WEAK AMENABILITY OF MODULE EXTENSIONS
OF BANACH ALGEBRAS

YONG ZHANG

Abstract. We start by discussing general necessary and sufficient conditions for a module extension Banach algebra to be \( n \)-weakly amenable, for \( n = 0, 1, 2, \ldots \). Then we investigate various special cases. All these case studies finally provide us with a way to construct an example of a weakly amenable Banach algebra which is not 3-weakly amenable. This answers an open question raised by H. G. Dales, F. Ghahramani and N. Grønbæk.

INTRODUCTION

Suppose that \( \mathfrak{A} \) is a Banach algebra, and that \( X \) is a Banach \( \mathfrak{A} \)-bimodule. A derivation from \( \mathfrak{A} \) into \( X \) is a linear operator \( D: \mathfrak{A} \to X \) satisfying
\[
D(ab) = D(a)b + aD(b) \quad (a, b \in \mathfrak{A}).
\]
A derivation \( D \) is inner if there is \( x_0 \in X \) such that \( D(a) = ax_0 - x_0a \) for \( a \in \mathfrak{A} \). The quotient space \( \mathcal{H}^1(\mathfrak{A}, X) \) of all continuous derivations from \( \mathfrak{A} \) into \( X \) modulo the subspace of inner derivations is called the first cohomology group of \( \mathfrak{A} \) with coefficients in \( X \). A Banach algebra \( \mathfrak{A} \) is said to be amenable if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\} \) for every Banach \( \mathfrak{A} \)-bimodule \( X \); here \( \mathfrak{A}^* \) denotes the Banach dual module of \( \mathfrak{A} \). The algebra \( \mathfrak{A} \) is said to be weakly amenable if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\} \), and is called \( n \)-weakly amenable, for an integer \( n \geq 0 \), if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\} \), where \( \mathfrak{A}^{(n)} \) is the \( n \)-th dual module of \( \mathfrak{A} \) when \( n \geq 1 \), and is \( \mathfrak{A} \) itself when \( n = 0 \). The algebra \( \mathfrak{A} \) is said to be permanently weakly amenable if it is \( n \)-weakly amenable for all \( n \geq 1 \).

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [22] (see also [7], [9], [11]–[16], [21] and [24]). Dales, Ghahramani and Grønbæk initiated the study of \( n \)-weak amenability of Banach algebras in their recent paper [10], where they revealed many important properties of this sort of Banach algebra. An interesting problem concerning this class of Banach algebras is the relation between \( n \)-weak amenability and \( m \)-weak amenability for different integers \( n \) and \( m \). For instance, if \( \mathfrak{A} \) is a commutative Banach algebra, then the assertion that \( \mathfrak{A} \) is weakly amenable is equivalent to saying that it is permanently weakly amenable ([1] Theorem 1.5]); but, for noncommutative Banach algebras, things are different—we only know that \( (n + 2) \)-weak amenability implies \( n \)-weak amenability.
amenability for $n \geq 1$ ([10, Proposition 1.2]), and weak amenability does not imply 2-weak amenability ([10, Theorems 5.1 and 5.2]). After investigating varieties of classical Banach algebras, Dales, Ghahramani and Grønbæk raised and left open the following question in [10]: Does weak amenability imply 3-weak amenability?

This paper is designed to answer the preceding question. We will construct a counterexample to the question. For this purpose, we study $n$-weak amenability of the module extension Banach algebra $A \oplus X$, the $l_1$-direct sum of a Banach algebra $A$ and a nonzero Banach $A$-module $X$ with the algebra product defined as follows:

$$(a, x) \cdot (b, y) = (ab, ay + xb) \quad (a, b \in A, \ x, y \in X).$$

Some aspects of algebras of this form have been discussed in [2] and [10]. We choose this class of Banach algebras to investigate for the preceding question because this class is neither too small nor is it too large; it contains permanently weakly amenable Banach algebras (see Section 6), and it contains no amenable Banach algebras due to [8, Lemma 2.7], since $X$ is a complemented nilpotent ideal in the algebra. If $A$ has both left and right approximate identities and they are also, respectively, left and right approximate identities for $X$, then $A \oplus X$ cannot be pointwise approximately biprojective (see [30]). The class of module extension Banach algebras also includes the natural triangular Banach algebra whose amenability has been investigated in [12]. We will give some comment on the latter algebra in Section 2.

This paper is organized as follows: in Section 1 we study the construction of module actions of $2m$-th dual algebras on $2m$-th dual modules. This extends the corresponding discussion in [10]. In Section 2 we give the main theorems which deal with the necessary and sufficient conditions for $A \oplus X$ to be $n$-weakly amenable. Section 3 discusses various techniques for lifting derivations. These will be applied in Section 4 to give the proofs of the main theorems. These will be applied in Section 4 to give the proofs of the main theorems. Sections 5 and 6 deal with the special cases of $X = A, A^{\ast}$ and $X_0$, where $X_0$ denotes an $A$-bimodule with the right module action trivial. In Section 7 we first discuss the condition for $A \oplus (X_1 + X_2)$ to be weakly amenable, where $+$ denotes the $l_1$ direct sum (of modules). Then, we give an example of a weakly amenable Banach algebra of this form and prove that it is not 3-weakly amenable. This finally answers the preceding open question in the negative.

Since $(A \oplus X)^{\ast} = (0 \oplus X)^{\perp} + (A \oplus 0)^{\perp}$, where $+$ denotes the direct module $l_\infty$-sum, and $(0 \oplus X)^{\perp}$ (respectively, $(A \oplus 0)^{\perp}$) is isometrically isomorphic to $A^\ast$ (respectively, $X^\ast$) as $A$-bimodules, for convenience, in this paper we simply identify the corresponding terms and write:

$$(A \oplus X)^\ast = A^\ast + X^\ast.$$

Similarly, we will identify the underlying space of the $n$-th conjugate $(A \oplus X)^{(n)}$ with $A^{(n)} + X^{(n)}$. The sum is an $l_1$-sum when $n$ is even and is an $l_\infty$-sum when $n$ is odd.

1. Bimodule actions of $A^{(2m)}$ on $X^{(2m)}$

Suppose that $A$ is a Banach algebra, and $X$ is a Banach $A$-bimodule. According to [11] pp. 27 and 28$, X^\ast$ is a Banach $A^{\ast\ast}$-bimodule, where $A^{\ast\ast}$ is equipped with the first Arens product. The module actions are successively defined as follows.
First, for $x \in X$, $f \in X^*$, $\phi \in X^{**}$ and $u \in \mathfrak{A}^{**}$, define $\phi f$, $fx \in \mathfrak{A}^*$ and $uf \in X^*$ by
\[
\langle a, \phi f \rangle = \langle fa, \phi \rangle, \quad \langle a, fx \rangle = \langle xa, f \rangle \quad (a \in \mathfrak{A}),
\]
\[
\langle x, uf \rangle = \langle fx, u \rangle \quad (x \in X).
\]
Then, for $\phi \in X^{**}$ and $u \in \mathfrak{A}^{**}$, define $u\phi \in X^{**}$ by
\[
\langle f, u\phi \rangle = \langle \phi f, u \rangle, \quad \langle f, \phi u \rangle = \langle uf, \phi \rangle \quad (f \in X^*).
\]
These give the left and right $\mathfrak{A}^{**}$-module actions on $X^{**}$. Also, the definition for $uf$ with $u \in \mathfrak{A}^{**}$ and $f \in X^*$ gives a left Banach $\mathfrak{A}^{**}$-module action on $X^*$. When $u = a \in \mathfrak{A}$, all the above $\mathfrak{A}^{**}$-module actions coincide with the $\mathfrak{A}$-module actions on the corresponding dual modules $X^*$ and $X^{**}$. Moreover, it is readily seen that, with these module actions, the first Arens product on $(\mathfrak{A} \oplus X)^{**}$ may be represented by
\[
(u, \phi) \cdot (v, \psi) = (uv, u\psi + \phi v) \quad (u, v \in \mathfrak{A}^{**}, \phi, \psi \in X^{**}).
\]

Viewing $\mathfrak{A}^{(2m)}$ as a new $\mathfrak{A}$ and $X^{(2m)}$ as a new $X$, the preceding procedure will successively define $X^{(2m+2)}$ as a Banach $\mathfrak{A}^{(2m+2)}$-bimodule. Here, and throughout the paper, the first Arens product is consistently assumed on each $\mathfrak{A}^{(2n)}$. Since some relations arising from the procedure are important for later use, we now give the definition in detail as follows.

Suppose that the bimodule action of $\mathfrak{A}^{(2m)}$ on $X^{(2m)}$ has been defined, where $m \geq 1$. Then in a natural way, $X^{(2m+k)}$, $k \geq 1$, is a Banach $\mathfrak{A}^{(2m)}$-bimodule with the module multiplications $uA$ and $Au \in X^{(2m+k)}$, for $A \in X^{(2m+k)}$ and $u \in \mathfrak{A}^{(2m)}$, defined by
\[
\langle \gamma, uA \rangle = \langle \gamma u, A \rangle, \quad \langle \gamma, Au \rangle = \langle u\gamma, A \rangle \quad (\gamma \in X^{(2m+k-1)}).
\]
If $u = a \in \mathfrak{A}$, these module actions coincide with $\mathfrak{A}$-module actions on $X^{(2m+k)}$.

Then, for $F \in X^{(2m+1)}$ and $\Phi \in X^{(2m+2)}$, define $F\Phi$, $\Phi F \in \mathfrak{A}^{(2m+1)}$ by
\[
\langle u, F\Phi \rangle = \langle F, \Phi u \rangle = \langle uF, \Phi \rangle \quad (u \in \mathfrak{A}^{(2m)}).
\]

Throughout this paper, for a Banach space $Y$ and an element $y \in Y$, $\hat{y}$ always denotes the image of $y$ in $Y^{**}$ under the canonical mapping. When $F \in X^{(2m+1)}$ and $\phi \in X^{(2m)}$, we denote $F\hat{\phi}$ by $F\phi$ and $\hat{\phi}F$ by $\phi F$. It is easy to check that
\[
\langle u, F\phi \rangle = \langle \phi u, F \rangle, \quad \langle u, \phi F \rangle = \langle u\phi, F \rangle \quad (u \in \mathfrak{A}^{(2m)}).
\]

By using the canonical image of $F$ or $\Phi$ in the appropriate $2l$-th dual space of the space that it belongs to, we can then signify a meaning for $F\Phi$ and $\Phi F$ for every $F \in X^{(2m+1)}$ and $\Phi \in X^{(2m)}$; they are elements of $\mathfrak{A}^{(2k+1)}$, where $k = \max\{m, n - 1\}$.

Now for $\mu \in \mathfrak{A}^{(2m+2)}$ and $F \in X^{(2m+1)}$, we define $\mu F \in X^{(2m+1)}$ by
\[
\langle \phi, \mu F \rangle = \langle F\phi, \mu \rangle \quad (\phi \in X^{(2m)}).
\]
This actually defines a left Banach $\mathfrak{A}^{(2m+2)}$-module action on $X^{(2m+1)}$.

Finally, for $\mu \in \mathfrak{A}^{(2m+2)}$ and $\Phi \in X^{(2m+2)}$, define $\Phi \mu \in X^{(2m+2)}$ by
\[
\langle F, \mu \phi \rangle = \langle \Phi F, \mu \rangle, \quad \langle F, \Phi \mu \rangle = \langle \mu F, \Phi \rangle \quad (F \in X^{(2m+1)}).
\]
These finally define the $\mathfrak{A}^{(2m+2)}$-module actions on $X^{(2m+2)}$ and, therefore, complete our definition.
If $\lim u_\alpha = \mu$ in $\sigma(\mathfrak{A}^{(2m+2)}, \mathfrak{A}^{(2m+1)})$ and $\lim \phi_\beta = \Phi$ in $\sigma(X^{(2m+2)}, X^{(2m+1)})$, where $(u_\alpha) \subset \mathfrak{A}^{(2m)}$ and $(\phi_\beta) \subset X^{(2m)}$, and $\sigma(Y^*, Y)$ denotes the weak* topology on $Y^*$, then

$$\mu \Phi = \lim \lim_{\beta} u_\alpha \phi_\beta, \quad \Phi \mu = \lim \lim_{\alpha} \phi_\beta u_\alpha \quad \text{in} \quad \sigma(X^{(2m+2)}, X^{(2m+1)}).$$

For $\mu \in \mathfrak{A}^{(2m+2)}$ and $\phi \in X^{(2m)}$, since $\mu \phi = \mu \hat{\phi}$, $\phi \mu = \hat{\phi} \mu$, we have

$$\langle F, \mu \phi \rangle = \langle \phi F, \mu \rangle, \quad \langle F, \phi \mu \rangle = \langle F \phi, \mu \rangle \quad (F \in X^{(2m+1)}).$$

One can also easily check the relations

$$u \hat{f} = \hat{u} f = (uf)^*, \quad \hat{f} \phi = (f \phi)^*, \quad \hat{\phi} u = (\phi u)^*,$$

where $f \in X^{(2m-1)}$, $\phi \in X^{(2m)}$ and $u \in \mathfrak{A}^{(2m)} (m \geq 1)$. Therefore, each product agrees with those previously defined.

Concerning dual module morphisms, we have the following.

**Lemma 1.1.** Suppose that $X$ and $Y$ are Banach $\mathfrak{A}$-bimodules. Then, for every continuous $\mathfrak{A}$-bimodule morphism $\tau \colon X \to Y$ and for each $m \geq 1$, $\tau^{(2m)} \colon X^{(2m)} \to Y^{(2m)}$, the $2m$-th dual operator of $\tau$ is an $\mathfrak{A}^{(2m)}$-bimodule morphism.

**Proof.** It suffices to prove the lemma in the case where $m = 1$. However, for this simple case, the proof is straightforward if we note that $\tau^{**}$ is weak*-weak* continuous.

In the following, to avoid involving unnecessarily complicated notation, for an element $y$ in a Banach space $Y$, we will use the same notation $y$ to represent its canonical image in any of the $2m$-th dual spaces $Y^{(2m)}$.

Take $\mathfrak{A}^{(n)} \rightharpoonup X^{(n)}$ as the underlying space of $(\mathfrak{A} \oplus X)^{(n)}$. From induction, by using the relations in (1.1) and (1.2), one can verify that the $(\mathfrak{A} \oplus X)$-bimodule actions on $(\mathfrak{A} \oplus X)^{(n)}$ are formulated as follows:

$$\begin{aligned}
(a, x) \cdot (a^{(n)}, x^{(n)}) &= \begin{cases}
(aa^{(n)} + xa^{(n)}, ax^{(n)}), & \text{if } n \text{ is odd}; \\
(aa^{(n)}, ax^{(n)} + xa^{(n)}), & \text{if } n \text{ is even},
\end{cases} \\
(a^{(n)}, x^{(n)}) \cdot (a, x) &= \begin{cases}
(a^{(n)}a + x^{(n)}x, x^{(n)}a), & \text{if } n \text{ is odd}; \\
(a^{(n)}a, a^{(n)}x + x^{(n)}a), & \text{if } n \text{ is even},
\end{cases}
\end{aligned}$$

where $(a, x) \in \mathfrak{A} \oplus X$ and $(a^{(n)}, x^{(n)}) \in \mathfrak{A}^{(n)} \rightharpoonup X^{(n)} = (\mathfrak{A} \oplus X)^{(n)}$.

2. **Main theorems**

Suppose that $\mathfrak{A}$ is a Banach algebra, and $X$ is a Banach $\mathfrak{A}$-bimodule. For $n$-weak amenability of the Banach algebra $\mathfrak{A} \oplus X$, we have the following main results, whose proofs will be given in Section 4.

**Theorem 2.1.** For $m \geq 0$, $\mathfrak{A} \oplus X$ is $(2m+1)$-weakly amenable if and only if the following conditions hold:

1. $\mathfrak{A}$ is $(2m+1)$-weakly amenable;
2. $\mathcal{H}^1(\mathfrak{A}, X^{(2m+1)}) = \{0\};$
3. for every continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \rightarrow \mathfrak{A}^{(2m+1)}$, there is $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ for $a \in \mathfrak{A}$ and $\Gamma(x) = xF - Fx$ for $x \in X$;

4. the only continuous $\mathfrak{A}$-bimodule morphism $T: X \rightarrow X^{(2m+1)}$ for which $xT(y) + T(x)y = 0$ for every $x, y \in X$ in $\mathfrak{A}^{(2m+1)}$ is $T = 0$.

**Theorem 2.2.** For $m \geq 0$, $\mathfrak{A} \oplus X$ is $2m$-weakly amenable if and only if the following conditions hold:

1. the only continuous derivations $D: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m)}$ for which there is a continuous operator $T: X \rightarrow X^{(2m)}$ such that $T(ax) = D(a)x + aT(x)$ and $T(xa) = xD(a) + T(x)a$ are the inner derivations;

2. $\mathcal{H}^1(\mathfrak{A}, X^{(2m)}) = \{0\}$;

3. the only continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \rightarrow \mathfrak{A}^{(2m)}$ for which $x\Gamma(y) + \Gamma(x)y = 0$ for every $x, y \in X$ in $X^{(2m)}$ is zero;

4. for every continuous $\mathfrak{A}$-bimodule morphism $T: X \rightarrow X^{(2m)}$, there exists $u \in \mathfrak{A}^{(2m)}$ for which $au = ua$ for every $a \in \mathfrak{A}$ and $T(x) = xu - ux$ for $x \in X$.

**Remark 2.3.** A simple calculation shows that, when $m = 0$, condition 3 in Theorem 2.1 is equivalent to the following:

- there is no nonzero continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \rightarrow \mathfrak{A}^*$.

For the general case, condition 3 in Theorem 2.1 is equivalent to the following:

3'. if $\Gamma: X \rightarrow \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism, then $\Gamma(X) \subseteq \mathfrak{A}^\perp$ and there is $G \in X^{(2m+1)} \cap \mathfrak{A}^\perp$ for which $aG - Ga = 0$ in $X^{(2m+1)}$ for every continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X \rightarrow \mathfrak{A}^{(2m+1)}$ (i.e., $\Gamma(x) = xG - GX$ for each $x \in X$).

**Proposition 2.4.** Suppose that condition 4 of Theorem 2.1 holds for an $m \geq 0$. Then, span$(\mathfrak{A}X + X\mathfrak{A})$ is dense in $X$.

**Proof.** Assume, towards a contradiction, that span$(\mathfrak{A}X + X\mathfrak{A})$ is not dense in $X$. Take a nonzero element $F \in X^* \cap (\mathfrak{A}X + X\mathfrak{A})^\perp$, and define $T: X \rightarrow X^*$ by $T(x) = F(x)F$.

Since $F|_{\mathfrak{A}X + X\mathfrak{A}} = 0$, it is easy to see that $T$ is a nonzero, continuous $\mathfrak{A}$-bimodule morphism and that $\mathfrak{A}T(X) = T(X)\mathfrak{A} = \{0\}$. Also, for $x, y \in X$, we have $xT(y) = T(x)y = 0$ in $\mathfrak{A}^*$ since $T(X) \subseteq (\mathfrak{A}X)^\perp \cap (X\mathfrak{A})^\perp$. This shows that condition 4 of Theorem 2.1 does not hold for $m = 0$. So it does not hold for all $m \geq 0$. This is a contradiction.

**Corollary 2.5.** For $m = 0$, condition 4 in Theorem 2.1 is equivalent to the following:

4'. span$(\mathfrak{A}X + X\mathfrak{A})$ is dense in $X$ and there is no nonzero $\mathfrak{A}$-bimodule morphism $T: X \rightarrow X^*$ satisfying $\langle x, T(y) \rangle + \langle y, T(x) \rangle = 0$ for $x, y \in X$.

**Proof.** Suppose that condition 4 in Theorem 2.1 holds. From the preceding proposition, span$(\mathfrak{A}X + X\mathfrak{A})$ is dense in $X$. If the $\mathfrak{A}$-bimodule morphism $T: X \rightarrow X^*$ satisfies

$$\langle x, T(y) \rangle + \langle y, T(x) \rangle = 0$$

for $x, y \in X$,

then, for every $a \in \mathfrak{A}$,

$$\langle a, xT(y) + T(x)y \rangle = \langle ax, T(y) \rangle + \langle y, T(ax) \rangle = 0.$$

This shows that $xT(y) + T(x)y = 0$ for $x, y \in X$. Therefore, $T = 0$ and so 4' holds.
Conversely, if \( 4^0 \) holds, and \( T: X \to X^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism satisfying \( xT(y) + T(x)y = 0 \) in \( \mathfrak{A}^* \), then, for every \( x = ax_1 + x_2b \in \mathfrak{A}X + X\mathfrak{A} \) and \( y \in X \), we have
\[
\langle x, T(y) \rangle + \langle y, T(x) \rangle = \langle ax_1T(y) + T(x_1)y + (b, T(y)x_2 + yT(x_2)) = 0.
\]
Since \( \text{span}(\mathfrak{A}X + X\mathfrak{A}) \) is dense in \( X \), this implies that \( \langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 \) for all \( x, y \in X \). Hence \( T = 0 \), and so condition 4 of Theorem 2.1 holds for \( m = 0 \). \( \square \)

Suppose that \( \mathfrak{A} \) and \( \mathfrak{B} \) are Banach algebras, and let \( \mathcal{M} \) be a Banach \( \mathfrak{A}, \mathfrak{B} \)-module. The algebra \( \mathcal{T} \) with the triangular matrix structure
\[
\mathcal{T} = \begin{pmatrix} \mathfrak{A} & \mathcal{M} \\ 0 & \mathfrak{B} \end{pmatrix}
\]
is called a triangular Banach algebra. The sum and product on \( \mathcal{T} \) are given by the usual \( 2 \times 2 \) matrix operations and obvious internal module actions. The norm on \( \mathcal{T} \) is
\[
\left\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right\| = \|a\|_\mathfrak{A} + \|m\|_\mathcal{M} + \|b\|_\mathfrak{B}.
\]
Denote by \( \mathfrak{A}+\mathfrak{B} \) the direct \( l_1 \)-sum Banach algebra of \( \mathfrak{A} \) and \( \mathfrak{B} \), and view \( \mathcal{M} \) as an \( (\mathfrak{A}+\mathfrak{B}) \)-bimodule with the module actions given by
\[
(a, b) \cdot m = am, \quad m \cdot (a, b) = mb, \quad a \in \mathfrak{A}, \ b \in \mathfrak{B}, \ m \in \mathcal{M}.
\]
Then \( \mathcal{T} \) is isometrically isomorphic to the module extension Banach algebra \( (\mathfrak{A}+\mathfrak{B}) \oplus \mathcal{M} \). With this setting and some calculations, one sees that Theorems 2.1 and 2.2 imply some main results in [12]. For instance, if \( \mathfrak{A} \) and \( \mathfrak{B} \) are unital and \( \mathcal{M} \) is a unital \( \mathfrak{A}, \mathfrak{B} \)-module, then \( \mathcal{T} \) is weakly amenable if and only if both \( \mathfrak{A} \) and \( \mathfrak{B} \) are weakly amenable. In fact, the condition can be weakened further to the following: there exist a bounded approximate identity of \( \mathfrak{A} \) and a bounded approximate identity of \( \mathfrak{B} \) that are also, respectively, left and right approximate identities for \( \mathcal{M} \).

3. Lifting derivations

In this section we give several lemmas concerning the lifting of derivations (and module morphisms) from \( \mathfrak{A} \) (or \( X \)) into \( \mathfrak{A}^{(n)} \) or \( X^{(n)} \) to derivations from \( \mathfrak{A} \oplus X \) into \( (\mathfrak{A} \oplus X)^{(n)} \).

Lemma 3.1. Suppose that \( \Gamma: X \to \mathfrak{A}^{(2m+1)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Then \( \overline{\Gamma}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)} \), defined by
\[
\overline{\Gamma}((a, x)) = (\Gamma(x), 0),
\]
is a continuous derivation. The derivation \( \overline{\Gamma} \) is inner if and only if there exists \( F \in X^{(2m+1)} \) such that \( aF - Fa = 0 \) and \( \Gamma(x) = xF - Fx \) for \( a \in \mathfrak{A} \) and \( x \in X \).

Proof. It is straightforward to check that \( \overline{\Gamma} \) is a continuous derivation. Noting that \( (\Gamma(x), 0) = \overline{\Gamma}((0, x)) \) and \( \overline{\Gamma}((a, 0)) = (0, 0) \), one can also see easily that the element \( F \in \mathfrak{A}^{(2m+1)} \) described in the lemma exists if \( \overline{\Gamma} \) is inner.

Conversely, if such an element \( F \) exists, then
\[
\overline{\Gamma}((a, x)) = (\Gamma(x), 0) = (xF - Fx, aF - Fa) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x),
\]
showing that \( \overline{\Gamma} \) is inner. \( \square \)
A similar proof gives the following lemma.

**Lemma 3.2.** Suppose that $T : X \rightarrow X^{(2m)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism. Then $\overline{T} : \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m)}$, defined by

$$\overline{T}((a, x)) = (0, T(x)),$$

is a continuous derivation. The derivation $\overline{T}$ is inner if and only if there exists $u \in \mathfrak{A}^{(2m)}$ such that $ua = au$ for $a \in \mathfrak{A}$, and $T(x) = xu - ux$ for all $x \in X$.

Concerning dual operators we have the following.

**Lemma 3.3.** Suppose that $k > 0$ is an integer, and that $D : \mathfrak{A} \rightarrow X^{(k)}$ is a (continuous) derivation. Then, for every integer $m \geq 0$, $D^{(2m+1)} : X^{(k+2m+1)} \rightarrow \mathfrak{A}^{(2m+1)}$, the $(2m+1)$-th dual operator of $D$, satisfies

$$D^{(2m+1)}(aF) = aD^{(2m+1)}(F) - (D(a)F)|_{\mathfrak{A}^{(2m)}},$$

$$D^{(2m+1)}(Fa) = D^{(2m+1)}(F)a - (FD(a))|_{\mathfrak{A}^{(2m)}},$$

for $a \in \mathfrak{A}$ and $F \in X^{(k+2m+1)}$.

**Proof.** The lemma is true for $m = 0$ because

$$\langle b, D^*(aF) \rangle = \langle D(b)a, F \rangle = \langle D(ba - bD(a), F \rangle = \langle b, aD^*(F) - D(a)F \rangle$$

and

$$\langle b, D^*(Fa) \rangle = \langle aD(b), F \rangle = \langle D(ab) - D(a)b, F \rangle = \langle b, D^*(F)a - FD(a) \rangle,$$

for $a, b \in \mathfrak{A}$ and $F \in X^{(k+1)}$.

For $m > 0$, from Proposition 1.7 of [10], $D^{(2m)} : \mathfrak{A}^{(2m)} \rightarrow X^{(k+2m)}$ is a (continuous) derivation; here we take the first Arens product in each $\mathfrak{A}^{(2m)}$. Then, the above shows that $D^{(2m+1)} = (D^{(2m)})^* : X^{(k+2m+1)} \rightarrow (\mathfrak{A}^{(2m)})^*$ satisfies

$$D^{(2m+1)}(uF) = uD^{(2m+1)}(F) - (D^{(2m)}(u)F)|_{\mathfrak{A}^{(2m)}},$$

and

$$D^{(2m+1)}(Fu) = D^{(2m+1)}(F)u - (FD^{(2m)}(u))|_{\mathfrak{A}^{(2m)}},$$

for $u \in \mathfrak{A}^{(2m)}$ and $F \in X^{(k+2m+1)}$. In particular, when $u = a \in \mathfrak{A}$, these give the formulae in the lemma.

**Lemma 3.4.** Let $m$ be an integer. Suppose that $D : \mathfrak{A} \rightarrow X^{(2m+1)}$ is a (continuous) derivation. Then $\overline{D} : \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$, defined by

$$\overline{D}((a, x)) = (-D^{(2m+1)}(x), D(a))$$

for $(a, x) \in \mathfrak{A} \oplus X$, is also a (continuous) derivation. Moreover,

1. if $\overline{D}$ is inner, then so is $D$;
2. if $D$ is inner, then there exists a (continuous) derivation $\tilde{D} : \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\tilde{D}((a, 0)) = 0$ $(a \in \mathfrak{A})$ and for which $\overline{D} - \tilde{D}$ is inner.
Proof. For $a, b \in \mathfrak{A}$ and $x, y \in X$, we have, from Lemma 3.3

$$\overline{D}((a, x) \cdot (b, y)) = \overline{D}((ab, ay + xb)) = \left(-D(2^{m+1})(ay + xb), D(ab)\right)$$

$$= \left(-[aD(2^{m+1})y]_\mathfrak{A}(2^{m}), D(ab) + aD(b)\right)$$

Thus, $\overline{D}$ is a (continuous) derivation.

Finally, if $x$ is a (continuous) derivation from $\mathfrak{A}$ into $\mathfrak{A}$, we define $\overline{D}$ as follows:

$$\overline{D}((a, x) \cdot (b, y)) = \overline{D}((ab, ay + xb)) = \left(-D(2^{m+1})(ay + xb), D(ab)\right)$$

Therefore, $\overline{D}$ is a (continuous) derivation.

Next, if $\overline{D} is inner, then for some $u \in \mathfrak{A}(2^{m+1})$ and $F \in X(2^{m+1})$, we have

$$\overline{D}((a, x) \cdot (b, y)) = \overline{D}((u, F) - (u, F) \cdot (a, x)).$$

This shows that $D(a) = aF - Fa$ for all $a \in \mathfrak{A}$, and hence $D$ is inner.

Conversely, if $D$ is inner, then there exists $F \in X(2^{m+1})$ such that $D(a) = aF - Fa$ for $a \in \mathfrak{A}$. Let $T : X \to \mathfrak{A}(2^{m+1})$ be defined by

$$T(x) = -D(2^{m+1})(x) - (xF - Fx) \quad (x \in X),$$

and let $\overline{T} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)(2^{m+1})$ be defined by

$$\overline{T}((a, x)) = (T(x), 0) \quad ((a, x) \in \mathfrak{A} \oplus X).$$

Then

$$\overline{T}((a, x)) = (T(x), 0) \quad ((a, x) \in \mathfrak{A} \oplus X).$$

for $(a, x) \in \mathfrak{A} \oplus X$. Therefore, $\overline{D} - \overline{T}$ is an inner derivation. This in turn implies that $\overline{T}$ is a (continuous) derivation. So $\overline{D} = \overline{T}$ satisfies all the requirements. This completes the proof. 

If $D$ is a (continuous) derivation from $\mathfrak{A}$ into $\mathfrak{A}(2^{m+1})$, $m \geq 0$, we define $\overline{D} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)(2^{m+1})$ by

$$\overline{D}((a, x)) = (D(a), 0).$$

If $D$ is a (continuous) derivation from $\mathfrak{A}$ into $X(2^{m})$, $m \geq 0$, we define $\overline{D} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)(2^{m})$ by

$$\overline{D}((a, x)) = (0, D(a)).$$

If $T : X \to \mathfrak{A}(2^{m})$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ in $X(2^{m})$ for $x, y \in X$, we define $\overline{T} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)(2^{m})$ by

$$\overline{T}((a, x)) = (T(x), 0).$$

Finally, if $T : X \to X(2^{m+1})$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ for $x, y \in X$, we define $\overline{T} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)(2^{m+1})$ by

$$\overline{T}((a, x)) = (0, T(x)).$$
Lemma 3.5. The operators $\overline{D}$ and $\overline{T}$ defined above are (continuous) derivations. Furthermore, the derivation $\overline{D}$ is inner if and only if $D$ is inner, and $\overline{T}$ is inner if and only if $T = 0$.

4. Proofs of the main theorems

We first prove Theorem 2.1.

Proof. Denote by $\Delta_1$ the projection from $(\mathfrak{A} \oplus X)^{(2m+1)}$ onto $\mathfrak{A}^{(2m+1)}$ with kernel $X^{(2m+1)}$. Let $\Delta_2$ be the projection $id - \Delta_1$: $(\mathfrak{A} \oplus X)^{(2m+1)} \to X^{(2m+1)}$, and let $\tau_1$: $\mathfrak{A} \to \mathfrak{A} \oplus X$ be the inclusion mapping (i.e., $\tau_1(a) = (a, 0)$). Then $\Delta_1$ and $\Delta_2$ are $\mathfrak{A}$-bimodule morphisms, and $\tau_1$ is an algebra homomorphism.

We now prove the sufficiency in Theorem 2.1. Suppose that conditions 1–4 hold. Suppose also that $D$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. Then $D \circ \tau_1$: $\mathfrak{A} \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. This implies that $\Delta_1 \circ D \circ \tau_1$: $\mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ and $\Delta_2 \circ D \circ \tau_1$: $\mathfrak{A} \to X^{(2m+1)}$ are continuous derivations. By conditions 1 and 2, they are inner. Therefore, $D \circ \tau_1$ is inner. From Lemmas 3.5 and 3.4

$$\overline{D} \circ \tau_1 = \Delta_1 \circ D \circ \tau_1 + \Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$$

is a continuous derivation, and there is a continuous derivation $\hat{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\hat{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$ and such that $D \circ \tau_1 - \hat{D}$ is inner. On the other hand,

$$(D - \overline{D} \circ \tau_1)((a, 0)) = D((a, 0)) - \overline{D} \circ \tau_1((a, 0))$$

$$= D \circ \tau_1(a) - D \circ \tau_1(a) = 0 \quad (a \in \mathfrak{A}).$$

Let $\tilde{D} = D - \overline{D} \circ \tau_1 + \hat{D}$. Then $\tilde{D}$ is a continuous derivation from $\mathfrak{A} \oplus X$ into $\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\tilde{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$. Moreover,

$$\tilde{D}((0, ax)) = \tilde{D}((0, 0) \cdot (0, x)) = (a, 0) \cdot \tilde{D}((0, x)) = a \tilde{D}((0, x)) \quad (a \in \mathfrak{A}, x \in X),$$

and

$$\tilde{D}((0, xa)) = \tilde{D}((0, x) \cdot (a, 0)) = \tilde{D}((0, x))a \quad (a \in \mathfrak{A}, x \in X).$$

Denote by $\tau_2$: $X \to \mathfrak{A} \oplus X$ the inclusion mapping given by $\tau_2(x) = (0, x)$ ($x \in X$). Then $\tilde{D} \circ \tau_2$: $X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism. From condition 3, there exists $F \in X^{(2m+1)}$ for which $\Delta_1 \circ \hat{D} \circ \tau_2(x) = x F - F x$, and $a F - Fa = 0$ for $x \in X$ and $a \in \mathfrak{A}$. Since

$$(0, 0) = \tilde{D}((0, 0)) = \tilde{D}((0, x) \cdot (0, y))$$

$$= \tilde{D}((0, x)) \cdot (0, y) + (0, x) \cdot \tilde{D}((0, y))$$

$$= ((\Delta_2 \circ \hat{D}((0, x)))(0, y) + (0, x)[\Delta_2 \circ \hat{D}((0, y)), 0])$$

$$= ((\Delta_2 \circ \hat{D} \circ \tau_2(x))(0, y) + (0, x)[\Delta_2 \circ \hat{D} \circ \tau_2(y), 0]),$$

we have

$$(\Delta_2 \circ \hat{D} \circ \tau_2(x))(0, y) + x(\Delta_2 \circ \hat{D} \circ \tau_2(y)) = 0 \quad (x, y \in X).$$
From condition 4, \( \Delta_2 \circ \hat{D} \circ \tau_2 = 0 \). Thus,
\[
\hat{D}((a, x)) = \hat{D}((0, x)) = \hat{D} \circ \tau_2(x)
\]
\[
= (\Delta_1 \circ \hat{D} \circ \tau_2(x), \Delta_2 \circ \hat{D} \circ \tau_2(x))
\]
\[
= (xF - Fx, 0) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x).
\]
We have that \( \hat{D} \) is inner. Thus \( D = \hat{D} + (\overline{D \circ \tau_1} - \hat{D}) \) is inner. This proves that \( \mathfrak{A} \oplus X \) is \((2m + 1)\)-weakly amenable.

Necessity: Suppose that \( \mathfrak{A} \oplus X \) is \((2m + 1)\)-weakly amenable. Then from Lemmas 4.5 and 3.4, \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2m+1)}) = \{0\} \) and \( \mathcal{H}^1(\mathfrak{A}, X^{(2m+1)}) = \{0\} \). Therefore, conditions 1 and 2 hold. Furthermore, Lemma 3.1 gives condition 3, and Lemma 3.5 shows that condition 4 holds. This completes the proof of Theorem 2.1. \( \square \)

We now turn to the proof of Theorem 2.2.

Proof. Denote by \( \tau_1 \) and \( \tau_2 \) the inclusion mappings described in the preceding proof from, respectively, \( \mathfrak{A} \) and \( X \) into \( \mathfrak{A} \oplus X \), and denote by \( \Delta_1 \) and \( \Delta_2 \) the natural projections from \( (\mathfrak{A} \oplus X)^{(2m)} \) onto \( \mathfrak{A}^{(2m)} \) and \( X^{(2m)} \), respectively. These are \( \mathfrak{A} \)-bimodule morphisms.

To prove the sufficiency we assume that conditions 1–4 in Theorem 2.2 hold.
Let \( D: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m)} \) be a continuous derivation. Then \( \Delta_1 \circ D \circ \tau_1: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m)} \) and \( \Delta_2 \circ D \circ \tau_1: \mathfrak{A} \rightarrow X^{(2m)} \) are continuous derivations.

Claim 1: \( \Delta_1 \circ D \circ \tau_2: X \rightarrow \mathfrak{A}^{(2m)} \) is trivial.
Let \( \Gamma = \Delta_1 \circ D \circ \tau_2 \). To prove claim 1, by condition 3 it suffices to show that \( \Gamma \) is an \( \mathfrak{A} \)-bimodule morphism satisfying \( x\Gamma(y) + \Gamma(x)y = 0 \) in \( X^{(2m)} \) for \( x, y \in X \). In fact,
\[
0 = D((0, 0)) = D((0, x) \cdot (0, y)) = D((0, x)) \cdot (0, y) + (0, x) \cdot D((0, y))
\]
\[
= (0, \Gamma(x)y) + (0, x\Gamma(y)).
\]
Thus, \( x\Gamma(y) + \Gamma(x)y = 0 \). On the other hand,
\[
\Gamma(ax) = \Delta_1 \circ D((a, 0, x)) = \Delta_1 \circ D((a, 0, 0))
\]
\[
= \Delta_1((a, 0, 0) \cdot (0, x) + (a, 0) \cdot D((0, x)))
\]
\[
= \Delta_1((a, 0) \cdot D((0, x))) = \Delta_1(aD \circ \tau_2(x)) = a\Gamma(x).
\]
Similarly, \( \Gamma(xa) = \Gamma(x)a \) and so \( \Gamma \) is an \( \mathfrak{A} \)-bimodule morphism. Therefore, claim 1 is true.

Now let \( T = \Delta_2 \circ D \circ \tau_2: X \rightarrow X^{(2m)} \), and set \( D_1 = \Delta_1 \circ D \circ \tau_1: \mathfrak{A} \rightarrow \mathfrak{A}^{(2m)} \).

Claim 2: \( T(ax) = D_1(a)x + aT(x) \) and \( T(xa) = xD_1(a) + T(x)a \) for \( a \in \mathfrak{A} \) and \( x \in X \).
In fact, from claim 1,
\[
(0, T(ax)) = D((0, ax)) = D((a, 0, x)) = D((a, 0)) \cdot (0, x) + (a, 0) \cdot D((0, x))
\]
\[
= (0, D_1(a)x) + a(0, T(x)) = (0, D_1(a)x) + aT(x).
\]
Similarly, \( (0, T(xa)) = (0, xD_1(a) + T(x)a) \), for \( a \in \mathfrak{A} \) and \( x \in X \). Thus, claim 2 is true.

Therefore, by condition 1, \( D_1 = \Delta_1 \circ D \circ \tau_1 \) is inner. Suppose that \( u \in \mathfrak{A}^{(2m)} \) satisfies \( D_1(a) = au - ua \) for \( a \in \mathfrak{A} \). Let \( T_1: X \rightarrow X^{(2m)} \) be defined by \( T_1(x) = xu - ux \).
for \( x \in X \). Then \( T - T_1 : X \to X^{(2m)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. In fact, from claim 2, for \( a \in \mathfrak{A} \) and \( x \in X \),

\[
(T - T_1)(ax) = T(ax) - T_1(ax) = (D_1(a)x + aT(x)) - (axu - uax) \\
= (au - ua)x + aT(x) - (axu - uax) \\
= a(ux - xu) + aT(x) = a(T - T_1)(x).
\]

Similarly, \( T - T_1 \) is a right \( \mathfrak{A} \)-module morphism. From condition 4, there is a \( v \in \mathfrak{A}^{(2m)} \) such that \( av = va \) for \( a \in \mathfrak{A} \), and \( (T - T_1)(x) = xv - vx \) for \( x \in X \). From Lemma 3.2 we have that

\[
\overline{T - T_1} : (a, x) \mapsto (0, (T - T_1)(x)), \quad \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}
\]

is an inner derivation. Since \( \Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \to X^{(2m)} \) is a continuous derivation, it is inner by condition 2. From Lemma 3.5

\[
\Delta_2 \circ D \circ \tau_1 : (a, x) \mapsto (0, \Delta_2 \circ D \circ \tau_1(a)), \quad \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}
\]

is also inner. Using claim 1, we now have

\[
D((a, x)) = (D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x)) \\
= \Delta_2 \circ D \circ \tau_1((a, x)) + (\overline{T - T_1})((a, x)) + (D_1(a), T_1(x)).
\]

Since

\[
(D_1(a), T_1(x)) = (au - ua, xu - ux) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x),
\]

for \( a \in \mathfrak{A} \) and \( x \in X \), it gives an inner derivation from \( \mathfrak{A} \oplus X \) into \( (\mathfrak{A} \oplus X)^{(2m)} \). Hence as a sum of three inner derivations, \( D \) is inner. This shows that under conditions 1–4 of Theorem 2.2 \( \mathfrak{A} \oplus X \) is \( 2m \)-weakly amenable.

Now we prove the necessity. Suppose that \( \mathfrak{A} \oplus X \) is \( 2m \)-weakly amenable. Let \( D : \mathfrak{A} \to \mathfrak{A}^{(2m)} \) be a continuous derivation with the property given in condition 1. Then \( \overline{D} : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)} \) defined by

\[
\overline{D}(a, x) = (D(a), T(x)), \quad (a, x) \in \mathfrak{A} \oplus X,
\]

is a continuous derivation and hence is inner. This implies that \( D \) is inner, and so condition 1 holds. The other conditions are consequences of Lemma 3.5 and Lemma 3.2.

The proof is complete. \( \square \)

5. The algebras \( \mathfrak{A} \oplus \mathfrak{A} \) and \( \mathfrak{A} \oplus \mathfrak{A}^* \)

In this and the following section we consider several concrete cases. This section deals mainly with the two cases \( X = \mathfrak{A} \) and \( X = \mathfrak{A}^* \) as Banach \( \mathfrak{A} \)-bimodules.

We first note that, if \( \mathfrak{A} \) is not amenable, then there is a Banach \( \mathfrak{A} \)-bimodule \( X \) such that \( \mathcal{H}^l(\mathfrak{A}, X^*) \neq \{0\} \). From Theorem 2.1 for this \( X \), \( \mathfrak{A} \oplus X \) is not weakly amenable. In fact, the Banach algebra \( \mathfrak{A} \oplus X \) is never weakly amenable when \( X = \mathfrak{A}^* \), as implied in the following proposition.

**Proposition 5.1.** Suppose that \( \mathfrak{A} \) is a Banach algebra. Then \( \mathfrak{A} \oplus \mathfrak{A}^* \) is not \( n \)-weakly amenable for every \( n \geq 0 \).
Proof. From Proposition 1.2 of [10], it suffices to prove the cases of $n = 0$, $n = 1$ and $n = 2$. Note that condition 3 holds does not hold, because the identity mapping from $X (= \mathfrak{A}^*)$ onto $\mathfrak{A}^*$ is a nonzero, continuous $\mathfrak{A}$-bimodule morphism. So the proposition is true for $n = 1$.

For $n = 2m$ with $m = 0$ or $m = 1$, if condition 4 in Theorem 2.2 holds for $X = \mathfrak{A}^*$, then the operator $T$ described in this condition has the property that $T(f) \in \mathfrak{A}^+$ for $f \in X$. In fact, for $a \in \mathfrak{A}$, we have

\[
\langle a, T(f) \rangle = \langle a, f \cdot u - u \cdot f \rangle = \langle af - fa, u \rangle = \langle f, ua - au \rangle = 0.
\]

But $X = \mathfrak{A}^*$ certainly does not annihilate $\mathfrak{A}$. So, as $\mathfrak{A}$-bimodule morphisms, the identity mapping (in the case $m = 0$) from $X$ onto $X$ and the inclusion mapping (in the case $m = 1$) from $X$ into $X^{**}$ do not satisfy condition 4. Consequently, $\mathfrak{A} \oplus \mathfrak{A}^*$ is not $2m$-weakly amenable for $m = 0$ and 1.

Now we consider the case that $X = \mathfrak{A}$. To avoid any confusion, from now on, when we regard $\mathfrak{A}$ as an $\mathfrak{A}$-bimodule, we will use the notation $A$ instead of $\mathfrak{A}$. If $X = A$, condition 4 in Theorem 2.2 never holds for any integer $m$ (the canonical embedding is a nonzero morphism). It turns out that $\mathfrak{A} \oplus A$ is never $2m$-weakly amenable for any $m \geq 0$. If $\mathfrak{A}$ is commutative, for the same reason we can conclude more as in the next proposition. Recall that an $\mathfrak{A}$-bimodule $X$ is symmetric if $ax = xa$ for $a \in \mathfrak{A}$ and $x \in X$.

**Proposition 5.2.** Suppose that $\mathfrak{A}$ is a commutative Banach algebra. Then for every nonzero, symmetric $\mathfrak{A}$-bimodule $X$, $\mathfrak{A} \oplus X$ is not $2m$-weakly amenable.

**Proof.** Let $X$ be symmetric. Then $xu = ux$ for $u \in \mathfrak{A}(2m)$ and $x \in X$. Since the canonical embedding from $X$ into $X(2m)$ is a nontrivial $\mathfrak{A}$-bimodule morphism, condition 4 in Theorem 2.2 does not hold for such a module $X$. \qed

But $\mathfrak{A} \oplus A$ can be weakly amenable. Before giving an example let us go through some relation identities for corresponding elements of $A(n)$ and $\mathfrak{A}(n)$. Suppose that $\phi \in A(n)$. We denote the same element in $\mathfrak{A}(n)$ by $\tilde{\phi}$.

**Lemma 5.3.** Suppose that $\mathfrak{A}$ is a Banach algebra, and let $m \geq 0$. Then, for $\phi, \psi \in A(2m)$ and $F \in A(2m+1)$, we have

\[
(\tilde{\phi} \psi)^\sim = \tilde{\phi} \psi = (\phi \psi)^\sim, \quad \phi F = (\tilde{\phi} F)^\sim = \tilde{\phi} \tilde{F}, \quad F \phi = (F \tilde{\phi})^\sim = \tilde{F} \phi.
\]

**Proof.** It is straightforward to check the identities for the case $m = 0$. Then, an induction on $m$ completes the proof for the general case. \qed

A special case of Lemma 5.3 is the following group of identities which will be used in the proof of the next theorem:

\[
(a \phi)^\sim = a \tilde{\phi}, \quad (\phi a)^\sim = \tilde{\phi} a, \quad xF = (\tilde{x} F)^\sim = \tilde{x} \tilde{F}, \quad F x = (F \tilde{x})^\sim = \tilde{F} \tilde{x},
\]

where $a \in \mathfrak{A}$, $x \in A$, $\phi \in A(2m)$ and $F \in A(2m+1)$. From these identities, we also see that, for $X = A$ and $m \geq 0$, condition 3 in Theorem 2.2 holds if and only if there is no nonzero $\mathfrak{A}$-bimodule morphism $T$ from $A$ into $A(2m+1)$, and that, if this is the case, then condition 4 holds automatically. Moreover, with $X = A$, conditions 1 and 2 of Theorem 2.2 are the same.
**Theorem 5.4.** For a Banach algebra \( \mathfrak{A} \):

1. if \( \text{span}\{ab - ba; \, a, b \in \mathfrak{A}\} \) is not dense in \( \mathfrak{A} \), then \( \mathfrak{A} \oplus A \) is not weakly amenable;
2. if \( \text{span}\{ab - ba; \, a, b \in \mathfrak{A}\} \) is dense in \( \mathfrak{A} \), then \( \mathfrak{A} \oplus A \) is weakly amenable, provided that \( \mathfrak{A} \) is weakly amenable and has a bounded approximate identity.

**Proof.** By condition 1 of Theorem 2.1, without loss of generality, we can assume that \( \mathfrak{A} \) is weakly amenable for both cases. If \( \text{span}\{ab - ba; \, a, b \in \mathfrak{A}\} \) is not dense in \( \mathfrak{A} \), then there exists \( f \in \mathfrak{A}^* \) such that \( f \neq 0 \) and \( \langle ab - ba, f \rangle = 0 \) for all \( a, b \in \mathfrak{A} \). So \( af = fa \) for all \( a \in \mathfrak{A} \). Then \( T: A \to \mathfrak{A}^* \), defined by

\[
T(x) = \hat{x}f = f\hat{x},
\]

is an \( \mathfrak{A} \)-bimodule morphism. According to Proposition 1.3 of [10], \( \mathfrak{A}^2 \), the linear span of all product elements \( ab, \, a, b \in \mathfrak{A} \), is dense in \( \mathfrak{A} \). So there are \( a, b \in \mathfrak{A} \) such that \( \langle ab, f \rangle \neq 0 \). This implies that \( T \neq 0 \). Therefore, condition 3 does not hold. As a consequence, \( \mathfrak{A} \oplus A \) is not weakly amenable.

If \( \text{span}\{ab - ba; \, a, b \in \mathfrak{A}\} \) is dense in \( \mathfrak{A} \), and \( \mathfrak{A} \) has a bounded approximate identity \((e_i)\), then, for every given continuous \( \mathfrak{A} \)-bimodule morphism \( T: A \to \mathfrak{A}^* \), we have \( T(a) = af = fa \), where \( f \) is a weak* cluster point of \( (T(e_i)) \). Therefore, \( \langle ab - ba, f \rangle = 0 \) for all \( a, b \in \mathfrak{A} \). This shows that \( f = 0 \) and hence \( T = 0 \). Thus conditions 3 and 4 in Theorem 2.1 hold for \( m = 0 \). The other two conditions hold automatically for \( m = 0 \). So, from Theorem 2.1 the second statement of the theorem is true.

From case 1 of Theorem 5.4 we immediately have the following corollary.

**Corollary 5.5.** If \( \mathfrak{A} \) is a commutative Banach algebra, then \( \mathfrak{A} \oplus A \) is not weakly amenable.

Let \( \mathcal{H} \) be an infinite-dimensional Hilbert space. According to a classical result due to Halmos (Theorem 8 of [18]), every element in \( B(\mathcal{H}) \) can be written as a sum of two commutators (see also [4] and [3]). Together with the fact that \( B(\mathcal{H}) \) has an identity and, as a \( C^* \)-algebra, is weakly amenable [17], from Theorem 5.4 we see that \( B(\mathcal{H}) \oplus B(\mathcal{H}) \) is weakly amenable. Later in this section we will see that it is in fact \((2m + 1)\)-weakly amenable.

**Proposition 5.6.** Suppose that \( V = \text{span}\{au - ua; \, u \in \mathfrak{A}^*, \, a \in \mathfrak{A}\} \) is not dense in \( \mathfrak{A}^* \oplus \mathfrak{A}^* \mathfrak{A} \) (if \( \mathfrak{A} \) has an identity, this is equivalent to saying that \( V \) is not dense in \( \mathfrak{A}^* \)). Then \( \mathfrak{A} \oplus A \) is not 3-weakly amenable.

**Proof.** In fact, in this case \( \mathfrak{A}^* \mathfrak{A} \not\subseteq cl(V) \), since otherwise it would follow that both \( \mathfrak{A}^* \mathfrak{A} \) and \( \mathfrak{A}^* \mathfrak{A} \) were in \( cl(V) \), and then \( cl(V) \supseteq \mathfrak{A}^* \mathfrak{A} \), which contradicts the assumption that \( V \) is not dense in \( \mathfrak{A}^* \).

Hence, from the Hahn-Banach Theorem, there exists \( F \in \mathfrak{A}^* \) such that \( F|_V = 0 \), but \( F \neq 0 \) on \( \mathfrak{A}^* \mathfrak{A} \). This implies that \( aF = Fa \) for all \( a \in \mathfrak{A} \) and \( aF \neq 0 \) for some \( a \in \mathfrak{A} \). Define \( T: A \to \mathfrak{A}^* \) by \( T(x) = \hat{x}F = F\hat{x} \). Then, \( T \) is a non-zero, continuous \( \mathfrak{A} \)-bimodule morphism from \( A \) into \( \mathfrak{A}^* \). Therefore, condition 3 in Theorem 2.1 does not hold for \( m = 1 \). This shows that \( \mathfrak{A} \oplus A \) is not 3-weakly amenable.

Regarding the open question of whether weak amenability implies 3-weak amenability, Theorem 5.4 and Proposition 5.6 suggest that one might find a counterexample in the Banach algebras of the form \( \mathfrak{A} \oplus A \). Unfortunately, \( B(\mathcal{H}) \) cannot be
a candidate. We can see this from the next two lemmas. The following lemma is basically Theorem 8 in [18], but we have highlighted some of its features which will be useful for our purposes.

**Lemma 5.7.** Suppose that \( \mathcal{H} \) is an infinite-dimensional Hilbert space. Then there are two elements \( Q_0 \) and \( S_0 \) in \( B(\mathcal{H}) \) such that, for each \( B \in B(\mathcal{H}) \), there exist \( P_B \in B(\mathcal{H}) \) and \( R_B \in B(\mathcal{H}) \) with \( \|P_B\| \leq \|B\| \) and \( \|R_B\| \leq \|B\| \) for which

\[
B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B).
\]

**Proof.** For an infinite-dimensional Hilbert space \( \mathcal{H} \), there exists an isometry \( \eta \): \( \mathcal{H} \to \sum_{i=1}^{\infty} \mathcal{H}_i \), where \( \sum_{i=1}^{\infty} \) denotes an \( l_2 \) direct sum and each \( \mathcal{H}_i \) is a copy of \( \mathcal{H} \).

Let \( Q: \mathcal{H} \to \sum_{i=1}^{\infty} \mathcal{H}_i \) and \( S: \sum_{i=1}^{\infty} \mathcal{H}_i \to \sum_{i=1}^{\infty} \mathcal{H}_i \) be the bounded operators given by the infinite matrices

\[
Q = \begin{pmatrix}
I \\
0 \\
0 \\
\vdots
\end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
0 & I & 0 & \cdots \\
0 & 0 & I & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Let \( Q_0 = \eta^{-1} \circ Q \) and \( S_0 = \eta^{-1} \circ S \circ \eta \). Then \( Q_0, S_0 \in B(\mathcal{H}) \). Given an element \( B \in B(\mathcal{H}) \), let \( P: \sum_{i=1}^{\infty} \mathcal{H}_i \to \mathcal{H} \) and \( R: \sum_{i=1}^{\infty} \mathcal{H}_i \to \sum_{i=1}^{\infty} \mathcal{H}_i \) be the bounded operators given by the infinite matrices

\[
P = \begin{pmatrix}
B & 0 & 0 & \cdots \\
0 & B & 0 & \cdots \\
0 & 0 & B & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix}
0 & B & 0 & \cdots \\
0 & 0 & B & \cdots \\
0 & 0 & 0 & B & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Take \( P_B = P \circ \eta \) and \( R_B = \eta^{-1} \circ R \circ \eta \). Then \( P_B, R_B \in B(\mathcal{H}) \) and \( \|P_B\| \leq \|B\| \), \( \|R_B\| \leq \|B\| \). We have that \( B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B) \). \( \square \)

The following result on the \( 2n \)-th dual of \( B(\mathcal{H}) \) seems not to be known.

**Lemma 5.8.** For every integer \( n \geq 0 \),

\[
B(\mathcal{H})^{(2n)} = \text{span}\{ au - ua; \ a \in B(\mathcal{H}), u \in B(\mathcal{H})^{(2n)} \}.
\]

**Proof.** By taking weak* limits and using induction, one can show the result immediately from Lemma 5.7. \( \square \)

**Proposition 5.9.** For each integer \( m \geq 0 \), \( B(\mathcal{H}) \oplus B(\mathcal{H}) \) is \((2m + 1)\)-weakly amenable but is not \( 2m \)-weakly amenable.

**Proof.** First, as a \( C^* \)-algebra, \( B(\mathcal{H}) \) is permanently weakly amenable. So conditions 1 and 2 of Theorem 2.1 hold for \( X = \mathfrak{A} = B(\mathcal{H}) \) and \( m \geq 0 \). To show that conditions 3 and 4 also hold, it suffices to prove that every continuous \( B(\mathcal{H}) \)-bimodule morphism \( T \) from \( B(\mathcal{H}) \) into \( B(\mathcal{H})^{(2m+1)} \) is trivial.

In fact, letting \( e \) be the identity of \( B(\mathcal{H}) \) and \( F = T(e) \), we have \( T(a) = aF = Fa \) for all \( a \in B(\mathcal{H}) \). Therefore, for all \( u \in B(\mathcal{H})^{(2m)} \), we have \( \langle au - ua, F \rangle = 0 \). From Lemma 5.8 this implies that \( F = 0 \). Hence \( T = 0 \). Therefore, \( B(\mathcal{H}) \oplus B(\mathcal{H}) \) is \((2m + 1)\)-weakly amenable for \( m \geq 0 \).
On the other hand, we have indicated in the paragraph before Proposition 5.2 that $\mathfrak{A} \oplus A$ is never $2m$-weakly amenable. So $B(H) \oplus B(H)$ is not $2m$-weakly amenable for $m \geq 0$. This completes the proof.

Remark 5.10. Denote by $K(H)$ the algebra of compact operators on $H$. Using Theorem 1 of [29] one can also prove that $K(H) \oplus K(H)$ and $B(H) \oplus K(H)$ are $(2m+1)$- (but not $2m$-) weakly amenable. On the other hand, it is interesting to recall Proposition 2.4 which implies that $K(H) \oplus B(H)$ is not weakly amenable.

6. The algebra $\mathfrak{A} \oplus X_0$

In this section we consider the case that the module action on one side of $X$ is trivial. We denote by $X_0$ (respectively, $\mathfrak{M}$) specifically the $\mathfrak{A}$-bimodules with right (respectively, left) module action trivial. We observe that, when $X = X_0$, conditions 3 and 4 in Theorem 2.1 are reduced, respectively, to the following:

1. for each continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X_0 \rightarrow \mathfrak{A}(2m+1)$, there is $F \in X_0^{(2m+1)}$ such that $Fa = 0$ for $a \in \mathfrak{A}$ and $\Gamma(x) = xF$ for $x \in X_0$;  
2. $\mathfrak{A}X_0$ is dense in $X_0$.

Also, conditions 1, 3 and 4 in Theorem 2.2 are reduced, respectively, to the following:

1. every continuous derivation $D: \mathfrak{A} \rightarrow \mathfrak{A}(2m)$ with the property that there is a continuous operator $T: X_0 \rightarrow X_0^{(2m)}$ such that $T(ax) = D(a)x + aT(x)$ for $a \in \mathfrak{A}$ and $x \in X_0$ is inner;
2. the only continuous $\mathfrak{A}$-bimodule morphism $\Gamma: X_0 \rightarrow \mathfrak{A}(2m)$ satisfying $\Gamma(x) = 0$ for $(x, y) \in X_0^{(2m)}$ is zero;
3. for every continuous $\mathfrak{A}$-bimodule morphism $T: X_0 \rightarrow X_0^{(2m)}$, there exists $u \in \mathfrak{A}(2m)$ such that $au = ua$ for $a \in \mathfrak{A}$ and $T(x) = ux$ for $x \in X_0$.

Proposition 6.1. Suppose that $\mathfrak{A}$ is a $(2m+1)$-weakly amenable Banach algebra with a bounded approximate identity and satisfying that $\mathfrak{A}\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$. Then, $\mathfrak{A} \oplus X_0$ is $(2m+1)$-weakly amenable if and only if $\mathfrak{A}X_0$ is dense in $X_0$.

Proof. Since $\mathfrak{A}$ has a bounded approximate identity, from Proposition 1.5 of [29], condition 2 in Theorem 2.1 always holds for $X = X_0$. If $\mathfrak{A}\mathfrak{A}^{(2m)} = \mathfrak{A}^{(2m)}$, then there is no nonzero, continuous $\mathfrak{A}$-bimodule morphism $T: X_0 \rightarrow \mathfrak{A}^{(2m+1)}$, since such a morphism must satisfy $\langle au, T(x) \rangle = \langle u, T(xa) \rangle = 0$ for $a \in \mathfrak{A}, u \in \mathfrak{A}^{(2m)}$. So condition 3 holds automatically.

For $m = 0$, the above proposition yields the following.

Corollary 6.2. Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus X_0$ is weakly amenable if and only if $\mathfrak{A}X_0$ is dense in $X_0$.

A dual result to Corollary 6.2 is as follows.

Corollary 6.3. Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity. Let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-bimodule with left module action trivial. Then, $\mathfrak{A} \oplus \mathfrak{M}$ is weakly amenable if and only if $\mathfrak{A}\mathfrak{M}$ is dense in $\mathfrak{M}$.

View $\mathfrak{A}$ as a left $\mathfrak{A}$-module and then impose a trivial right $\mathfrak{A}$-module action on it. This results in a Banach $\mathfrak{A}$-bimodule. We denote it by $A_0$. Suppose that $\phi \in A_0^{(n)}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We denote the same element in \( \mathfrak{A}^{(n)} \) by \( \tilde{\phi} \). Similarly to Lemma 5.3 one can check that the following equalities hold:

\[
(u\phi)^\sim = u\tilde{\phi}, \quad \phi u = 0, \quad \phi F = \tilde{\phi}F, \\
F\phi = 0, \quad uF = 0, \quad (Fu)^\sim = \tilde{F}u,
\]

where \( u \in \mathfrak{A}^{(2m)}, \phi \in A_0^{(2m)}, F \in A_0^{(2m+1)} (m \geq 0) \).

**Proposition 6.4.** Suppose that \( \mathfrak{A} \) is a \((2m+1)\)-weakly amenable Banach algebra with a bounded approximate identity. Then \( \mathfrak{A} \oplus A_0 \) is \((2m+1)\)-weakly amenable.

**Proof.** As in the proof of Proposition 6.1 it suffices to verify conditions 3’ and 4’. Condition 4’ holds since \( \mathfrak{A} \) has a left bounded approximate identity for \( A_0 \). Let \( (x_\alpha) \subset A_0 \) be a net such that \( (\bar{x}_\alpha) \) is a bounded approximate identity for \( \mathfrak{A} \). If \( \Gamma: A_0 \to \mathfrak{A}^{(2m+1)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism, we let \( \tilde{F} \) be a weak* cluster point of \( (\Gamma(x_\alpha)) \). Let the element in \( A_0^{(2m+1)} \) corresponding to \( \tilde{F} \) be \( \bar{F} \). Then \( \bar{F} \) satisfies the requirement in condition 3’.

Concerning \( 2m \)-weak amenability, we have the following.

**Proposition 6.5.** Let \( m \geq 1 \), and suppose that \( \mathfrak{A} \) is a commutative \( 2m \)-weakly amenable Banach algebra with a bounded approximate identity. Then \( \mathfrak{A} \oplus A_0 \) is \( 2m \)-weakly amenable.

**Proof.** It suffices to show that conditions 3’’ and 4’’ hold. Suppose that an \( \mathfrak{A} \)-bimodule morphism \( \Gamma: A_0 \to \mathfrak{A}^{(2m)} \) satisfies \( \Gamma(x)0 = 0 \) in \( A_0^{(2m)} \) \( (x, y \in A_0) \). Then

\[
0 = (\Gamma(x)y)^\sim = \Gamma(x)\tilde{y} = \tilde{y}\Gamma(x) = \tilde{y}\Gamma(x) \quad (x, y \in A_0).
\]

This implies that \( \Gamma(xa) = 0 \) for \( a \in \mathfrak{A} \) and \( x \in A_0 \). So \( \Gamma(x) = 0 \) for all \( x \in A_0 \). Therefore, condition 3’’ holds.

Assume that \( T: A_0 \to A_0^{(2m)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Let \( u \) be a weak* cluster point of \( (T(x_i)) \), where \( (\bar{x}_i) \) is a bounded approximate identity for \( \mathfrak{A} \). Let \( u = \tilde{u} \). Then, \( T(x) = \lim T(\bar{x}_i) = \bar{x}u \). However, \( (\bar{x}u)^\sim = \bar{x}\setminus \bar{u} = \bar{x}u = (ux)^\sim \). Hence \( T(x) = ux \). On the other hand, \( u \in A_0 \) since \( \mathfrak{A} \) is commutative. Condition 4’’ holds.

Although we have already had an example of a Banach algebra which is \((2m+1)\)-weakly amenable but not \( 2m \)-weakly amenable (see Proposition 5.9), another known example is the nuclear algebra \( \mathcal{N}(E) \) with \( E \) a reflexive Banach space having the approximation property ([10] Corollary 5.4)], we end this section by giving one more example of a weakly amenable Banach algebra which is not \( 2 \)-weakly amenable.

Suppose that \( \mathfrak{A} \) is a weakly amenable Banach algebra with a bounded approximate identity and satisfying that \( \mathfrak{A}^* \neq \mathfrak{A}^* \mathfrak{A} \) (an example is \( \mathfrak{A} = L^1(G) \) with \( G \) a non-SIN locally compact group; see [28] and [25] for the reference of SIN groups, and Theorem 32.44 of [20] as well as [26] for the property we need here). Without loss of generality, we assume that \( \mathfrak{A}^* \not\subset \mathfrak{A}^* \).

**Example 6.6.** For the above Banach algebra \( \mathfrak{A} \), \( \mathfrak{A} \oplus A_0 \) is weakly amenable but is not \( 2 \)-weakly amenable.
Proof. From Proposition 6.4, \( \mathfrak{A} \oplus A_0 \) is weakly amenable. We show that condition 3\(_0^\prime\) does not hold for \( m = 1 \). Take a \( \phi \in \mathfrak{A}^{**} \) for which \( \phi|_{\mathfrak{A}X} = 0 \) but \( \phi|_{\mathfrak{A} \cdot \mathfrak{A}} \neq 0 \) (notice that by Cohen’s factorization theorem, \( \mathfrak{A}^* \) is closed in \( \mathfrak{A}^* \)). Then \( \phi a = 0 \) for all \( a \in \mathfrak{A} \) and \( a\phi \neq 0 \) for some \( a \in \mathfrak{A} \). Let \( T: A_0 \to \mathfrak{A}^{**} \) be defined by \( T(x) = \ddot{x}\phi \). Then \( T \) is a continuous \( \mathfrak{A} \)-bimodule morphism and \( T \neq 0 \). Since

\[
(T(x)y)\sim = T(x)\ddot{y} = (\ddot{x}\phi)\ddot{y} = \ddot{x}(\phi\ddot{y}) = 0,
\]

we have \( T(x)y = 0 \) for all \( x, y \in A_0 \). Therefore, condition 3\(_0^\prime\) is not satisfied. \( \square \)

7. Weak amenability does not imply 3-weak amenability

Suppose that \( X_1 \) and \( X_2 \) are two Banach \( \mathfrak{A} \)-bimodules. We denote by \( X_1 + X_2 \) the direct module sum of \( X_1 \) and \( X_2 \), i.e., the \( l_1 \) direct sum of \( X_1 \) and \( X_2 \) with the module actions given by \( a(x_1, x_2) = (ax_1, ax_2) \), \( (x_1, x_2)a = (x_1a, x_2a) \). For this module we have the following equality:

\[
(x_1, x_2) \cdot (f_1^*, f_2^*) = x_1 f_1^* + x_2 f_2^* \quad ((x_1, x_2) \in X_1 + X_2, (f_1^*, f_2^*) \in (X_1 + X_2)^*)
\]

In this section we shall first study the weak amenability of the Banach algebra \( \mathfrak{A} \oplus (X_1 + X_2) \). Then we shall give an example of a weakly amenable Banach algebra of this form which is not 3-weakly amenable.

Lemma 7.1. Suppose that \( \mathfrak{A} \oplus X_1 \) and \( \mathfrak{A} \oplus X_2 \) are weakly amenable. Then the following are equivalent:

(i) \( \mathfrak{A} \oplus (X_1 + X_2) \) is weakly amenable;
(ii) there is no nonzero, continuous \( \mathfrak{A} \)-bimodule morphism \( \gamma: X_1 \to X_1^* \);
(iii) there is no nonzero, continuous \( \mathfrak{A} \)-bimodule morphism \( \eta: X_2 \to X_2^* \).

Proof. Suppose that (i) holds. We show that (ii) also holds. Indeed, suppose that \( \gamma: X_1 \to X_2^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Consider the continuous \( \mathfrak{A} \)-bimodule morphism \( T: X_1 + X_2 \to (X_1 + X_2)^* \) defined by

\[
T((x_1, x_2)) = (-\gamma(x_2), \gamma(x_1)), \quad (x_1, x_2) \in X_1 + X_2.
\]

For \( (x_1, x_2), (y_1, y_2) \in X_1 + X_2 \), and \( a \in \mathfrak{A} \), we have

\[
\langle a, (x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) \rangle = \langle a, -x_1\gamma(y_2) + x_2\gamma(y_1) \rangle + \langle a, -\gamma(x_2)y_1 + \gamma(x_1)y_2 \rangle
\]

\[
= \langle a, -\gamma(x_1)y_2 + x_2\gamma(y_1) \rangle + \langle a, -x_2\gamma(y_1) + \gamma(x_1)y_2 \rangle = 0.
\]

So \( (x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0 \). Then, from condition 4 of Theorem 2.1, \( T = 0 \). Thus \( \gamma = 0 \). As a consequence, (ii) holds.

To prove that (ii) implies (iii), we suppose that \( \eta: X_2 \to X_1^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Then \( \gamma: X_1 \to X_2^* \) defined by \( \gamma = \eta|_{X_1} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Therefore, \( \gamma = 0 \). This implies that \( \eta^* = 0 \) since \( \eta^* \) is weak*-weak* continuous and \( X_1 \) is weak* dense in \( X_1^* \). Thus, \( \eta = 0 \), showing that (iii) holds. Similarly, one can prove that (iii) implies (ii).

Finally, we prove that (ii) + (iii) implies (i). Because \( \mathfrak{A} \oplus X_1 \) and \( \mathfrak{A} \oplus X_2 \) are weakly amenable, conditions 1–3 of Theorem 2.1 hold automatically for \( X = X_1 + X_2 \) and \( m = 0 \). We show that condition 4 also holds. Suppose that \( T: X \to X^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism satisfying

\[
(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0 \quad ((x_1, x_2), (y_1, y_2) \in X).
\]
Let $P_i: X^* \to X_i^*$ be the natural projections and let $\tau_i: X_i \to X$ be the natural embeddings, $i = 1, 2$. Then, by taking $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$ separately, we have

$$x_1 \cdot P_1 \circ T \circ \tau_1(y_1) + P_1 \circ T \circ \tau_1(x_1) \cdot y_1 = 0,$$
$$x_2 \cdot P_2 \circ T \circ \tau_2(y_2) + P_2 \circ T \circ \tau_2(x_2) \cdot y_2 = 0,$$

for all $x_i, y_i \in X_i$, $i = 1, 2$. So we have $P_i \circ T \circ \tau_i = 0$ by applying condition 4 of Theorem 2.1 to the weakly amenable Banach algebras $A \oplus X_i$, $i = 1, 2$. Furthermore, (ii) and (iii) imply that $P_1 \circ T \circ \tau_2: X_2 \to X_1^*$ and $P_2 \circ T \circ \tau_1: X_1 \to X_2^*$ are trivial. Therefore, we have $T = 0$. Condition 4 of Theorem 2.1 holds for $X = X_1 + X_2$. From Theorem 2.1 $A \oplus (X_1 + X_2)$ is weakly amenable. This completes the proof.

**Proposition 7.2.** The algebra $A \oplus (X_1 + X_2)$ is weakly amenable if and only if both $A \oplus X_1$ and $A \oplus X_2$ are weakly amenable and condition (ii) or condition (iii) in Lemma 7.1 holds.

**Proof.** If $A \oplus (X_1 + X_2)$ is weakly amenable, then conditions 1–4 of Theorem 2.1 hold for this algebra. It follows that these conditions also hold for the algebras $A \oplus X_1$ and $A \oplus X_2$. So the latter two are also weakly amenable. The rest has been given in Lemma 7.1.

In the remainder of the paper we focus on constructing an example of a weakly amenable Banach algebra which is not 3-weakly amenable. Recall that we always equip $A^{(2m)}$ with the first Arens product. The following lemma has been proved in [31].

**Lemma 7.3.** Suppose that $A$ is a left (right) ideal in $A^{**}$. Then it is also a left (respectively, right) ideal in $A^{(2m)}$ for all $m \geq 1$.

Suppose that $B$ is a Banach algebra and $A = B^{**}$. If $B$ is an ideal in $B^{**}$, then a natural way to make $B$ an $A$-bimodule is using (the first) Arens product to give the module actions. In this way $B^{**}$ is an $A^{**}$-bimodule. For $b \in B \subset B^{**}$ and $u \in A^{**}$, the module coupling $u \cdot b$ and $b \cdot u$ result in elements of $B^{**}$. Since $B \subset B^{(4)} (= A^{**})$, we can also consider the products $ub$ and $bu$ in the sense of Arens in $B^{(4)}$. But, from the above lemma, $ub, bu \in B \subset B^{**}$. It is routine to check that, as elements in $B^{**}$, $u \cdot b = ub$ and $b \cdot u = bu$.

From this point on, $H$ will denote an infinite-dimensional, separable Hilbert space, $B(H)$ will denote the Banach algebra of all bounded operators on $H$, and $K(H)$ the ideal of all compact operators on $H$. It is well known that, with any Arens product, $K(H)^{**} = B(H)$ (see [27] p. 103 for details).

**Lemma 7.4.** There is an element $a_0 \in B(H)$ such that $a_0 \notin K(H)$, $a_0$ is not right invertible in $B(H)$ and $a_0 K(H)$ is dense in $K(H)$.

**Proof.** Let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis of $H$. Let $a_0 \in B(H)$ be defined by

$$a_0(e_i) = \begin{cases} i e_i & \text{if } i \text{ is even;} \\ e_i & \text{if } i \text{ is odd.} \end{cases}$$

Clearly, $a_0 \notin K(H)$. Also, $a_0$ is neither right nor left invertible because any one-sided inverse of $a_0$ must satisfy

$$a_0^{-1}(e_i) = \begin{cases} i e_i & \text{if } i \text{ is even;} \\ e_i & \text{if } i \text{ is odd,} \end{cases}$$
which cannot be a bounded operator.

For each $n \geq 1$, denote by $V_n$ the subspace of $\mathcal{H}$ generated by $\{e_1, e_2, \ldots, e_n\}$, and let $P_n$ be the orthogonal projection from $\mathcal{H}$ onto $V_n$. Then, from Corollary II.4.5 of [8], for every $k \in K(\mathcal{H})$ and $\varepsilon > 0$, there is $n = n(k, \varepsilon)$, such that $\|P_n \circ k - k\| < \varepsilon$. For this $n = n(k, \varepsilon)$, let $b_n \in B(\mathcal{H})$ be defined by

$$b_n(e_i) = \begin{cases} i e_i & \text{if } i \leq n \text{ and } i \text{ is even;} \\ e_i & \text{if } i \leq n \text{ and } i \text{ is odd;} \\ 0 & \text{for } i > n. \end{cases}$$

Then $a_0 \circ b_n = P_n$ and $a_0 \circ b_n \circ P_n = P_n^2 = P_n$. Let $k_n = b_n \circ P_n \circ k$. Then $k_n \in K(\mathcal{H})$, and $a_0 \circ k_n = P_n \circ k$. Also, $\|a_0 \circ k - k\| = \|P_n \circ k - k\| < \varepsilon$. Since $k \in K(\mathcal{H})$ and $\varepsilon > 0$ are arbitrarily given, this shows that $a_0K(\mathcal{H})$ is dense in $K(\mathcal{H})$.

For the element $a_0$ in the above lemma, $a_0B(\mathcal{H})$ is a proper right ideal of $B(\mathcal{H})$ since the identity $1 \notin a_0B(\mathcal{H})$. The closure of $a_0B(\mathcal{H})$ is also a proper right ideal of $B(\mathcal{H})$ ([8] p. 46). So there is $F \in B(\mathcal{H})^*$ such that $F \neq 0$ but $Fa_0 = 0$. Then, $FB(\mathcal{H}) \neq \{0\}$ is a right $B(\mathcal{H})$-submodule of $B(\mathcal{H})^*$. Take

$$X_0 = (K(\mathcal{H})_0, \text{ and } \mathcal{Y} = \mathcal{O}(\text{cl}(FB(\mathcal{H}))).$$

Then we have the following example.

**Example 7.5.** $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is weakly amenable but not 3-weakly amenable.

**Proof.** Clearly, we have $B(\mathcal{H})X_0 = X_0$ and $\mathcal{Y}B(\mathcal{H}) = \mathcal{Y}$. By Corollaries [6,2] and [8], the Banach algebras $B(\mathcal{H})X_0$ and $B(\mathcal{H}) \oplus \mathcal{Y}$ are weakly amenable.

Suppose that $T: \mathcal{Y} \rightarrow X_0$ is a continuous $B(\mathcal{H})$-bimodule morphism. We prove that $T$ is trivial. Let $f = T(F)$. Then $f a_0 = T(F a_0) = 0$, and so $(a_0 K(\mathcal{H}), f) = \{0\}$. We then have $f = 0$ since $a_0K(\mathcal{H})$ is dense in $K(\mathcal{H})$. This shows that $T(F) = 0$ and hence $T(FB(\mathcal{H})) = \{0\}$. Thus, $T = 0$. From Proposition [7,2], $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is weakly amenable.

To prove that $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is not 3-weakly amenable, we show that it fails condition 4 of Theorem [7,2] for $m = 1$. Since

$$(X_0)^{**} = \mathcal{O}(K(\mathcal{H})^{**}) = \mathcal{O}(B(\mathcal{H})^*) \supset \mathcal{Y},$$

there exists a nonzero $B(\mathcal{H})$-bimodule morphism from $\mathcal{Y}$ into $(X_0)^{**}$ (e.g., the inclusion mapping). Let $\tau: \mathcal{Y} \rightarrow (X_0)^{**}$ be such a morphism, and let $\Delta: (X_0)^{**} \rightarrow (X_0)^*$ be the projection with the kernel $X_0^\perp$. Take $T = \Delta \circ \tau: \mathcal{Y} \rightarrow X_0$. From the preceding paragraph, we have that $T = 0$. So

$$\langle x, \tau(y) \rangle = \langle x, T(y) \rangle = 0 \quad (y \in \mathcal{Y}, x \in X_0).$$

Now let $\Gamma: X_0 + \mathcal{Y} \rightarrow (X_0 + \mathcal{Y})^{**}$ be the continuous $B(\mathcal{H})$-bimodule morphism defined by

$$\Gamma((x, y)) = (\tau(y), 0).$$

Then, for $(x_1, y_1), (x_2, y_2) \in X_0 + \mathcal{Y}$, and $u \in B(\mathcal{H})^{**}$, we have

$$\langle u, (x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) \rangle$$

$$= \langle (u \cdot x_1, 0), (\tau(y_2), 0) \rangle + \langle (0, y_2 \cdot u), (\tau(y_1), 0) \rangle$$

$$= \langle u \cdot x_1, \tau(y_2) \rangle = \langle u x_1, \tau(y_2) \rangle = 0.$$
Here we used the fact that \( u \cdot x_1 = ux_1 \in X_0 \) (see the paragraph following Lemma 7.3). So
\[
(x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) = 0
\]
for all \((x_1, y_1), (x_2, y_2) \in X_0^+ \). But \( \Gamma \neq 0 \); so condition 4 of Theorem 2.1 does not hold for \( m = 1 \) and \( X = X_0^+ \). As a consequence, \( B(\mathcal{H}) \oplus (X_0^+ \oplus Y) \) is not 3-weakly amenable.

**Acknowledgement**

The author thanks Professor F. Ghahramani, who brought to him the problem of whether weak amenability implies 3-weak amenability and suggested he should look for a counterexample amongst Banach algebras of the form \( \mathfrak{A} \oplus X \). He also appreciates valuable discussions about this topic with Professors H. G. Dales, J. Duncan, N. Grønbæk, and G. A. Willis.

**References**


Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
E-mail address: zhangy@cc.umanitoba.ca