A NOTE ON MEYERS’ THEOREM IN $W^{k,1}$

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Abstract. Lower semicontinuity properties of multiple integrals

$$u \in W^{k,1}(\Omega; \mathbb{R}^d) \mapsto \int_{\Omega} f(x, u(x), \cdots, \nabla^k u(x)) \, dx$$

are studied when $f$ may grow linearly with respect to the highest-order derivative, $\nabla^k u$, and admissible $W^{k,1}(\Omega; \mathbb{R}^d)$ sequences converge strongly in $W^{k-1,1}(\Omega; \mathbb{R}^d)$. It is shown that under certain continuity assumptions on $f$, convexity, 1-quasiconvexity or $k$-polyconvexity of

$$\xi \mapsto f(x_0, u(x_0), \cdots, \nabla^{k-1} u(x_0), \xi)$$

ensures lower semicontinuity. The case where $f(x_0, u(x_0), \cdots, \nabla^{k-1} u(x_0), \cdot)$ is $k$-quasiconvex remains open except in some very particular cases, such as when $f(x, u(x), \cdots, \nabla^k u(x)) = h(x)g(\nabla^k u(x))$.

1. Introduction

In a classical paper Meyers [26] proved that $k$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$u \mapsto \int_{\Omega} f(x, u(x), \cdots, \nabla^k u(x)) \, dx$$

with respect to weak convergence (weak* convergence if $p = \infty$) in the Sobolev space $W^{k,p}(\Omega; \mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand $f$, thus extending to the case $k > 1$ the notion of quasi-convexity introduced by Morrey when $k = 1$. Here $\Omega$ is an open, bounded subset of $\mathbb{R}^N$, with $N \geq 1$, and $k, d \in \mathbb{N}$, $1 \leq p \leq \infty$. Meyers’ theorem uses results of Agmon, Douglis and Nirenberg [1] concerning Poisson kernels for elliptic equations. Fusco [22] later gave a simpler proof using De Giorgi’s Slicing Lemma. He also extended the result

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to Carathéodory integrands when $p = 1$, while the case $p > 1$ has been recently established by Guidorzi and Poggiolini [24] under the Lipschitz condition

$$
|f(x, v, \xi) - f(x, \nu, \xi_1)| \leq C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})|\xi - \xi_1|
$$

(note that this condition is automatically satisfied for $k = 1$ and $k = 2$, see [25], and by Braides, Fonseca and Leoni in [8], who obtained a general relaxation result in $W^{k,p}(\Omega; \mathbb{R}^d)$ with respect to weak convergence.

In most applications, the lower semicontinuity results mentioned above are completely satisfactory when $p > 1$, since bounded sequences in $W^{k,p}(\Omega; \mathbb{R}^d)$ admit weakly convergent subsequences. However, when $p = 1$, due to loss of reflexivity of the space $W^{k,1}(\Omega; \mathbb{R}^d)$ one can only conclude that an energy bounded sequence $\{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d)$ with

$$
\sup_n \|u_n\|_{W^{k,1}} < \infty
$$

admits a subsequence (not relabelled) such that

$$
(1.1) \quad u_n \rightharpoonup u \quad \text{in} \quad W^{k-1,1}(\Omega; \mathbb{R}^d),
$$

where $u \in W^{k-1,1}(\Omega; \mathbb{R}^d)$ and $\nabla^{k-1}u$ is a vector-valued function of bounded variation. In this paper we seek to establish lower semicontinuity in the space $W^{k,1}(\Omega; \mathbb{R}^d)$ under this natural notion of convergence.

When $k = 1$ the scalar case $d = 1$ has been extensively treated, while the vectorial case $d > 1$ was first studied by Fonseca and Müller in [19], who proved (sequential) lower semicontinuity in $W^{1,1}(\Omega; \mathbb{R}^d)$ of a functional

$$
\quad u \mapsto \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx
$$

with respect to strong convergence in $L^1(\Omega; \mathbb{R}^d)$ (see also [3], [20], [17], [18] and the references contained therein). The approach in [19] is based on blow-up and truncation methods.

Similar truncation techniques have been used quite successfully in the study of existence and qualitative properties of solutions of second-order elliptic equations and systems (see e.g. [7] and the references contained within). Their main drawback lies in the fact that they cannot be easily extended to truncated gradients or higher-order derivatives. This may explain in part why several important results for second-order elliptic equations have no analog for higher-order equations.

The main result of this paper extends Meyers’ Theorem to the case where weak convergence in $W^{k,1}(\Omega; \mathbb{R}^d)$ is replaced by (1.1) together with a weak form of coercivity of the convex, 1-quasiconvex or $k$-polyconvex density $f$ (see Theorems 1.2, 1.3, and 1.6 below). We start with the case where $f$ depends essentially only on $x$ and on the highest-order derivatives, that is, $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require one to truncate the initial sequence $\{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d)$. Using the notation and terminology introduced in Section 2, we state the following:

**Theorem 1.1.** Let $f : \Omega \times E_{d[k-1]}^d \times E_k^d \to [0, \infty)$ be a Borel integrand. Suppose that for all $(x_0, v_0) \in \Omega \times E_{d[k-1]}^d$ and $\varepsilon > 0$ there exist $\delta_0 > 0$ and a modulus of continuity $\rho$, with $\rho(s) \leq C_0(1 + s)$ for $s > 0$ and for some $C_0 > 0$, such that

$$
(1.2) \quad f(x_0, v_0, \xi) - f(x, v, \xi) \leq \varepsilon(1 + f(x, v, \xi)) + \rho(|v - v_0|)
$$
for all \( x \in \Omega \) with \( |x - x_0| \leq \delta_0 \), and for all \( (\mathbf{v}, \xi) \in E^d_{[k-1]} \times E^d_0 \). Assume also that one of the following three conditions is satisfied:
(a) \( f(x_0, \mathbf{v}_0, \cdot) \) is \( k \)-quasiconvex in \( E^d_0 \) and
\[
C_1 |\xi| - C_1 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1 (1 + |\xi|) \quad \text{for all } \xi \in E^d_0,
\]
where \( C_1 > 0 \);
(b) \( f(x_0, \mathbf{v}_0, \cdot) \) is 1-quasiconvex in \( E^d_0 \) and
\[
0 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1 (1 + |\xi|) \quad \text{for all } \xi \in E^d_0,
\]
where \( C_1 > 0 \);
(c) \( f(x_0, \mathbf{v}_0, \cdot) \) is convex in \( E^d_0 \).

Let \( u \in BV^k(\Omega; \mathbb{R}^d) \), and let \( \{u_n\} \) be a sequence of functions in \( W^{k,1}(\Omega; \mathbb{R}^d) \) converging to \( u \) in \( W^{k-1,1}(\Omega; \mathbb{R}^d) \). Then
\[
\int_{\Omega} f(x, u, \ldots, \nabla^k u) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} f(x, u_n, \ldots, \nabla^k u_n) \, dx.
\]

Here \( \nabla^k u \) is the Radon–Nikodym derivative of the distributional derivative \( D^k u \) of \( \nabla^{k-1} u \), with respect to the \( N \)-dimensional Lebesgue measure \( \mathcal{L}^N \). An important class of integrands which satisfy (1.2) of Theorem 1.1 is given by
\[
f = f(x, \xi) := h(x) g(\xi),
\]
where \( h(x) \) is a nonnegative lower semicontinuous function and \( g \) is a nonnegative function that satisfies (a) or (b) or (c). The case where \( h(x) \equiv 1 \) and \( g \) satisfies condition (a) was proved by Amar and De Cicco [2]. Theorem 1.1 extends a result of Fonseca and Leoni [17] (Theorem 1.7 in [17]) to higher-order derivatives, where the statement is exactly that of Theorem 1.1 setting \( k = 1 \) and excluding part (a). Related results when \( k = 1 \) were obtained previously by Serrin [28] in the scalar case \( d = 1 \) and by Ambrosio and Dal Maso [4] in the vectorial case \( d > 1 \) (see also Fonseca and Müller [19], [20]). Even in the simple case where \( f = f(\xi) \) it is not known if Theorem 1.1(a) still holds without the coercivity condition
\[
f(\xi) \geq \frac{1}{C_1} |\xi| - C_1.
\]

The main tool in the proof of Theorem 1.1 used also in an essential way in subsequent results, is the blow-up method introduced by Fonseca and Müller [19], [20], which reduces the domain \( \Omega \) to a ball and the target function \( u \) to a polynomial.

When the integrand \( f \) depends on the full set of variables in an essential way, the situation becomes significantly more complicated, since one needs to truncate gradients and higher-order derivatives in order to localize lower-order terms.

The following theorem was proved for \( k = 1 \) by Fonseca and Leoni in [17] (Theorem 1.8). Here we extend the result to the higher-order case.

**Theorem 1.2.** Let \( f : \Omega \times E^d_{[k-1]} \times E^d_0 \to [0, \infty) \) be a Borel integrand, with \( f(x, \mathbf{v}, \cdot) \) 1-quasiconvex in \( E^d_0 \). Suppose that for all \( (x_0, \mathbf{v}_0) \in \Omega \times E^d_{[k-1]} \) either \( f(x_0, \mathbf{v}_0, \cdot) \equiv 0 \), or for every \( \varepsilon > 0 \) there exist \( C, \delta_0 > 0 \) such that
\[
f(x_0, \mathbf{v}_0, \xi) - f(x, \mathbf{v}, \xi) \leq \varepsilon(1 + f(x, \mathbf{v}, \xi)),
\]
and
\[
C|\xi| - \frac{1}{C} \leq f(x_0, \mathbf{v}_0, \xi) \leq C(1 + |\xi|)
\]
for all \((x, v) \in \Omega \times E_{k-1}^d\) with \(|x-x_0| + |v-v_0| \leq \delta_0\) and for all \(\xi \in E_k^d\).

Let \(u \in BV^k(\Omega; \mathbb{R}^d)\), and let \(\{u_n\}\) be a sequence of functions in \(W^{k,1}(\Omega; \mathbb{R}^d)\) converging to \(u\) in \(W^{k-1,1}(\Omega; \mathbb{R}^d)\). Then

\[
\int_\Omega f(x, u, \ldots, \nabla^k u) \, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n) \, dx.
\]

A standing open problem is to decide whether Theorem 1.5 continues to hold under the weaker assumption that \(f(x, v, \cdot)\) is \(k\)-quasiconvex, which is the natural assumption in this context.

In the scalar case \(d = 1\) (that is, when \(u\) is an \(\mathbb{R}\)-valued function), and for first-order gradients, i.e., \(k = 1\), condition (1.6) can be eliminated; see Theorem 1.1 in [17]. In particular, in [17] Fonseca and Leoni have shown the following result:

**Proposition 1.3** (cf. [17], Corollary 1.2). Let \(g : \mathbb{R}^N \to [0, \infty)\) be a convex function, and let \(h : \Omega \times \mathbb{R} \to [0, \infty)\) be a lower semicontinuous function. If \(u \in BV(\Omega; \mathbb{R})\) and \(\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R})\) converges to \(u\) in \(L^1(\Omega; \mathbb{R})\), then

\[
\int_\Omega h(x, u)g(\nabla u) \, dx \leq \liminf_{n \to \infty} \int_\Omega h(x, u_n)g(\nabla u_n) \, dx.
\]

It is interesting to observe that the analog of this result is false when \(k \geq 2\).

**Theorem 1.4.** Let \(\Omega := (0,1)^N, N \geq 3\), and let \(h \) be a smooth cut-off function on \(\mathbb{R}\) with \(0 \leq h \leq 1\), \(h(u) = 1\) for \(u \leq \frac{1}{2}\), \(h(u) = 0\) for \(u \geq 1\). There exists a sequence of functions \(\{u_n\}\) in \(W^{2,1}(\Omega; \mathbb{R})\) converging to zero in \(W^{1,1}(\Omega; \mathbb{R})\) such that \(\{\|\Delta u_n\|_{L^1(\Omega; \mathbb{R})}\}\) is uniformly bounded and

\[
\limsup_{n \to \infty} \int_\Omega h(u_n)(1 - \Delta u_n)^+ \, dx < \int_\Omega h(0) \, dx.
\]

As in Theorem 1.1 conditions (1.5) and (1.6) can be considerably weakened if we assume that \(f(x, v, \cdot)\) is convex rather than \(1\)-quasiconvex. Indeed, we have the following result:

**Theorem 1.5.** Let \(f : \Omega \times E_{k-1}^d \times E_k^d \to [0, \infty]\) be a lower semicontinuous function, with \(f(x, v, \cdot)\) convex in \(E_k^d\). Suppose that for all \((x_0, v_0) \in \Omega \times E_{k-1}^d\) either \(f(x_0, v_0, \cdot) \equiv 0\), or there exist \(C_1, \delta_0 > 0\), and a continuous function \(g : B(x_0, \delta_0) \times B(v_0, \delta_0) \to E_k^d\) such that

\[
f(x, v, g(x, v)) \in L^\infty(B(x_0, \delta_0) \times B(v_0, \delta_0); \mathbb{R}),
\]

(1.7)

\[
f(x, v, \xi) \geq C_1 |\xi| - \frac{1}{C_1}
\]

(1.8)

for all \((x, v) \in \Omega \times E_{k-1}^d\) with \(|x - x_0| + |v - v_0| \leq \delta_0\) and for all \(\xi \in E_k^d\). Let \(u \in BV^k(\Omega; \mathbb{R}^d)\), and let \(\{u_n\}\) be a sequence of functions in \(W^{k,1}(\Omega; \mathbb{R}^d)\) converging to \(u\) in \(W^{k-1,1}(\Omega; \mathbb{R}^d)\). Then

\[
\int_\Omega f(x, u, \ldots, \nabla^k u) \, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n) \, dx.
\]

Theorem 1.5 was obtained by Fonseca and Leoni for the case \(k = 1\) in Theorem 1.1 of [18]. It is interesting to observe that without a condition of the type (1.7), Theorem 1.5 is false in general. This has been recently proved by Černý and Malý in [12].
The proofs of Theorems 1.1(b) and (c), 1.2 and 1.3 can be deduced easily from the corresponding ones in [17] and [18], where \( k = 1 \). It suffices to write
\[
\int_{\Omega} f(x, u(x), \ldots, \nabla^k u(x)) \, dx =: \int_{\Omega} F(x, v(x), \nabla v(x)) \, dx
\]
with \( v := (u, \ldots, \nabla^{k-1} u) \), and then to perturb the new integrand \( F \) in order to recover the full coercivity conditions necessary to apply the results in [17], [18].

This approach cannot be used for \( k \)-polyconvex integrands, and a new proof is needed to treat this case. Thus Theorem 1.1(a) and Theorem 1.6 below are the only truly genuine higher-order results, in that they cannot be reduced in a trivial way to a first-order problem.

For each \( \xi \in E^d_k \) let \( M(\xi) \in \mathbb{R}^r \) be the vector whose components are all the minors of \( \xi \).

**Theorem 1.6.** Let \( h : \Omega \times E^d_{[k-1]} \times \mathbb{R}^r \to [0, \infty) \) be a lower semicontinuous function, with \( h(x, v, \cdot) \) convex in \( \mathbb{R}^r \). Suppose that for all \( (x_0, v_0) \in \Omega \times E^d_{[k-1]} \) either \( h(x_0, v_0, \cdot) \equiv 0 \), or there exist \( C, \delta_0 > 0 \), and a continuous function \( g : B(x_0, \delta_0) \times B(v_0, \delta_0) \to \mathbb{R}^r \) such that
\[
(1.9) \quad h(x, v, g(x, v)) \in L^\infty (B(x_0, \delta_0) \times B(v_0, \delta_0); \mathbb{R}),
\]
\[
(1.10) \quad h(x, v, v) \geq C|v| - \frac{1}{C}
\]
for all \( (x, v) \in \Omega \times E^d_{[k-1]} \) with \( |x-x_0| + |v-v_0| \leq \delta_0 \) and for all \( v \in \mathbb{R}^r \). Let \( u \in BV^k(\Omega; \mathbb{R}^d) \), and let \( \{u_n\} \) be a sequence of functions in \( W^{k,p}(\Omega; \mathbb{R}^d) \) that converges to \( u \) in \( W^{k-1,1}(\Omega; \mathbb{R}^d) \), where \( p \) is the minimum between \( N \) and the dimension of \( E^d_{[k-1]} \). Then
\[
\int_{\Omega} h(x, u, \ldots, \nabla^{k-1} u, M(\nabla^k u)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} h(x, u_n, \ldots, \nabla^{k-1} u_n, M(\nabla^k u_n)) \, dx.
\]

Theorem 1.6 is closely related to a result of Ball, Currie and Olver [6], where it was assumed that
\[
h(x, v, v) \geq \gamma (|v|) - \frac{1}{C},
\]
with
\[
\frac{\gamma (s)}{s} \to \infty \quad \text{as} \quad s \to \infty.
\]

Also, as stated above and with \( k = 1 \), Theorem 1.6 was proved by Fonseca and Leoni in [18], Theorem 1.4.

### 2. Preliminaries

We start with some notation. Here \( \Omega \subset \mathbb{R}^N \) is an open, bounded subset; \( \mathcal{L}^N \) and \( \mathcal{H}^{N-1} \) are, respectively, the \( N \)-dimensional Lebesgue measure and the \( (N-1) \)-dimensional Hausdorff measure in \( \mathbb{R}^N \). Let \( Q \) be the the unit cube \((-1/2, 1/2)^N\) and set \( Q(x_0, \varepsilon) := x_0 + \varepsilon Q \).

For each \( j \in \mathbb{N} \) the symbol \( \nabla^j u \) stands for the vector-valued function whose components are all derivatives of order \( j \) of \( u \). If \( u \) is \( C^\infty \), then for \( j \geq 2 \) we have...
that $\nabla^j u(x) \in E^d_j$, where $E^d_j$ stands for the space of symmetric $j$-linear maps from $\mathbb{R}^N$ into $\mathbb{R}^d$. We set $E^d_0 := \mathbb{R}^d$, $E^d_1 := \mathbb{R}^{d \times N}$ and

$$E^d_{[j-1]} := E^d_0 \times \cdots \times E^d_{j-1}, \quad E^d_{[0]} := E^d_0.$$  

For any integer $k \geq 2$ we define

$$BV^k(\Omega; \mathbb{R}^d) := \{ u \in W^{k-1,1}(\Omega; \mathbb{R}^d) : \nabla^{k-1} u \in BV(\Omega; E^d_{k-1}) \},$$

where $\nabla^j u$ is the Radon–Nikodým derivative of the distributional derivative $D^j u$ of $\nabla^{j-1} u$, with respect to the $N$–dimensional Lebesgue measure $\mathcal{L}^N$.

We recall that a function $f : E^d_k \to \mathbb{R}$ is said to be $k$-quasiconvex if

$$f(\xi) \leq \int_Q f(\xi + \nabla^k w(y)) \, dy$$

for all $\xi \in E^d_k$ and all $w \in C^\infty_0(Q; \mathbb{R}^d)$.

The following theorem was proved in the case $k = 1$ by Ambrosio and Dal Maso [4], while Fonseca and Müller [19] treated general integrands of the form $f = f(x, u, \nabla u)$, but their argument requires coercivity. The case $k \geq 2$ is due to Amar and De Cicco [2]. For completeness we give a proof for all $k \geq 1$.

**Proposition 2.1.** Let $f : E^d_k \to [0, \infty)$ be a $k$-quasiconvex function such that

$$0 \leq f(\xi) \leq C (1 + |\xi|)$$

for all $\xi \in E^d_k$. Moreover, when $k \geq 2$ assume that

$$f(\xi) \geq C_1 |\xi|$$

for $|\xi|$ large.

If $\{u_n\}$ is a sequence of functions in $W^{k,1}(Q; \mathbb{R}^d)$ converging to 0 in $W^{k-1,1}(Q; \mathbb{R}^d)$, then

$$f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx.$$  

**Proof.** We start with the case $k \geq 2$. Without loss of generality, we may assume that

$$\liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx = \lim_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx < \infty,$$

so that by condition (2.2),

$$K := \sup_n \int_Q |\nabla^k u_n| \, dx < \infty.$$  

Let $\varepsilon > 0$, $M \in \mathbb{N}$, and decompose $L := Q \setminus (1-\varepsilon)Q$ into $M$ layers with mutually disjoint interiors, $L_i := \alpha_{i+1}Q \setminus \alpha_iQ$, so that

$$1 - \varepsilon = \alpha_1 < \alpha_2 < \ldots < \alpha_M < 1 =: \alpha_{M+1}.$$  

Since

$$\sum_{i=1}^M \int_{L_i} (1 + |\nabla^k u_n|) \, dx \leq 1 + K$$

for all $n \in \mathbb{N}$, there exist $i_n \in \{1, \ldots, M\}$ and a subsequence of $\{u_n\}$ (not relabelled) such that

$$\int_{L_{i_n}} (1 + |\nabla^k u_n|) \, dx \leq \frac{1 + K}{M}$$

for all $n \in \mathbb{N}$.  

\[ \boxed{ (2.3) } \]
Let \( \varphi \in C^\infty_c(Q; [0, 1]) \) with \( \varphi(x) = 1 \) in \( \alpha_i, Q \), \( \varphi(x) = 0 \) if \( x \notin \alpha_{i+1}, Q \). Since \( f \) is \( k \)-quasiconvex,

\[
f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k (\varphi u_n)) \, dx
\]

\[
\leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx + \int_{Q \setminus \alpha_{i+1}, Q} f(0) \, dx
\]

\[
+ C \limsup_{n \to \infty} \int_{L_{i+1}} (1 + |\nabla^k (\varphi u_n)|) \, dx,
\]

where we have used (2.1). As \( u_n \to 0 \) in \( W^{k-1,1}(Q; \mathbb{R}^d) \) strongly, we have

\[
\limsup_{n \to \infty} \int_{L_{i+1}} (1 + |\nabla^k (\varphi u_n)|) \, dx \leq \limsup_{n \to \infty} \int_{L_{i+1}} (1 + |\nabla^k u_n|) \, dx \leq \frac{1 + K}{M}
\]

by (2.1). We conclude that

\[
(1 - \varepsilon)^N f(0) \leq \alpha_{i+1} f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx + \frac{1 + K}{M},
\]

and the result now follows by letting first \( \varepsilon \to 0^+ \) and then \( M \to \infty \).

Next we consider the case where \( k = 1 \). Let \( \varepsilon > 0 \), fix \( n \in \mathbb{N} \), set

\[
M_n := \left[ n \int_Q (1 + |\nabla u_n|) \, dx \right] + 1,
\]

where \([\cdot]\) denotes the integer part, and decompose \( L := Q \setminus (1 - \varepsilon) Q \) into \( M_n \) layers with mutually disjoint interiors, \( L_i^{(n)} := \alpha_{i+1}^{(n)} \setminus \alpha_{i}^{(n)}, Q \), so that

\[
1 - \varepsilon = \alpha_{1}^{(n)} < \alpha_{2}^{(n)} < \ldots < \alpha_{M_n}^{(n)} < 1 =: \alpha_{M+1}^{(n)}
\]

and, in addition, \( \alpha_{i+1}^{(n)} - \alpha_{i}^{(n)} = \frac{2M_n}{M}, i = 1, \ldots, M_n \). Let \( \varphi_i^{(n)} \in C^\infty_c(Q; [0, 1]) \) with \( \varphi_i^{(n)}(x) = 1 \) in \( \alpha_i^{(n)}, Q \), \( \varphi_i^{(n)}(x) = 0 \) if \( x \notin \alpha_{i+1}^{(n)}, Q \), \( ||\nabla \varphi_i^{(n)}|| \leq \frac{2M_n}{\varepsilon} \), \( i = 1, \ldots, M_n \). We have

\[
\int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + \int_{Q \setminus \alpha_{i+1}^{(n)}, Q} f(0) \, dx
\]

\[
+ C \int_{L_i^{(n)}} (1 + |\nabla u_n|) \, dx + C \frac{2M_n}{\varepsilon} \int_{L_i^{(n)}} |u_n| \, dx.
\]

Thus

\[
\frac{1}{M_n} \sum_{i=1}^{M_n} \int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + \int_{Q \setminus \alpha_1^{(n)}, Q} f(0) \, dx
\]

\[
+ C \frac{M_n}{\varepsilon} \int_{Q \setminus \alpha_1^{(n)}, Q} (1 + |\nabla u_n|) \, dx + C \frac{2M_n}{\varepsilon} \int_{Q \setminus \alpha_1^{(n)}, Q} |u_n| \, dx
\]

\[
\leq \int_Q f(\nabla u_n) \, dx + O(\varepsilon) + C \frac{M_n}{\varepsilon} \int_{Q \setminus \alpha_1^{(n)}, Q} |u_n| \, dx.
\]

We may, therefore, find \( i = i(n, \varepsilon) \in \{1, \ldots, M\} \) such that, in view of the quasi-convexity of \( f \),

\[
f(0) \leq \int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + O(\varepsilon) + \frac{C}{n} \int_{Q \setminus \alpha_1^{(n)}, Q} |u_n| \, dx,
\]

and the conclusion follows by letting \( n \to \infty \) and then \( \varepsilon \to 0^+ \). \( \square \)
Proposition 2.2. Let \( h : \mathbb{R}^\tau \to [0, \infty) \) be a convex function such that
\[
h(v) \to \infty \quad \text{as} \quad |v| \to \infty.
\]
Let \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \), and let \( \{u_n\} \) be a sequence of functions in \( W^{1,p}(\Omega; \mathbb{R}^d) \) that converges to \( u \) in \( L^1(\Omega; \mathbb{R}^d) \), where \( p = \min\{d, N\} \). Then
\[
\int_\Omega h(\mathcal{M}(\nabla u)) \, dx \leq \liminf_{n \to \infty} \int_\Omega h(\mathcal{M}(\nabla u_n)) \, dx.
\]

Proposition 2.2 has been proved by Dal Maso and Sbordone (cf. Theorem 2.2 in [14]) using Cartesian currents, and by Fusco and Hutchinson (cf. Theorem 2.6 in [23]).

Next we present an approximation result for convex functions.

Proposition 2.3. Let \( M \) be a closed set of \( \mathbb{R}^p \), and let \( V \) be an reflexive and separable Banach space. Let \( f : M \times V \to (0, +\infty] \) be a sequentially lower semicontinuous function, convex in the last variable and such that there exists a continuous function \( v_0 : M \to V \) with
\[
(f(\cdot, v_0(\cdot)))^+ \in L^\infty_{\text{loc}}(M; \mathbb{R}).
\]
Then there exist two sequences of continuous functions
\[
a_j : M \to \mathbb{R}, \quad b_j : M \to V^*,
\]
where \( V^* \) is the dual space of \( V \), such that
\[
f(t, v) = \sup_j (a_j(t) + \langle b_j(t), v \rangle)^+
\]
for all \( t \in M \) and \( v \in V \).

Proposition 2.3 was proved by Fonseca and Leoni in [18], following closely the argument of Ambrosio in [3], who studied the case where (2.4) is replaced by the assumption that \( f(\cdot, v_0(\cdot)) \) is continuous.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Without loss of generality, we may assume that
\[
\liminf_{n \to \infty} \int_\Omega f(x, u_n(x), \ldots, \nabla^k u_n(x)) \, dx = \lim_{n \to \infty} \int_\Omega f(x, u_n(x), \ldots, \nabla^k u_n(x)) \, dx < \infty.
\]
Passing to a subsequence, if necessary, there exists a nonnegative Radon measure \( \mu \) such that
\[
f(x, u_n(x), \ldots, \nabla^k u_n(x)) \mathcal{L}^N | \Omega \overset{\text{weak}^*}{\rightharpoonup} \mu
\]
as \( n \to \infty \), weakly* in the sense of measures. We claim that
\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} \geq f(x_0, u(x_0), \ldots, \nabla^k u(x_0))
\]
for \( \mathcal{L}^N \) a.e. \( x_0 \in \Omega \). If (3.1) holds, then the conclusion of the theorem follows immediately. Indeed, let \( \varphi \in C_c(\Omega; \mathbb{R}) \), \( 0 \leq \varphi \leq 1 \). We have
\[
\lim_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n) \, dx \geq \liminf_{n \to \infty} \int_\Omega \varphi f(x, u_n, \ldots, \nabla^k u_n) \, dx
\]
\[
= \int_\Omega \varphi \, d\mu \geq \int_\Omega \varphi \frac{d\mu}{d\mathcal{L}^N} \, dx \geq \int_\Omega \varphi f(x, u_n, \ldots, \nabla^k u_n) \, dx.
\]
By letting $\varphi \to 1$, and using the Lebesgue Monotone Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem it suffices to show (3.1).

Take $x_0 \in \Omega$ such that

$$
\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} < \infty,
$$

and set

$$
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0, \varepsilon)} |u(x) - T_k(x)| \frac{1}{|x - x_0|^k} dx = 0,
$$

where

$$
T_k(x) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,
$$

and set

$$
v_0 := (u(x_0), \ldots, \nabla^{k-1}u(x_0)).
$$

Choosing $\varepsilon_m \searrow 0$ such that $\mu(\partial Q(x_0, \varepsilon_m)) = 0$, then

$$
\lim_{m \to \infty} \frac{\mu(Q(x_0, \varepsilon_m))}{\varepsilon_m^N} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{\varepsilon_m^N} \int_{Q(x_0, \varepsilon_m)} f(x, u_n, \ldots, \nabla^k u_n) dx
$$

$$
= \lim_{m \to \infty} \lim_{n \to \infty} \int_Q f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_n(x_0 + \varepsilon_m y), \nabla T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^{k-1} \nabla^k w_n(x_0 + \varepsilon_m y, \ldots, \nabla^k w_n(x_0 + \varepsilon_m y)) dy,
$$

where

$$
w_n(x_0 + \varepsilon_m y) := u_n(x_0 + \varepsilon_m y) - T_{k-1}(x_0 + \varepsilon_m y).
$$

Clearly $w_n \in W^{k,1}(Q; \mathbb{R}^d)$, and, by (3.2),

$$
\lim_{m \to \infty} \lim_{n \to \infty} ||w_n - w_0||_{W^{k-1,1}(Q; \mathbb{R}^d)} = 0,
$$

where

$$
w_0(y) := \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0) y^\alpha.
$$

By a standard diagonalization argument, we may extract a subsequence $w_m := w_{n_m}$ that converges to $w_0$ in $W^{k-1,1}(Q; \mathbb{R}^d)$, such that $\nabla^j w_m \to \nabla^j w_0$ pointwise a.e. for $j = 0, \ldots, k - 1$, and

$$
\frac{d\mu}{d\mathcal{L}^N}(x_0)
$$

(3.3)

$$
= \lim_{m \to \infty} \int_Q f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) dy.
$$

By condition (1.2), for all $\varepsilon > 0$ and for $m$ large enough,

$$
(1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon
$$

$$
\geq \lim_{m \to \infty} \left( \int_Q f(x_0, u(x_0), \ldots, \nabla^{k-1}u(x_0), \nabla^k w_m(y)) dy - \int_Q \rho(|z_m(y)|) dy \right),
$$
where
\[
  z_m(y) := (T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m w_m(y), \ldots, \nabla^{k-1} T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m \nabla^{k-1} w_m(y)) - v_0.
\]

By Fatou’s Lemma, and since \( \rho \) is continuous with \( \rho(0) = 0 \), we have
\[
  C_0 - \limsup_{m \to \infty} \int_Q \rho(|z_m(y)|) \, dy = \liminf_{m \to \infty} \int_Q [C_0(1 + |z_m(y)|) - \rho(|z_m(y)|)] \, dy \\
  \geq \int_Q \liminf_{m \to \infty} [C_0(1 + |z_m(y)|) - \rho(|z_m(y)|)] \, dy = C_0,
\]
and so
\[
  \int_Q \rho(|z_m(y)|) \, dy \to 0 \quad \text{as} \quad m \to \infty.
\]

Thus
\[
  (1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{m \to \infty} \int_Q f(x_0, v_0, \nabla^{k} w_m(y)) \, dy.
\]

If \( g(\xi) := f(x_0, v_0, \xi) \) satisfies condition (a), then use Proposition 2.1 and if either condition (b) or (c) holds, then apply Theorem 1.7 in [17] to conclude that
\[
  (1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq f(x_0, u(x_0), \ldots, \nabla^{k} u(x_0)),
\]
and it suffices to let \( \varepsilon \to 0^+ \).

**Proof of Theorem 1.2.** Theorem 1.2 can be easily deduced from Theorem 1.8 in [17]. It suffices to write
\[
  \int_{\Omega} f(x, u(x), \ldots, \nabla^{k} u(x)) \, dx =: \int_{\Omega} F(x, v(x), \nabla v(x)) \, dx
\]
with \( v := (u, \ldots, \nabla^{k-1} u) \). Note, however, that the coercivity condition (1.6) for \( F \) now reads
\[
  F(x_0, v_0, \eta) \geq C|\eta_k| - \frac{1}{C},
\]
where
\[
  \eta = (\eta_1, \ldots, \eta_k) \in E_1^d \times \cdots \times E_k^d \quad \text{and} \quad F(x, v, \eta) := f(x, v, \eta_k).
\]
In order to be in position to apply Theorem 1.8 we need to ensure full coercivity. Due to the strong convergence of admissible sequences \( \{u_n\} \) in \( W^{k-1,1}(\Omega; \mathbb{R}^d) \), and therefore of \( \{v_n\} \) in \( L^1 \left( \Omega; E_{[k-1]}^d \right) \), it suffices to consider
\[
  F_\varepsilon(x, v, \eta) := F(x, v, \eta) + \varepsilon \chi_A(x, v)(|\eta_1, \ldots, \eta_{k-1}|).
\]
where \( A := \{ (x, v) \in \Omega \times E^d_{[k-1]} : f(x, v, \cdot) \neq 0 \} \). Theorem 1.8 in [17] now yields
\[
\liminf_{n \to \infty} \int_{\Omega} f(x, u_n, \ldots, \nabla^k u_n) \, dx = \liminf_{n \to \infty} \int_{\Omega} F(x, v_n, \nabla v_n) \, dx \\
\geq \int_{\Omega} F(x, v, \nabla v) \, dx - \varepsilon \int_{\Omega} \left| (\nabla u, \ldots, \nabla^{k-1} u) \right| \, dx \\
= \int_{\Omega} f(x, u, \ldots, \nabla^k u) \, dx \\
- \varepsilon \int_{\Omega} \left| (\nabla u, \ldots, \nabla^{k-1} u) \right| \, dx.
\]

Let \( \varepsilon \to 0^+ \).

4. Proof of Theorem 1.4

Throughout this section we assume that \( N \geq 3 \).

**Lemma 4.1.** Let \( D \) be a cube with \( |D| \leq 1 \). Then there exist constants \( C > 0 \) and \( \lambda \in (0,1) \), depending only on \( N \), a function \( u \in W^{2,\infty}(D; \mathbb{R}) \) with compact support in \( D \), and sets \( A, E, G \subset D \), with \( A \cup E \cup G = D \) and \( |E| \leq \lambda |D| \), such that
\[
\|\Delta u\|_{L^1(D; \mathbb{R})} \leq C|D|, \quad \|u\|_{W^{1,1}(D; \mathbb{R})} \leq C|D|^{1+\frac{1}{N}},
\]
\[
\Delta u = 1 \quad \text{on} \ A,
\]
\[
u = 0 \quad \text{on} \ E, \quad \nu \geq 1 \quad \text{on} \ G.
\]

**Proof.** After a translation we may assume that there exists \( B(0,R) \subset D \) such that
\[
C^{-1}R^N \leq |D| \leq CR^N, \quad R \in (0,1/2),
\]
for some \( C > 0 \). We search for a radial function of type
\[
u(x) := \varphi(|x|),
\]
where \( \varphi \) is a \( C^2 \)-function on \( (0,\infty) \) such that
\[
\varphi(t) = 0 \quad \text{for} \ t \geq R,
\]
\[
\varphi'(0+) = 0.
\]
Further, we want that for some \( a > 0 \),
\[
\Delta u(x) = \begin{cases} \ -a & \text{if } |x| < r, \\
\ 1 & \text{if } r < |x| < R, \end{cases}
\]
where \( r \) is determined by the equation
\[
r^{2-N}R^N = 2N(N-2). \tag{4.7}
\]
Note that \( r \in (0, R) \), because \( R < 1 \) and \( N \geq 3 \). In order to find \( a \) and \( \varphi \) satisfying (4.1), (4.5) and (4.6), we note that
\[
\Delta u(x) = \varphi''(|x|) + |x|^{-1} (N-1) \varphi'(|x|), \quad \text{for } |x| \neq 0,
\]
or, equivalently,
\[
\Delta u(x) = t^{1-N}(t^{N-1} \varphi'(t))', \quad \text{where } t = |x|.
\]
On the interval \( (r, R) \), (4.6) now yields
\[
t^{N-1} \varphi'(t) = t^{N-1}.
\]
and thus, by \((4.4)\),
\[
\varphi'(t) = \frac{t}{N} \left(1 - \frac{R^N}{t^N}\right).
\]
On the interval \((0, r)\), and in view of \((4.6)\), we have
\[
(t^{N-1} \varphi'(t))' = -at^{N-1},
\]
which, together with \((4.5)\), implies that
\[
(\varphi'(t))' = -\frac{at^N}{N},
\]
We have
\[
-\frac{ar}{N} = \varphi'(r-) = \varphi'(r+) = \frac{r}{N} \left(1 - \frac{R^N}{r^N}\right),
\]
and thus
\[
(4.10)
a = \left(\frac{R^N}{r^N} - 1\right).
\]
Now the function \(u\) is uniquely determined by its properties. Obviously we have
\[
(4.1)\qquad \text{Proof of Theorem 1.4.}
\]
We set \(\Omega = (0, 1)\), and we construct the \(1\)-periodic sequence \(\{u_n\}\) as follows: divide \(\Omega\) into small cubes \(D_\alpha\) of measure \(\frac{1}{n^N}\), \(\alpha \in I_n\), where the set of indices \(I_n\) has cardinality \(n^N\). On each \(D_\alpha\), we construct \(u_n\) as indicated in Lemma 1.1 and denote by \(A_\alpha, E_\alpha, G_\alpha\) the corresponding sets. Then \(u_n \to 0\) in \(W^{1,1}(\Omega; \mathbb{R})\), because
\[
\|u_n\|_{W^{1,1}(\Omega; \mathbb{R})} = \sum_{\alpha \in I_n} \|u_n\|_{W^{1,1}(D_\alpha; \mathbb{R})} \leq n^N C \left(\frac{1}{n^N}\right)^{1+\frac{1}{r^N}} \to 0 \quad \text{as } n \to \infty,
\]
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and $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ is uniformly bounded since

$$\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})} = \sum_{\alpha \in I_n} \|\Delta u_n\|_{L^1(D_{\alpha};\mathbb{R})} \leq n^N C \frac{1}{n^N} = C.$$ 

Consider the functional

$$F(v) := \int_{\Omega} h(v)(1 - \Delta v)^+ \, dx.$$ 

For $\alpha \in I_n$ we have by (4.1)–(4.3),

$$h(u_n) = 1 \quad \text{and} \quad \Delta u_n = 0 \quad \text{on } E_\alpha,$$

$$\Delta u_n = 1 \quad \text{on } A_\alpha,$$

$$h(u_n) = 0 \quad \text{on } G_\alpha,$$

and thus

$$\int_{D_{\alpha}} h(u_n)(1 - \Delta u_n)^+ \, dx = |E_\alpha| \leq \lambda |D_\alpha|.$$ 

Summing up over $\alpha \in I_n$, we conclude that

$$\int_{\Omega} h(u_n)(1 - \Delta u_n)^+ \, dx \leq \lambda < 1 = F(0).$$

**Remark 4.2.** We cannot obtain an a priori bound on $\|u_n\|_{W^{2,1}(\Omega;\mathbb{R})}$, because the function

$$h(v)(1 - \text{trace } \xi)^+$$

is convex in the last variable, and, after adding a small multiple of $|\xi|$, the corresponding functional is lower semicontinuous on $W^{1,1}(\Omega;\mathbb{R})$ according to Theorem 1.5. A direct heuristic computation using the notation of Lemma 4.1 yields

$$\int_Q |\nabla^2 u| \, dx \sim \int_0^R \left( t^{N-1} |\varphi''(t)| + t^{N-2} |\varphi'(t)| \right) \, dt$$

$$\sim \left( \int_0^R t^{N-1} \frac{R^N}{r^N} \, dt + \int_0^R \frac{R^N}{t} \, dt \right)$$

$$\sim R^N \log R,$$

and so an inequality of the type

$$\|\nabla^2 u\|_{L^1(Q;\mathbb{R}^d)} \leq C|Q|$$

will not hold.

5. **Proof of Theorems 1.5 and 1.6**

**Proof of Theorem 1.5.** As in the proof of Theorem 1.2, it is easy to obtain Theorem 1.5 from Theorem 1.1 in [18] by considering $v := (u, \ldots, \nabla^{k-1} u)$ and the reformulated functionals

$$\int_{\Omega} \left( F(x, v(x), \nabla v(x)) + \varepsilon \chi_A(x, v(x)) \left| (\nabla v)_1, \ldots, (\nabla v)_{k-1} \right| \right) \, dx,$$

with

$$\nabla v := ((\nabla v)_1, \ldots, (\nabla v)_k) \in E_1^d \times \cdots \times E_k^d.$$
Observe that, as opposed to Theorems 1.1(b) and (c), 1.2 and 1.3, Theorem 1.6 cannot be deduced easily from the analogous result already obtained in the case where k = 1, i.e., Theorem 1.4 in [18]. Indeed, there is no obvious way of perturbing the new integrand $H(x, v, \mathcal{M}(\nabla v)) := h(x, u, \ldots, \nabla^{k-1}u, \mathcal{M}(\nabla^k u))$, with $v := (u, \ldots, \nabla^{k-1}u)$, in such a way that (1.10) is satisfied for the perturbed integrand, i.e.,

$$H_\varepsilon(x, v, \mathcal{M}(\nabla v)) \geq C_\varepsilon |\mathcal{M}(\nabla v)| - \frac{1}{C_\varepsilon}$$

and $H_\varepsilon \geq H$, for all $(x, v, \xi) := h(x, v, \mathcal{M}(\xi))$. We proceed as in the proof of Theorem 1.1 until we reach (3.3); precisely,

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \liminf_{m \to \infty} \int_\Omega f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) \, dy,$$

where now $w_m \in W^{k,p}(Q; \mathbb{R}^d)$, and $|w_m - w_0|_{W^{k-1,1}(Q; \mathbb{R}^d)} \to 0$ as $m \to \infty$, where

$$w_0(y) := \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)y^\alpha.$$ 

If $f(x_0, v_0, \cdot) \equiv 0$, with $v_0 := (u(x_0), \ldots, \nabla^{k-1}u(x_0))$, then there is nothing to prove. Otherwise, let $\delta_0 > 0$ be given by (1.9) and (1.10). Setting

$$Q_m := \{y \in Q : |(w_m(y), \ldots, \nabla^{k-1}w_m(y))| \leq \delta_0/(2\varepsilon_m)\},$$

by (5.1) and (1.10) we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{m \to \infty} \int_{Q_m} f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) \, dy$$

$$\geq C \limsup_{m \to \infty} \int_{Q_m} |\mathcal{M}(\nabla^k w_m(y))| \, dy - 1/C,$$

and so there exists a constant $K > 0$ such that

$$\int_{Q_m} |\mathcal{M}(\nabla^k w_m(y))| \, dy \leq K \quad \text{for all } m \in \mathbb{N}.$$ 

By Proposition 2.3 with $M = (x_0 + \varepsilon_1 Q) \times B(v_0, \delta_0/2)$ and $V = \mathbb{R}^7$, in view of (1.9) there exist two sequences of continuous functions

$$a_j : M \to \mathbb{R}, \quad b_j : M \to \mathbb{R}$$

such that

$$h(x, v, \eta) = \sup_j (a_j(x, v) + b_j(x, v) \cdot \eta)^+$$

for all $(x, v) \in M$ and $\eta \in \mathbb{R}^7$. Define

$$h_j(x, v, \eta) := (a_j(x, v) + b_j(x, v) \cdot \eta)^+, \quad f_j(x, v, \xi) := h_j(x, v, \mathcal{M}(\xi)).$$
Clearly $h_j$ is continuous, convex in $\eta$, and
\begin{equation}
0 \leq h_j(x, v, \eta) \leq C_j(|\eta| + 1),
\end{equation}
for all $(x, v) \in M$ and $\eta \in \mathbb{R}^r$, where
\[ C_j := \max\{|a_j(x, v)| + |b_j(x, v)| : (x, v) \in M\} \]
Fix $\varepsilon > 0$ and find $0 < \delta_j \leq \delta_0/2$ such that
\[ |a_j(x, v) - a_j(x_0, v_0)| + |b_j(x, v) - b_j(x_0, v_0)| \leq \varepsilon \]
for all $(x, v) \in (x_0 + \delta_j \overline{Q}) \times \overline{B(v_0, \delta_j)}$. Since the function $s \mapsto s^+$ is Lipschitz continuous with Lipschitz constant 1, we have
\begin{equation}
|f_j(x, v, \xi) - f_j(x_0, v_0, \xi)| \leq |a_j(x, v) - a_j(x_0, v_0)| + |b_j(x, v) - b_j(x_0, v_0)| |\mathcal{M}(\xi)| \leq \varepsilon(1 + |\mathcal{M}(\xi)|)
\end{equation}
for all $(x, v) \in (x_0 + \delta_j \overline{Q}) \times \overline{B(v_0, \delta_j)}$ and all $\xi \in E^d_k$. By (5.1) and for any $j \in \mathbb{N}$ we obtain
\begin{equation}
\frac{d\mu}{d\mathcal{L}^n}(x_0) \geq \liminf_{m \to \infty} \int_{Q_m} f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m w_m(y), \ldots, \nabla^k w_m(y)) \, dy
\end{equation}
\begin{equation}
\geq \liminf_{m \to \infty} \int_{Q_m} f_j(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m w_m(y), \ldots, \nabla^k w_m(y)) \, dy
\end{equation}
\begin{equation}
\geq \liminf_{m \to \infty} \left( \int_{Q_m} f_j(x_0, v_0, \nabla^k w_m(y)) \, dy - \varepsilon - \varepsilon \int_{Q_m} |\mathcal{M}(\nabla^k w_m(y))| \, dy \right)
\end{equation}
\begin{equation}
\geq \liminf_{m \to \infty} \int_{Q_m} f_j(x_0, v_0, \nabla^k w_m(y)) \, dy - \varepsilon - \varepsilon K,
\end{equation}
where we have used (5.3) and (5.2). Define
\[ z_m(y) := (\varepsilon_m^{-1} w_m(y), \ldots, \varepsilon_m \nabla^{k-2} w_m(y)), \quad u_m(y) := \nabla^{k-1} w_m(y). \]
Fix an integer $P \in \mathbb{N}$ such that $e^P > 1 + \|\nabla^{k-1} w_0\|_\infty$. For $m$ sufficiently large, say $m \geq m_P$, we have $e^{2P+1} \leq \delta_0/(2\varepsilon_m)$; so in view of (5.2) we may find $i_m \in \{P + 1, \ldots, 2P\}$ such that
\[ \{y \in Q : e^{i_m} \leq |(z_m(y), u_m(y))| \leq e^{i_{m+1}}\} \subset Q_m \]
and
\[ \int_{\{y \in Q : e^{i_m} \leq |(z_m(y), u_m(y))| \leq e^{i_{m+1}}\}} (1 + |\mathcal{M}(\nabla^k w_m(x))|) \, dx \leq \frac{1 + K}{P}. \]
\end{equation}
Since $\{P + 1, \ldots, 2P\}$ is a finite set, we may find $i_P \in \{P + 1, \ldots, 2P\}$ such that
\begin{equation}
\int_{\{y \in Q : e^{i_P} \leq |(z_m(y), u_m(y))| \leq e^{i_{P+1}}\}} (1 + |\mathcal{M}(\nabla^k w_m(x))|) \, dx \leq \frac{1 + K}{P}
\end{equation}
for infinitely many indices $m \in \mathbb{N}$. From now until the end of the proof we assume without loss of generality that the whole sequence satisfies (5.7).
Set
\[ v_m(y) := G((|z_m(y), u_m(y))|) u_m(y) \]
and

\[ D_m := \{ y \in Q : |(z_m(y), u_m(y))| < e^{i\rho} \}, \]
\[ D_m^n := \{ y \in Q : e^{i\rho} \leq |(z_m(y), u_m(y))| \leq e^{i\rho+1} \}, \]
\[ D_m^p := \{ y \in Q : |(z_m(y), u_m(y))| > e^{i\rho+1} \}, \]

where

\[ G(s) := \begin{cases} 1 & \text{if } s < e^{i\rho}, \\ \frac{e^{i\rho+1} - s}{e^{i\rho+1} - e^{i\rho}} & \text{if } e^{i\rho} \leq s \leq e^{i\rho+1}, \\ 0 & \text{if } s > e^{i\rho+1}. \end{cases} \]

Note that \(|D_m^- \cup D_m^+| = |\{ y \in Q : |(z_m(y), u_m(y))| \geq e^{i\rho} \}|

(5.8)

\[ \leq |\{ y \in Q : |(z_m(y), u_m(y)) - (0, \nabla^{k-1}w_0(y))| \geq 1 \}| \]
\[ \leq |z_m|_{L_1(Q)} + |u_m - \nabla^{k-1}w_0|_{L_1(Q)} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \]

where we have used the fact that \(e^{i\rho} > 1 + \|\nabla^{k-1}w_0\|_\infty\). Also,

(5.9)

\[ |\nabla z_m| = \left| (\varepsilon_m^{-1} \nabla w_m, \ldots, \varepsilon_m^{-1} \nabla w_m) \right| \]
\[ \leq \varepsilon_m \left| (\varepsilon_m^{-1} \nabla w_m, \varepsilon_m^{-2} \nabla w_m, \ldots, \varepsilon_m^{-k+1} \nabla w_m) \right| = \varepsilon_m |(z_m, u_m)|. \]

We claim that

(5.10)

\[ |M(\nabla v_m(y))| \leq C \left( 1 + \varepsilon_m e^{i\rho+1} \right) |M(\nabla^k w_m(y))|. \]

In view of the definition of \( g \) this is immediate for \( x \in D_m \cup D_m^+ \). Thus it remains

to assert (5.10) in \( D_m^- \). We have

\[ \nabla v_m = \left( G \left( |(z_m, u_m)| \right) I + G' \left( |(z_m, u_m)| \right) \frac{u_m \otimes u_m}{|(z_m, u_m)|} \right) \nabla u_m \]
\[ + G' \left( |(z_m, u_m)| \right) \frac{u_m \otimes z_m}{|(z_m, u_m)|} \nabla z_m, \]

where \( I \) is the identity matrix. Since in \( D_m^- \),

\[ |GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|}| + |G' \frac{u_m \otimes z_m}{|(z_m, u_m)|}| \leq C, \]

we have

\[ |M_l \left( \left( GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|} \right) \nabla u_m \right)| \]
\[ \leq |M_l \left( GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|} \right)| |M_l (\nabla u_m)| \leq C |M_l (\nabla u_m)|, \]

and

\[ |M_l \left( G' \frac{u_m \otimes z_m}{|(z_m, u_m)|} \nabla z_m \right)| \leq |M_l \left( G' \frac{u_m \otimes z_m}{|(z_m, u_m)|} \right)| |M_l (\nabla z_m)| \]
\[ \leq \begin{cases} 0 & l > 1, \\ C |M_l (\nabla z_m)| & l = 1, \end{cases} \]
where $\mathcal{M}_l(X)$ is the vector whose components are all the minors of $X$ of order $l$. Here we have used the facts that

$$|\mathcal{M}_l(X + Y)| \leq C \sum_{i=0}^{l} |\mathcal{M}_i(X)| \|\mathcal{M}_{l-i}(Y)||,$$

that

$$|\mathcal{M}_l(X Y)| \leq |\mathcal{M}_l(X)| |\mathcal{M}_l(Y)|,$$

and that $u_m \otimes z_m$ is a rank-one matrix. Then, in view of (5.4),

$$|\mathcal{M}_l(\nabla v_m(y))| \leq C \sum_{i=0}^{l} |\mathcal{M}_i\left(\left(G + G' \frac{u_m \otimes u_m}{|z_m, u_m|}\right) \nabla u_m\right)|$$

$$|\mathcal{M}_{l-i}\left(G' \frac{u_m \otimes z_m}{|z_m, u_m|}\nabla z_m\right)|$$

$$\leq C (|\mathcal{M}_l(\nabla u_m)| + |\mathcal{M}_{l-1}(\nabla u_m)| |\mathcal{M}_1(\nabla z_m)|)$$

$$\leq C (|\mathcal{M}_l(\nabla u_m)| + \epsilon_m (|z_m, u_m| \|\mathcal{M}_{l-1}(\nabla u_m)|).$$

Since $|(z_m, u_m)| \leq e^{i\epsilon+1}$ for $x \in D_m$, we conclude that (5.10) holds.

Since $D_m \subset Q_m$, it follows from (5.6) that

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \geq \liminf_{m \to \infty} \int_{D_m} f_j(x_0, v_0, \nabla v_m) \, dy - \epsilon - \epsilon K.$$  

By (5.4) and (5.8),

$$\int_{D_m^+} f_j(x_0, v_0, \nabla v_m) \, dy \leq C_j \int_{D_m^+} \left(1 + |\mathcal{M}(\nabla v_m)|\right) \, dy = C_j |D_m^+| \to 0,$$

while from (5.4), (5.10) and (5.7), and taking $m > m_P$ so that $\epsilon_m e^{p+1} < 1$,

$$\int_{D_m} f_j(x_0, v_0, \nabla v_m) \, dy \leq C_j \int_{D_m} \left(1 + |\mathcal{M}(\nabla v_m)|\right) \, dy$$

$$\leq C C_j \int_{D_m} \left(1 + |\mathcal{M}(\nabla^k w_m(y))|\right) \, dy$$

$$\leq C C_j \frac{1 + K}{P^\epsilon}.$$  

Consequently, in view of (5.11), (5.12) and (5.13),

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \geq \liminf_{m \to \infty} \int_Q f_j(x_0, v_0, \nabla v_m) \, dy - \epsilon - \epsilon K - C C_j \frac{1 + K}{P^\epsilon},$$

and by (5.2), (5.11), and in view of the fact that $v_m \equiv 0$ in $D_m^+$,

$$\sup_m \int_Q |\mathcal{M}(\nabla v_m)| \, dy \leq K_1 < \infty,$$

where $K_1$ is independent of $m$ and $j$. Define

$$h_{j, \epsilon}(v) := h_j(x_0, v_0, v) + \epsilon |v|,$$

$$f_{j, \epsilon}(\xi) := h_{j, \epsilon}(\mathcal{M}(\xi)).$$
Then by (5.14), (5.15), and Proposition 2.2,

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{m \to \infty} \int_Q h_{j,\varepsilon}(\mathcal{M}(|\nabla v_m(y)|)) \, dy - \varepsilon - \varepsilon(K + K_1) - CC_j \frac{1 + K}{P}
\]

\[
\geq h_{j,\varepsilon}(\mathcal{M}(|\nabla^k w_0(x_0)|)) - \varepsilon - \varepsilon(K + K_1) - CC_j \frac{1 + K}{P}
\]

\[
= f_j(x_0, u(x_0), \cdots, \nabla^k u(x_0)) + \varepsilon \left| \mathcal{M}(|\nabla^k w_0(x_0)|) \right| - \varepsilon - \varepsilon(K + K_1).
\]

Letting first \( P \to \infty \), then taking the supremum in \( j \) yields

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \cdots, \nabla^k u(x_0)) + \varepsilon \left| \mathcal{M}(|\nabla^k w_0(x_0)|) \right| - \varepsilon - \varepsilon(K + K_1),
\]

by (5.3). To complete the proof it suffices to let \( \varepsilon \to 0^+ \).

\[\square\]

**References**


A NOTE ON MEYERS’ THEOREM IN $W^{k,1}$


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