra $k'Ge$ is a matrix algebra over k'. The action of A on G induces an action on $k'Ge$. As $k'Ge$ is a direct summand of $k'G$ and $char(k') = q = |A|$, it follows that $k'Ge$ has an $A-$stable $k'-$basis. Thus [4] (or [23, (28.7)]) implies that $k'C_G(A)Br_A(e)$ is a matrix algebra over k'. Thus $Br_A(e)$ determines an absolutely irreducible character η of $C_G(A)$. The argument in [1] shows that η occurs with multiplicity prime to q in the restriction to $C_G(A)$ of χ. Thus η is the Glauberman correspondent of χ by 2.1(ii), which implies the result. □

The next result is known as the Fourier inversion formula:

Proposition 2.5 (cf. [22, 6.2, Prop. 11])**.** *Let G be a finite group. Assume that $\mathcal{K}$ contains a primitive $|G|$-th root of unity. Let $\chi \in \mathrm{Irr}_{\mathcal{K}}(G)$ and let $e(\chi)$ be its associated central primitive idempotent in $\mathcal{K}G$. For any $s \in \mathcal{K}Ge(\chi)$ we have*

$$s = \frac{\chi(1)}{|G|}\sum_{x\in G}\chi(x^{-1}s)x.$$

Proof. It suffices to check this for an element s of the form $s = ye(\chi)$ for some $y \in G$. Since $e(\chi)$ is the central primitive idempotent associated with χ, we have $\chi(x^{-1}ye(\chi)) = \chi(x^{-1}y)$. Thus $\sum_{x\in G}\chi(x^{-1}ye(\chi))x = \sum_{x\in G}\chi(x^{-1}y)xy^{-1}y = \frac{|G|}{\chi(1)}e(\chi)y$, from which the result follows. □

If H is a subgroup of a finite group G, we say that a function $\lambda : H \to \mathcal{O}$ is *G-stable*, if $\lambda(h) = \lambda(h')$ for any two elements h, h' in H which are conjugate in G. We conclude this section with the following statement on the extension of stable linear characters.

Proposition 2.6. *Let G be a finite group and let P be a Sylow p-subgroup of G. Let $\lambda : P \to \mathcal{O}^\times$ be a $G-$stable linear character of P. Then λ extends to a linear character of G.*

Proof. By [9, Ch. 7, Theorem 3.4], the intersection $P \cap [G,G]$ of P with the commutator subgroup of G is generated by the set of commutators $xux^{-1}u^{-1}$, where $u \in P$ and $x \in G$ such that $xux^{-1} \in P$. Since λ is $G-$stable, we have $P \cap [G,G] \subseteq \ker(\lambda)$. Thus λ induces a linear character of the Sylow p-subgroup $P/(P\cap[G,G])$ of the abelian group $G/[G,G]$, which extends therefore to a character of $G/[G,G]$, and its inflation to G in turn extends λ. □

3. Quoted results on $\mathcal{O}^\times-$groups

As in [10], we follow the terminology of [18, section 5]. An $\mathcal{O}^\times-$*group* is a group $\hat{G}$ endowed with a group monomorphism $\mathcal{O}^\times \longrightarrow \hat{G}$ whose image lies in $Z(\hat{G})$; we usually denote by $\hat{\lambda}$ the image of $\lambda \in \mathcal{O}^\times$ in $\hat{G}$, if no confusion arises, and by $\check{G}$ the *opposite* $\mathcal{O}^\times-$group, which as an abstract group is equal to $\hat{G}$, but endowed with the homomorphism $\mathcal{O}^\times \longrightarrow \check{G}$ sending $\lambda \in \mathcal{O}^\times$ to $\check{\lambda} = (\hat{\lambda})^{-1}$.

In our context, $\mathcal{O}^\times-$groups arise from the action of a finite group G on a matrix algebra S over $\mathcal{O}$ by algebra automorphisms (we write ${}^xs$ for the action of $x \in G$

on $s \in S$): by the Skolem-Noether Theorem, any automorphism of S is inner, and hence, for any $x \in G$ there is an $s \in S$ such that ${}^x t = sts^{-1}$ for all $t \in S$. The group $\hat{G}$ of all such pairs $(x, s) \in G \times S^\times$ satisfying ${}^x t = sts^{-1}$ for all $t \in S$ becomes an $\mathcal{O}^\times-$group with the group homomorphism sending $\lambda \in \mathcal{O}^\times$ to $\hat{\lambda} = (1_G, \lambda 1_S)$. We call $\hat{G}$ the $\mathcal{O}^\times-$*group defined by the action of* G *on* S; clearly $\hat{G}$ is a central $\mathcal{O}^\times-$extension of G, since there is an obvious short exact sequence of groups

$$1 \longrightarrow \mathcal{O}^\times \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1 \ .$$

The *twisted group algebra* $\mathcal{O}_*\hat{G}$ associated with the $\mathcal{O}^\times-$group $\hat{G}$ is the quotient of the group algebra $\mathcal{O}\hat{G}$ by the ideal generated by the set of elements $\lambda.g - 1_{\mathcal{O}}.(\hat{\lambda}g)$, where λ runs over $\mathcal{O}^\times$ and g runs over $\hat{G}$ (cf. [18, 5.12] or [23, 10.4]). If for any $x \in G$ we choose some inverse image $\hat{x}$ of x in $\hat{G}$, the image of the set $\{\hat{x}\}_{x \in G}$ in $\mathcal{O}_*\hat{G}$ is an $\mathcal{O}-$basis; in particular, the algebra $\mathcal{O}_*\hat{G}$ is $\mathcal{O}-$free of rank $|G|$. Moreover, for any two $x, y \in G$, there is a unique $\lambda_{x,y} \in \mathcal{O}^\times$ such that $\hat{x}\hat{y} = \hat{\lambda}_{x,y}\widehat{xy}$, because both $\hat{x}\hat{y}$ and $\widehat{xy}$ are inverse images of xy in $\hat{G}$. The map $\lambda : G \times G \longrightarrow \mathcal{O}^\times$ sending (x, y) to $\lambda_{x,y}$ is a $2-$cocycle which determines $\hat{G}$ as an $\mathcal{O}^\times-$group. While λ itself is not uniquely determined by $\hat{G}$ (it obviously depends on the choice of the $\hat{x}$) its class $\overline{\lambda} \in H^2(G, \mathcal{O}^\times)$ is uniquely determined by $\hat{G}$ (see, e.g., [25, §6.6] for standard properties relating central group extensions of a finite group to its second cohomology group). We use the analogous terminology for $k^\times$ instead of $\mathcal{O}^\times$. Since k is perfect, by [21, Ch. II, Section 4, Prop. 8],

3.1. *there is a canonical group isomorphism* $\mathcal{O}^\times \cong (1 + J(\mathcal{O})) \times k^\times$ *compatible with the inclusion* $1 + J(\mathcal{O}) \subset \mathcal{O}^\times$ *and the canonical surjection* $\mathcal{O}^\times \to k^\times$.

Suppose now that $S = M_n(\mathcal{O})$ for some positive integer n which is prime to p, let G be a finite group acting on S and let $\hat{G}$ be the $\mathcal{O}^\times-$group defined by the action of G on S. Then the following hold:

3.2. *For any* $x \in G$ *there is* $s_x \in S^\times$ *such that* $\det(s_x) = 1$ *and* $\hat{x} = (x, s_x) \in \hat{G}$. *The order of* s_x *in* $S^\times$ *is then finite and divides* mn, *where* m *is the order of* x. *Moreover, if there is any other* $s'_x \in S^\times$ *such that* $\det(s'_x) = 1$ *and* $(x, s'_x) \in \hat{G}$, *there is a unique* n*-th root of unity* ζ *such that* $s'_x = \zeta s_x$.

Proof. If s is some element in $S^\times$ such that $(x, s) \in \hat{G}$, then by [21, Ch. II, Section 4, Prop. 7] there is $\tau \in \mathcal{O}^\times$ such that $\tau^n = \det(s)$. It suffices to take $s_x = \tau^{-1}s$. As $(s_x)^m$ acts trivially on S, we have $(s_x)^m = \mu 1_S$ for some $\mu \in \mathcal{O}^\times$. Then $1 = \det((s_x)^m) = \mu^n$ implies $(s_x)^{mn} = 1_S$. Similarly, $s_x^{-1}s'_x = \zeta 1_S$ for a unique $\zeta \in \mathcal{O}^\times$, and whence $1 = \det(s_x^{-1}s'_x) = \zeta^n$. □

Consider now the $2-$cocycle λ defined by $\hat{x}\hat{y} = \hat{\lambda}_{x,y}\widehat{xy}$ for this particular choice of inverse images $\hat{x}$ of $x \in G$ in $\hat{G}$. Denote again by $\overline{\lambda}$ its image in $H^2(G, \mathcal{O}^\times)$.

3.3. *For any two* $x, y \in G$, *we have* $(\lambda_{x,y})^n = 1$.

In other words, the $2-$cocycle λ has values in the group μ_n of n-th roots of unity in $\mathcal{O}^\times$. This is because if we look at what the equality $\hat{x}\hat{y} = \hat{\lambda}_{x,y}\widehat{xy}$ means in the second component, we get $s_x s_y = \lambda_{x,y} s_{xy}$. As both $s_x s_y$ and s_{xy} have determinant 1, this forces $(\lambda_{x,y})^n = 1$. In terms of cohomology groups, statement 3.3 is equivalent to:

3.4. *The class $\overline{\lambda} \in H^2(G, \mathcal{O}^\times)$ is in the image of the map $H^2(G, \mu_n) \longrightarrow H^2(G, \mathcal{O}^\times)$ induced by the inclusion of the group μ_n of n-th roots of unity in $\mathcal{O}^\times$.*

The map $H^2(G, \mu_n) \to H^2(G, \mathcal{O}^\times)$ need not be injective; its kernel is the image of the connecting homomorphism $H^1(G, \mathcal{O}^\times) \to H^2(G, \mu_n)$ of the long exact cohomology sequence associated with the short exact "Kummer" sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathcal{O}^\times \xrightarrow{(-)^n} \mathcal{O}^\times \longrightarrow 1,$$

where $(-)^n$ is the group homomorphism taking an element of $\mathcal{O}^\times$ to its n-th power; this is a surjective map as k is algebraically closed and n is prime to p. Moreover, the subgroup μ_n of $\mathcal{O}^\times$ is contained in the canonical image of $k^\times$ in $\mathcal{O}^\times$. This implies that the image $\overline{\lambda}$ of the $2-$cocycle λ in $H^2(G, \mathcal{O}^\times)$ is trivial if and only if its image in $H^2(G, k^\times)$ is trivial. In other words, the $\mathcal{O}^\times-$group $\hat{G}$ is already determined by the $k^\times-$group $\tilde{G}$ obtained from the action of G on the matrix algebra $k \underset{\mathcal{O}}{\otimes} S$. Precisely:

3.5. *We have an isomorphism of $\mathcal{O}^\times-$groups $\hat{G} \cong (1 + J(\mathcal{O})) \times \tilde{G}$, where the $\mathcal{O}^\times-$group structure on the right side is given via the canonical isomorphism 3.1.*

Any central extension of a finite $p-$group P by the $p'-$group μ_n splits canonically. Thus 3.2, 3.3 imply the following statement (cf. [23, (21.5)]):

3.6. *If P is a finite $p-$group acting on S, there is a unique group homomorphism $\sigma : P \to S^\times$ lifting the action of P on S with $\mathrm{Im}(\sigma) \subset \ker(\det : S^\times \to \mathcal{O}^\times)$; in particular, the $\mathcal{O}^\times-$group $\hat{P}$ defined by the action of P on S is canonically isomorphic to the trivial extension $\mathcal{O}^\times \times P$.*

The canonical isomorphism of $\mathcal{O}^\times-$groups $\mathcal{O}^\times \times P \cong \hat{P}$ in 3.6 maps $(\lambda, u) \in \mathcal{O}^\times \times P$ to $(u, \lambda\sigma(u))$. If we choose in the situation of 3.6 a free $\mathcal{O}-$module V of rank n and an isomorphism $S \cong \mathrm{End}_{\mathcal{O}}(V)$, then V becomes an $\mathcal{O}P-$module through σ. Recall from [4] that V is called an *endo-permutation $\mathcal{O}P-$module*, if S is a permutation $\mathcal{O}P-$module; that is, if S has a $P-$stable $\mathcal{O}-$basis (with respect to the action ${}^u s = \sigma(u)s\sigma(u^{-1})$ for any $u \in P$, $s \in S$). In that case, the Brauer quotient $S(P) = S^P/(\sum_{Q<P} S_Q^P + J(\mathcal{O})S^P)$ is again a matrix algebra over k. Whence if G acts on both S and P in such a way that σ becomes a $G-$map, then the action of G on S induces an action of G on $S(P)$ which in turn defines a central $k^\times-$extension of G. Dade's theorem on the splitting of fusion in endo-permutation modules, in its version given by Puig [17], implies that:

3.7. *If S is endowed with a group homomorphism $\sigma : P \to S^\times$ for some finite $p-$group P on which G acts such that $\sigma({}^x u) = {}^x\sigma(u)$ for all $x \in G$ and $u \in P$ and such that $k \underset{\mathcal{O}}{\otimes} S$ has a $P-$stable $k-$basis, then the $k^\times-$groups defined by the actions of G on $k \underset{\mathcal{O}}{\otimes} S$ and $S(P)$ are isomorphic.*

The proof of 3.7 in [17] is far from being elementary; it goes by induction on the order of P, using Dade's classification [5] of endo-permutation modules over finite abelian $p-$groups. It is therefore worthwhile to point out, that the proof is easy in the particular case where G acts trivially on P. In that case, if we consider again for any $x \in G$ an element $\hat{x} = (x, s_x)$ belonging to the $\mathcal{O}^\times-$group $\hat{G}$ defined by the action of G on S, we have $s_x \in S^P$. Denote by $\overline{s}_x$ the image of s_x in $S(P)$ under the

Brauer homomorphism $Br_P : S^P \to S(P)$. Since Br_P is an algebra homomorphism, the equality $s_x s_y = \lambda_{x,y} s_{x,y}$ implies the equality $\overline{s}_x \overline{s}_y = \overline{\lambda}_{x,y} \overline{s}_{x,y}$, for any $x, y \in G$, where $\overline{\lambda}_{x,y}$ is the canonical image of $\lambda_{x,y}$ in k. It is only this particular case of 3.7 which is used implicitly in the proof of 4.5. In view of uniqueness issues, we recall the following obvious statement:

3.8. *For any $\mathcal{O}^\times-$automorphism α of $\hat{G}$ inducing the identity on G there is a unique linear character $\lambda : G \to \mathcal{O}^\times$ of G such that $\alpha(\hat{x}) = \lambda(x)\hat{x}$ for all $x \in G$ and any inverse image $\hat{x}$ of x in $\hat{G}$.*

If P is a p-subgroup of G, then by 3.6, the inverse image $\hat{P}$ of P in $\hat{G}$ is canonically isomorphic to $P \times \mathcal{O}^\times$, and through this isomorphism, we view $\mathcal{O}_*\hat{G}$ as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule.

3.9. *As $\mathcal{O}P$-$\mathcal{O}P$-bimodules, $\mathcal{O}_*\hat{G}$ and $\mathcal{O}G$ are isomorphic.*

Proof. Let $y \in G$ and denote by $\hat{y}$ some inverse image of y in $\hat{G}$. If $u \in P$ is such that $v = y^{-1}uy \in P$, then $\hat{y}^{-1}(u, \sigma(u))\hat{y} = (v, \sigma(v))$ by the uniqueness of σ. Thus $\mathcal{O}[P\hat{y}P] \cong \mathcal{O}[PyP]$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodules. By taking the direct sum of these isomorphisms with y running over a set of representatives of the P-P-double cosets in G one gets 3.9. □

4. The reduced case

The proof of Theorem 1.1 goes essentially in two steps, following Puig's algebra theoretic version [16] of Fong's reduction in [7]: first, we consider the "reduced" case, where the $A-$stable block b of the $p-$solvable group G is actually a block of the maximal normal subgroup $O_{p'}(G)$ of order prime to p. Second, we show in section 5, that the general case can be obtained from the "reduced" case by an induction argument.

Let G be a finite $p-$solvable group and b a $(G \rtimes A)-$stable block of $O_{p'}(G)$. By [10, 3.3], b is a block of $\mathcal{O}G$ having the Sylow p-subgroups of G as defect groups. Therefore, if C contains some Sylow p-subgroup P of G, Watanabe's Theorem 2.2 applies and determines a block $w(b)$ of $\mathcal{O}C$ by the condition $\pi(G, A)(\mathrm{Irr}_{\mathcal{K}}(G, b)) = \mathrm{Irr}_{\mathcal{K}}(C, w(b))$. But the Glauberman correspondence yields another way to produce a block of $\mathcal{O}C$: since b is a block of the $p'-$group $O_{p'}(G)$, it corresponds to a unique $(G \rtimes A)-$stable irreducible character β of $O_{p'}(G)$. Thus its Glauberman correspondent $\gamma = \pi(O_{p'}(G), A)(\beta)$ is an irreducible character of $O_{p'}(G) \cap C$. The latter group is shown to be equal to $O_{p'}(C)$, and hence γ determines a $C-$stable block e of $O_{p'}(C)$. Again by [10, 3.3], e is then also a block of $\mathcal{O}C$ having P as defect group. The next result shows that $e = w(b)$, providing an alternative proof of the statements (i), (ii) of Watanabe's Theorem in this particular case.

Theorem 4.1. *Let G be a finite $p-$solvable group, A a solvable subgroup of $\mathrm{Aut}(G)$ of order prime to $|G|$. Set $C = C_G(A)$ and suppose that C contains a Sylow p-subgroup P of G. Let b be a $G \rtimes A-$stable block of $O_{p'}(G)$. Denote by β the unique irreducible character of $O_{p'}(G)$ associated with b, set $\gamma = \pi(O_{p'}(G), A)(\beta)$, and denote by e the block of $O_{p'}(G) \cap C = O_{p'}(C)$ determined by γ.*

(i) b is a block of $\mathcal{O}G$ having P as defect group, and b is of principal type (i.e., $Br_Q(b)$ is a block of $kC_G(Q)$ for any subgroup Q of P).

(ii) e is a block of $\mathcal{O}C$ having P as defect group, and e is of principal type.

(iii) The Brauer categories of b, e, viewed as blocks of $\mathcal{O}G$, $\mathcal{O}C$, respectively, are equivalent.

(iv) We have $\pi(G,A)(\mathrm{Irr}_{\mathcal{K}}(G,b)) = \mathrm{Irr}_{\mathcal{K}}(C,e)$, or equivalently, $e = w(b)$.

(v) If b is the principal block of $\mathcal{O}G$, then e is the principal block of $\mathcal{O}C$, and the algebras $\mathcal{O}Gb$ and $\mathcal{O}Ce$ are equal.

Before we prove 4.1, we recall some well-known results on $p-$solvable groups.

Lemma 4.2 ([9])**.** *Let G be a finite $p-$solvable group.*

(i) For any Sylow p-subgroup Q of $O_{p',p}(G)$ we have $C_G(Q) \subset O_{p',p}(G)$.

(ii) For any p-subgroup R of G we have $O_{p'}(N_G(R)) = O_{p'}(C_G(R)) = O_{p'}(G) \cap C_G(R)$.

Proof. (i) is well-known (see [9, Ch. 6, Theorem 3.3]). (ii) The image of $N_G(R)$ in $G/O_{p'}(G)$ is precisely the normaliser of the image of R in $G/O_{p'}(G)$, because the orders of R and $O_{p'}(G)$ are coprime. Thus we may assume that $O_{p'}(G) = \{1\}$. Note that then $C_G(Q) = Z(Q)$ by (i). Setting $X = O_{p'}(N_G(R))$, we have to show that $X = \{1\}$. The group XR normalises Q, and $C_Q(R)$ is normal in $N_G(R)$. Thus X and $C_Q(R)$ normalise each other, and hence commute elementwise as their orders are coprime. By Thompson's $A \times B$ Lemma, ([9, Theorem 5.3.4]), X and Q commute elementwise. But $C_G(Q) = Z(Q)$, and so $X = \{1\}$, which proves (ii). $\square$

A finite group satisfying 4.2(i) is called *$p-$constrained*. Our proof shows that 4.2(ii) also holds for a p-constrained group G. It is easy to check that the proof of the following proposition also works under the weaker hypothesis that G and thus $G \rtimes A$ are $p-$constrained.

Proposition 4.3. *Let G be a finite $p-$solvable group and let A be a subgroup of $\mathrm{Aut}(G)$ of order prime to $|G|$. Set $C = C_G(A)$. Denote by Q a Sylow p-subgroup of $O_{p',p}(G)$. Suppose that $Q \subset C$. Then the following hold.*

(i) For any p-subgroup R of C we have $N_G(R) = O_{p'}(C_G(R))N_C(R)$.

(ii) $G = O_{p'}(G)N_C(Q)$.

(iii) $O_{p'}(C) = O_{p'}(G) \cap C$.

(iv) The inclusions of the subgroups $N_C(Q)$, $N_G(Q)$ and C in G induce isomorphisms $N_C(Q)/O_{p'}(N_C(Q)) \cong C/O_{p'}(C) \cong G/O_{p'}(G) \cong N_G(Q)/O_{p'}(N_G(Q))$.

Proof. Observe that all statements in the proposition hold trivially, if p does not divide the order of G. Thus we may assume that p divides the order of G, and hence p does not divide the order of A. We show now that then $O_{p'}(G \rtimes A) = O_{p'}(G)A$. Since A acts trivially on Q and $G \rtimes A$ is again $p-$solvable, by 4.2(i) we have $A \subset C_{G\rtimes A}(Q) \subset O_{p',p}(G \rtimes A)$; thus $A \subset O_{p'}(G \rtimes A)$. Therefore $O_{p'}(G \rtimes A) = (O_{p'}(G \rtimes A) \cap G)A = O_{p'}(G)A$.

(i) By 4.2(ii), applied to $G \rtimes A$, we have $O_{p'}(N_{G\rtimes A}(R)) = O_{p'}(G \rtimes A) \cap C_{G\rtimes A}(R) = (O_{p'}(G)A) \cap (C_G(R)A) = (O_{p'}(G) \cap C_G(R))A = O_{p'}(C_G(R))A$, where the last equality comes again from 4.2(ii). Both R, A are characteristic in RA since their orders are coprime. Therefore $N_{G\rtimes A}(RA) = N_{G\rtimes A}(R) \cap N_{G\rtimes A}(A) = N_{G\rtimes A}(R) \cap (CA) = N_C(R)A$. By the Schur-Zassenhaus Theorem ([9, Theorem 6.2.1]) applied to RA in $O_{p'}(C_G(R))(RA)$, together with the Frattini argument, we get $N_{G\rtimes A}(R) = O_{p'}(C_G(R))(N_C(R)A)$. Intersecting this equality with G yields (i).

(ii) By the Frattini argument, we have $G = O_{p'}(G)N_G(Q)$, and by (i), we have $N_G(Q) = O_{p'}(C_G(Q))N_C(Q)$. Since $C_G(Q) \subset O_{p',p}(G)$, it follows that $O_{p'}(C_G(Q)) \subset O_{p'}(G)$, which proves (ii).

(iii) Intersecting the equality in (ii) with C yields $C = (O_{p'}(G) \cap C)N_C(Q)$, and dividing the equality in (ii) by $O_{p'}(G)$ yields $O_{p'}(N_C(Q)) = O_{p'}(G) \cap N_C(Q)$, from which (iii) follows.

(iv) is a straightforward consequence of the previous statements. □

Proof of 4.1. (i) is a restatement of the (well-known) result [10, 3.3]. Since the argument is short, we give it for the convenience of the reader.

Set $N = O_{p',p}(G)$. Since any block of N is a linear combination of $p'-$elements of N it lies in $\mathcal{O}O_{p'}(G)$, and therefore b is a block of N having Q as a defect group. If b' is any block of G such that $b'b = b'$, then Q is contained in a defect group of b' (see Dade [3]); in particular, $Br_Q(b') \neq 0$. By the same argument, if $b - b' \neq 0$ then $Br_Q(b - b') \neq 0$. However, $Br_Q(b)$ is an $N_G(Q)-$stable block of $kN_N(Q)$, thus still a block of $kN_G(Q)$, since any block of $kN_G(Q)$ lies in $kC_G(Q) = kC_N(Q)$. Therefore $Br_Q(b') = Br_Q(b)$, which forces $b' = b$ by the previous argument, and so b is indeed a block of G. Now P acts by conjugation on $\mathcal{O}O_{p'}(G)b$, which is a matrix algebra of rank prime to p having a $P-$stable $\mathcal{O}-$basis (as it is a direct summand of $\mathcal{O}O_{p'}(G)$) and therefore P stabilises at least one element of any such $P-$stable $\mathcal{O}-$basis of $\mathcal{O}O_{p'}(G)b$. This means $Br_P(\mathcal{O}O_{p'}(G)b) \neq 0$, and hence $Br_P(b) \neq 0$. It follows that P is a defect group of b as a block of G.

In order to see that b is of principal type, let R be a subgroup of P. Then $Br_R(b)$ is a block of $N_{O_{p'}(G)R}(R) = C_{O_{p'}(G)}(R)R = O_{p'}(C_G(R))R$. Again, as $Br_R(b)$ involves only $p'-$elements, it is in fact a block of $O_{p'}(C_G(R))$ which is $C_G(R)-$stable as b is $G-$stable. Therefore, by the first statement applied to $Br_R(b)$ and $C_G(R)$ instead of b and G, respectively, $Br_R(b)$ is a block of $C_G(R)$.

(ii) We show that γ is $C-$stable. In view of 2.1(i) we proceed by induction on $|A|$, reducing the situation to the case where A is cyclic of prime order q. Then γ is $C-$stable, as it is the unique irreducible constituent of $\mathrm{Res}^{O_{p'}(G)}_{C\cap O_{p'}(G)}(\beta)$ occurring with a multiplicity prime to q. Thus the block e of $O_{p'}(C) = C \cap O_{p'}(G)$ is $C-$stable, and whence (ii) follows from (i).

(iii) is a direct consequence of Proposition 4.3(i) together with the fact that both b and e are of principal type.

(iv) Since A acts trivially on P, all irreducible characters of G belonging to b and of C belonging to e are $A-$stable (cf. [24, Prop. 1]). Let $\chi \in \mathrm{Irr}_{\mathcal{K}}(G)^A$ and set $\psi = \pi(G, A)(\chi)$. By [11, 13.29], we have $\langle \mathrm{Ind}^C_{O_{p'}(C)}(\gamma), \psi\rangle \neq 0$ if and only if $\langle \mathrm{Ind}^G_{O_{p'}(G)}(\beta), \chi\rangle \neq 0$. Using Frobenius' reciprocity, this means $\langle \gamma, \mathrm{Res}^C_{O_{p'}(C)}(\psi)\rangle \neq 0$ if and only if $\langle \beta, \mathrm{Res}^G_{O_{p'}(G)}(\chi)\rangle \neq 0$. This in turn means that χ belongs to b if and only if ψ belongs to e, since the restriction to $O_{p'}(G)$ of any irreducible character of G belonging to b is a multiple of β, while β occurs in the restriction of no other irreducible character of G; similarly for C and e. This concludes the proof of (iv).

(v) If b is the principal block of OG, then e is the principal block of $\mathcal{O}C$, and thus (v) follows from 4.3(ii). □

Using Puig's method in [16, Proposition 2] (which generalises Fong's reduction in [7]), one gets the structure of the block algebra $\mathcal{O}Gb$ in 4.1.

Theorem 4.4. *Let G be a finite group, let N be a normal p'-subgroup of G, let b be a $G-$stable block of N and set $S = \mathcal{O}Nb$. Denote by $\hat{G}$ the $\mathcal{O}^\times-$group opposite to that obtained from the action of G on S. Set $\hat{L} = \hat{G}/N$, where we identify the element y of N to its canonical image (y, yb) in $\hat{G}$. There is a unique algebra*

isomorphism

$$\mathcal{O}Gb \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}$$

mapping xb *to* $s \otimes \overline{(x,s)}$ *with inverse mapping* $t \otimes \overline{(x,s)}$ *to* $ts^{-1}x$, *where* $x \in G$, $t \in S$ *and* $s \in S^\times$ *such that* $(x,s) \in \hat{G}$, *and where* $\overline{(x,s)}$ *is the image of* (x,s) *in* $\hat{L}$.

Proof. Straightforward verification. □

With the notation of the previous theorem, let P be a Sylow p-subgroup of G. By 3.6, for any $u \in P$, there is a canonical $\sigma(u) \in S^\times$ with determinant 1 such that $(u, \sigma(u)) \in \hat{G}$. Thus S becomes an interior $P-$algebra with structural mapping sending $u \in P$ to $\sigma(u)$, and $\mathcal{O}_*\hat{L}$ becomes an interior $P-$algebra with structural homomorphism sending $u \in P$ to $\overline{(u, \sigma(u))}$. In this way, the algebra isomorphism $\mathcal{O}Gb \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}$ in 4.4 becomes an isomorphism of interior $P-$algebras. If we write $S = \mathrm{End}_{\mathcal{O}}(U)$ for some $\mathcal{O}-$module U, then U becomes an $\mathcal{O}P-$module through σ. In fact, U is then an endo-permutation $\mathcal{O}P$-module because S is a direct summand of $\mathcal{O}N$, hence a permutation $\mathcal{O}P-$module with respect to the action of P by conjugation. Since S is stable under $G-$conjugation it follows that U is fusion-stable; more precisely, $\mathrm{Res}^P_R(U) \cong \mathrm{Res}_\varphi(U)$ for any subgroup R of P and any group homomorphism $\varphi : R \to P$ for which there is an element $x \in G$ satisfying $\varphi(r) = xrx^{-1}$ for all $r \in R$.

Of course, Theorem 4.4 applies also to the block $\mathcal{O}Ce$ in 4.1, yielding an isomorphism $\mathcal{O}Ce \cong T \otimes \mathcal{O}_*\tilde{L}$, where $T = \mathcal{O}O_{p'}(C)e$ and $\tilde{L} = \tilde{C}/O_{p'}(C)$ is obtained from the $\mathcal{O}^\times-$group $\tilde{C}$ opposite to that defined by the action of C on T. What we will show next, is that $\hat{L} \cong \tilde{L}$ in such a way, that the canonical images of P in $\hat{L}$ and $\tilde{L}$ are preserved. The proof consists of an explicit choice of $2-$cocycles λ, μ defining the central $\mathcal{O}^\times-$extension $\hat{L}$, $\tilde{L}$. The point is that it is not sufficient to show that the classes of these two $2-$cocycles coincide in $H^2(G, \mathcal{O}^\times)$, because even though this would imply that there is an isomorphism of $\mathcal{O}^\times-$groups $\hat{L} \cong \tilde{L}$, it would not give information about what happens to the elements of P under such an isomorphism.

Theorem 4.5. *With the notation of 4.1, let* Q *be a Sylow* p*-subgroup of* $O_{p',p}(G)$ *contained in* P. *Set* $S = \mathcal{O}O_{p'}(G)b$ *and* $T = \mathcal{O}O_{p'}(C)e$. *Denote by* $\hat{G}$ *and* $\tilde{C}$ *the* $\mathcal{O}^\times-$*groups opposite to that defined by the actions of* G *on* S *and of* C *on* T, *respectively. Set* $\hat{L} = \hat{G}/O_{p'}(G)$ *and* $\tilde{L} = \tilde{C}/O_{p'}(C)$. *Set* $m = |G||O_{p'}(G)|$ *and let* μ_m *be the subgroup of* $\mathcal{O}^\times$ *of all* m*-th roots of unity.*

For any $x \in G$, *choose* $s_x \in S^\times$ *such that* $(x, s_x) \in \hat{G}$ *and* $\det(s_x) = 1$; *for any* $y \in C$, *choose* $t_y \in T^\times$ *such that* $(y, t_y) \in \tilde{C}$ *and* $\det(t_y) = 1$. *Set*

$$G' = \{(x, \lambda s_x) | x \in G, \lambda \in \mu_m\} \text{ and } C' = \{(y, \lambda t_y) | y \in C, \lambda \in \mu_m\} \ .$$

Denote by $\sigma : P \to S^\times$ *and* $\tau : P \to T^\times$ *the unique group homomorphisms such that* $(u, \sigma(u)) \in \hat{G}$, $(u, \tau(u)) \in \tilde{C}$, *and* $\det(\sigma(u)) = 1 = \det(\tau(u))$ *for all* $u \in P$. *Then the following hold.*

(i) G' *is a finite* $A-$*stable subgroup of* $\hat{G}$; *the exponent of* G' *divides* m *and we have* $\hat{G} = \mathcal{O}^\times \cdot G'$ *and* G' *contains* $\{(x, xb) | x \in O_{p'}(G)\}$ *and* $\{(u, \sigma(u)) | u \in P\}$.

(ii) $C_{G'}(A) = \{(y, \lambda s_y) | y \in C, \lambda \in \mu_m\}$ *and* $G' = O_{p'}(G)C_{G'}(A)$.

(iii) C' *is a finite subgroup of* $\tilde{C}$; *the exponent of* C' *divides* m *and we have* $\tilde{C} = \mathcal{O}^\times \cdot C'$ *and* C' *contains* $\{(y, ye) | y \in O_{p'}(C)\}$ *and* $\{(u, \tau(u)) | u \in P\}$.

(iv) There is a group isomorphism $\Phi : C' \cong C_{G'}(A)$ which makes the following obvious diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_m & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \Phi} & & \| & & \\ 1 & \longrightarrow & \mu_m & \longrightarrow & C_{G'}(A) & \longrightarrow & C & \longrightarrow & 1 \end{array}$$

commutative, such that $\Phi(y, ye) = (y, yb)$ for any $y \in O_{p'}(C)$ and $\Phi(u, \tau(u)) = (u, \sigma(u))$ for any $u \in P$.

Before we prove this theorem, let us note a consequence of statement (iv): since the group isomorphism $C' \cong C_{G'}(A)$ preserves the canonical images of $O_{p'}(C)$ as well as of P in C' and G', we can reformulate (iv) as follows.

Corollary 4.6. *With the notation of 4.5, the inclusion $C \subseteq G$ lifts to an $\mathcal{O}^\times$-isomorphism $\tilde{L} \cong \hat{L}$ which preserves the canonical images of P elementwise.*

From this and 4.4 follows already that $\mathcal{O}Gb$ and $\mathcal{O}Ce$ are Morita equivalent, and that there is a Morita equivalence given by a bimodule with endo-permutation source; this is because the group isomorphism in 4.6 preserves P and $S = \mathrm{End}_{\mathcal{O}}(U)$, $T = \mathrm{End}_{\mathcal{O}}(V)$ for some endo-permutation $\mathcal{O}P-$modules U, V (whose module structure is given by σ and τ, respectively). It remains to see, that we can choose this Morita equivalence in such a way, that it induces the Glauberman correspondence on characters.

Proof of Theorem 4.5. By 4.3 we have $G = CO_{p'}(G)$; thus $\hat{G} = \hat{C}O_{p'}(G)$, where $\hat{C}$ is the $\mathcal{O}^\times-$group opposite to that defined by the action of C on S, and where we again identify $O_{p'}(G)$ to its canonical image $\{(x, xb)\}_{x \in O_{p'}(G)}$ in $\hat{G}$. We have to show that

4.5.1. *There is an isomorphism of $\mathcal{O}^\times-$groups $\hat{C} \cong \tilde{C}$ which lifts the identity on C and which preserves the canonical images of P and $O_{p'}(C)$.*

Proceeding along a composition series of A, we may assume that

4.5.2. *A is cyclic of prime order q, where q is a prime which does not divide the order of G.*

For the computations that follow, keep in mind that $\hat{G}$ is the $\mathcal{O}^\times-$group *opposite* to that defined by the action of G on S; in other words, for any $(x, s) \in \hat{G}$ and any $\lambda \in \mathcal{O}^\times$ we have $\lambda(x, s) = (x, \lambda^{-1}s)$. By 3.3, the associated $2-$cocycles have values in groups of units; more precisely,

4.5.3. *For any $x, y \in G$ there is an $|O_{p'}(G)|$-th root of unity $\lambda_{x,y}$ satisfying $s_x s_y = (\lambda_{x,y})^{-1} s_{xy}$.*

This shows that G' is a finite subgroup of $\hat{G}$; the statement on the exponent of G' follows from 3.2. If $x \in O_{p'}(G)$, then $xb = \lambda s_x$ for some $\lambda \in \mathcal{O}^\times$; hence $\det(xb) = \lambda^n$ and thus $\lambda^m = 1$ (cf. 3.2). Thus (i) holds. Similarly,

4.5.4. *For any $x, y \in C$ there is an $|O_{p'}(C)|$-th root of unity $\mu_{x,y}$ satisfying $t_x t_y = (\mu_{x,y})^{-1} t_{xy}$.*

As above, this implies (iii). To see that (ii) holds, let $y \in C$ and $a \in A$. By the Skolem-Noether Theorem, the automorphism of S induced by a is inner, and thus both s_y and ${}^a(s_y)$ have determinant one. Since they both act as y on S, they differ by some $|O_{p'}(G)|$-th root of unity, by 3.2. As a has order dividing q it follows that s_y and ${}^a(s_y)$ have to be equal. Thus $(y, s_y) \in C_{G'}(A)$, which implies (ii).

Recall that β denotes the unique irreducible character of $O_{p'}(G)$ associated with b and that $\gamma = \pi(O_{p'}(G), A)(\beta)$ is the unique irreducible character of $O_{p'}(C)$ associated with e. The Fourier inversion formula 2.5 applied to $O_{p'}(G)$ and β implies that

4.5.5. *for any $x \in C$ we have* $s_x = \frac{\beta(1)}{|O_{p'}(G)|} \sum_{y \in O_{p'}(G)} \beta(s_x y^{-1}) y$.

We have a similar formula for the elements t_x, using the character γ. Given our choice of s_x, for any $x \in C$ and any $y \in O_{p'}(G)$, the element $s_x y^{-1}$ has a finite order in $S^\times$ dividing m (cf. 3.2 applied to the element xy^{-1}), and thus $\beta(s_x y^{-1}) \in \mathbb{Z}[\zeta]$, where ζ is a primitive m-th root of unity in $\mathcal{K}$. Let $\mathcal{O}'$ be the local subring of $\mathcal{K}$ obtained from localising $\mathbb{Z}[\zeta]$ at a maximal ideal containing q. Clearly $\mathcal{O}'$ has the residue field $k' = \mathbb{F}_q[\zeta]$ of characteristic q.

The formula 4.5.5 and its analogue for the elements t_x implies that

4.5.6. *for any $x \in C$ we have $s_x \in (\mathcal{O}' O_{p'}(G) b)^A$ and $t_x \in \mathcal{O}' O_{p'}(C) e$.*

It is the statement 4.5.6 which makes it possible to apply the Brauer construction with respect to A. Note that by 2.4 we have $Br_A(b) = \bar{e}$, where $\bar{e}$ is the image of e in $k' O_{p'}(C)$. We compare for $x \in C$ the image $Br_A(s_x)$ in $k' O_{p'}(C)\bar{e}$ to the canonical image of t_x in $k' O_{p'}(C)\bar{e}$. Both elements act as x on this matrix algebra, whence they differ by a scalar. Since both elements have finite order, they differ actually by a root of unity; that is,

4.5.7. *for any $x \in C$ there is a unique root of unity ν_x of order dividing m such that $Br_A(s_x) = \nu_x Br_A(t_x)$*

Any such root lifts uniquely to $\mathcal{O}'$, and we denote this abusively by ν_x again. Comparing the equations 4.5.3, 4.5.4 and 4.5.7 we will show that

4.5.8. *there is a unique $\mathcal{O}^\times-$isomorphism $\hat{C} \cong \tilde{C}$ mapping (x, t_x) to $\nu_x(x, s_x)$ for any $x \in C$.*

To see that this is a group homomorphism, let $x, y \in C$. Write $s_x s_y = (\lambda_{x,y})^{-1} s_{xy}$. Applying Br_A to this equation yields, by 4.5.3 and 4.5.7, the equation

$$\lambda_{x,y} \nu_x \nu_y Br_A(t_x t_y) = \nu_{xy} Br_A(t_{xy}).$$

Using 4.5.4 implies that

4.5.9. *for any $x, y \in C$ we have $\nu_x \nu_y \lambda_{x,y} = \nu_{xy} \mu_{x,y}$.*

This shows that the $\mathcal{O}^\times-$linear map defined in 4.5.8 maps

$$(x, t_x)(y, t_y) = \mu_{x,y}(xy, t_{xy})$$

to $\nu_{xy}\mu_{x,y}(xy, s_{x,y}) = \nu_x \nu_y \lambda_{x,y}(xy, s_{xy}) = \nu_x \nu_y (x, s_x)(y, s_y)$, which is the product of the images of (x, t_x), (y, t_y). This shows that the map in 4.5.8 is a group homomorphism. We show next, that this group homomorphism preserves the canonical images of $O_{p'}(C)$. Precisely, we are going to show that

4.5.10. *the group isomorphism 4.5.8 maps (x, xe) to (x, xb) for any $x \in O_{p'}(C)$.*

To see this, observe that if $x \in O_{p'}(C)$, then the elements s_x and xb differ by a scalar since they both act as x on S; say $s_x = \delta_x xb$ for some $\delta_x \in \mathcal{O}$. Both s_x and xb have finite order dividing m, and therefore δ_x is a root of unity belonging to $\mu_m \subset \mathcal{O}'$. Similarly, $t_x = \epsilon_x xe$ for some root of unity $\epsilon_x \in \mu_m$. Denote by $\bar{b}$ and $\bar{e}$ the canonical images of b and e in $k'O_{p'}(G)$ and $k'O_{p'}(C)$, respectively. By 2.4, we have $Br_A(\bar{b}) = Br_A(\bar{e})$, and thus $Br_A(x\bar{b}) = x\bar{e}$ for any $x \in C$. Now 4.5.7 shows that $Br_A(s_x) = \nu_x Br_A(t_x) = \nu_x \epsilon_x x\bar{e}$. Together we get $\delta_x = \nu_x \epsilon_x$. Thus the homomorphism in 4.5.8 maps $(x, xe) = \epsilon_x(x, t_x)$ to $\epsilon_x \nu_x(x, s_x) = \epsilon_x \nu_x \delta_x^{-1}(x, xb) = (x, xb)$, which proves 4.5.10. It remains to investigate the effect of the group homomorphism 4.5.8 on the images of the elements of P in $\tilde{C}$ and $\hat{C}$. For that, we may assume that $s_u = \sigma(u)$ and $t_u = \tau(u)$ for any $u \in P$. Then $\lambda_{u,v} = 1 = \mu_{u,v}$ for any $u, v \in P$. Hence $\nu_u \nu_v = \nu_{uv}$ by 4.5.9. In other words, the map $\nu : P \to \mathbb{Z}[\zeta]^\times$ sending $u \in P$ to $\nu(u) = \nu_u$ is a linear character of P. This character is $C-$stable: by the uniqueness properties of σ and τ we have ${}^y\sigma(u) = \sigma({}^yu)$ and ${}^y\tau(u) = \tau({}^yu)$ for any $u \in P$ and any $y \in C$ such that ${}^yu \in P$, and so $\nu(u) = \nu({}^yu)$ by 4.5.7. Thus ν extends, by 2.6, to a linear character of C, abusively still denoted by ν, such that $O_{p'}(C) \subseteq \ker(\nu)$. Now the map sending (x, t_x) to $\nu(x)^{-1}\nu_x(x, s_x)$, where $x \in C$, induces a group isomorphism $\tilde{C} \cong \hat{C}$ which fulfills 4.5.1. □

Proof of 1.1 in the reduced case. We use the notation of 4.1 and 4.5. By 4.5 (iv), we have a commutative diagram of finite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_m & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 1 \\ & & \| & & {\scriptstyle \iota'}\downarrow & & \downarrow{\scriptstyle \iota} & & \\ 1 & \longrightarrow & \mu_m & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where ι is the inclusion, ι' is the isomorphism $\Phi : C' \cong C_{G'}(A)$ from 4.5 (iv) followed by the inclusion $C_{G'}(A) \subseteq G'$.

By the notation in 4.1, we have $\gamma = \pi(O_{p'}(G), A)(\beta)$. Consider β as $\mathcal{O}$-linear function on $\mathcal{O}O_{p'}(G)$ and set $\beta'(x, s) = \beta(s)$ for any $(x, s) \in G'$. Then β' is an A-stable irreducible character of G' which extends β through the canonical embedding $O_{p'}(G) \to G'$. Similarly, define $\gamma' \in \mathrm{Irr}_{\mathcal{K}}(C')$ by setting $\gamma'(y, t) = \gamma(t)$ for any $(y, t) \in C'$; as before, γ' extends γ through the canonical embedding $O_{p'}(C) \to C'$.

Since β' extends β, it follows from 2.3 (i) that $\pi(G', A)(\beta') \circ \Phi$ is a character of C' which extends γ. Therefore, there is a linear character $\lambda : C' \to \mathcal{O}^\times$ such that $O_{p'}(C) \subseteq \ker(\lambda)$ and such that

$$\pi(G', A)(\beta) \circ \Phi = \lambda\gamma' \ .$$

We are going to show that $\mu_m \subseteq \ker(\lambda)$. If $\zeta \in \mu_m$, then $\beta'(1, \zeta b) = \zeta \deg(\beta)$, and since $\pi(G', A)(\beta')$ appears in the restriction of β' to $C_{G'}(A)$, we also have $\pi(G', A)(\beta')(1, \zeta b) = \zeta \deg(\pi(G', A)(\beta')) = \zeta \deg(\gamma) = \gamma'(1, \zeta e)$. Since $\Phi(1, \zeta e) = (1, \zeta b)$, it follows that $\mu_m \subseteq \ker(\lambda)$. Moreover, since the exponent of C' divides m, it follows that $\mathrm{Im}(\lambda) \subseteq \mu_m$, and thus the map $\kappa = \lambda^{-1}\iota' : C' \to G'$ sending $(y, t) \in C'$ to $\lambda(y, t)^{-1}\iota'(y, t)$ is still an injective group homomorphism with image $C_{G'}(A)$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_m & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 1 \\ & & \| & & {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle \iota} & & \\ 1 & \longrightarrow & \mu_m & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

By the construction of κ, we still have $\kappa(y, ye) = (y, yb)$ for any $y \in O_{p'}(C)$. Note that κ need no longer preserve the images of P, because

$$\kappa(u, \tau(u)) = \lambda(u, \tau(u))^{-1}(u, \sigma(u))$$

for any $u \in P$. The reason why we replace ι' by κ is that we have now

$$\pi(G', A)(\beta) \circ \kappa = \gamma' \ .$$

The group homomorphism κ induces still an isomorphism of $\mathcal{O}^\times$-groups $\mu : \tilde{L} \cong \hat{L}$, which in turn induces a commutative diagram of $\mathcal{O}$-algebras

$$\begin{array}{ccc} \mathcal{O}C' & \longrightarrow & \mathcal{O}_*\tilde{L} \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ \mathcal{O}G' & \longrightarrow & \mathcal{O}_*\hat{L} \end{array}$$

where we denote abusively the induced algebra homomorphisms by κ and μ again, and where the horizontal maps are the canonical surjections.

We have thus a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Ce$ given by composing the Morita equivalence between $\mathcal{O}Gb \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}$ and $\mathcal{O}_*\hat{L}$, the algebra isomorphism $\mu : \mathcal{O}_*\tilde{L} \cong \mathcal{O}_*\hat{L}$, and the Morita equivalence between $\mathcal{O}_*\tilde{L}$ and $\mathcal{O}Ce \cong T \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\tilde{L}$. We are going to show that this Morita equivalence satisfies 1.1. The $\mathcal{O}Gb$-$\mathcal{O}Ce$-bimodule M inducing this equivalence is explicitly given by

$$M = (U \underset{\mathcal{O}}{\otimes} (\mathcal{O}_*\hat{L})_\mu) \underset{\mathcal{O}_*\tilde{L}}{\otimes} (\mathcal{O}_*\tilde{L} \underset{\mathcal{O}}{\otimes} V^*) \cong U \underset{\mathcal{O}}{\otimes} (\mathcal{O}_*\hat{L})_\mu \underset{\mathcal{O}}{\otimes} V^* \ ,$$

where $S = \mathrm{End}_{\mathcal{O}}(U)$ and $T = \mathrm{End}_{\mathcal{O}}(V)$, and where the left and right module structure on M is given via the isomorphisms $\mathcal{O}Gb \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}$ and $\mathcal{O}Ce \cong T \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\tilde{L}$ from 4.4.

We show that M has ΔP as vertex and a fusion-stable endo-permutation $\mathcal{O}\Delta P$-source. The $\mathcal{O}$-modules U and V are fusion-stable endo-permutation $\mathcal{O}P-$modules (through the group homomorphisms $\sigma : P \to S^\times$ and $\tau : P \to T^\times$, respectively, as explained before 4.5). Denote by λ_P the rank one $\mathcal{O}P-$module whose character is the restriction of λ to P. All three modules U, V, λ_P have ranks prime to p. Thus $U \underset{\mathcal{O}}{\otimes} \lambda_P^* \underset{\mathcal{O}}{\otimes} V^*$ has an indecomposable direct summand W with vertex ΔP. Since all of U, V, λ_P are fusion-stable endo-permutation modules, so is W. It suffices to see that W is a source of M. It follows from 3.9 that there is an isomorphism of $\mathcal{O}P$-$\mathcal{O}P$-bimodules $\mathcal{O}_*\hat{L} \cong \mathcal{O}L$, where $L = G/O_{p'}(G)$. Thus the restriction of M to $P \times P$ is isomorphic to the direct sum of the modules $U \underset{\mathcal{O}}{\otimes} \mathcal{O}[PyP]_\mu \underset{\mathcal{O}}{\otimes} V^*$, where we denote abusively the restriction of μ to P by the same letter, and where y runs over a set of representatives of the P-P-double cosets in L. For any such y, we have $\mathcal{O}[PyP] \cong \mathrm{Ind}_R^{P\times P}(\mathcal{O})$ as $\mathcal{O}(P \times P)$-modules, where $R = \{(u, y^{-1}uy) | u \in P \cap yPy^{-1}\}$, and thus $\mathcal{O}[PyP]_\mu \cong \mathrm{Ind}_R^{P\times P}(\lambda_P^*)$. Therefore $U \underset{\mathcal{O}}{\otimes} \mathcal{O}[PyP]_\mu \underset{\mathcal{O}}{\otimes} V^* \cong \mathrm{Ind}_R^{P\times P}(U \underset{\mathcal{O}}{\otimes} \lambda_P^* \underset{\mathcal{O}}{\otimes} V^*)$, and hence a vertex of M has order at most $|P|$. For $y = 1$ we get that $\mathrm{Ind}_{\Delta P}^{P\times P}(U \underset{\mathcal{O}}{\otimes} \lambda_P^* \underset{\mathcal{O}}{\otimes} V^*)$ is isomorphic to a direct summand of M restricted to $P \times P$. Thus ΔP is a vertex and W a source of M.

We show next, that on characters, the equivalence given by M induces the Glauberman correspondence. Arguing by induction, we may assume that A is a q-group for some prime q not dividing the order of G.

Let $\chi \in \mathrm{Irr}_{\mathcal{K}}(G,b)$. Then $\chi = \beta \otimes \chi_{\hat{L}}$ for a unique irreducible character $\chi_{\hat{L}}$ of $\mathcal{O}_*\hat{L}$ via the isomorphism $\mathcal{O}Gb \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}$. That is,

$$\chi(x) = \beta(s)\chi_{\hat{L}}(\overline{(x,s)}) \ ,$$

where $x \in G$ and $s \in S^\times$ such that $(x,s) \in \hat{G}$, and where $\overline{(x,s)}$ is the canonical image of (x,s) in $\mathcal{O}_*\hat{L}$.

Through the algebra isomorphism μ, the character $\chi_{\hat{L}}$ corresponds to an irreducible character $\chi_{\tilde{L}}$ of $\mathcal{O}_*\tilde{L}$. The character $\eta = \gamma \otimes \chi_{\tilde{L}}$ corresponding to $\chi_{\tilde{L}}$ via the isomorphism $\mathcal{O}Ce \cong T \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\tilde{L}$ is given, as above, by

$$\eta(y) = \gamma(t)\chi_{\tilde{L}}(\overline{(y,t)}) \ ,$$

where $(y,t) \in \tilde{C}$. Consider now the inflation χ' of χ to G' via the canonical surjection $G' \to G$ and the inflation η' to C' of η. The preceding equalities now read

$$\chi'(x,s) = \beta'(x,s)\chi_{\hat{L}}(\overline{(x,s)})$$

for any $(x,s) \in G'$, and similarly,

$$\eta'(y,t) = \gamma'(y,t)\chi_{\tilde{L}}(\overline{(y,t)})$$

for any $(y,t) \in C'$. Here we now use the particular choice of κ: since $\pi(G',A)(\beta') \circ \kappa = \gamma'$, the character γ' appears with multiplicity prime to q in the restriction $\beta' \circ \kappa$ of β' through κ (cf. 2.1(ii)). Since μ is precisely the algebra isomorphism induced by κ, it follows that η' appears with multiplicity prime to q in the restriction $\chi' \circ \kappa$ of χ' through κ. As χ', η' are just the inflations of χ, η, it follows that η appears with multiplicity prime to q in the restriction $\mathrm{Res}^G_C(\chi)$. By 2.1(ii), this is equivalent to $\eta = \pi(G,A)(\chi)$.

The statements (i) and (ii) follow. If b is the principal block of G, then e is the principal block of C, because the Glauberman correspondent of the trivial character of G is the trivial character of C. Moreover, by 4.3 (ii), we have $G = CO_{p'}(G)$, which implies the equality $\mathcal{O}Gb = \mathcal{O}Ce$ as $O_{p'}(G)$ is the kernel of the principal block of G. □

Remark 4.7. A Morita equivalence between two blocks having isomorphic defect groups which is given by a bimodule with a fusion-stable endo-permutation source always induces an isotypy. This yields an alternative proof of the fact that the Glauberman correspondence is indeed an isotypy in the situation of Theorem 1.1.

5. The reduction step

Theorem 5.1 below shows that the "reduction step" as described in [10, 3.1] is both $A-$equivariant and compatible with the Glauberman correspondence. It shows in particular, that in order to prove Theorem 1.1, we may assume that b is actually a $G-$invariant block of the maximal normal subgroup $O_{p'}(G)$ of order prime to p in G.

Theorem 5.1. *Let G be a finite $p-$solvable group, A a solvable group of automorphisms of G of order prime to $|G|p$, and b be an $A-$stable block of $\mathcal{O}G$ having a defect group which is centralised by A.*

There is an $A-$stable subgroup H of G and an $H \rtimes A-$stable block e of $O_{p'}(H)$ such that the following hold:

(i) We have $O_{p'}(G) \subset H$ and $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$ as interior $G-$algebras and as $A-$algebras; in particular, e is an $A-$stable block of H and we have $b = \mathrm{Tr}_H^G(e)$.

(ii) Any Sylow p-subgroup of H is a defect group of e as a block of H and of the block b of G; moreover, there is a Sylow p-subgroup P of H contained in $C_H(A)$.

(iii) The Brauer categories of $\mathcal{O}Gb$ and $\mathcal{O}He$ are equivalent and the block e of H is of principal type; that is, $Br_Q(e)$ is a block of $kC_H(Q)$ for any subgroup Q of P.

If we set $C = C_G(A)$, $D = C_H(A) = C \cap H$ and denote by $w(b)$, $w(e)$ the blocks of $\mathcal{O}C$, $\mathcal{O}D$ corresponding to b, e, respectively, through the Glauberman correspondence, then moreover the following hold:

(iv) We have $O_{p'}(D) = O_{p'}(H) \cap C$, and $w(e)$ is a $D-$stable block of $\mathcal{O}O_{p'}(D)$.

(v) We have $\mathcal{O}Cw(b) \cong \mathrm{Ind}_D^C(\mathcal{O}Dw(e))$ as interior $C-$algebras; in particular, $w(b) = \mathrm{Tr}_D^C(w(e))$.

(vi) The Brauer categories of $\mathcal{O}Cw(b)$ and $\mathcal{O}Dw(e)$ are equivalent; moreover, the block $w(e)$ of $\mathcal{O}D$ is of principal type.

(vii) For any $\psi \in \mathrm{Irr}_{\mathcal{K}}(H, e)$ we have $\pi(G, A)(\mathrm{Ind}_H^G(\psi)) = \mathrm{Ind}_D^C(\pi(H, A)(\psi))$; that is, the Glauberman correspondence commutes with induction.

(viii) If b is the principal block of $\mathcal{O}G$, then $w(b)$ is the principal block of $\mathcal{O}C$, and we have $G = H$ and $C = D$.

Proof. If d is a block of $\mathcal{O}O_{p'}(G)$ such that $bd \neq 0$, then $b({}^x d) \neq 0$ for any $x \in G$. Thus, by denoting by H the stabiliser in G of d, the sum $\mathrm{Tr}_H^G(d)$ of all different $G-$conjugates of d is a $G-$stable idempotent satisfying $b\mathrm{Tr}_H^G(d) = b$. Thus G acts transitively on the set of blocks d of $\mathcal{O}O_{p'}(G)$ satisfying $bd \neq 0$. By Glauberman's lemma [11, 13.8], there is an $A-$stable block d of $\mathcal{O}O_{p'}(G)$ such that $bd \neq 0$. Denote still by H the stabiliser of d in G. If $G = H$ then $C = D$ and $b = d$, and whence all statements of the theorem follow from 4.1.

We may therefore assume that H is a proper subgroup of G. Then $\mathcal{O}G\mathrm{Tr}_H^G(d) \cong \mathrm{Ind}_H^G(\mathcal{O}Hd)$. Thus the map Tr_H^G induces an isomorphism of the centers $Z(\mathcal{O}Hd) \cong Z(\mathcal{O}G\mathrm{Tr}_H^G(d))$, and therefore determines a unique block e of $\mathcal{O}H$ such that $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$. Clearly H and e are $A-$stable as d is A-stable, and we have $O_{p'}(G) \subset H$. By standard properties of induction of interior algebras (cf. [18]), any defect group of $\mathcal{O}He$ is also a defect group of $\mathcal{O}Gb$, and the Brauer categories of $\mathcal{O}Gb$, $\mathcal{O}He$ are equivalent. Moreover, H acts transitively on the set of defect groups of $\mathcal{O}He$, and whence, again by Glauberman's lemma [11, 13.8], there is an $A-$stable defect group P of $\mathcal{O}He$. Since there is some defect group of $\mathcal{O}Gb$ on which A acts trivially, and since $C = C_G(A)$ acts, by [11, 13.9], transitively on the set of $A-$stable defect groups of $\mathcal{O}Gb$, it follows that $P \subseteq C_G(A)$.

Therefore the statements (i), (ii), (iii) follow now by induction.

Using 4.3(iii) we also get $O_{p'}(D) = O_{p'}(H) \cap D$. By 4.1 applied to e instead of b, the idempotent $w(e)$ is a $D-$stable block of $\mathcal{O}O_{p'}(D)$. This shows (iv). Statement (vii) is clear by the induction theorem 2.3. Since $w(b)$ and $w(e)$ are the sums of the centrally primitive idempotents in $\mathcal{K}Cw(b)$ and $\mathcal{K}Dw(e)$, a straightforward computation shows, that this implies $w(b) = \mathrm{Tr}_D^C(w(e))$, which in turn shows (v). Statement (vi) follows then from (v). By (i), the algebra $\mathcal{O}Gb$ is isomorphic to the matrix algebra $M_{[G:H]}(\mathcal{O}Hc)$; in particular, $[G : H]$ divides the degree of any

$\chi \in \mathrm{Irr}_{\mathcal{K}}(G, b)$. Thus, if b is the principal block, we have $G = H$. But then we are in the reduced case of 4.1, and so (viii) follows from 4.1 (v). □

Proof of Theorem 1.1 in the general case. Theorem 5.1 implies that in order to prove Theorem 1.1, we may assume that we are in the situation of Theorem 4.1. In that case, the proof follows from the developments in section 4, which completes the proof of Theorem 1.1. □

Remark 5.2. By combining 4.4 and 5.1 one gets the main result of Puig [19] on the structure of the source algebras of blocks of $p-$solvable groups, with the exception of the uniqueness statements in [19]. If G is a finite $p-$solvable group and b a block of $\mathcal{O}G$, then by 5.1 (with $A = 1$) there is a subgroup H of G and an $H-$stable block e of $\mathcal{O}O_{p'}(H)$ such that $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$. By a standard result on the induction of interior algebras due to Puig, the block algebras $\mathcal{O}He$ and $\mathcal{O}Gb$ have isomorphic source algebras. Thus we may assume that we are in the situation of 4.4 (with $O_{p'}(G)$ instead of N). The structure of the source algebras in that case follows from the structure of the block algebra as described in 4.4.

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