DIFFERENTIAL OPERATORS
ON A POLARIZED ABELIAN VARIETY

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ABSTRACT. Let $L$ be an ample line bundle over a complex abelian variety $A$. We show that the space of all global sections over $A$ of $\text{Diff}^n_A(L, L)$ and $S^n(\text{Diff}^1_A(L, L))$ are both of dimension one. Using this it is shown that the moduli space $M_X$ of rank one holomorphic connections on a compact Riemann surface $X$ does not admit any nonconstant algebraic function. On the other hand, $M_X$ is biholomorphic to the moduli space of characters of $X$, which is an affine variety. So $M_X$ is algebraically distinct from the character variety if $X$ is of genus at least one.

1. INTRODUCTION

Let $X$ be a compact connected Riemann surface of genus at least one. A holomorphic connection on a holomorphic line bundle $L$ over $X$ is a first-order differential operator

$$D : L \rightarrow K_X \otimes L$$

satisfying the Leibniz rule, which says $D(fs) = fD(s) + \partial f \otimes s$, where $f$ is a locally defined holomorphic function and $s$ is a local holomorphic section of $L$. Let $M_X$ denote the moduli space of all rank one holomorphic connections on $X$. In other words, $M_X$ parametrizes isomorphism classes of pairs of the form $(L, D)$, where $D$ is a holomorphic connection on $L$. The space $M_X$ is a smooth quasi-projective variety of dimension $2g$, where $g$ is the genus of $X$.

Since any holomorphic connection on a Riemann surface is flat, the monodromy map identifies $M_X$ with the character variety $R := \text{Hom}(\pi_1(X), \mathbb{C}^*)$. This identification is in fact a biholomorphism between $M_X$ and $R$. We show that $R$ is not algebraically isomorphic to $M_X$. More precisely, while $R$ is an affine variety, $M_X$ does not have any nonconstant function (Theorem 3.2).

Let $A$ be a complex abelian variety and $L$ an ample line bundle over $A$. By $\text{Diff}^1_A(L, L)$ we denote the sheaf of differential operators of order one on $L$.

In Theorem 2.3 we prove that

$$\dim H^0(A, S^n(\text{Diff}^1_A(L, L))) = 1$$

for all $n \geq 1$. As a corollary we have (Corollary 2.10)

$$\dim H^0(A, \text{Diff}^n_A(L, L)) = 1.$$
In [6], using the method of the present paper, we prove a Torelli theorem for the moduli space of $\tau$-connections on a compact Riemann surface.

2. Differential operators and connections

Let $A$ be a complex abelian variety. Fix an ample line bundle $L$ over $A$.

For $n > 0$, let $\text{Diff}^n_A(L, L)$ denote the vector bundle over $A$ defined by the sheaf of differential operators of order $n$ on $L$. So,

$$\text{Diff}^0_A(L, L) = \text{Hom}_\mathcal{O}(L, L) = \mathcal{O}$$

and $\text{Diff}^{n-1}_A(L, L)$ is a subbundle of $\text{Diff}^n_A(L, L)$ in an obvious way. More precisely, there is an exact sequence

$$0 \longrightarrow \text{Diff}^{n-1}_A(L, L) \longrightarrow \text{Diff}^n_A(L, L) \xrightarrow{\sigma_n} S^n(TA) \longrightarrow 0,$$

where $\sigma_n$ is the symbol homomorphism and $S^n(TA)$ is the $n$-th symmetric power of the tangent bundle. By convention, the $0$-th symmetric power of a vector bundle is the trivial line bundle.

The $n$-th symmetric power of the homomorphism $\sigma_1$ in (2.1) gives an exact sequence

$$0 \longrightarrow S^{n-1}(\text{Diff}^1_A(L, L)) \longrightarrow S^n(\text{Diff}^1_A(L, L)) \xrightarrow{\sigma_n} S^n(TA) \longrightarrow 0,$$

of vector bundles. The vector bundle $S^{n-1}(\text{Diff}^1_A(L, L))$ is realized as a subbundle using the composition

$$S^{n-1}(\text{Diff}^1_A(L, L)) \xrightarrow{\alpha} S^{n-1}(\text{Diff}^1_A(L, L)) \otimes \text{Diff}^1_A(L, L) \xrightarrow{\beta} S^n(\text{Diff}^1_A(L, L)),$$

where $\alpha$ is defined using the inclusion of $\mathcal{O}$ in $\text{Diff}^1_A(L, L)$ in the exact sequence (2.1) and $\beta$ is the symmetrization.

Theorem 2.3. For $n \geq 1$, the homomorphism

$$H^0(A, S^0(\text{Diff}^1_A(L, L))) = H^0(A, \mathcal{O}) \longrightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

obtained using (2.2) repeatedly is an isomorphism.

Proof. Consider the long exact sequence of cohomologies

$$H^0(A, S^{n-1}(\text{Diff}^1_A(L, L))) \longrightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

(2.4) $$\longrightarrow H^0(A, S^n(TA)) \xrightarrow{h_n} H^1(A, S^{n-1}(\text{Diff}^1_A(L, L)))$$

obtained from (2.2). To prove the theorem, it suffices to show that the above homomorphism $h_n$ is injective for all $n \geq 1$. Indeed, if $h_n$ is injective, then the injective map

$$H^0(A, S^{n-1}(\text{Diff}^1_A(L, L))) \longrightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

is also surjective.

A connected homomorphism, like $h_n$ in (2.4), is the cup product by the extension class for the corresponding short exact sequence. So we need to understand the extension class

$$\mathcal{C}_n \in H^1(A, \text{Hom}(S^n(TA), S^{n-1}(\text{Diff}^1_A(L, L))))$$
for the exact sequence (2.2). Using the homomorphism $S^{n-1}(\sigma_1)$ in (2.2), the cohomology class $C_n$ gives

$$C_n \in H^1(A, \text{Hom}(S^n(TA), S^{n-1}(TA))).$$

This cohomology class $C_n$ is clearly the extension class for the exact sequence

$$0 \rightarrow S^{n-1}(\text{Diff}_A(L, L))/S^{n-2}(\text{Diff}_A(L, L)) = S^{n-1}(TA)$$

obtained from (2.2). Before we describe $C_n$, we need to identify the extension class for the exact sequence from which (2.2) is built, namely, the one obtained by setting $n = 1$ in (2.1).

The first step would be to show that the extension class for the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \text{Diff}_A(L, L) \rightarrow TA \rightarrow 0$$

is the first Chern class of $L$. Although this fact is well known, we give brief details of the argument.

We fix a convention. For our convenience, the first Chern class will always denote $2\pi\sqrt{-1}$ times the standard rational class. So, for example, $c_1(L) \in H^2(J, 2\pi\sqrt{-1}\mathbb{Z})$.

Let $\{U_i\}_{i \in I}$ be a covering of $A$ by analytic open sets and

$$\phi_i : L|_{U_i} \rightarrow \mathcal{O}_{U_i}$$

be local trivializations of $L$. The composition $\phi_j \circ (\phi_i)^{-1}$ is a multiplication by a function on $U_i \cap U_j$. This function will be denoted by $\phi_{i,j}$.

Using $\phi_i$ and the differentiation action of $TU_i$ on $\mathcal{O}_{U_i}$, we have a splitting

$$\psi_i : TU_i \rightarrow \text{Diff}_A(L|_{U_i}, L|_{U_i})$$

of the symbol map. The difference $\psi_j - \psi_i$ on $U_i \cap U_j$ factors as a composition homomorphism

$$T(U_i \cap U_j) \xrightarrow{\gamma} \mathcal{O}_{U_i \cap U_j} \xrightarrow{\text{Diff}_A(L|_{U_i \cap U_j}, L|_{U_i \cap U_j})},$$

and the one-form $\gamma$ on $U_i \cap U_j$ coincides with $d\phi_{i,j}/\phi_{i,j}$. Therefore, the one-cocyle $\{d\phi_{i,j}/\phi_{i,j}\}_{i,j \in I}$ represents the extension class in $H^1(A, \Omega^1_A)$ for the exact sequence (2.7). On the other hand, $\{d\phi_{i,j}/\phi_{i,j}\}$ represents the Chern class $c_1(L)$.

Since the exact sequence (2.2) is simply the $n$-th symmetric power of (2.7), the extension class $C_n$ in (2.5) is also $c_1(L)$. To explain this, first note that the cup product of $c_1(L) \in H^1(A, \Omega^1_A)$ with the identity automorphism of $S^n(TA)$ is a cohomology class

$$c \in H^1(A, \text{Hom}(S^n(TA), \Omega^1_A \otimes S^n(TA))).$$

Using the contraction $\Omega^1_A \otimes S^n(TA) \rightarrow S^{n-1}(TA)$, the cohomology class $c$ gives

$$C_n' \in H^1(A, \text{Hom}(S^n(TA), S^{n-1}(TA))).$$

The extension class $C_n$ in (2.5) coincides with $C_n'$. Indeed, since the extension class for (2.7) is $c_1(L)$, this is an immediate consequence of the fact that (2.2) is the symmetric power of (2.7).

Take a translation invariant $(1,1)$-form $\omega$ on the abelian variety $A$ such that $\omega$ represents the first Chern class $c_1(L)$. It is easy to see that there is exactly one
such form. Since $L$ is ample, the form $\omega$ must be positive. In other words, the homomorphism
\begin{equation}
\tilde{\omega} : TA \longrightarrow \Omega_A^{0,1}
\end{equation}
that sends any $v \in T_pA$ to the contraction of $\omega(p)$ with $v$ is an isomorphism.

Since $TA$ is trivial, any section of $S^n(TA)$ is invariant under translations in $A$. Therefore, the $\hat{\omega}$ invariant under the translations. Furthermore, since $\tilde{\omega}$ is the homomorphism obtained, in an obvious fashion, from $A, S$.

Theorem 2.3.

obtained from (2.1) is also surjective. Therefore, we have the following corollary of

Taking a nonzero section
\begin{equation}
H^{\ast}(A, S^0(TA))(\sigma) \rightarrow H^1(A, S^{n-1}(TA))
\end{equation}
with values in $S^{n-1}(TA)$. We noted earlier that $C_n$ in (2.5) coincides with $C_n'$. Therefore, the $S^{n-1}(TA)$-valued $(0, 1)$-form $\xi$ represents the cohomology class $\overline{S}^{n-1}(\sigma_1) \circ h_n(\xi) \in H^1(A, S^{n-1}(TA))$

in Dolbeaut cohomology, where $h_n$ is the connecting homomorphism in (2.4) and

$\overline{S}^{n-1}(\sigma_1) : H^1(A, S^{n-1}(\text{Diff}_A^1(L, L))) \rightarrow H^1(A, S^{n-1}(TA))$

is the homomorphism obtained, in an obvious fashion, from $S^{n-1}(\sigma_1)$ in (2.2).

Since both $\omega$ and $\xi$ are invariant under the translations in $A$, the form $\overline{\xi}$ is also invariant under the translations. Furthermore, since $\tilde{\omega}$ in (2.8) is an isomorphism and $\xi \neq 0$, we have $\overline{\xi} \neq 0$. From this it follows that the cohomology class in $H^1(A, S^{n-1}(TA))$ represented by $\overline{\xi}$ is nonzero. To see this, note that $\omega$ being positive defines a Kähler structure on $A$. In order to prove that the cohomology class in $H^1(A, S^{n-1}(TA))$ represented by $\overline{\xi}$ is nonzero, it suffices to show that the form $\overline{\xi}$ is harmonic for the Dolbeaut complex for $S^{n-1}(TA)$. However, since the Kähler form is translation invariant, $\overline{\xi}$ being translation invariant must be harmonic.

We already noted that the Dolbeaut cohomology class represented by $\overline{\xi}$ coincides with $\overline{S}^{n-1}(\sigma_1) \circ h_n(\xi)$. Since this class is nonzero, $h_n(\xi)$ must be nonzero. In other words, the homomorphism $h_n$ in (2.4) is injective. We noted earlier that the injectivity of $h_n$ proves the theorem. Therefore, the proof of the theorem is complete.

For $n \geq 1$, consider the exact sequence
\begin{equation}
0 \rightarrow \text{Diff}^{-1}_A(L, L)/\text{Diff}^{-2}_A(L, L) = S^{n-1}(TA)
\end{equation}
\begin{equation}
\rightarrow \text{Diff}^0_A(L, L)/\text{Diff}^{-2}_A(L, L) \rightarrow S^n(TA) \rightarrow 0
\end{equation}
obtained from (2.1), where $\text{Diff}^{-1}_A(L, L)$ denotes 0. It is known that the exact sequence (2.8) is isomorphic to the exact sequence (2.6). Therefore, the injectivity of the homomorphism $h_n$ in (2.4) implies that the connecting homomorphism
\begin{equation}
H^0(A, S^n(TA)) \rightarrow H^1(A, S^{n-1}(TA))
\end{equation}
in the long exact sequence of cohomologies for (2.9) is also injective. Consequently, the injective homomorphism
\begin{equation}
H^0(A, \text{Diff}^{-1}_A(L, L)) \rightarrow H^0(A, \text{Diff}^0_A(L, L))
\end{equation}
obtained from (2.1) is also surjective. Therefore, we have the following corollary of Theorem 2.3.
Corollary 2.10. The inclusion
\[ H^0(A, \mathcal{O}) \rightarrow H^0(A, \text{Diff}^n_A(L, L)) \]
obtained from (2.1) is an isomorphism for all \( n \geq 0 \).

Consider the exact sequence
\[ 0 \rightarrow \Omega^1_A \rightarrow \text{Diff}^1_A(L, L)^* \xrightarrow{\tau} \mathcal{O} \rightarrow 0, \tag{2.11} \]
which is the dual of (2.7). We will denote by \( \mathcal{T} \) the image of the section of \( \mathcal{O} \)
defined by the constant function 1. The subset of the total space of the vector bundle \( \text{Diff}^1_A(L, L)^* \)
defined by the inverse image \( \tau^{-1}(\mathcal{T}) \) will be denoted by \( C(L) \).

Let \( p : C(L) \rightarrow A \) be the obvious projection. The exact sequence (2.11) shows that for any point \( x \in A \),
the inverse image \( p^{-1}(x) \) is an affine space for the holomorphic cotangent space \( (\Omega^1_A)_x \).

Let \( U \subset A \) be an open subset and \( \theta \) a holomorphic section over \( U \) of the fiber bundle \( C(L) \).
Such a section \( \theta \) defines a holomorphic connection on \( L|_U \) \[\text{[1]}\]. The exact sequence (2.7) for a holomorphic line bundle over a complex manifold is known as the Atiyah exact sequence.
A splitting of the Atiyah exact sequence is a holomorphic connection \[\text{[1]}\]. A section \( \theta \) of \( C(L) \) over \( U \) clearly gives a splitting over \( U \) of the exact sequence (2.7).

The subset \( C(L) \subset \text{Diff}^1_A(L, L)^* \) being a Zariski open set has a natural algebraic structure. By \( \mathcal{O}_{C(L)} \) we will denote the structure sheaf of this algebraic variety.

Proposition 2.13. For the variety \( C(L) \),
\[ H^0(C(L), \mathcal{O}_{C(L)}) = \mathbb{C} \]
or, in other words, there is no nonconstant algebraic function on \( C(L) \).

Proof. Let \( P := P\text{Diff}^1_A(L, L)^* \) be the projective bundle over \( A \) consisting of lines in \( \text{Diff}^1_A(L, L)^* \).
Similarly, \( P' := P\Omega^1_A \) denotes the projective bundle defined by the lines in \( \Omega^1_A \). Using the inclusion of \( \Omega^1_A \) in \( \text{Diff}^1_A(L, L)^* \) in (2.11), we have \( P' \) as a subbundle of the projective bundle \( P \). Let \( P_0 := P - P' \) be the complement. It is easy to see that \( P_0 \) is naturally identified with \( C(L) \). The identification is defined by the obvious projection to \( P \) of the complement of the zero section in \( \text{Diff}^1_A(L, L)^* \).

Since the quotient bundle \( \text{Diff}^1_A(L, L)^*/\Omega^1_A \) is trivial, the divisor \( P' \) on \( P \) is the divisor of the tautological line bundle \( \mathcal{O}_P(1) \) over \( P \). So a meromorphic function on \( P \) with pole of order \( d \) along \( P' \) is a section of \( \mathcal{O}_P(d) \). Therefore, it suffices to prove that \[ \dim H^0(P, \mathcal{O}_P(d)) = 1 \]
for all \( d \geq 0 \).

Let \( \gamma \) denote the projection of \( P \) to \( A \). Taking direct image to \( A \), we have the identification
\[ H^0(P, \mathcal{O}_P(d)) = H^0(A, \gamma_*\mathcal{O}_P(d)) = H^0(A, S^d(\text{Diff}^1_A(L, L))). \]
Now Theorem 2.3 implies that \( \dim H^0(P, \mathcal{O}_P(d)) = 1 \) for \( d \geq 0 \). This completes the proof of the proposition. \[\square\]
In the next section we will specialize to Jacobians of curves.

3. Rank one connections on a curve

Let \( X \) be a connected smooth projective curve over \( \mathbb{C} \) or, equivalently, a compact connected Riemann surface. The genus \( g \) of \( X \) is assumed to be positive. Fix once and for all a point \( x_0 \in X \). Let \( J := \text{Pic}^0(X) \) be the Jacobian of \( X \). We will denote by \( \Theta \) the line bundle over \( J \) defined by the divisor that consists of all \( L \) with

\[
H^0(X, \mathcal{O}_X((g-1)x_0) \otimes L) \neq 0.
\]

It is known that \( \Theta \) is ample. More precisely, it defines a principal polarization on \( J \).

Let \( M_X \) denote the moduli space of rank one holomorphic connections on \( X \). In other words, \( M_X \) parametrizes pairs of the form \((L, D)\), where \( L \) is a holomorphic line bundle over \( X \) and \( D \) is a holomorphic connection on \( L \). Since \( \dim X = 1 \), any holomorphic connection on \( X \) is flat. The moduli space of holomorphic connections on a smooth projective variety has been constructed in [5]. In particular, \( M_X \) is a quasi-projective variety.

Let \( \phi : M_X \to J \) be the forgetful morphism. So \( \phi \) sends a pair \((L, D)\) to \( L \).

Let \( R \) denote the character variety \( \text{Hom}(\pi_1(X), \mathbb{C}^*) \) of the fundamental group. If we fix generators of the fundamental group \( \pi_1(X) \), then \( R \) gets identified with the \( 2g \)-fold self-product \( \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* \).

By associating its monodromy to a flat connection, the space \( R \) gets identified with \( M_X \). More precisely, for \( D \in M_X \), this identification associates \( D \) with the element in \( R \) that sends any \( g \in \pi_1(X) \) to the holonomy of \( D \) around \( g \). This identification of \( M_X \) with \( R \) is biholomorphic but not necessarily algebraic [5]. In fact, we will see that \( M_X \) is not algebraically isomorphic to \( R \).

Since \( R \) is a product of copies of \( \mathbb{C}^* \), it is an affine variety. In particular, there are many nonconstant functions on \( R \). In view of that, the following theorem shows that \( M_X \) is not algebraically isomorphic to \( R \).

**Theorem 3.2.** For the variety \( M_X \),

\[
\dim H^0(M_X, \mathcal{O}_{M_X}) = 1,
\]

where \( \mathcal{O}_{M_X} \) denotes the structure sheaf.

**Proof.** Set the pair \((A, L)\) is Section 2 to be \((J, \Theta)\). Consider the fiber bundle

\[
p : \mathcal{C}(\Theta) \to J
\]

constructed in (2.12). In view of Proposition 2.13, the theorem follows immediately from the following proposition.

**Proposition 3.3.** The fiber bundle \( \mathcal{C}(\Theta) \) over \( J \) defined by \( p \) is algebraically isomorphic to \( M_X \) defined in (3.1).

**Proof.** We already remarked that \( \mathcal{C}(\Theta) \) is an affine bundle over \( J \) for the cotangent bundle, that is, any fiber of \( p \) is an affine space for the cotangent space at that point. Now note that \( M_X \) is also an affine bundle for the cotangent bundle. Indeed, the space of holomorphic connections on a degree zero line bundle over \( X \) is an affine
space for $H^0(X, K_X)$, where $K_X$ denotes the holomorphic cotangent bundle. On the other hand, $H^0(X, K_X)$ are the fibers $\Omega^1_J$.

Affine bundles for the cotangent bundle are classified by $H^1(J, \Omega^1_J)$. We will quickly recall how a cohomology class is associated to an affine bundle.

Let $q : Z \to J$ be an affine bundle for $\Omega^1_J$. Let $\{U_i\}_{i \in I}$ be a covering of $J$ by analytic open subsets and

$$\psi_i : U_i \to Z|_{U_i}$$

holomorphic sections. Since the fibers of $Z$ are affine spaces, $\psi_j - \psi_i$ is a holomorphic section of $\Omega^1_{U_i \cap U_j}$. These one-forms $\{\psi_j - \psi_i\}_{i, j \in I}$ define a cocycle. Let $\beta_Z \in H^1(J, \Omega^1_J)$ be the corresponding cohomology class. It is easy to see that another affine bundle $Z'$ will be holomorphically isomorphic to $Z$ if $\beta_Z$ coincides with the corresponding cohomology class $\beta_{Z'}$ for $Z'$. If these two affine bundles are analytically isomorphic, then from the GAGA principle of [4], it follows that they must be algebraically isomorphic.

If $\beta_Z \neq 0$ and $\beta_{Z'} = \lambda \beta_Z$, where $\lambda \in \mathbb{C}^*$, then also the two fiber bundles $Z$ and $Z'$ are algebraically isomorphic. However, if $\lambda \neq 1$, then there will be no isomorphism preserving the affine space structures. Nevertheless, there will be an isomorphism $h : Z' \to Z$ of fiber bundles satisfying the identity $h(z + \theta) = h(z) + \lambda \theta$, where $\theta \in \Omega^1_J$.

Let $\beta_p$ (respectively, $\beta_q$) be the cohomology class in $H^1(J, \Omega^1_J)$ associated to $\mathcal{C}(\Theta)$ (respectively, $\mathcal{M}_X$). We will show that both $\beta_p$ and $2\beta_q$ coincide with $c_1(\Theta)$.

In the proof of Theorem 2.3, we have seen that the extension class for the Atiyah exact sequence (2.7) for $\Theta$ coincides with $c_1(\Theta)$. We already noted that any section $\psi : U \to \mathcal{C}(\Theta)|_U$ as in (3.4) gives a splitting over $U$ of the Atiyah exact sequence for $\Theta$. Consequently, $\beta_p$ coincides with $c_1(\Theta)$.

Let

$$f : J \to M_X$$

be a $C^\infty$ section of the map $\phi$ in (3.1). The obstruction to the holomorphicity of the map $f$ gives a form $\omega_f$ on $J$ of type $(1, 1)$. This form $\omega_f$ can be described as follows. For any point $z \in J$, let

$$df(z) : T^R_z J \to T^R_{f(z)} M_X$$

be the homomorphism of real tangent spaces given by the differential of $f$. Let

$$J_z : T^R_z J \to T^R_z J$$

be the almost complex structure of $J$ at $z$. Similarly, the almost complex structure of $M_X$ at $f(z)$ will be denoted by $J_{f(z)}$. Now, for any $v \in T^R_z J$, the difference

$$J_{f(z)} \circ df(z)(v) - df(z) \circ J_z(v)$$

is an element of $(T^R_z J)^*$. Indeed, this is an immediate consequence of the fact that the kernel of the differential homomorphism

$$d\phi(f(z)) : T^R_{f(z)} M_X \to T^R_z J$$

is identified with $(T^R_z J)^*$ using the affine space structure of the fibers of $\phi$. The resulting homomorphism $T^R_z J \to (T^R_z J)^*$ that sends any $v$ to $J_{f(z)} \circ df(z)(v) - df(z) \circ J_z(v)$ defines the $(1, 1)$-form $\omega_f$.

The cohomology class in $H^1(J, \Omega^1_J)$ represented by $\omega_f$ coincides with $\beta_p$. In fact, this is the Dolbeault analog of the earlier construction of the cohomology class $\beta_Z$. 

Any holomorphic line bundle over $X$ of degree zero admits a unique unitary flat connection. Let $f$ be the map that associates to any $L$ in $J$ the unitary flat connection on $L$. From \[2\,\text{Theorem 2.11}\] we know that $2\omega_f$ coincides with the pullback, using $f$, of a certain natural symplectic form on $M_X$. The symplectic form on $M_X$ in question is the one defined in \[3\] on the representation space $R$. On the other hand, the pullback of this symplectic form coincides with $c_1(\Theta)$. This is well known; the details can be found in \[2\].

Therefore, both $\beta_p$ and $2\beta_p$ coincide with $c_1(\Theta)$. This completes the proof of the proposition.

We already noted that Proposition 3.3 completes the proof of Theorem 3.2. Therefore, the proof of Theorem 3.2 is complete.

Let $Y$ be another compact connected Riemann surface. Let $M_Y$ denote the moduli space of rank one holomorphic connections on $Y$. Let $\Omega(X)$ (respectively, $\Omega(Y)$) denote the natural symplectic form on $M_X$ (respectively, $M_Y$) constructed in \[3\].

**Proposition 3.5.** If there is an algebraic isomorphism of $M_X$ with $M_Y$ that takes the symplectic form $\Omega(X)$ to $\Omega(Y)$, then the Riemann surface $X$ is isomorphic to $Y$.

*Proof.* The Torelli theorem says that if the Jacobian of $X$ is isomorphic to the Jacobian of $Y$ as a principally polarized abelian variety, then $X$ is isomorphic to $Y$. The principal polarization in question is the one given by theta. The proposition will be proved by recovering the Jacobian of $X$, along with its polarization, from the symplectic variety $(M_X, \Omega(X))$.

Let $\phi_Y : M_Y \rightarrow \text{Pic}^0(Y)$ be the projection defined in (3.1) for $Y$.

There is no nonconstant algebraic map from the affine line to an abelian variety. This is an immediate consequence of the fact that there is no nonzero holomorphic one-form on the projective line. Therefore, any algebraic isomorphism

$$\psi : M_X \rightarrow M_Y$$

induces an isomorphism $\overline{\psi} : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ which is determined by the identity

$$\phi_Y \circ \psi = \overline{\psi} \circ \phi,$$

where $\phi$ is as in (3.1). Consequently, $M_X$ determines both $\text{Pic}^0(X)$ and the projection $\phi$.

Take a $C^\infty$ section

$$f : \text{Pic}^0(X) \rightarrow M_X$$

(as in the proof of Proposition 3.3) of the projection $\psi$. As in the proof of Proposition 3.3, let $\omega_f$ denote the $(1,1)$-form on $\text{Pic}^0(X)$ given by the obstruction to the holomorphicity of $f$. If $f_0 : \text{Pic}^0(X) \rightarrow M_X$ is another section of $\psi$, then it is easy to check that

(3.6)

$$\omega_f - \omega_{f_0} = \overline{\partial}(f - f_0).$$

Note that using the affine bundle structure of $M_X$, the difference $f - f_0$ defines a $(1,0)$-form on $\text{Pic}^0(X)$.

Set $f_0$ to be the section that sends any line bundle $L$ in $\text{Pic}^0(X)$ to the (unique) unitary flat connection on $L$. In the proof of Proposition 3.3 we saw that $\omega_{f_0}$
represents $c_1(\Theta)/2$. Therefore, the identity (3.6) implies that the cohomology class in $H^1(\text{Pic}^0(X), \Omega^1_{\text{Pic}^0(X)})$ represented by the form $2\omega_f$ coincides with $c_1(\Theta)$.

Therefore, the algebraic variety $M_X$ equipped with the symplectic form $\Omega(X)$ determines the principally polarized abelian variety $(\text{Pic}^0(X), c_1(\Theta))$. This completes the proof of the proposition.

Since only the cohomology class represented by the symplectic form is used, $X$ is isomorphic to $Y$ if there is an isomorphism of $M_X$ with $M_Y$ that takes the cohomology class for the symplectic form $\Omega(Y)$ to that for $\Omega(X)$.

References