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ATTRACTORS FOR GRAPH CRITICAL RATIONAL MAPS

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Abstract. We call a rational map f *graph critical* if any critical point either belongs to an invariant finite graph G, or has minimal limit set, or is non-recurrent and has limit set disjoint from G. We prove that, for any conformal measure, either for almost every point of the Julia set $J(f)$ its limit set coincides with $J(f)$, or for almost every point of $J(f)$ its limit set coincides with the limit set of a critical point of f.

1. Introduction

The central question in the Dynamical Systems Theory is about the long-term behavior of orbits. In particular, it is important to know ω-limit sets for typical orbits. In this paper we address this type of question for a class of rational maps on the Riemann sphere, understanding "typical" in terms of conformal measures.

Let us describe the ideas motivating our research. A conceptually important approach to the problem of long-term behavior of orbits was suggested by Milnor in [M1], where he introduced the notion of an *attractor*. One can think of an attractor as the limit set assumed on a set of points of positive measure. In later papers (see, e.g., [BL1], [BL2], [BL3], [BL4], [L2], which are closer to our work but certainly do not exhaust the list of all main papers on this popular subject), attractors have been thoroughly studied for interval maps. It was established that to a great extent the number and type of attractors of interval maps is related to the behavior of the trajectories of their critical points (see also [BM], where the attractors and critical limit sets of negative Schwarzian interval maps are studied in great detail). Here by a critical limit set we mean the limit set of a critical point.

Clearly, the topology of the space is important for this type of results, so already in real dimension two, in particular for complex rational maps, the problem becomes harder to tackle. However, certain topological properties are shared by interval maps and some classes of rational maps. Judging from the fact that critical limit sets play a central role for interval maps, it is reasonable to consider rational maps for which critical points belong to invariant graphs (real one-dimensional branched manifolds) as a class of maps for which similar results could hold. In other words, we want to replace one-dimensionality of the space by one-dimensionality of a set containing critical limit sets, and see if the attractors of such maps can be described.

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On the other hand, a different, more dynamical than topological, property of critical limit sets also can be sufficient to describe attractors. Recently it has been proven (in [BMO], where results of [Ma] were used) that for a conformal measure μ either almost every point has limit set coinciding with the Julia set, or almost every point has limit set contained in the union of the limit sets of recurrent critical points. Thus, in the latter case, if all critical limit sets are minimal, it is easy to see that for almost every point the limit set is exactly one of the critical limit sets.

The class of graph critical maps, which we introduce later in this section and study throughout the paper, contains classes of rational maps mentioned in the preceding two paragraphs. Let us stress that our idea is to have a purely topological or dynamical set of conditions which singles out the family of rational maps we want to deal with. Rational maps are already smooth enough, so to draw conclusions about their limit behavior we do not need to add more smoothness properties.

The paper [BL1] may be considered a prototype for out work. Some ideas and the general approach to the problem are certainly shared by both papers. However, we had to come up with new tools in order to implement those general ideas.

To fix terminology and notation, recall that for a continuous map T of a compact Hausdorff space X to itself and a point $x \in X$ the *orbit of* x is the sequence $(f^n(x))_{n=0}^{\infty}$ (we denote it $\mathrm{orb}(x)$ and sometimes consider it to be a set rather than a sequence), and the *ω-limit set* of x is the set of all accumulation points of $\mathrm{orb}(x)$. We denote it by $\omega(x)$ and usually call it simply the *limit set of* x.

If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map, we denote by $C(f) = C$ the set of its critical points and by $P(f) = \bigcup_{c \in C(f)} \omega(c)$ its *postcritical* set. A periodic orbit is called a *cycle*. For a point x let $B(x, r)$ be the open disk of radius r centered at x (*r-disk*) and $\overline{B}(x, r)$ the corresponding closed disk (*closed r-disk*). A *Jordan disk* is a set U, homeomorphic to an open disk with $\overline{U}$ homeomorphic to a closed disk, such that U is the interior of $\overline{U}$; closed Jordan disks are closures of the open ones. Also, a (closed) Jordan disk *around* a point z has interior containing z. Note that a component of the inverse image of an open Jordan disk is a Jordan disk. We call a set A *minimal* if the map restricted to this set is minimal (i.e., the orbit of every point of A is dense in A). We say that f is *exact* (for a measure μ) if for any set B such that $B = f^{-n}(B_n)$ for $n = 0, 1, 2, \dots$ and measurable sets B_n, either B or its complement has measure 0. Clearly, an exact map is ergodic.

Let us list some known results about limit sets of points for rational maps. By [Su2] the limit set of a point in the Fatou set is either an attracting or parabolic cycle, or a simple closed curve on which the map is conjugate to an irrational rotation. Hence, we restrict our attention to the limit sets of points from the Julia set $J(f)$.

The first results in this direction were obtained in [L1], where it is proven that if $J(f) \neq \widehat{\mathbb{C}}$, then the limit set of Lebesgue almost every point in $J(f)$ is contained in $P(f)$. Also, when the Julia set has positive Lebesgue measure, it was shown in [McM1] that if the limit set of Lebesgue almost every point in $J(f)$ is not contained in $P(f)$, then $J(f) = \widehat{\mathbb{C}}, f$ is ergodic and the limit set of almost every point is the entire $\widehat{\mathbb{C}}$. In [Ba] it was shown that in the second case the map is also exact and conservative.

These papers deal with the Lebesgue measure. One may hope to generalize their results to *conformal* measures. A measure μ on $J(f)$ is *conformal* (for f) if for an *exponent* $\alpha > 0$ we have $\mu(f(A)) = \int_A |f'(z)|^{\alpha} d\mu$ whenever $f|_A$ is 1-to-1 (by [Su1]

f has at least one conformal measure), and indeed it turns out that results similar to those of [L1], [McM1], [Ba] hold for conformal measures too. Note that since we assume that the conformal measure is supported by $J(f)$, the Lebesgue measure is usually not conformal.

Now, for $x \in \widehat{\mathbb{C}}$ and $n > 0$ consider the supremum $r_n(x)$ of all r such that $B(f^n(x), r)$ can be pulled back to x univalently. By this we understand that there are sets $V_i \ni f^i(x)$ for $i = 0, \dots, n$ such that $V_n = B(f^n(x), r)$ and V_{i-1} is a connected component of $f^{-1}(V_i)$, and f is univalent on all V_i, $i < n$. Then $r_n(x) > 0$ if all points $x, \dots, f^{n-1}(x)$ are not critical; otherwise define $r_n(x)$ as 0. Moreover, there exists a critical point c_n belonging to the boundary of one of the sets V_{m_n}, where $r = r_n(x)$. We call (c_n, m_n) a *generating pair* for $r_n(x)$.

If $x \in J(f)$ and $r_n(x) \not\to 0$, then x is called *(C-)reluctant*. The set of all such points is denoted by $\mathrm{Rlc}(f)$ (reluctant points are also called *conical*, see e.g. [DMNU], and the set $\mathrm{Rlc}(f)$ is also called the *radial Julia set of* f, see [McM2]; those points are also discussed in Section 8.3 of [LM]). If $x \in J(f)$ and $r_n(x) \to 0$, the point x is called *(C-)persistent*. There are trivial cases when a point is persistent, e.g., if it is eventually mapped into a critical point or into a parabolic periodic point (whose orbit by the Fatou theorem is the limit set of some critical point). The set of all persistent points is denoted by $\mathrm{Prs}(f)$. By definition, $\mathrm{Prs}(f) \subset J(f)$ and $\mathrm{Rlc}(f) \subset J(f)$. Also, let $P_r(f)$ be the union of the limit sets of recurrent critical points of f.

Theorem 1.1 ([BMO]). *The following holds for a conformal measure μ for f.*

1. *If $\mu(\mathrm{Rlc}(f)) > 0$, then $\mathrm{supp}(\mu) = J(f)$, $\mu(\mathrm{Rlc}(f)) = 1$, μ is non-atomic, the map f is exact, conservative, and $\omega(x) = J(f)$ for μ-a.e. point $x \in J(f)$.*
2. *If $\mu(\mathrm{Rlc}(f)) = 0$, then for μ-a.e. point $x \in J(f)$ either $\omega(x) \subset P_r(f)$ or x is an eventual preimage of a critical or parabolic point of f.*

If we want to apply this theorem to the Lebesgue measure m instead of a conformal measure, there are two cases possible. If $m(J(f)) = 0$, then the statements are vacuous. If $m(J(f)) > 0$, then the restriction of m to $J(f)$ is a conformal measure and we can apply the theorem. The same will apply to our Main Theorem, stated later in this section.

The aim of Theorem 1.1 is to deal with conformal measures with *no assumptions* on maps. Stronger results for maps with specific assumptions are known: if $J(f)$ is expanding ([Bo], [Su1], [W]) or if f is from a certain class of unimodal polynomials [Pr], then f is ergodic with respect to μ. Also, results close to Theorem 1.1(1) were obtained in [GPS] for Lebesgue measure and could actually be extended onto conformal measures by a slight modification of methods of [GPS]. For a rational map with no recurrent critical points in the Julia set, it is shown in [U] that for any nonatomic conformal measure and almost every z, the limit set $\omega(z)$ is $J(f)$. Finally, other related results concerning conformal measures are obtained in [P] under certain hyperbolic assumptions.

We study limit sets of persistent points (which is justified by Theorem 1.1) for the class of rational maps which we now define. A *graph* is a compact one-dimensional branched manifold (not necessarily connected). A rational map f of degree at least 2 is *graph critical* if for some f-invariant graph G (not necessarily smooth) every critical point of f either has minimal limit set, or is non-recurrent and has limit set disjoint from G, or belongs to G.

Two obvious examples of graph critical maps are: rational maps with real coefficients and real critical points, and rational maps with all critical points having finite limit sets. To give less obvious examples, consider a polynomial f with a locally connected Julia set containing all critical points of f. Then one can prove that all critical points of f belong to an invariant graph if and only if all critical points of f are eventually mapped into points at which two external rays land (see [BLe]).

Main Theorem (Theorem 4.13)**.** *For a graph critical map f and a conformal measure, exactly one of the following holds.*

1. *For almost every $x \in J(f)$, $\omega(x) = J(f)$.*
2. *For almost every $x \in J(f)$, $\omega(x) = \omega(c)$ for some critical point c of f.*

This theorem is a corollary to Theorem 1.1 and the following main technical result of the paper.

Theorem 4.12. *For a graph critical map f and a persistent point $x \in J(f)$, $\omega(x) = \omega(c)$ for some critical point c of f.*

The Main Theorem allows us to answer (for graph critical maps) the questions about *primitive attractors* in the sense of the Milnor ([M1]). For a given measure μ (in our case, any conformal measure, including the case when the Julia set has positive Lebesgue measure and μ is the restriction of the Lebesgue measure to the Julia set), they are the sets A such that $\mu(\{x : A = \omega(x)\}) > 0$. In Case 1 of the Main Theorem, there is only one primitive attractor, namely $J(f)$. In Case 2 all primitive attractors are of the form $\omega(c)$ for a critical point c of f; so there may be several primitive attractors, but their number is no greater than $2 \deg f - 2$.

We would like to make one final remark concerning the proofs. As we have explained, our definition of graph-critical maps is supposed to accommodate several possible types of behavior of critical points. Still, as an introductory step, the reader may think of a simpler situation of graph-critical functions in the narrow sense, i.e. rational functions whose critical points all belong to an invariant graph. In this case one can simply omit certain parts of the proofs.

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2. Contraction principles for graph maps

Throughout this section G denotes a graph and $f : G \to G$ a continuous map. The *open n-od* is the set of all points z in the open unit disk with z^n real and positive; its closure is called the *closed n-od* or just the *n-od*. A point $x \in G$ has a neighborhood homeomorphic to the open n-od for some $n = n(x)$ with x corresponding to the origin. The number n here is called the *valence* of x (in G) and denoted by $\mathrm{val}_G(x)$. If $n = 1$, then x is called an *endpoint* of G (then x has a neighborhood in G homeomorphic to $[0, 1)$, where x corresponds to 0). If $n > 2$, then x is called a *branching point* of G. The branching points and endpoints are called *vertices*. The set of vertices of G is finite. Let d be a metric in G.

An important role for interval maps is played by the Contraction Principle (it can be deduced from the "spectral decomposition" [B1] or proved directly [MMS]). A similar fact holds for continuous graph maps (we call it the First Contraction Principle). It can also be deduced from the "spectral decomposition" (obtained

for graph maps in [B2]), but we give a direct proof and include in the statement some useful facts. In addition, we prove a Second Contraction Principle, which seems to be new (although it follows from [B2] as well). First we need to introduce appropriate language.

By the *diameter* of a set A we mean $\operatorname{diam} A = \sup_{x,y\in A} d(x,y)$; so we can speak of the diameter of any set ($\operatorname{diam}(\emptyset) = 0$). If (A_i) is a sequence of sets with $\operatorname{diam}(A_i) \to 0$ and for some (and hence for any) sequence $x_i \in A_i$ we have $x_i \to x$, then we say that A_i *converges* to x and write $A_i \to x$. If $A \subset G$ is a set, x is a periodic point of period k, and $f^{nk}(A) \to x$, then we say that the orbit of A *(f-)converges* to $\operatorname{orb}(x)$ (clearly, $\omega(y) = \operatorname{orb}(x)$ for all $y \in A$). By continuity, the orbit of A f-converges to a cycle if and only if it f^m-converges to a cycle for some m.

Consider an important special case. If P is a cycle of period k, then for every point $b \in P$ we denote by N_b the union of all connected sets that contain b and whose orbits f^k-converge to b. By continuity, in the definition of the set N_b we may also use any multiple of k as a period of b; the resulting set N_b will be the same. If b is periodic and there exists a point $a \in \operatorname{orb}(b)$ such that N_a is non-degenerate, then we call b a *sink*. Equivalently, a periodic point b is a *sink* if there exists a connected non-degenerate set that intersects $\operatorname{orb}(b)$ and whose orbit converges to $\operatorname{orb}(b)$. Note that we understand the notion topologically; in the smooth case, only attracting and sometimes neutral cycles can be sinks.

A set A will be called *roaming* if it is connected and $f^i(A) \cap f^j(A) = \emptyset$ for every $i \neq j$. If additionally the orbit of A does not converge to a cycle, we will call it *wandering*. Our use of this term agrees with its common use in one-dimensional dynamics (wandering intervals, wandering domains). Note that for a roaming set A we have $\operatorname{diam}(f^i(A)) \to 0$. If A is connected and either it is roaming, or its image under some iterate of f is a periodic point, or its orbit converges to the orbit of a sink, we say that A is *waning*. We choose this name since clearly if A is waning, then $\operatorname{diam}(f^i(A)) \to 0$. In fact, from the First Contraction Principle below it follows that for a connected A this property is equivalent to A being waning. Note that a roaming, wandering or waning set may well be a singleton (that is, consist of one point).

Finally, a subset $J \subset G$ is called an *interval* if it is homeomorphic to an interval of the real line.

Lemma 2.1. *Let $I \subset G$ be a connected set such that $I \cap f(I) \neq \emptyset$ and*

$$\liminf_{n\to\infty} \operatorname{diam}(f^n(I)) = 0.$$

Then the set $K_0^\infty = \bigcup_{j=0}^\infty f^j(I)$ is connected, and its orbit converges to a fixed point a of f (so $\lim_{n\to\infty} \operatorname{diam}(f^n(I)) = 0$). Moreover, $K_0^\infty \subset N_a$.

Proof. Since I intersects $f(I)$, then also $f^n(I)$ intersects $f^{n+1}(I)$ for each n. Therefore the sets $K_m^n = \bigcup_{j=m}^n f^j(I)$ (including the case $n = \infty$) are connected.

Since $\liminf_{n\to\infty} \operatorname{diam}(f^n(I)) = 0$, there is a sequence $k_n \to \infty$ with $f^{k_n}(I) \to a$, $a \in G$. Then $f^{k_n+1}(I) \to f(a)$, and since $\operatorname{diam}(f^{k_n}(I) \cup f^{k_n+1}(I)) \to 0$ we get $f(a) = a$.

Assume first that there is an N such that $f^{N+1}(I) \subset K_0^N$. Then by induction $f^s(I) \subset K_0^N$ for every $s > N$, and therefore $f^{k_n+s}(I) \subset K_{k_n}^{k_n+N}$. Thus, $K_m^\infty \subset K_{k_n}^{k_n+N}$ for any $m \geq k_n$. However, $K_{k_n}^{k_n+N} \to a$, and therefore $f^m(K_0^\infty) = K_m^\infty \to a$.

Assume now that such an N does not exist. Suppose that $a \in K_0^\infty$. This means that $a \in f^M(I)$ for some M. Since $f(a) = a$, we get $a \in f^n(I)$ for all $n \geq M$. Among the components of the intersection of a sufficiently small neighborhood U of a with K_0^n there is one that contains a; call it W_n. It is homeomorphic to the $l(n)$-od. The numbers $l(n)$ may increase from time to time, but they eventually stabilize. Thus, there is an $N \geq M$ such that for $n \geq N$ the set W_n is homeomorphic to the $l(N)$-od; so W_N is a neighborhood of a in K_0^∞. If n is sufficiently large, the set $f^{k_n}(I)$ has a very small diameter and contains a. Since it is contained in K_0^∞, it is also contained in W_N, which is in turn contained in K_0^N, a contradiction. Hence, $a \notin K_0^\infty$.

Hence, taking into account that the valence of a is finite, by replacing the sequence (k_n) with its subsequence we may assume that all sets $f^{k_n}(I)$ are contained in an interval $(a, b]$ for some $b \in G$ and that there are no branching points of G in $(a, b]$. Since K_0^∞ is connected and contains all $f^{k_n}(I)$, we can choose b in such a way that $(a, b] \subset K_0^\infty$. Then $a \notin f((a, b])$ while $f((a, b]) \cap (a, b] \neq \emptyset$ (the latter follows from $f^{k_n}(I) \cap f^{k_n+1}(I) \neq \emptyset$); so for any $x \in (a, b]$ we can say whether it gets mapped towards a or away from a. The third possibility, that $f(x) = x$, is excluded since then all $f^r(I)$ for r sufficiently large would contain x, which contradicts $f^{k_n}(I) \to a$.

By continuity, either all points of $(a, b]$ are mapped away from a or all are mapped towards a. Assume first that they are mapped away from a. Let $m > 0$ and let $f^m(I) \subset (a, b)$ be a sufficiently small interval close enough to a. Since $f^{m-1}(I) \cap f^m(I) \neq \emptyset$, then $f^{m-1}(I)$ contains points of $(a, b]$ close to a. On the other hand, $f^{m-1}(I)$ cannot contain points between $f^m(I)$ and b, because they would not be mapped into $f^m(I)$. Hence the set $f^{m-1}(I)$ is also contained in $(a, b]$ and is closer to a than $f^m(I)$. Repeating this argument m times, we get that I is contained in $(a, b]$ and is closer to a than $f^m(I)$. On the other hand, by the assumption, $f^{k_n}(I) \to a$; so we can choose n such that $f^{k_n}(I)$ is closer to a than I, a contradiction.

Thus, all points of $(a, b]$ are mapped towards a. Since $f^N(I) \subset (a, b]$ for $N = k_1$, we get $f^n(K_N^\infty) \to a$. Taking into account that $f^N(K_0^\infty) = K_N^\infty$, we get $f^n(K_0^\infty) \to a$.

In such a way we have proved that in all cases $f^n(K_0^\infty) \to a$ and a belongs to the closure of K_0^∞. Hence, since the set K_0^∞ is connected, it is contained in N_a. $\square$

Note that in the above lemma, a is a sink unless $f^n(I) = \{a\}$ for some n. Thus, as a corollary, we get the First Contraction Principle.

Corollary 2.2 (First Contraction Principle)**.** *Let $I \subset G$ be a connected set such that* $\liminf_{n\to\infty} \operatorname{diam}(f^n(I)) = 0$. *Then I is waning.*

Remark 2.3. In the above situation, if I is not roaming, a is not a sink and $f^n(I)$ consists of more than one point, then $f^n(I)$ is disjoint from the orbit of a.

Our Second Contraction Principle is related to the First Contraction Principle, but deals with sequences of sets rather than with one set. It can also be deduced from the "spectral decomposition" ([B2]). To state it we need some definitions.

Let I be a connected subset of G. Then we can speak of the endpoints of I (they are the endpoints of the graph $\overline{I}$ and the points of $\overline{I} \setminus I$). We would like to be able to use similar notions as for the case of intervals of a real line, when we speak of an open or closed interval, and this indicates only whether we include endpoints or not. In our case the notion of a closed set gives us what we want: if I contains all

its endpoints, then it is closed. However, the topological notion of an open set may give us what we do not want. For example, if G has the shape of the letter "T", then we would like its horizontal segment without endpoints to be open, but the branching point belongs to its boundary. Moreover, we would not like the vertical segment with the lower endpoint included to be open, but it is. Thus, we say that a connected set I is *endless* if it does not contain any of its endpoints.

Now we need a notion of convergence for intervals in G. Since we do not really care about the endpoints, we assume that the limit is endless (except when it is a singleton). Thus, if all I_n are intervals and I is either an endless interval with distinct endpoints or a singleton, we will write $I_n \to I$ if the Hausdorff distance between I_n and I goes to zero.

When speaking about terms of a sequence, by "almost all" we will mean "all but a finite number".

Proposition 2.4 (Second Contraction Principle)**.** *Let (I_n) be a sequence of intervals such that $I_n \to I$, and let (m_n) be a sequence of numbers such that*

$$\operatorname{diam}(f^{m_n}(I_n)) \to 0.$$

Assume that $f^m(I)$ is not a singleton for any m. Assume also that there are $k \geq 0$ and $l > 0$ such that $f^k(I) \cap f^{k+l}(I) \neq \emptyset$. Then the following properties hold.

(a) *For all sufficiently large n we have $f^k(I_n) \cap f^{k+l}(I_n) \neq \emptyset$.*
(b) *For some sink a with $f^l(a) = a$ and any interval J contained in infinitely many intervals I_n, the set $\bigcup_{i=0}^{\infty} f^{k+il}(J)$ is contained in N_a and its orbit converges to* $\operatorname{orb}(a)$.
(c) $f^k(\overline{I}) \subset \overline{N_a}$.

Proof. Suppose that $m_n \not\to \infty$. Then we can choose a subsequence with $m_n = m$; hence by continuity $f^m(I)$ is a singleton and we get a contradiction. Therefore $m_n \to \infty$.

Since $f^k(I) \cap f^{k+l}(I) \neq \emptyset$, there are points $x, y \in I$ such that $f^k(x) = f^{k+l}(y)$. Observe that x and y are not endpoints of I, because we agreed that I is an endless interval. Hence the fact that $I_n \to I$ implies that x and y belong to almost all of the intervals I_n; so for all sufficiently large n we have $f^k(I_n) \cap f^{k+l}(I_n) \neq \emptyset$. This proves (a).

To prove (b), assume that the interval J is contained in infinitely many of the intervals I_n. Since any closed interval contained in I is contained in almost all intervals I_n, we may assume that $f^k(J) \cap f^{k+l}(J) \neq \emptyset$. Since $\operatorname{diam}(f^{m_n}(I_n)) \to 0$ and $m_n \to \infty$, we have $\liminf_{s\to\infty} \operatorname{diam}(f^s(J)) = 0$; so by Lemma 2.1 the set $\bigcup_{i=0}^{\infty} f^{k+il}(J)$ is contained in N_a and its orbit converges to $\operatorname{orb}(a)$ with $f^l(a) = a$. Moreover, $\bigcup_{i=0}^{\infty} f^{k+il}(J)$ is contained in N_a. Note that a different choice of J yields the same point a, since two intervals close to I intersect each other. Also, a is a sink because no image of I is a singleton, and we can always enlarge J so that $f^{k+l}(J)$ and therefore N_a are not degenerate.

Since $f^k(\overline{J}) \to f^k(\overline{I})$ as $J \to I$, we see that $f^k(\overline{I}) \subset \overline{N_a}$, which proves (c). □

In the above proposition one can replace intervals by general connected sets. However, this requires introduction of a rather complicated notion of convergence for connected sets in order to avoid anomalies due to two or more endpoints of the limit coinciding. Since we will apply this proposition only for intervals, we decided to spare the reader those details.

Remark 2.5. By the definition, the orbits of all points of N_a converge to $\mathrm{orb}(a)$. Therefore in the above situation if the orbit of a point $z \in f^k(\overline{I})$ does not converge to $\mathrm{orb}(a)$, then z belongs to the finite set $\overline{N_a} \setminus N_a$. Moreover, N_a and therefore $\overline{N_a}$ are f^l-invariant; hence the entire orbit of z under f^l is contained in the finite set $\overline{N_a} \setminus N_a$. Thus, z is (pre)periodic (by this we understand periodic or preperiodic). The orbit of z is disjoint from $\mathrm{orb}(a)$ since it does not converge to $\mathrm{orb}(a)$.

Proposition 2.6. *Let (I_n) be a sequence of connected sets and let (m_n) be a sequence of nonnegative integers such that* $\mathrm{diam}(f^{m_n}(I_n)) \to 0$. *Assume that for some points x, y and sequences of nonnegative integers (i_n), (j_n) we have $f^{i_n}(x) \in I_n$, $f^{j_n}(y) \in I_n$ and $f^{i_n}(x) \to x'$, $f^{j_n}(y) \to y'$. Then* $\mathrm{diam}(f^n(K)) \to 0$ *for some connected set K containing x' and y' (so, by the First Contraction Principle, K is waning), and hence $\omega(x') = \omega(y')$.*

Proof. If $x' = y'$, then there is nothing to prove. Hence, we assume that $x' \neq y'$. We can replace each I_n by a smaller set (we will call it also I_n) which is a closed interval with endpoints $f^{i_n}(x)$ and $f^{j_n}(y)$. Then we can replace the sequences (i_n) and (j_n) by subsequences (we will keep their names, too) such that the intervals I_n converge to an endless interval I whose endpoints are x' and y'. Then either some image of I is a singleton, or I is roaming, or the Second Contraction Principle applies. As K we take the closure of I. In the first two cases $\mathrm{diam}(f^n(I)) \to 0$; so also $\mathrm{diam}(f^n(K)) \to 0$.

Let us consider the third case. We have to prove that then $\mathrm{diam}(f^n(K)) \to 0$. To this end we consider several possibilities. First, if for infinitely many n we have $x' \in I_n$ and $y' \in I_n$, then $K \subset I_n$, and by statement (b) of the Second Contraction Principle, the orbit of K converges to $\mathrm{orb}(a)$; so $\mathrm{diam}(f^n(K)) \to 0$.

In the remaining case, for almost all n at least one of the points x', y' does not belong to I_n. We may assume that x' does not belong to I_n for infinitely many n's. Then, by replacing the sequence (i_n) by a subsequence, we get $x' \notin I_n$ for all n (equivalently, $f^{i_n}(x) \in I$). We have $i_n \to \infty$, because $f^{i_n}(x) \to x' \notin I_n$. Set $k = 0$, $l = i_2 - i_1$. Then for every interval J contained in infinitely many I_n's such that $f^{i_1}(x), f^{i_2}(x) \in J$ (in particular, for all compact intervals $J \subset I$ which are close enough to I) we have $J \cap f^l(J) \neq \emptyset$, and thus by statement (b) of the Second Contraction Principle the connected set $J' = \bigcup_{i=0}^{\infty} f^{il}(J)$ is contained in N_a and its orbit converges to $\mathrm{orb}(a)$ for some sink a with $f^l(a) = a$. We may assume that the f^l-orbit of J' converges to a and a does not depend on the choice of J. In particular, for any point $z \in I$ its f^l-orbit converges to a, which is the main conclusion of this paragraph we will need below.

Let us again replace the sequences (i_n) and (j_n) by subsequences and, if necessary, replace the points x, y by their forward iterates, so that all numbers i_n, j_n are multiples of l. Then $f^{i_n}(x) \to x' = a$ (we have $x' = a$, since all $f^{i_n}(x)$ belong to the f^l-orbit of $f^{i_1}(x) \in I$). Now, if $f^{j_n}(y) \in I$ for some n, then similarly $f^{j_n}(y) \to y' = a$, a contradiction with $x' \neq y'$. Thus, we may assume that $f^{j_n}(y) \notin I$ for all n, and hence $y' \in I_n$ for almost all n.

Choose $b \in I$ close to a and let $J = [b, y']$ be a subinterval of I. Then $J \subset I_n$ for almost all n. Our previous arguments apply to J; so the f^l-orbit of the connected set $J' = \bigcup_{i=0}^{\infty} f^{il}(J)$ converges to a. Moreover, since all i_n are multiples of l and $f^{i_n}(x) \in I$, we see that J' contains a small semineighborhood of a in I. When we choose $\hat{b}$ inside this semineighborhood, then for the corresponding $\hat{J} = [\hat{b}, y']$ we

have $\hat{J}' = \bigcup_{i=0}^{\infty} f^{il}(\hat{J}) \supset I$. As we have shown, the orbit of $\hat{J}'$ converges to a, and thus the orbit of I also converges to a. This completes the proof. □

3. Limit sets of followed points

In this section we establish conditions under which the limit set of a point coincides with one of the limit sets from some finite collection. They serve as a tool to get our further results, and hopefully could be applied to other dynamical systems too. Throughout this section $T : X \to X$ is a continuous map of a metric compact space X with metric d, and $C \subset X$ is a finite set. In this non-smooth situation one can still call a periodic point a of period m *repelling (topologically)* if in some metric d_1 equivalent to d, for some $\varepsilon > 0$ and any point $x \neq a$ which is at most ε away from a, we have $d_1(T^m(x), a) > d_1(x, a)$. If in the proofs we use the fact that some point is repelling, we will assume that our metric is already modified as above.

Now we introduce our Basic Setup. It consists of definitions and notation, and depends on a choice of a point $x \in X$ and a set C. We give it a special name since we will have to refer to it several times.

Basic Setup. Suppose that $x \in X$ and for every integer $i \geq 0$ an integer $m_i \in [0, i]$ and a point $c_i \in C$ are chosen. Then we use the following definitions and notation (for simplicity we sometimes skip the dependence on x).

1. Denote by C_x the set of points of C for which the sequence of numbers i such that $c_i = c$ is infinite.
2. Write $z \succ y$ if $\omega(z) \supset \omega(y)$; let C'_x be the set of all $\succ$-maximal elements of C_x.
3. For a given $c \in C_x$, if the sequence of numbers m_i with $c_i = c$ does not tend to ∞, then we call this case *bounded* (for c); otherwise we call the case *unbounded* (for c).
4. A pair of points $(T^r(x), T^{r-m_i}(c_i))$, $m_i \leq r \leq i$, is called an *i-pair*. It is called an *(i, ε)-pair* if $d(T^i(x), T^{i-m_i}(c_i)) < \varepsilon$.
5. If $c \in C_x$, then the set of all accumulation points of the sequence $(T^{m_i}(x))$, where $c_i = c$, will be denoted by L_c. Clearly, $L_c \subset \overline{\mathrm{orb}(x)}$. Moreover, in the unbounded case for c we have $L_c \subset \omega(x)$.
6. A pair of points (x', c') that is the limit for some sequence of i-pairs with $i \to \infty$ and $c_i = c \in C$ with $\omega(c)$ not minimal is called a *limiting pair*.

Note that if $c \in C_x$, $\omega(c)$ is not minimal and $z \in L_c$, then (z, c) is a limiting pair.

We will also often invoke the following condition; so we call it basic.

Basic Condition. $d(T^i(x), T^{i-m_i}(c_i)) \to 0$ as $i \to \infty$.

With the Basic Setup, we will say that x is *C-followed* if the Basic Condition holds and for any limiting pair (x', c') we have $\omega(x') = \omega(c')$.

Clearly, a Basic Setup for the set C gives us a Basic Setup for the set C_x. Thus, if x is C-followed, then it is C_x-followed.

The simplest Basic Setup that we will use is when x is a persistent point of a rational map f (see the Introduction). Then for any i we have a generating pair (c_i, m_i) for r_i, and by the definition of a persistent point the Basic Condition is satisfied.

We begin with a well-known result from topological dynamics, whose proof we include for the sake of completeness.

Lemma 3.1. *Let $x \in X$, $M > 0$, and let $K \subset \omega(x)$ be a compact set such that $T^M(W \cap \omega(x)) \subset K$ for some open set $W \supset K$. Then $\omega(x) = \bigcup_{i=0}^{M-1} T^i(K)$. In particular, if $M = 1$ then $K = \omega(x)$. Moreover, finite limit sets are cycles.*

Proof. By the continuity of T and the compactness of X, for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that if $d(a, b) < \delta$, then $d(T^M(a), T^M(b)) < \varepsilon$. If ε is sufficiently small, then the 2ε-neighborhood of K is contained in W. Take N such that if $n > N$, then $T^n(x)$ is in the δ-neighborhood of $\omega(x)$.

Denote by K_ε the ε-neighborhood of K. Assume that $n > N$ and $T^n(x) \in K_\varepsilon$. There is $z \in \omega(x)$ such that $d(T^n(x), z) < \delta$, and since $\delta < \varepsilon$, this z is in W. By the assumptions, $T^M(z) \in K$, and so $T^{n+M}(x) \in K_\varepsilon$. By induction, $T^{n+iM}(x) \in K_\varepsilon$ for all $i \geq 0$; so the ω-limit set of x for T^M is contained in K_ε. Since ε can be arbitrarily small, the ω-limit set of x for T^M is contained in K. Thus, $\omega(x) \subset \bigcup_{i=0}^{M-1} T^i(K)$. The reverse inequality follows from the assumption that $K \subset \omega(x)$.

By applying the above results to the case of a finite $\omega(x)$ (with K consisting of one point), we see that it is a cycle. □

In the next lemma we establish easy facts from the theory of dynamical systems, and draw conclusions related to points for which the Basic Condition holds. For a set D, denote by $\mathrm{orb}(D)$ the union of the orbits of all points of D and by $\omega(D)$ the union of limit sets of all points of D.

Lemma 3.2. *The following properties hold.*

1. *If $\omega(z) = P$ is a topologically repelling cycle, then z is eventually mapped into P.*
2. *If $\mathrm{orb}(z)$ is infinite and periodic points in X are topologically repelling, then $\omega(z)$ is infinite.*
3. *Suppose the Basic Condition holds for a set C and a point $x \in X$. Then:*
   (a) *$\omega(x) \subset \omega(C_x)$,*
   (b) *if $\omega(x)$ is infinite, then it is contained in the union of all infinite limit sets of points of C_x, and*
   (c) *if $c \in C_x$ and $\omega(c)$ is minimal, then $\omega(c) \subset \omega(x)$, and if $c \in C'_x$, then $\omega(c) = \omega(x)$.*

Proof. (1) If we replace T by an iterate of T, then we may assume $P = \{a\}$ to be a fixed point. Suppose that z is not eventually mapped into a. Then the orbit of z is infinite. Choose ε so that for any point $x \neq a$ which is at most ε away from a we have $d(T(x), a) > d(x, a)$. Since the orbit of z is infinite, there are arbitrarily large n such that $0 < d(T^n(z), a) < \varepsilon$. For any such n there exists $m > n$ such that $d(T^m(z), a) \geq \varepsilon$. Thus, there exists a sequence $m_i \to \infty$ such that $d(f^{m_i}(z), a) \geq \varepsilon$, a contradiction with $\omega(z) = \{a\}$.

(2) Follows from (1) and Lemma 3.1.

(3) (a) We have $\omega(x) \subset \overline{\bigcup_{c \in C} \mathrm{orb}(c)}$. Hence all points of $\omega(x)$ not belonging to $\omega(C)$ must belong to the orbits of points of C. Let $y \in \omega(x) \setminus \omega(C)$. Since f maps $\omega(x)$ onto itself, for any i there exists a point $y_i \in \omega(x)$ with $T^i(y_i) = y$. Then $y_i \notin \omega(C)$, and hence y_i belongs to the orbit of a point of C for any i. Since C is finite, y_j is periodic for some j, and thus $y_j \in \omega(C)$, a contradiction. Since the Basic Condition holds for the set C_x as well, we get the desired conclusion.

(3) (b) Let $\omega(x)$ be infinite. Denote by A the union of all infinite limit sets of points of C_x and by B the union of all finite limit sets $\omega(c)$, $c \in C_x$, non-disjoint

from $\omega(x)$ (and hence such that $\omega(c) \subset \omega(x)$). Then by (3)(a) $\omega(x) \subset A \cup B$. By Lemma 3.1, all periodic points that belong to $\omega(x)$ are not isolated in $\omega(x)$, and hence all points of B are not isolated in $\omega(x) \setminus B \subset A$. Since A is closed, this implies that $\omega(x) \subset A$.

(3) (c) If $c \in C_x$, then by the Basic Condition the intersection $\overline{\mathrm{orb}(c)} \cap \omega(x)$ is non-empty. If y belongs to this intersection, then $\omega(y) = \omega(c)$ because of the minimality of $\omega(c)$, while $\omega(y) \subset \omega(x)$ because $y \in \omega(x)$. Thus, $\omega(c) \subset \omega(x)$, as desired.

Now let $c \in C'_x$, and let $\omega(c)$ be minimal. Then, for any $d \in C_x$, either $\omega(c)$ and $\omega(d)$ are disjoint or $\omega(c) = \omega(d)$. Thus, by the claim (a), if $\omega(c) \neq \omega(x)$, then $\omega(c)$ is an open and closed (in the subspace topology) non-trivial subset of $\omega(x)$ whose image is contained in it; however by Lemma 3.1 this is impossible. Hence, $\omega(x) = \omega(c)$. □

The next lemma lists some useful properties of C-followed points and partially relies upon Lemma 3.2(3).

Lemma 3.3. *Suppose that x is C-followed. Then the following hold.*

1. *$\omega(x) = \omega(C_x) = \omega(C'_x)$, so that the collection of possible limit sets of C-followed points consists of at most $2^{\operatorname{card} C} - 1$ sets $\{\omega(D) : D \subset C, D \neq \emptyset\}$.*
2. *For any point $z \in \omega(x)$, if $\omega(z) \supset \omega(c)$ for some $c \in C'_x$, then $z \in \omega(z) = \omega(c)$.*
3. *For any $c \in C'_x$ we have $\overline{\mathrm{orb}(c)} \cap \omega(x) = \omega(c)$.*
4. *If for $c \in C'_x$ an unbounded case holds and if $z \in L_c$, then $z \in \omega(z) = \omega(c)$ (so that, in particular, $L_c \subset \omega(c)$).*

Proof. (1) By Lemma 3.2(3)(a) $\omega(x) \subset \omega(C_x)$. On the other hand, $\omega(C_x) \subset \omega(x)$ because for any $c \in C_x$ either $\omega(c)$ is minimal, and then $\omega(c) \subset \omega(x)$ by Lemma 3.2(3)(c), or $\omega(c)$ is not minimal, and then $\omega(c) = \omega(z) \subset \omega(x)$ for any $z \in L_c \subset \overline{\mathrm{orb}(x)}$. Hence, $\omega(x) = \omega(C_x)$. The second part of the statement immediately follows.

(2) Suppose that $z \in \omega(x)$ and $\omega(z) \supset \omega(c)$ for some $c \in C'_x$. Then (1) implies that $z \in \omega(u)$ for some $u \in C_x$, and so $\omega(u) \supset \omega(z) \supset \omega(c)$. Thus, $u \succ c$. Since c is $\succ$-maximal, we see that $\omega(c) = \omega(u) = \omega(z) \ni z$.

(3) By (1) $\omega(c) \subset \omega(x)$, and so $\overline{\mathrm{orb}(c)} \cap \omega(x) \supset \omega(c)$. If $T^k(c) \in \omega(x)$ for some k, then by (2) $T^k(c) \in \omega(c)$, and hence $\overline{\mathrm{orb}(c)} \cap \omega(x) = \omega(c)$.

(4) Clearly, in the unbounded case $L_c \subset \omega(x)$. Consider now two cases. First, let $\omega(c)$ be minimal. Then by Lemma 3.2(3)(c) we have $\omega(x) = \omega(c)$; so $\omega(x)$ is minimal too, and thus $z \in \omega(z) = \omega(c)$. Now, let $\omega(c)$ be non-minimal. Then for any $z \in L_c$, (z, c) is a limiting pair; so $\omega(z) = \omega(c)$. Thus by (2) we get $z \in \omega(z) = \omega(c)$. □

Theorem 3.4. *If x is C-followed, then $\omega(x) = \omega(c)$ for some $c \in C$. Moreover, this c can be chosen in such a way that if the bounded case holds for it, then $T^m(x) \in L_c$ for some $m \in \mathbb{N}$, while in the unbounded case $L_c \subset \omega(x)$, each point of L_c is recurrent and has limit set $\omega(x)$.*

Proof. Suppose that for some $c \in C'_x$ the bounded case holds. Then there exist a number m and a sequence of $i \to \infty$ such that $c_i = c$ and $m_i = m$ for all elements of this sequence. This implies that $T^m(x) \in L_c$ and hence $\omega(x) = \omega(c)$ (by the definition of C-following if $\omega(c)$ is not minimal, or by Lemma 3.2(3)(c) if $\omega(c)$ is

minimal). This proves the theorem in the case when for some $c \in C'_x$ the bounded case holds.

For the rest of the proof we assume that for any $c \in C'_x$ the unbounded case holds. If for some $c \in C'_x$ the set $\omega(c)$ is minimal, then by Lemma 3.2(3)(c) we have $\omega(x) = \omega(c)$ and all the claims of the theorem about the unbounded case immediately follow. Hence, for the rest of the proof we assume also that for any $c \in C'_x$ the set $\omega(c)$ is not minimal.

Fix $u \in C'_x$. By Lemma 3.3(4), $L_u \subset \omega(u) \subset \omega(x)$ and $\omega(z) = \omega(u)$ for any $z \in L_u$ (so, in particular, if $\omega(u) = \omega(x)$, then the claims of the theorem about the unbounded case hold). If $\omega(u) \neq \omega(x)$, then for some $\gamma > 0$ there exists an arbitrarily large n with $d(T^n(x), \omega(u)) > 2\gamma$. We may assume that γ satisfies additionally the following conditions. By $d(A, B)$ we denote the infimum of $d(a, b)$ for $a \in A$, $b \in B$.

1. For any $c \in C'_x$ and any set L_w disjoint from $\omega(c)$ we have $d(\omega(c), L_w) > 2\gamma$.
2. For any $c, c' \in C$, if $\omega(c) \cap \omega(c') = \emptyset$, then $d(\overline{\mathrm{orb}(c)}, \overline{\mathrm{orb}(c')}) > 2\gamma$.
3. If $c, c' \in C$ and $c \notin \omega(c')$, then $d(c, \omega(c')) > 2\gamma$.

We draw a contradiction from this. Choose $\nu < \gamma$ such that $d(y, z) \leq \nu$ implies $d(T(y), T(z)) < \gamma$.

Step 1. *There exist sequences $i \to \infty$ and $r_i \to \infty$ with $r_i > m_i$ such that:*

1. *$c_i = u$, $T^{m_i}(x) \to a \in L_u \subset \omega(u)$ and $T^{r_i}(x) \to b \in \omega(x) \setminus \omega(u)$ (which always implies $r_i - m_i \to \infty$);*
2. *$d(T^j(x), \omega(u)) \leq \gamma$ for all $m_i \leq j \leq r_i$, and so $d(b, \omega(u)) \leq \gamma$.*

Indeed, the infinite sequence of numbers i with $c_i = u$ contains a subsequence along which $T^{m_i}(x) \to a$ for some a. Since we deal with the unbounded case, $m_i \to \infty$, and then by the definition and Lemma 3.3(4) $a \in L_u \subset \omega(u)$. Thus, the first part of claim (1) of Step 1 holds.

We may assume that $d(T^{m_i}(x), a) < \nu$. First let us show that for any sequence $r_i > m_i$ with $T^{r_i}(x) \to b \notin \omega(u)$ we must have $r_i - m_i \to \infty$. Indeed, otherwise, by continuity and because $T^{m_i}(x) \to a \in \omega(u)$, we conclude that $b \in \omega(u)$, a contradiction. Also, this implies that $r_i \to \infty$, and so $b \in \omega(x)$.

Let us now choose a sequence r_i with the required properties. By the properties of γ there exists a minimal $r_i > m_i$ such that $d(T^{r_i}(x), \omega(u)) > \nu$. By passing to a subsequence we may assume that $T^{r_i}(x) \to b \notin \omega(u)$. By the choice of r_i we have $d(T^{r_i - 1}(x), \omega(u)) \leq \nu$, and hence $d(T^{r_i}(x), \omega(u)) < \gamma$. Hence, $d(T^j(x), \omega(u)) \leq \gamma$ for $m_i \leq j \leq r_i$. This completes the proof of Step 1.

Step 2. *By refining our sequences we may get new sequences i and r_i such that $c_{r_i} = v \notin \omega(u)$ for all i, $T^{r_i}(x) \to b \in (\omega(x) \cap \overline{\mathrm{orb}(v)}) \setminus \omega(u)$ and $T^{m_{r_i}}(x) \to s \in L_v$.*

We can refine our sequences so that $c_{r_i} = v$ for all i and $T^{m_{r_i}}(x) \to s \in L_v$ (since $r_i \to \infty$, we have $v \in C_x$). Since $d(T^{r_i}(x), T^{r_i - m_{r_i}}(v)) \to 0$, we get $b \in \overline{\mathrm{orb}(v)}$. Hence, $b \in (\omega(x) \cap \overline{\mathrm{orb}(v)}) \setminus \omega(u)$. Clearly, $v \notin \omega(u)$. This completes the proof of Step 2.

Step 3. *The point v is not recurrent, $\omega(v) \subset \omega(u)$, and so $b = T^l(v) \in \overline{\mathrm{orb}(v)} \setminus \omega(v)$, where $l > 0$ is well-defined. We may refine our sequences further so that $r_i = m_{r_i} + l$, $b = T^l(s) = T^l(v)$ and $s \in \omega(x) \setminus \omega(u)$.*

Let us show that $\omega(v) \subset \omega(u)$. First suppose that $\omega(v)$ is minimal. If $\omega(v) \not\subset \omega(u)$, then $\omega(v)$ and $\omega(u)$ are disjoint, and so $d(\overline{\mathrm{orb}(v)}, \overline{\mathrm{orb}(u)}) > 2\gamma$ by the choice

of γ. On the other hand $b \in \overline{\mathrm{orb}(v)}$ and $d(b, \omega(u)) \leq \gamma$. This contradiction shows that in this case $\omega(v) \subset \omega(u)$. Now suppose that $\omega(v)$ is not minimal. We begin by showing that eventually $m_{r_i} \geq m_i$. Indeed, assume that there exist sequences from Step 2 with $m_{r_i} < m_i$ for all i. We may assume that $T^{m_i - m_{r_i}}(v) \to d$. Since $m_i < r_i$, we see that $(a = \lim T^{m_i}(x), d = \lim T^{m_i - m_{r_i}}(v))$ is a limiting pair. Then by the definition of C-following $\omega(d) = \omega(a)$, and since $a \in L_u$, $\omega(a) = \omega(u)$. Therefore $\omega(v) \supset \omega(d) = \omega(u)$. By $\succ$-maximality of u we conclude that $\omega(v) = \omega(u)$, and so v is also $\succ$-maximal. Now, by Lemma 3.3(3), $\overline{\mathrm{orb}(v)} \cap \omega(x) = \omega(v)$. Since by Step 2 $b \in (\omega(x) \cap \overline{\mathrm{orb}(v)}) \setminus \omega(u)$, we conclude that $b \in \omega(v) \setminus \omega(u)$, a contradiction.

Since $r_i \geq m_{r_i} \geq m_i$, by Step 1(2) we have $d(s, \omega(u)) \leq \gamma$. Since $s \in L_v$, the distance between L_v and $\omega(u)$ is at most γ. By the choice of γ then $\omega(u)$ and L_v are non-disjoint, i.e., there exists $z \in \omega(u) \cap L_v$. Since by the definition of C-following $\omega(v) = \omega(z)$, we get $\omega(v) \subset \omega(u)$.

Clearly, $b \in \omega(v)$ is then impossible because by Step 2 $b \notin \omega(u)$. Since by Step 2 $b \in \overline{\mathrm{orb}(v)}$, we conclude that $b = T^l(v) \in \overline{\mathrm{orb}(v)} \setminus \omega(v)$, and for some $\varepsilon > 0$ and any $k \neq l$ we have $d(b, T^k(v)) > \varepsilon$. Also, since $v \notin \omega(u)$, we get $d(v, \omega(u)) > \gamma$ by the choice of γ, while $d(b, \omega(u)) \leq \gamma$. Thus $b \neq v$, and hence $l > 0$. On the other hand, clearly $T^{r_i - m_{r_i}}(v) \to b$. We conclude that for a sufficiently large i from our sequence $r_i = m_{r_i} + l$. By continuity, $b = T^l(v) = T^l(s)$. If now $s \in \omega(u)$, then $b \in \omega(u)$, a contradiction. Hence, $s \notin \omega(u)$, while obviously $s \in \omega(x)$. The existence of $b \in \overline{\mathrm{orb}(v)} \setminus \omega(v)$ shows that v is not recurrent. This completes the proof of Step 3.

Thus, starting with sequences from Step 1, we can after a few refinements obtain sequences i (refined) and m_{r_i}. It is easy to see that they satisfy the properties from Step 1, with the sequence m_{r_i} playing the role of the sequence r_i and the point s playing the role of b. Indeed, by Step 1 we have $r_i - m_i \to \infty$, and by Step 3 we have $m_{r_i} = r_i - l$; so we may assume that $m_{r_i} > m_i$. Now, the properties from Step 1 obviously hold. However, there is an important difference between the initial sequences and the new ones: now for some number $l > 0$ we have $b = T^l(s) = T^l(v)$ with $v \in C_x$. Making Steps 2 and 3 again, we obtain a new number l' and find new points s', v' for which we have $s = T^{l'}(s') = T^{l'}(v')$, and so on by induction.

Now, there are only finitely many points in the set C; hence at some moment after finitely many (no more than $\operatorname{card} C$) steps in the process of refinements of the original sequence the same point from C will appear twice in the sequence of points $v, v', \ldots$. This implies that the point b is periodic and cannot belong to $\overline{\mathrm{orb}(v)} \setminus \omega(v)$, a contradiction which completes the proof. □

The next results show that under certain circumstances, modeling the situation of graph critical maps, a point is C-followed, which allows us to apply Theorem 3.4. The verification of the fact that a point is C-followed relies on both Contraction Principles from Section 2.

We say that two orbit segments $(x, \ldots, T^n(x))$ and $(y, \ldots, T^n(y))$ *ε-approximate each other* if $d(T^i(x), T^i(y)) \leq \varepsilon, 0 \leq i \leq n$. Suppose now that a graph $G \subset X$ is T-invariant, and there are a finite set $C \subset X$, a point $x \in X$ such that $\mathrm{orb}(x) \cap C = \emptyset$, and sequences $m_i \in \mathbb{N}$ and $c_i \in C$, $d_i \in \mathrm{orb}(C \cap G)$ such that the following properties hold:

1. $\omega(c)$ is minimal for any $c \in C \setminus G$ such that $c = c_i$ for infinitely many i's;
2. for every i, $m_i \leq i$ and $d(T^i(x), T^{i-m_i}(c_i)) \to 0$;

3. for any i such that $c_i \in C \cap G$ the orbit segment $(d_i, \dots, T^{i-m_i}(d_i))$ η_i-approximates the segment $(T^{m_i}(x), \dots, T^i(x))$, where $\eta_i \to 0$ and for some connected set $J_i \subset G$ we have $d_i, c_i \in J_i$ and $\operatorname{diam}(T^{i-m_i}(J_i)) \to 0$.

Although this situation looks specific, it is important for us since under certain circumstances it represents the dynamics of almost every point with respect to a conformal measure for a graph critical map. Therefore we give it a special name: if the point x has the above properties, then we say that x *nicely approaches* G.

One should understand the phrase "nicely approaches G" as short for "approaches G nicely if at all". In particular, if the orbit of x converges to a minimal limit set of a point $c \in C$ and $\omega(c)$ is disjoint from G, this is also counted as a nice approach to G (in this case (3) is void).

Now we can state the most important (for applications) result of this section.

Theorem 3.5. *Suppose that a point x nicely approaches G. Then x is C-followed, and therefore there exists $c(x) \in C_x$ such that $\omega(x) = \omega(c(x))$. In addition, the following hold.*

1. *If $\omega(x)$ is not minimal, then $c(x) \in C \cap G$, for any $z \in L_{c(x)}$ we have $\omega(z) = \omega(x) = \omega(c(x))$, and there exists a roaming set $J \subset G$ containing z and $c(x)$. Moreover, if for $c(x)$ the unbounded case holds, then z is recurrent.*
2. *If all points $T^n(x)$ nicely approach G and $\omega(x)$ is not minimal, then we can choose $c(x)$ in such a way that $c(x) \in C \cap G$ and there is a roaming set $J \subset G$ containing $c(x)$ and a recurrent point.*

Proof. We have all necessary components of the Basic Setup for x. To prove that x is C-followed, we consider limiting pairs. For a given limiting pair (x', c'), by passing to a subsequence $\{i_j\}$ we may assume that $d_{i_j} = T^{k_j}(s)$ for all i_j and for some $s \in D$. Thus, we may assume that $T^{l_j}(x) \to x', T^{l_j - m_{l_j}}(c) \to c'$, where $(T^{l_j}(x), T^{l_j - m_{l_j}}(c))$ is an i_j-pair with $c \in C_x \cap G$, $m_{i_j} \le l_j \le i_j$ and $i_j \to \infty$ as $j \to \infty$. By property (3) of nice approaching, $d(T^{l_j}(x), T^{l_j - m_{l_j} + k_j}(s)) \le \eta_{i_j} \to 0$, and hence $T^{l_j - m_{l_j} + k_j}(s) \to x' \in G$. Moreover, the connected set $T^{l_j - m_{i_j}}(J_{i_j})$ contains both $T^{l_j - m_{l_j}}(c)$ and $T^{l_j - m_{l_j} + k_j}(s)$ and has $T^{i_j - l_j}$-images with diameters converging to 0. Therefore, by Proposition 2.6, $\omega(x') = \omega(c')$. Hence, x is C-followed (and thus C_x-followed), and then $\omega(x) = \omega(c(x))$ for some $c(x) \in C_x$ by Theorem 3.4.

Assume that the set $\omega(x)$ is not minimal and $z \in L_{c(x)}$. Then $(z, c(x))$ is a limiting pair. From the definition of nice approaching, $c(x) \in C \cap G$ (otherwise $\omega(x)$ would be minimal) and there is a sequence of connected sets $J_i \subset G$ such that $d_i, c(x) \in J_i$, $d_i \to z$ and $\operatorname{diam}(T^{i-m_i}(J_i)) \to 0$. By Proposition 2.6, there exists a waning set $J \supset \{z, c(x)\}$. Now, if for $c(x)$ the unbounded case holds, then $z \in \omega(x)$ is a recurrent point by Theorem 3.4.

Recall that if a set is waning, then either it is roaming, or its image under some iterate of f is a periodic point, or its orbit converges to the orbit of a sink. Since $c(x) \in J$, in the last two cases $\omega(c(x))$ is minimal. However, $\omega(c(x)) = \omega(x)$ and we assumed that $\omega(x)$ is not minimal. Therefore J is roaming. This completes the proof of (1).

Suppose now that in addition all points $T^n(x)$ nicely approach G. This means that for every n and the point $T^n(x)$ the choice of sequences and points from the definition of nice approaching is made. For every n this provides us with the corresponding point $c^{(n)}(x)$ such that $\omega(x) = \omega(T^n(x)) = \omega(c^{(n)}(x))$, and since the

set C does not depend on n we may find a sequence of n with $c^{(n)}(x) = c$ for some point $c \in C$, and, as before, $c \in C \cap G$ because $\omega(x) = \omega(c)$ is not minimal.

Apply (1) to a point $z_n \in L_{c,n}$, where n comes from the sequence chosen above and $L_{c,n}$ is like L_c, but constructed for $T^n(x)$ instead of x (clearly, it is non-empty); in particular, then there is a roaming set $J_n \ni c, z_n$. If the unbounded case holds for c and some $T^n(x)$, by (1) z_n is recurrent and we are done, setting $J = J_n$. Otherwise, for any n from our sequence we have $z_n = T^{k_n}(x)$ with $k_n \geq n$, and we may assume that $z_n \to z \in \omega(x)$. By Proposition 2.6, there exists a waning set $J \supset \{c, z\}$, and so $\omega(c) = \omega(z)$. By the same argument as in the proof of (1), we see that J is roaming. Since $\omega(c) = \omega(x)$, we get that z is recurrent. This proves (2). □

Remark 3.6. In the situation from Theorem 3.5(2), since the set J is roaming and contains a recurrent point y, it is either a wandering set or a singleton. Thus, if it is not wandering, it is equal to $\{y\}$ and y is recurrent and (pre)periodic; so it must be periodic. In particular, in this case $\omega(y)$ is minimal.

4. C-persistent points for graph critical rational maps

In this section we apply the results of Section 3 and study C-persistent points of a graph critical map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where C is the set of critical points of f. Recall that the definitions of a graph critical map, conformal measure μ with exponent α, of various kinds of disks, of sets $\mathrm{Prs}(f)$ of persistent points and $\mathrm{Rlc}(f)$ of reluctant points and other more traditional sets related to rational maps were given in the Introduction.

Let us start with two lemmas that are not related to graphs. Let us denote by $\mathbf{0}$ the origin and by $\mathbb{D}$ the unit disk. The first lemma is well known, but for the sake of completeness we include its proof. Recall that we are using the spherical metric on $\widehat{\mathbb{C}}$ induced by the embedding of $\widehat{\mathbb{C}}$ into three-dimensional Euclidean space. The diameter of a set A in this metric will be denoted by $\mathrm{diam}(A)$.

Lemma 4.1. *For any $\varepsilon > 0$ and $\zeta < \mathrm{diam}(\widehat{\mathbb{C}}) = 2$, there exist $\delta > 0$ and $N \geq 0$ such that if $x \in J(f)$, $\eta > 0$, $n > 0$, and Jordan disks $V_n \supset V_n'$ around x satisfy the following conditions:*

1. *V_n is a component of $f^{-1}(f(V_n))$ and V_n' is a component of $f^{-1}(f(V_n'))$,*
2. *$f^{n-1}|_{f(V_n)} : f(V_n) \to B(f^n(x), \eta)$ and $f^{n-1}|_{f(V_n')} : f(V_n') \to B(f^n(x), \eta/2)$ are univalent and onto,*

and either $\eta < \delta$ or $n > N$ and $\eta < \zeta$, then $\mathrm{diam}(V_n') < \varepsilon$.

Proof. Let us call a pair of Jordan disks $V_n \supset V_n'$ around x satisfying (1) and (2) an (x, n, η)-pair. If the assertion of the lemma does not hold, then we can find sequences (x_n), (i_n), (η_n) and a sequence of (x_n, i_n, η_n)-pairs of Jordan disks (V_n, V_n') such that $\mathrm{diam}(V_n') \geq \varepsilon$ and $x_n \in J(f)$ for each n, and either $\eta_n \to 0$ or $i_n \to \infty$ and $\eta_n < \zeta$.

If (i_n) is bounded, then $(\mathrm{diam}(f^{i_n}(V_n')))$ is bounded away from 0; so (η_n) is also bounded away from 0. Therefore we may assume that $i_n \to \infty$. We can also assume that $\eta_n < \zeta$ for all n. Moreover, there is $\varepsilon' > 0$ such that $\mathrm{diam}(f(V_n')) \geq \varepsilon'$ for all n.

Let $\gamma_n : B(f^{i_n}(x_n), \eta_n) \to \mathbb{D}$ be a conformal map such that $\gamma_n(f^{i_n}(x_n)) = \mathbf{0}$. It is unique up to composition with rotations, and can be realized geometrically

as the projection from the point antipodal to $f^{i_n}(x_n)$ to a plane perpendicular to the radius through this point, followed by the isometry from this plane to $\mathbb{C}$. Then $\gamma_n(B(f^{i_n}(x_n), \eta_n/2))$ is also a disk. A simple computation shows that its radius is equal to $\sqrt{(4-\eta_n^2)/(16-\eta_n^2)}$. Thus it is bounded from below by a positive constant ξ depending only on ζ, and from above by $1/2$. Consider the sequence of maps $\varphi_n = f^{-i_n+1} \circ \gamma_n^{-1} : \mathbb{D} \to f(V_n)$, where branches of inverse maps are chosen appropriately.

By (2) and since $i_n \to \infty$, critical points of f^k for a given k do not belong to $f(V_n)$ for sufficiently large n. Thus we may assume that all sets $f(V_n)$ miss two given points, and therefore by Montel's Theorem the family $\{\varphi_n\}$ is normal. Thus we may assume that $\varphi_n \to \varphi$. Since $\operatorname{diam}(\varphi_n(B(\mathbf{0}, 1/2))) > \varepsilon'$, and the sequence (φ_n) converges uniformly on $B(\mathbf{0}, 1/2)$, the map φ is not constant. Thus we may assume that $\varphi_n(\mathbf{0}) = f(x_n) \to z \in J(f)$ and that for some $\varepsilon'' > 0$ we have $B(z, \varepsilon'') \subset \varphi_n(B(\mathbf{0}, \xi)) \subset f(V_n')$. However, the f^{i_n-1}-images of a given ball centered at $z \in J(f)$ have to approach the whole sphere. This means that $\eta_n/2 \to 2$, a contradiction. □

The next lemma is an introduction to our study of C-persistent points. Denote by $P'(f)$ the union of all infinite limit sets of critical points of f.

Lemma 4.2. 1. *The set* $\operatorname{Prs}'(f) = \operatorname{Prs}(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(C)$ *is invariant.*

2. *If* $x \in \operatorname{Prs}(f)$*, then* $\omega(x) \subset \omega(C)$*; moreover, if the orbit of* $x \in \operatorname{Prs}'(f)$ *is not attracted to a neutral cycle, then* $\omega(x)$ *is infinite and is contained in* $P'(f)$*.*

Proof. (1) Clearly, $\operatorname{Prs}'(f)$ is the set of points $x \in J(f)$ such that $r_l(x) \to 0$ and $r_l(x) \neq 0$. Assume that $x \in \operatorname{Prs}'(f)$, but $f(x) \notin \operatorname{Prs}'(f)$. Then for some $\delta > 0$ there exist a sequence $n_k \to \infty$ and Jordan disks W_k around $f(x)$ such that $f^{n_k}|_{W_k} : W_k \to B(f^{n_k+1}(x), \delta)$ is univalent and onto. Choose Jordan disks $W_k' \subset W_k$ around x so that $f^{n_k}|_{W_k'} : W_k' \to B(f^{n_k+1}(x), \delta/2)$ is univalent and onto. By Lemma 4.1, $\operatorname{diam}(W_k') \to 0$. Since $x \notin C$, for sufficiently big k we can find a Jordan disk W_k'' around x such that the map $f|_{W_k''} : W_k'' \to W_k'$ is univalent and onto. This shows that $r_{n_k+1}(x) \not\to 0$, a contradiction.

(2) By Lemma 3.2(3)(a), we have $\omega(x) \subset \omega(C)$. We suppose that the orbit of $x \in \operatorname{Prs}'(f)$ is not attracted by a neutral cycle, and prove that then $\omega(x)$ is infinite. Indeed, if $\omega(x)$ is finite, it has to be a cycle, and then the orbit of x is attracted to it, by the definition of attraction. Thus, this cycle is not neutral, and since it is contained in the Julia set, it is repelling. By Lemma 3.2(1) this implies that $f^n(x) = a$, where a is a repelling periodic point of f of period m for some m. Since x is not an eventual preimage of a critical point of f, there are an r-disk W around a and a Jordan disk V around x such that $f^m(W) \supset W = f^n(V)$ and $f^{m+n}|_V$ is univalent. This implies that $r_l(x) > r$ for all $l = n + mi, i \geq 0$, a contradiction. Hence $\omega(x)$ is infinite, and by Lemma 3.2(3)(b), $\omega(x) \subset P'(f)$. □

Now we prove some lemmas related to graphs. We will use intersections of disks with the graph G and their preimages. Since G is not necessarily smooth, for $x \in G$ and $\varepsilon > 0$ the intersection $\overline{B}(x, \varepsilon) \cap G$ can be complicated. Therefore we use Jordan disks rather than the usual round disks. We will call an open Jordan disk U a *(G)-elementary neighborhood* if $U \cap G$ is homeomorphic to the open n-od for some n and $\overline{U \cap G}$ is equal to $\overline{U} \cap G$, is homeomorphic to the closed n-od, and contains at most one branching point. Clearly, every point of G has an elementary neighborhood.

Lemma 4.3. *There exists an invariant graph $H \supset G$ such that $f^k(H) \subset G$ for some k, each critical point of f either belongs to H or has orbit disjoint from H, and the following property holds: if W is an H-elementary neighborhood and U is a component of $f^{-i}(W)$ such that*

1. *$f^j(U)$ is an H-elementary neighborhood for $1 \le j \le i$, and*
2. *$f^r(U)$ contains no critical points not belonging to H for $0 \le r < i$,*

then $U \cap H$ is connected.

Proof. We will construct H in several steps. During each step the new graph will contain the old graph and will be contained in the inverse image of the old graph under some iterate of f. This will give us $H \supset G$ and $f^k(H) \subset G$. We will also keep our graph at each stage invariant.

In the first step we replace G by its inverse image G_1 under a sufficiently high iterate of f, so that each critical point of f either belongs to G_1 or has orbit disjoint from G_1. Now we look at the valence of a given critical point from G_1 in $f^{-j}(G_1)$ for $j = 0, 1, \ldots$. It either stabilizes or goes to infinity. Thus, we can replace G_1 by its inverse image G_2 under a sufficiently high iterate of f, so that for each critical point $c \in G_2$ of f either its valence in $f^{-1}(G_2)$ is the same as in G_2 or its valence in G_2 is larger than 2 times the order of this point; so $f(c)$ is a branching point of G_2.

Now as H we choose an invariant graph such that $G_2 \subset H \subset f^{-j}(G_2)$ for some $j \ge 1$ and H has the minimal possible number of endpoints among such graphs. Since H contains G_1, each critical point of f either belongs to H or has orbit disjoint from H.

Let us prove the other property claimed in the lemma. We can use induction; so we need to prove it only for $i = 1$. Suppose that it does not hold for an H-elementary neighborhood W and a component U of $f^{-1}(W)$. Note that $f(U) = W$. The set $U \cap H$ is disconnected, but since by (2) there are no critical points in $U \setminus H$, the set $U \cap f^{-1}(H)$ is connected. Therefore there is a closed interval $I \subset U \cap f^{-1}(H)$ whose interior is disjoint from $U \cap H$ and whose endpoints a, b belong to two different components of $U \cap H$.

Set $H' = H \cup I$. If a or b is an endpoint of H, then $G_2 \subset H' \subset f^{-k-1}(G_2)$ and H' has fewer endpoints than H, a contradiction. Therefore a and b are branching points of H'. Let us show that $f(a)$ is a branching point of H. This is clear if a is not critical. If a is critical, then, by the construction of G_2, either the valence of a in H and $f^{-1}(H)$ is the same or $f(a)$ is a branching point of G_2. The first case is impossible, since the valence of a in H' is larger than in H; so the second case holds, and since $G_2 \subset H$, the point $f(a)$ is branching in H. Similarly, $f(b)$ is a branching point of H. Since they both belong to W, we see that $f(a) = f(b)$ is the only branching point of H in W. This implies that I contains a critical point eventually mapped into G, while all such critical points belong to G_1 by our construction. This contradiction completes the proof. □

Remark 4.4. In the situation from Lemma 4.3 the following hold.

1. If U contains more than one branching point of H, then it contains a critical point of f, because $f(U)$ is an elementary neighborhood.
2. If $f^{i-1}|_{f(U)}$ is univalent, then condition (2) holds automatically for $r > 0$, while condition (1) follows from the fact that $f^{i-1}(f(U))$ is an elementary

neighborhood and from Lemma 4.3 applied to the neighborhoods $f^j(U)$, $1 \leq j \leq i$, step by step.

Throughout the rest of this section we assume that our initial graph G is already extended (so $H = G$) and has the properties from Lemma 4.3 and Remark 4.4.

The following lemma establishes a simple property of continuous graph maps.

Lemma 4.5. *Let $f : G \to G$ be a continuous graph map and let $I \subset G$ be an open connected set such that the orbits of all points of I converge to cycles. Then the orbits of the endpoints of I also converge to cycles.*

Proof. If the diameter of $f^n(I)$ tends to 0, then the claim is obvious. Otherwise the images of I eventually intersect. Replacing f by an iterate of it, we may assume that $f(I) \cap I \neq \emptyset$. Then the union A of all images of I is a connected set, and the orbits of all points of A converge to cycles. Since the endpoints of I belong to the closure of A, it is enough to prove our claim for A instead of I.

There are finitely many endpoints of A. Those endpoints that are eventually mapped into A have orbits converging to cycles by the assumptions. Those that are not eventually mapped into A form a finite invariant set; so they are (pre)periodic. This completes the proof. □

As a consequence, we get the next lemma.

Lemma 4.6. *If R is a component of $G \setminus J(f)$, then the orbits of all its points converge to cycles and its endpoints are (pre)periodic.*

Proof. Clearly, R is contained in a Fatou domain, and by [Su2] some image $f^r(R)$ of R is contained in a periodic Fatou domain U. Replacing f by its iterate, we may assume that U is forward invariant. There are the following possibilities.

(a) The domain U is a Siegel disc or a Herman ring. Then the closure of the orbit of $f^r(R)$ contains a topological annulus, a contradiction.

(b) The domain U is the immediate basin for an attracting fixed point a or for one petal of a parabolic fixed point a with multiplier 1 (cf. [M2]). Then the orbits of all points of $f^r(R)$ converge to a, and it remains to consider the endpoints of R. By Lemma 4.5 their orbits converge to cycles. Since $G \cap J(f)$ is invariant, those cycles are contained in $G \cap J(f)$. By the Snail Lemma they are either repelling or parabolic, and since repelling and parabolic points are topologically repelling in $J(f)$, the only way this convergence can take place is when the endpoints of R are (pre)periodic. □

In the case when $J(f) \subset G$ let us discuss the local structure of $J(f)$ and G. Assume that $x \in J(f)$ has valence k in G. Then there are k (short) subintervals of G of the form $(y_i, x]$, whose interiors are pairwise disjoint. We call them *spokes at x*. A spoke $(y_i, x]$ is *good* if there is a sequence of points $z_n \in J(f) \cap (y_i, x]$ converging to x; otherwise it is of course *bad*. Note that if there is a bad spoke at x, then x is an endpoint of a component of $G \setminus J(f)$; so by Lemma 4.6 it is (pre)periodic.

Lemma 4.7. *Assume that $J(f) \subset G$. Let $x \in J(f)$ be not (pre)periodic. Then $\mathrm{val}_G(f(x)) \leq \mathrm{val}_G(x)$. If x is a critical point of f, then the inequality is strict.*

Proof. Each spoke at x is mapped by f to a spoke at $f(x)$ in a homeomorphic way, provided it is sufficiently short. Since neither x nor $f(x)$ is (pre)periodic, all spokes at x and $f(x)$ are good. Since $J(f)$ is fully invariant and $J(f) \subset G$, every good

spoke at $f(x)$ is the image of a good spoke at x. Thus, the number of spokes at $f(x)$ is not larger than the number of spokes at x; that is, $\mathrm{val}_G(f(x)) \leq \mathrm{val}_G(x)$. If x is a critical point of f, then several spokes at x are mapped to one spoke at $f(x)$; so the inequality is strict. □

Lemma 4.8. *Assume that $J(f) \subset G$. Then any endpoint of G that belongs to $J(f)$ is (pre)periodic, and any branching point of G that belongs to $J(f)$ is periodic.*

Proof. Let x be a vertex of G belonging to $J(f)$. If x is an endpoint of G, then by Lemma 4.7, all points $f^n(x)$ are endpoints of G. Since G has finitely many endpoints, x is (pre)periodic. If x is a branching point of G, then by Lemma 4.7, every element of $f^{-n}(x)$ is a branching point of G. The set $f^{-n}(x)$ is non-empty, since $J(f)$ is fully invariant and $J(f) \subset G$. Since G has finitely many branching points, x is periodic. □

Now we can prove the following result, important for our main theorem.

Proposition 4.9. *If c is a critical point of a graph critical rational function f, then $\omega(c) \neq J(f)$.*

Proof. Suppose that $\omega(c) = J(f)$ for some critical point c. Then c belongs to $J(f)$; so it is recurrent. Moreover, $J(f)$ is infinite and contains periodic points; so it is not minimal, and thus $\omega(c)$ is not minimal. Therefore, by the definition of graph critical maps, $c \in G$. Since G is invariant, we get $J(f) \subset G$.

If c is not a vertex of G, then $\mathrm{val}_G(c) = 2$ and by Lemma 4.7 $\mathrm{val}_G(f(c)) = 1$. Thus, either c or $f(c)$ is a vertex of G; so by Lemma 4.8 c is (pre)periodic, a contradiction. □

Let us introduce in greater detail an important and well-known tool, called *pulling back* (we have already referred to it in the Introduction). For a (closed) Jordan disk W around $f^n(x)$ whose boundary does not contain critical images, there exists a component of $f^{-n}(W)$ which is a (closed) Jordan disk around x. One can imagine the process of taking preimages of W step by step, which is usually referred to as *pulling back*: first we choose the component W' of $f^{-1}(W)$ containing $f^{n-1}(x)$, then the component of $f^{-1}(W')$ containing $f^{n-2}(x)$, and so on. The described sequence of preimages of W is called the *pull-back chain* of W along $x, \ldots, f^n(x)$, or just a *pull-back chain* (if $x, \ldots, f^n(x)$ are not specified). The number $n+1$ is then called the *length* of the chain, and the Jordan disks in the chain are called *pull-backs* of W. The supremum of all r such that the pull-backs of the open r-disk $B(f^n(x), r)$ along $x, \ldots, f^n(x)$ contain no critical points is denoted by $r_n(x)$. The closure of at least one of the pull-backs of $B(f^n(x), r_n(x))$ (say, corresponding to $f^{m_n(x)}(x)$) has to contain a critical point. Denote it by $c_n(x)$ (if there is more than one, choose one). The pair $(c_n(x), m_n(x))$ is said to *generate* $r_n(x)$. In fact, we have already used some of these well-known notions, and give the definitions here just for the sake of completeness.

Now our aim is to study the behavior of the orbit of a C-persistent point x. Recall that by the definition $x \in J(f)$. The easiest case is when the point x is eventually mapped to a critical point c. Then $\omega(x) = \omega(c)$. Assume now that $f^n(x) \notin C$ for $n \geq 0$. The limit behavior of the orbit of x depends on the behavior of the orbits of $c_n(x)$. As we noticed, in this case there is a natural Basic Setup. However, our main idea is to consider other Basic Setups for x. Basically, we choose a neighborhood W of $f^n(x)$ so that its boundary is disjoint from the orbits of all

critical points, and then consider the step-by-step process of pulling W back along $x, \dots, f^n(x)$. For us the important moment in this process is when the component of the preimage of W for the *first time* contains a critical point. That is, we get k pull-backs of W which contain no critical points, and then a Jordan disk which is the component of $f^{-k-1}(W)$ containing $f^{n-k-1}(x)$ and a critical point. At this moment we stop the process of pulling back, and call this the *critical pull-back* of W. The existence of such k is guaranteed if $\overline{B}(f^n(x), r_n(x)) \subset W$.

The next lemma is useful in the proof of Proposition 4.6. Recall that $P_r(f)$ is the union of limit sets of recurrent critical points of f. We will call a point z *precritical* if $f^n(z)$ is critical for some $n > 0$, and *preparabolic* if $f^n(z)$ is a periodic parabolic point for some $n > 0$.

Lemma 4.10 ([BMO])**.** *If $z \in \mathrm{Prs}(f)$ is neither precritical nor preparabolic, then $\omega(z) \subset P_r(f)$.*

Now our preparations are over and we can start looking at the limit sets of persistent points.

Proposition 4.11. *Assume that x is persistent, but neither precritical nor preparabolic. If $\omega(x) \not\subset G$, then $\omega(x)$ is minimal and equal to $\omega(c(x))$ for some $c(x) \in C$. If $\omega(x) \subset G$, then x nicely approaches G and $c(x) \in G$.*

Proof. Set $C \setminus G = C \setminus G$ and $C \cap G = C \cap G$. By Lemma 4.10, $\omega(x) \subset P_r(f)$.

Assume first that the set $\omega(x) \setminus G$ is non-empty. Then it is contained in the union of all limit sets of recurrent points of $C \setminus G$. By the definition of a graph critical map, any such limit set is minimal. That is, there exists a recurrent point $c \in C \setminus G$ such that $\omega(c) \cap (\omega(x) \setminus G) \neq \emptyset$. Since G is invariant, we have $\omega(c) \cap G = \emptyset$. Since any two different minimal sets are disjoint, we have $W \cap \omega(x) = \omega(c) \cap \omega(x)$ for some neighborhood W of $\omega(c) \cap \omega(x)$. Hence, by Lemma 3.1 we get $\omega(x) = \omega(c)$. Thus the proof in the case when $\omega(x) \not\subset G$ is complete.

Observe that if $c \in C \setminus G$, then either $\omega(c)$ is minimal, or c is not recurrent and $\omega(x)$ is disjoint from G. This implies that if $\omega(c), c \in C \setminus G$, is not minimal, then $\omega(x)$ is bounded away from G. Hence, if $\omega(x) \subset G$, then sufficiently small neighborhoods of points $f^n(x)$ contain no points from the orbits of points of $C \setminus G$ with non-minimal limit sets.

We claim that to show that x nicely approaches G, it is enough for any sufficiently small ε and all sufficiently big i to do one of the following.

(a) Find $c \in C \setminus G$ and $m_i \leq i$ such that $d(f^i(x), f^{i-m_i}(c)) < \varepsilon$.
(b) Find numbers l_i and $m_i \leq i$ and points $v_i, u_i \in C \cap G$ such that the segment $(f^{l_i}(v_i), \dots, f^{l_i+i-m_i}(v_i))$ ε-approximates the segment $(f^{m_i}(x), \dots, f^i(x))$ and $\mathrm{diam}(f^{i-m_i}(J_i)) < \varepsilon$ for some connected set $J_i \subset G$ containing both $f^{l_i}(v_i)$ and u_i.

Indeed, if (a) holds, then condition (1) of nice approaching is satisfied because by the above analysis for small ε (a) implies that $\omega(c)$ is minimal, (2) is automatic and (3) is void. On the other hand, if (b) holds, then condition (1) of the nice approaching is void while (2) and (3) follow directly from (b).

Let us choose some constants. By Lemma 4.1, there exists $\delta \in (0, \varepsilon)$ such that if $z \in J(f)$, $V \supset V'$ are Jordan disks around z, $\eta < \delta$, and n is such that $f^{n-1}|_{f(V)} : f(V) \to B(f^n(z), \eta)$ and $f^{n-1}|_{f(V')} : f(V') \to B(f^n(z), \eta/2)$ are univalent and onto, then $\mathrm{diam}(f^j(V')) < \varepsilon$ for $0 \leq j \leq n$.

Since every point of G has an elementary neighborhood of arbitrarily small diameter and G is compact, there exists a finite cover $\mathcal{W}$ of G by elementary neighborhoods of diameter less than ε. Let $\eta \in (0, \delta)$ be so small that 2η is smaller than the Lebesgue number of this cover. Choose M such that $r_n(x) < \eta/2$ for any $n \geq M$. Note that then $r_n(x) < \eta < \delta < \varepsilon$.

Let $i \geq M$. Then $r_i(x)$ is generated by some pair (c, m). If $c \in C \setminus G$, then we simply set $m_i = m$, $c_i(x) = c$, and (a) holds. Assume that $c \in C \cap G$. Then $r_i(x) = d(f^i(x), f^{i-m}(c)) < \eta/2$, and hence $f^{i-m}(c)$ belongs to the ball $B(f^i(x), \eta/2)$. By the choice of η there exists an elementary neighborhood W containing $B(f^{i-m}(x), \eta)$.

Let us pull W back along the orbit of x and consider its critical pull-back V (it exists, since after $i - m$ steps we get to c; however, it may happen that we meet a critical point earlier). Then for some number $m_i < i$ we have that $f^{i-m_i-1}|_{f(V)} : f(V) \to W$ is univalent and onto, while V contains a critical point u_i. If $u_i \in C \setminus G$, then we choose u_i as $c_i(x)$ and stop there, since (a) holds. Otherwise, $u_i \in C \cap G$. Choose $V' \subset V'' \subset V$ as the appropriate pull-backs of $B(f^i(x), \eta/2) \subset B(f^i(x), \eta) \subset W$ respectively. Since $\eta < \delta$, by our choice of δ we get $\operatorname{diam}(f^j(V')) < \varepsilon$ for all $0 \leq j \leq i - m_i$. Set $v_i = c$ and $l_i = m_i - m$. Then the segment $(f^{l_i}(v_i), \ldots, f^{l_i+i-m_i}(v_i))$ ε-approximates the segment $(f^{m_i}(x), \ldots, f^i(x))$. Moreover, since $c, u_i \in C \cap G \subset G$, the set $G \cap V$ contains $f^{l_i}(v_i)$ and u_i, and by Lemma 4.3 and Remark 4.4 it is connected. Thus, (b) holds.

Thus, in any case either (a) or (b) holds, and so we are done. □

One consequence of Proposition 4.11 is the main technical result of the paper, mentioned already in the Introduction.

Theorem 4.12. *For a graph critical map f and a persistent point $x \in J(f)$, $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ of f.*

Proof. Let x be a persistent point of f. If x is precritical or preparabolic, it is clear that $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ of f. Assume that x is neither precritical nor preparabolic. By Proposition 4.11, if $\omega(x) \not\subset G$, then $\omega(x) = \omega(c(x))$ for some critical point $c(x)$ of f; if $\omega(x) \subset G$, then x nicely approaches G, and by Theorem 3.5 we draw the same conclusion. □

We are ready now to prove the main result of the paper.

Theorem 4.13. *For a graph critical map f and a conformal measure, exactly one of the following holds.*

1. *For almost every $x \in J(f)$, $\omega(x) = J(f)$.*
2. *For almost every $x \in J(f)$, $\omega(x) = \omega(c)$ for some critical point c of f.*

Proof. By Theorems 4.12 and 1.1 we get that indeed (1) or (2) must hold. If both (1) and (2) hold simultaneously, then there exists a critical point c such that $\omega(c) = J(f)$, which is impossible for graph critical rational maps by Proposition 4.9. □

Moreover, from Proposition 4.11 we can get some additional consequences of a technical nature.

Theorem 4.14. *Consider a graph critical map with graph G and a persistent point x. Then either x is precritical, or $\omega(c(x))$ is minimal, or there is a wandering set $J \subset G$ containing $c(x)$ and a recurrent point.*

Proof. Assume that x is not precritical. If it is preparabolic, then $\omega(c(x))$ is minimal. If $\omega(x) \not\subset G$, then we get the same result by Proposition 4.11. Assume that $\omega(x) \subset G$. By Lemma 4.2, all points $f^n(x)$ are persistent, and by Proposition 4.11 we get that $f^n(x)$ nicely approaches G for all n. If additionally $\omega(x)$ is not minimal, then by Theorem 3.5(2) we can choose $c(x) \in C \cap G$ so that there exists a roaming set $J \subset G$ containing $c(x)$ and a recurrent point. By Remark 3.6, either J is wandering or $\omega(c(x))$ is minimal. □

To understand better the last option mentioned in the above theorem, let us observe that although our map is as smooth as a map can be, the graph G is not necessarily smooth. Therefore we cannot exclude existence of non-trivial wandering sets. In this case, of course, the recurrent point mentioned in the theorem must be a boundary point of J. If J is trivial (that is, $J = \{c(x)\}$), then $c(x)$ is recurrent.

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