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LINES TANGENT TO $2n-2$ SPHERES IN $\mathbb{R}^n$

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ABSTRACT. We show that for $n \geq 3$ there are $3 \cdot 2^{n-1}$ complex common tangent lines to $2n-2$ general spheres in $\mathbb{R}^n$ and that there is a choice of spheres with all common tangents real.

1. INTRODUCTION

We study the following problem from (real) enumerative geometry.

Given: $2n-2$ (not necessarily disjoint) spheres with centers $c_i \in \mathbb{R}^n$ and radii r_i, $1 \leq i \leq 2n-2$.

Question: In the case of finitely many common tangent lines, what is their maximum number?

The number of $2n-2$ spheres guarantees that in the generic case there is indeed a finite number of common tangent lines. In particular, for $n = 2$ the answer is 4 since two disjoint circles have 4 common tangents.

The reason for studying this question—which, of course, is an appealing and fundamental geometric question in itself—came from different motivations. An essential task in statistical analysis is to find the line that best fits the data in the sense of minimizing the maximal distance to the points (see, e.g., [3]). More precisely, the decision variant of this problem asks: Given $m, n \in \mathbb{N}$, $r > 0$, and a set of points $y_1, \ldots, y_m \in \mathbb{R}^n$, does there exist a line l in $\mathbb{R}^n$ such that every point y_i has Euclidean distance at most r from l. From the complexity-theoretical point of view, for fixed dimension the problem can be solved in polynomial time via quantifier elimination over the reals [5]. However, currently no practical algorithms focusing on exact computation are known for $n > 3$ (for approximation algorithms, see [3]).

From the algebraic perspective, for dimension 3 it was shown in [1, 14] how to reduce the algorithmic problem to an algebraic-geometric core problem: finding the real lines which all have the same prescribed distance from 4 given points; or, equivalently, finding the real common tangent lines to 4 given unit spheres in $\mathbb{R}^3$. This problem in dimension 3 was treated in [9].

Proposition 1. *Four unit spheres in $\mathbb{R}^3$ have at most 12 common tangent lines unless their centers are collinear. Furthermore, there exists a configuration with 12 different real tangent lines.*

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The same reduction idea to the algebraic-geometric core problem also applies to arbitrary dimensions, in this case leading to the general problem stated at the beginning.

From the purely algebraic-geometric point of view, this tangent problem is interesting for the following reason. In dimension 3, the formulation of the problem in terms of Plücker coordinates gives 5 quadratic equations in projective space $\mathbb{P}^5_{\mathbb{R}}$, whose common zeroes in $\mathbb{P}^5_{\mathbb{C}}$ include a 1-dimensional component at infinity (accounting for the "missing" $2^5 - 12 = 20$ solutions). Quite remarkably, as observed in [2], this excess component cannot be resolved by a single blow-up. Experimental results in [16] for $n = 4, 5$, and 6, indicate that for higher dimensions the generic number of solutions differs from the Bézout number of the straightforward polynomial formulation even more. We discuss this further in Section 5.

Our main result can be stated as follows.

Theorem 2. *Suppose $n \geq 3$.*

(a) *Let $c_1, \ldots, c_{2n-2} \in \mathbb{R}^n$ affinely span $\mathbb{R}^n$, and let $r_1, \ldots, r_{2n-2} > 0$. If the $2n-2$ spheres with centers c_i and radii r_i have only a finite number of complex common tangent lines, then that number is bounded by $3 \cdot 2^{n-1}$.*

(b) *There exists a configuration with $3 \cdot 2^{n-1}$ different real common tangent lines. Moreover, this configuration can be achieved with unit spheres.*

Thus the bound for real common tangents equals the (*a priori greater*) bound for complex common tangents; so this problem of common tangents to spheres is fully real in the sense of enumerative real algebraic geometry [15, 17]. We prove Statement (a) in Section 2 and Statement (b) in Section 3, where we explicitly describe configurations with $3 \cdot 2^{n-1}$ common real tangents. Figure 1 shows a configuration of 4 spheres in $\mathbb{R}^3$ with 12 common tangents (as given in [9]).

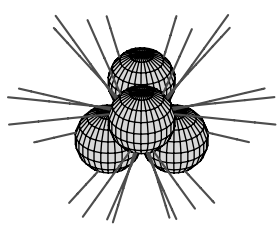


FIGURE 1. Spheres with 12 real common tangents

In Section 4, we show that there are configurations of spheres with affinely dependent centers having $3 \cdot 2^{n-1}$ *complex* common tangents; thus, the upper bound of Theorem 2 also holds for spheres in this special position. Megyesi [11] has recently shown that all $3 \cdot 2^{n-1}$ may be real. We also show that if the centers of the spheres are the vertices of the crosspolytope in $\mathbb{R}^{n-1}$, there will be at most 2^n common tangents, and if the spheres overlap but do not contain the centroid of the crosspolytope, then all 2^n common tangents will be real. We conjecture that when the centers are affinely dependent and all spheres have the same radius, then there will be at most 2^n real common tangents. Strong evidence for this conjecture is provided by Megyesi [10], who showed that there are at most 8 real common tangents to 4 unit spheres in $\mathbb{R}^3$ whose centers are coplanar but not collinear.

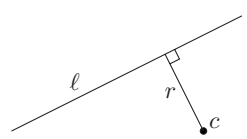
In Section 5, we put the tangent problem into the perspective of common tangents to general quadric hypersurfaces. In particular, we discuss the problem of common tangents to $2n-2$ smooth quadrics in *projective* n-space, and describe the excess component at infinity for this problem of spheres. In this setting, Theorem 2(a) implies that there will be at most $3 \cdot 2^{n-1}$ isolated common tangents to $2n-2$ quadrics in projective n-space, when the quadrics all contain the same (smooth) quadric in a given hyperplane. In particular, the problem of the spheres can be seen as the case when the common quadric is at infinity and contains no real points. We conclude with the question of how many of these common tangents may be real when the shared quadric has real points. For $n = 3$, there are 5 cases to consider, and for each, all 12 lines can be real [16]. Megyesi [11] has recently shown that all common tangents may be real, for many cases of the shared quadric.

2. Polynomial Formulation with Affinely Independent Centers

For $x, y \in \mathbb{C}^n$, let $x \cdot y := \sum_{i=1}^n x_i y_i$ denote their Euclidean dot product. We write x^2 for $x \cdot x$.

We represent a line in $\mathbb{C}^n$ by a point $p \in \mathbb{C}^n$ lying on the line and a direction vector $v \in \mathbb{P}^{n-1}_{\mathbb{C}}$ of that line. (For notational convenience we typically work with a representative of the direction vector in $\mathbb{C}^n \setminus \{0\}$.) If $v^2 \neq 0$ we can make p unique by requiring that $p \cdot v = 0$.

By definition, a line $\ell = (p, v)$ is tangent to the sphere with center $c \in \mathbb{R}^n$ and radius r if and only if it is tangent to the quadratic hypersurface $(x-c)^2 = r^2$, i.e., if and only if the quadratic equation $(p + tv - c)^2 = r^2$ has a solution of multiplicity two. When ℓ is real then this is equivalent to the metric property that ℓ has Euclidean distance r from c.



For any line $\ell \subset \mathbb{C}^n$, the algebraic tangent condition on ℓ gives the equation

$$\frac{[v \cdot (p-c)]^2}{v^2} - (p-c)^2 \; + r^2 \; = \; 0\,.$$

For $v^2 \neq 0$ this is equivalent to

$$v^2 p^2 - 2v^2 p \cdot c + v^2 c^2 - [v \cdot c]^2 - r^2 v^2 \; = \; 0\,. \tag{2.1}$$

To prove part (a) of Theorem 2, we can choose c_{2n-2} to be the origin and set $r := r_{2n-2}$. Then the remaining centers span $\mathbb{R}^n$. Subtracting the equation for the sphere centered at the origin from the equations for the spheres $1, \ldots, 2n-3$ gives the system

$$\begin{aligned} p \cdot v &= 0\,, \\ p^2 &= r^2\,, \\ 2v^2 p \cdot c_i &= v^2 c_i^2 - [v \cdot c_i]^2 - v^2(r_i^2 - r^2)\,, \qquad i = 1, 2, \ldots, 2n-3\,. \end{aligned} \tag{2.2}$$

Remark 3. Note that this system of equations does not have a solution with $v^2 = 0$. Namely, if we had $v^2 = 0$, then $v \cdot c_i = 0$ for all i. Since the centers span $\mathbb{R}^n$, this

would imply $v = 0$, contradicting $v \in \mathbb{P}^{n-1}_{\mathbb{C}}$. This validates our assumption that $v^2 \neq 0$ prior to (2.1).

Since $n \geq 3$, the bottom line of (2.2) contains at least n equations. We can assume $c_1, \ldots, c_n$ are linearly independent. Then the matrix $M := (c_1, \ldots, c_n)^{\mathrm{T}}$ is invertible, and we can solve the equations with indices $1, \ldots, n$ for p:

$$p \;=\; \frac{1}{2v^2} M^{-1} \begin{pmatrix} v^2 c_1^2 - [v \cdot c_1]^2 - v^2(r_1^2 - r^2) \\ \vdots \\ v^2 c_n^2 - [v \cdot c_n]^2 - v^2(r_n^2 - r^2) \end{pmatrix}. \tag{2.3}$$

Now substitute this expression for p into the first and second equation of the system (2.2), as well as into the equations for $i = n+1, \ldots, 2n-3$, and then clear the denominators. This gives $n-1$ homogeneous equations in the coordinate v, namely one cubic, one quartic, and $n-3$ quadrics. By Bézout's Theorem, this means that if the system has only finitely many solutions, then the number of solutions is bounded by $3 \cdot 4 \cdot 2^{n-3} = 3 \cdot 2^{n-1}$, for $n \geq 3$. For small values of n, these values are shown in Table 1. The value 12 for $n = 3$ was computed in [9], and the values for $n = 4, 5, 6$ were computed experimentally in [16].

TABLE 1. Maximum number of tangents in small dimensions

n	3	4	5	6	7
maximum # tangents	12	24	48	96	192

We simplify the cubic equation obtained by substituting (2.3) into the equation $p \cdot v = 0$ by expressing it in the basis $c_1, \ldots, c_n$. Let the representation of v in the basis $c_1, \ldots, c_n$ be

$$v \;=\; \sum_{i=1}^{n} t_i c_i$$

with homogeneous coordinates $t_1, \ldots, t_n$. Further, let $c'_1, \ldots, c'_n$ be a dual basis to $c_1, \ldots, c_n$; i.e., let $c'_1, \ldots, c'_n$ be defined by $c'_i \cdot c_j = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta function. By elementary linear algebra, we have $t_i = c'_i \cdot v$.

When expressing p in this dual basis, $p = \sum p'_i c'_i$, the third equation of (2.2) gives

$$p'_i \;=\; \frac{1}{v^2}\left(v^2 c_i^2 - [v \cdot c_i]^2 - v^2(r_i^2 - r^2)\right).$$

Substituting this representation of p into the equation

$$0 \;=\; 2v^2(p \cdot v) \;=\; 2v^2 \left(\sum_{i=1}^{n} p'_i c'_i\right) \cdot v \;=\; 2v^2 \sum_{i=1}^{n} p'_i t_i\,,$$

we obtain the cubic equation

$$\sum_{i=1}^{n} (v^2 c_i^2 - [v \cdot c_i]^2 - v^2(r_i^2 - r^2)) t_i \;=\; 0\,.$$

In the case that all radii are equal, expressing v^2 in terms of the t-variables yields

$$\sum_{1 \leq i \neq j \leq n} \alpha_{ij} t_i^2 t_j + \sum_{1 \leq i < j < k \leq n} 2\beta_{ijk} t_i t_j t_k \;=\; 0\,,$$

where

$$\begin{aligned}
\alpha_{ij} &= (\mathrm{vol}_2(c_i,c_j))^2 = \det\begin{pmatrix} c_i\cdot c_i & c_i\cdot c_j \\ c_j\cdot c_i & c_j\cdot c_j \end{pmatrix}, \\
\beta_{ijk} &= \det\begin{pmatrix} c_i\cdot c_j & c_i\cdot c_k \\ c_k\cdot c_j & c_k\cdot c_k \end{pmatrix} + \det\begin{pmatrix} c_i\cdot c_k & c_i\cdot c_j \\ c_j\cdot c_k & c_j\cdot c_j \end{pmatrix} \\
&\quad + \det\begin{pmatrix} c_j\cdot c_k & c_j\cdot c_i \\ c_i\cdot c_k & c_i\cdot c_i \end{pmatrix},
\end{aligned}$$

and $\mathrm{vol}_2(c_i,c_j)$ denotes the oriented area of the parallelogram spanned by c_i and c_j. In particular, if $0c_1\dots c_n$ constitutes a regular simplex in $\mathbb{R}^n$, then we obtain the following characterization.

Theorem 4. *Let $n \geq 3$. If $0c_1\dots c_n$ is a regular simplex and all spheres have the same radius, then the cubic equation expressed in the basis $c_1,\dots,c_n$ is equivalent to*

$$\sum_{1\leq i\neq j\leq n} t_i^2 t_j + 2\sum_{1\leq i<j<k\leq n} t_i t_j t_k = 0. \tag{2.4}$$

For $n=3$, this cubic equation factors into three linear terms; for $n\geq 4$ it is irreducible.

Proof. Let e denote the edge length of the regular simplex. Then the form of the cubic equation follows from computing $\alpha_{ij} = e^2(1\cdot 1 - 1/2\cdot 1/2) = 3e^2/4$, $\beta_{ijk} = 3e^2(1/2\cdot 1 - 1/2\cdot 1/2) = 3e^2/4$.

Obviously, for $n=3$ the cubic polynomial factors into $(t_1+t_2)(t_1+t_3)(t_2+t_3)$ (cf. [13, 9]). For $t\geq 4$, assume that there exists a factorization of the form

$$\left(t_1 + \sum_{i=2}^{n}\rho_i t_i\right)\left(\sum_{1\leq i\leq j\leq n}\sigma_{ij}t_i t_j\right)$$

with $\sigma_{12}=1$. Since (2.4) does not contain a monomial t_i^3, we have either $\rho_i=0$ or $\sigma_{ii}=0$ for $1\leq i\leq n$.

If there were more than one vanishing coefficient ρ_i, say $\rho_i=\rho_j=0$, then the monomials $t_i^2t_j$ could not be generated. So only two cases have to be investigated.

Case 1. $\rho_i\neq 0$ for $2\leq i\leq n$. Then $\sigma_{ii}=0$ for $1\leq i\leq n$. Furthermore, $\sigma_{ij}=1$ for $i\neq j$ and $\rho_i=1$ for all i. Hence, the coefficient of the monomial $t_1t_2t_3$ is 3, which contradicts (2.4).

Case 2. There exists exactly one coefficient $\rho_i=0$, say, $\rho_4=0$. Then $\sigma_{11}=\sigma_{22}=\sigma_{33}=0$, $\sigma_{44}=1$. Further, $\sigma_{ij}=1$ for $1\leq i<j\leq 3$ and $\rho_i=1$ for $1\leq i\leq 3$. Hence, the coefficient of the monomial $t_1t_2t_3$ is 3, which is again a contradiction. □

3. Real Lines

In the previous section, we have given the upper bound of $3\cdot 2^{n-1}$ for the number of complex solutions to the tangent problem. Now we complement this result by providing a class of configurations leading to $3\cdot 2^{n-1}$ real common tangents. Hence, the upper bound is tight, and is achieved by real tangents.

There are no general techniques known to find and prove configurations with a maximum number of real solutions in enumerative geometry problems like the one studied here. For example, for the classical enumerative geometry problem of 3264

conics tangent to five given conics (dating back to Steiner in 1848 [19] and solved by Chasles in 1864 [4]) the existence of five real conics with all 3264 *real* was only recently established ([12] and [7, §7.2]).

Our construction is based on the following geometric idea. For 4 spheres in $\mathbb{R}^3$ centered at the vertices $(1,1,1)^{\mathrm{T}}$, $(1,-1,-1)^{\mathrm{T}}$, $(-1,1,-1)^{\mathrm{T}}$, $(-1,-1,1)^{\mathrm{T}}$ of a regular tetrahedron, there are [9]

- 3 different real tangents (of multiplicity 4) for radius $r=\sqrt{2}$;
- 12 different real tangents for $\sqrt{2}<r<3/2$;
- 6 different real tangents (of multiplicity 2) for $r=3/2$.

Furthermore, based on the explicit calculations in [9], it can be easily seen that the symmetry group of the tetrahedron acts transitively on the tangents. By this symmetry argument, all 12 tangents have the same distance d from the origin. In order to construct a configuration of spheres with many common tangents, say, in $\mathbb{R}^4$, we embed the centers via

$$(x_1,x_2,x_3)^{\mathrm{T}} \;\longmapsto\; (x_1,x_2,x_3,0)^{\mathrm{T}}$$

into $\mathbb{R}^4$ and place additional spheres with radius r at $(0,0,0,a)^{\mathrm{T}}$ and $(0,0,0,-a)^{\mathrm{T}}$ for some appropriate value of a. If a is chosen in such a way that the centers of the two additional spheres have distance r from the above tangents, then, intuitively, all common tangents to the six four-dimensional spheres are located in the hyperplane $x_4=0$ and have multiplicity 2 (because of the two different possibilities of signs when perturbing the situation). By perturbing this configuration slightly, the tangents are no longer located in the hyperplane $x_4=0$, and therefore the double tangents are forced to split. The idea also generalizes to dimension $n\geq 5$.

Formally, suppose that the $2n-2$ spheres in $\mathbb{R}^n$ all have the same radius, r, and the first four have centers

$$\begin{aligned}
c_1 &:= (\;\;1,\;\;1,\;\;1,\;0,\dots,0)^{\mathrm{T}},\\
c_2 &:= (\;\;1,-1,-1,\;0,\dots,0)^{\mathrm{T}},\\
c_3 &:= (-1,\;\;1,-1,\;0,\dots,0)^{\mathrm{T}},\\
c_4 &:= (-1,-1,\;\;1,\;0,\dots,0)^{\mathrm{T}}
\end{aligned}$$

at the vertices of a regular tetrahedron inscribed in the 3-cube

$$(\pm1,\pm1,\pm1,0,\dots,0)^{\mathrm{T}}.$$

We place the subsequent centers at the points $\pm a e_j$ for $j=4,5,\dots,n$, where $e_1,\dots,e_n$ are the standard unit vectors in $\mathbb{R}^n$.

Theorem 5. *Let $n\geq 4$, $r>0$, $a>0$, and $\gamma := a^2(n-1)/(a^2+n-3)$. If*

$$(r^2-3)\,(3-\gamma)\,(a^2-2)\,(r^2-\gamma)\,\left((3-\gamma)^2+4\gamma-4r^2\right)\;\neq\;0\,, \tag{3.1}$$

then there are exactly $3\cdot 2^{n-1}$ different lines tangent to the $2n-2$ spheres. If

$$a^2>2,\quad \gamma<3,\quad \textit{and}\quad \gamma\;<\;r^2\;<\;\gamma+\tfrac{1}{4}\,(3-\gamma)^2\,, \tag{3.2}$$

then all these $3\cdot 2^{n-1}$ lines are real. Furthermore, this system of inequalities defines a nonempty subset of the (a,r)-plane.

Given values of a and r satisfying (3.2), we may scale the centers and parameters by $1/r$ to obtain a configuration with unit spheres, proving Theorem 2 (b).

Remark 6. The set of values of a and r which give all solutions real is nonempty. To show this, we calculate

$$\gamma \;=\; \frac{a^2(n-1)}{a^2+n-3} \;=\; (n-1)\left(1-\frac{n-3}{a^2+n-3}\right), \tag{3.3}$$

which implies that γ is an increasing function of a^2. Similarly, set $\delta := \gamma+(3-\gamma)^2/4$, the upper bound for r^2. Then

$$\frac{d}{d\gamma}\,\delta \;=\; \frac{d}{d\gamma}\left(\frac{\gamma+(3-\gamma)^2}{4}\right)p \;=\; 1+\frac{\gamma-3}{2},$$

and so δ is an increasing function of γ when $\gamma > 1$. When $a^2 = 2$, we have $\gamma = 2$; so δ is an increasing function of a in the region $a^2 > 2$. Since when $a = \sqrt{2}$, we have $\delta = \frac{9}{4} > \gamma$, the region defined by (3.2) is nonempty.

Moreover, we remark that the region is qualitatively different in the cases $n = 4$ and $n \geq 5$. For $n = 4$, γ satisfies $\gamma < 3$ for any $a > \sqrt{2}$. Hence, $\delta < 3$ and $r < \sqrt{3}$. Thus the maximum value of 24 real lines may be obtained for arbitrarily large a. In particular, we may choose the two spheres with centers $\pm ae_4$ disjoint from the first four spheres. Note, however, that the first four spheres do meet, since we have $\sqrt{2} < r < \sqrt{3}$.

For $n \geq 5$, there is an upper bound to a. The upper and lower bounds for r^2 coincide when $\gamma = 3$; so we always have $r^2 < 3$. Solving $\gamma = 3$ for a^2, we obtain $a^2 < 3(n-3)/(n-4)$. When $n = 5$, Figure 2 displays the discriminant locus (defined by (3.1)) and shades the region consisting of values of a and r for which all solutions are real.

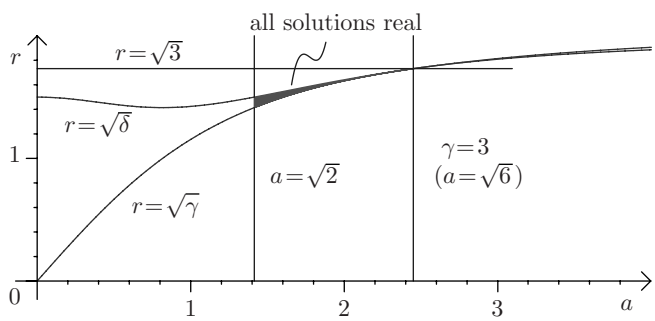


FIGURE 2. Discriminant locus and values of a, r giving all solutions real

Proof of Theorem 5. We prove Theorem 5 by treating a and r as parameters and explicitly solving the resulting system of polynomials in the coordinates $(p, v) \in \mathbb{C}^n \times \mathbb{P}^{n-1}_{\mathbb{C}}$ for lines in $\mathbb{C}^n$. This shows that there are $3 \cdot 2^{n-1}$ *complex* lines tangent to the given spheres, for the values of the parameters (a, r) given in Theorem 5. The inequalities (3.2) describe the parameters for which all solutions are real. □

First consider the equations (2.1) for the line to be tangent to the spheres with centers $\pm ae_j$ and radius r:

$$\begin{aligned} v^2p^2 - 2av^2p_j + a^2v^2 - a^2v_j^2 - r^2v^2 &= 0, \\ v^2p^2 + 2av^2p_j + a^2v^2 - a^2v_j^2 - r^2v^2 &= 0. \end{aligned}$$

Taking their sum and difference (and using $av^2 \neq 0$), we obtain

$$p_j = 0, \qquad 4 \leq j \leq n, \tag{3.4}$$

$$a^2 v_j^2 = (p^2 + a^2 - r^2)v^2, \qquad 4 \leq j \leq n. \tag{3.5}$$

Subtracting the equations (2.1) for the centers $c_1, \ldots, c_4$ pairwise gives

$$4v^2(p_2 + p_3) \;=\; -4(v_1 v_3 + v_1 v_2)$$

(for indices 1,2) and analogous equations. Hence,

$$p_1 \;=\; -\frac{v_2 v_3}{v^2}, \qquad p_2 \;=\; -\frac{v_1 v_3}{v^2}, \qquad p_3 \;=\; -\frac{v_1 v_2}{v^2}.$$

Further, $p \cdot v = 0$ implies $v_1 v_2 v_3 = 0$. Thus we have 3 symmetric cases. We treat one, assuming that $v_1 = 0$. Then we obtain

$$p_1 \;=\; -\frac{v_2 v_3}{v^2}, \qquad p_2 = p_3 = 0.$$

Hence, the tangent equation (2.1) for the first sphere becomes

$$v^2 p_1^2 - 2v^2 p_1 + 3v^2 - (v_2 + v_3)^2 - r^2 v^2 \;=\; 0\,.$$

Using $0 = v^2 p_1 + v_2 v_3$, we obtain

$$v_2^2 + v_3^2 \;=\; v^2(p_1^2 + 3 - r^2)\,. \tag{3.6}$$

The case $j = 4$ of (3.5) gives $a^2 v_4^2 = v^2(p_1^2 + a^2 - r^2)$, since $p_2 = p_3 = 0$. Combining these, we obtain

$$v_2^2 + v_3^2 \;=\; a^2 v_4^2 + v^2(3 - a^2)\,.$$

Using $v^2 = v_2^2 + v_3^2 + (n-3)v_4^2$ yields

$$(a^2 - 2)(v_2^2 + v_3^2) \;=\; v_4^2(3(a^2 + n - 3) - a^2(n-1)).$$

We obtain

$$(a^2 - 2)(v_2^2 + v_3^2) \;=\; v_4^2(a^2 + n - 3)(3 - \gamma)\,, \tag{3.7}$$

where $\gamma = a^2(n-1)/(a^2 + n - 3)$.

Note that $a^2 + n - 3 > 0$ for $n > 3$. If neither $3 - \gamma$ nor $a^2 - 2$ are zero, then we may use this to compute

$$\begin{aligned}(a^2 + n - 3)(3 - \gamma)v^2 \;&=\; [(a^2 + n - 3)(3 - \gamma) + (n-3)(a^2 - 2)](v_2^2 + v_3^2)\\ &=\; (a^2 + n - 3)(v_2^2 + v_3^2)\,,\end{aligned}$$

and so

$$(3 - \gamma)v^2 \;=\; v_2^2 + v_3^2\,. \tag{3.8}$$

Substituting (3.8) into (3.6) and dividing by v^2 gives

$$p_1^2 \;=\; r^2 - \gamma\,. \tag{3.9}$$

Combining this with $v^2 p_1 + v_2 v_3 = 0$, we obtain

$$p_1(v_2^2 + v_3^2) + (3 - \gamma)v_2 v_3 \;=\; 0\,. \tag{3.10}$$

Summarizing, we have n linear equations

$$v_1 \;=\; p_2 \;=\; p_3 \;=\; p_4 \;=\; \cdots \;=\; p_n \;=\; 0\,,$$

and $n - 4$ simple quadratic equations

$$v_4^2 \;=\; v_5^2 \;=\; \cdots \;=\; v_n^2\,,$$

and the three more complicated quadratic equations, (3.7), (3.9), and (3.10).

We now solve these last three equations. We solve (3.9) for p_1, obtaining

$$p_1 \;=\; \pm\sqrt{r^2-\gamma}\,.$$

Then we solve (3.10) for v_2 and use (3.9), obtaining

$$v_2 \;=\; -\frac{3-\gamma\pm\sqrt{(3-\gamma)^2-4(r^2-\gamma)}}{2p_1}\,v_3\,.$$

Finally, (3.7) gives

$$v_4\sqrt{a^2+n-3} \;=\; \pm\sqrt{\frac{a^2-2}{3-\gamma}(v_2^2+v_3^2)}\,.$$

Since $v_3=0$ would imply $v=0$ and hence contradict $v\in\mathbb{P}^{n-1}_{\mathbb{C}}$, we see that $v_3\neq 0$. Thus we can conclude that when none of the following expressions

$$r^2-3\,,\;\; 3-\gamma\,,\;\; a^2-2\,,\;\; r^2-\gamma\,,\;\; (3-\gamma)^2+4\gamma-4r^2$$

vanish, there are $8=2^3$ different solutions to the last 3 equations. For each of these, the simple quadratic equations give 2^{n-4} solutions; so we see that the case $v_1=0$ contributes 2^{n-1} different solutions, each of them satisfying $v_2\neq 0$, $v_3\neq 0$. Since there are three symmetric cases, we obtain $3\cdot 2^{n-1}$ solutions in all, as claimed.

We complete the proof of Theorem 5 and determine which values of the parameters a and r give all these lines real. We see that

(1) p_1 is real if $r^2-\gamma>0$.
(2) Given that p_1 is real, v_2/v_3 is real if $(3-\gamma)^2+4\gamma-4r^2>0$.
(3) Given this, v_4/v_3 is real if $(a^2-2)/(3-\gamma)>0$.

Suppose the three inequalities above are satisfied. Then all solutions are real, and (3.8) implies that $3-\gamma>0$, and so we also have $a^2-2>0$. This completes the proof of Theorem 5. □

4. Affinely Dependent Centers

In our derivation of the Bézout number $3\cdot 2^{n-1}$ of common tangents for Theorem 2, it was crucial that the centers of the spheres affinely spanned $\mathbb{R}^n$. Also, the construction in Section 3 of configurations with $3\cdot 2^{n-1}$ real common tangents had centers affinely spanning $\mathbb{R}^n$. When the centers are affinely dependent, we prove the following result.

Theorem 7. *For $n\geq 4$, there are $3\cdot 2^{n-1}$ complex common tangent lines to $2n-2$ spheres whose centers are affinely dependent, but otherwise general. There is a choice of such spheres with 2^n real common tangent lines.*

Remark 8. Theorem 7 extends the results of [9, Section 4], where it is shown that when $n=3$, there are 12 complex common tangents. Megyesi [10] has shown that there is a configuration with 12 real common tangents, but that the number of tangents is bounded by 8 for the case of unit spheres. For $n\geq 4$, we are unable either to find a configuration of spheres with affinely dependent centers and equal radii having more than 2^n real common tangents, or to show that the maximum number of real common tangents is less than $3\cdot 2^{n-1}$. Similar to the case $n=3$, it might be possible that the case of unit spheres and the case of spheres with general radii might give different maximum numbers.

Remark 9. Megyesi [11] recently showed that there are $2n-2$ spheres with affinely dependent centers having all $3 \cdot 2^{n-1}$ common tangents real. Furthermore, all but one of the spheres in his construction have equal radii.

By Theorem 2, $3 \cdot 2^{n-1}$ is the upper bound for the number of complex common tangents to spheres with affinely dependent centers. Indeed, if there were a configuration with more common tangents, then—since the system is a complete intersection—perturbing the centers would give a configuration with affinely independent centers and more common tangent lines than allowed by Theorem 2.

By this discussion, to prove Theorem 7 it suffices to give $2n-2$ spheres with affinely dependent centers having $3 \cdot 2^{n-1}$ complex common tangents and also such a configuration of $2n-2$ spheres with 2^n real common tangents. For this, we use spheres with equal radii whose centers are the vertices of a perturbed crosspolytope in a hyperplane. We work with the notation of Sections 2 and 3.

Let $a \neq -1$ and suppose we have spheres with equal radii r and centers at the points

$$ae_2, \ -e_2, \quad \text{and} \quad \pm e_j, \quad \text{for } 3 \leq j \leq n\,.$$

Then we have the equations

$$p \cdot v = 0, \tag{4.1}$$

$$f := v^2(p^2 - 2ap_2 + a^2 - r^2) - a^2 v_2^2 = 0, \tag{4.2}$$

$$g := v^2(p^2 + 2p_2 + 1 - r^2) - v_2^2 = 0, \tag{4.3}$$

$$v^2(p^2 \pm 2p_j + 1 - r^2) - v_j^2 = 0, \qquad 3 \leq j \leq n\,. \tag{4.4}$$

As in Section 3, the sum and difference of the equations (4.4) for the spheres with centers $\pm e_j$ give

$$\begin{aligned} p_j &= 0, \\ v^2(p^2 + 1 - r^2) &= v_j^2. \end{aligned} \qquad 3 \leq j \leq n\,.$$

Thus we have the equations

$$\begin{aligned} p_3 = p_4 = \cdots = p_n &= 0, \\ v_3^2 = v_4^2 = \cdots = v_n^2 &. \end{aligned} \tag{4.5}$$

Similarly, we have

$$\begin{aligned} f + ag &= (1+a)\left[v^2(p^2 - r^2 + a) - av_2^2\right] = 0, \\ f - a^2 g &= (1+a)v^2\left[(1-a)(p^2 - r^2) + 2ap_2\right] = 0. \end{aligned}$$

As before, $v^2 \neq 0$: If $v^2 = 0$, then (4.3) and (4.4) imply that $v_2 = \cdots = v_n = 0$. With $v^2 = 0$, this implies that $v_1 = 0$ and hence $v = 0$, contradicting $v \in \mathbb{P}^{n-1}_{\mathbb{C}}$. By (4.5), we have $p^2 = p_1^2 + p_2^2$, and so we obtain the system of equations in the variables p_1, p_2, v_1, v_2, v_3:

$$\begin{aligned} p_1 v_1 + p_2 v_2 &= 0, \\ (1-a)(p_1^2 + p_2^2 - r^2) + 2ap_2 &= 0, \\ v^2(p_1^2 + p_2^2 - r^2 + a) - av_2^2 &= 0, \\ v^2(p_1^2 + p_2^2 - r^2 + 1) - v_3^2 &= 0. \end{aligned} \tag{4.6}$$

(For notational sanity, we do not yet make the substitution $v^2 = v_1^2 + v_2^2 + (n-2)v_3^2$.)

We assume that $a \neq 1$ and will treat the case $a = 1$ at the end of this section. Using the second equation of (4.6) to cancel the terms $v^2(p_1^2 + p_2^2)$ from the third equation and dividing the result by a, we can solve for p_2:

$$p_2 \;=\; \frac{(1-a)(v^2 - v_2^2)}{2v^2}\,.$$

If we substitute this into the first equation of (4.6), we may solve for p_1:

$$p_1 \;=\; -\frac{(1-a)(v^2 - v_2^2)v_2}{2v^2 v_1}\,.$$

Substitute these into the second equation of (4.6), clear the denominator $(4v_1^2 v^4)$, and remove the common factor $(1-a)$ to obtain the sextic

$$(1-a)^2(v_1^2 + v_2^2)(v^2 - v_2^2)^2 \;-\; 4r^2 v_1^2 v^4 \;+\; 4a v_1^2 v^2 (v^2 - v_2^2) \;=\; 0\,. \tag{4.7}$$

Subtracting the third equation of (4.6) from the fourth equation and recalling that $v^2 = v_1^2 + v_2^2 + (n-2)v_3^2$, we obtain the quadratic equation

$$(1-a)v_1^2 + v_2^2 + [(n-3) - a(n-2)]\, v_3^2 \;=\; 0\,. \tag{4.8}$$

Consider the system consisting of the two equations (4.7) and (4.8) in the homogeneous coordinates v_1, v_2, v_3. Any solution to this system gives a solution to the system (4.6), and thus gives 2^{n-3} solutions to the original system (4.1)–(4.4).

These last two equations (4.7) and (4.8) are polynomials in the squares of the variables v_1^2, v_2^2, v_3^2. If we substitute $\alpha = v_1^2, \beta = v_2^2$, and $\gamma = v_3^2$, then we have a cubic and a linear equation, and any solution α, β, γ to these with nonvanishing coordinates gives 4 solutions to the system (4.7) and (4.8): $(v_1, v_2, v_3)^{\mathrm{T}} := (\alpha^{1/2}, \pm\beta^{1/2}, \pm\gamma^{1/2})^{\mathrm{T}}$, since v_1, v_2, v_3 are homogeneous coordinates.

Solving the linear equation in α, β, γ for β and substituting into the cubic equation gives a homogeneous cubic in α and γ whose coefficients are polynomials in a, n, r.[1] The discriminant of this cubic is a polynomial with integral coefficients of degree 16 in the variables a, n, r having 116 terms. Using a computer algebra system, it can be verified that this discriminant is irreducible over the rational numbers. Thus, for any fixed integer $n \geq 3$, the discriminant is a nonzero polynomial in a, r. This implies that the cubic has 3 solutions for general a, r and any integer n. Since the coefficients of this cubic similarly are nonzero polynomials for any n, the solutions α, β, γ will be nonzero for general a, r and any n. We conclude:

> For any integer $n \geq 3$ and general a, r, there will be $3 \cdot 2^{n-1}$ complex common tangents to spheres of radius r with centers
>
> $$ae_2, \;\; -e_2, \quad \text{and} \quad \pm e_j, \quad \text{for } 3 \leq j \leq n\,.$$

We return to the case when $a = 1$, i.e., the centers are the vertices of the crosspolytope $\pm e_j$ for $j = 2, \ldots, n$. Then our equations (4.5) and (4.6) become

$$\begin{aligned} p_2 \;=\; p_3 \;=\; \cdots \;=\; p_n \;&=\; 0, \\ v_2^2 \;=\; v_3^2 \;=\; \cdots \;=\; v_n^2 \;&, \\ p_1 v_1 \;&=\; 0, \\ v^2(p_1^2 - r^2 + 1) - v_2^2 \;&=\; 0. \end{aligned} \tag{4.9}$$

[1] Maple V.5 code verifying this and other explicit calculations presented in this manuscript is available at `www.math.umass.edu/~sottile/pages/spheres`.

As before, $v^2 = v_1^2 + (n-1)v_2^2$. We solve the last two equations. Any solution they have (in $\mathbb{C}^1 \times \mathbb{P}^1_{\mathbb{C}}$) gives rise to 2^{n-2} solutions, by the second list of equations $v_3^2 = \cdots = v_n^2$. By the penultimate equation $p_1 v_1 = 0$, one of p_1 or v_1 vanishes. If $v_1 = 0$, then the last equation becomes

$$(n-1)v_2^2(p_1^2 - r^2 + 1) \;=\; v_2^2\,.$$

Since $v_2 = 0$ implies $v^2 = 0$, we have $v_2 \neq 0$ and so we may divide by v_2^2 and solve for p_1 to obtain

$$p_1 \;=\; \pm\sqrt{r^2 - 1 + \frac{1}{n-1}}\,.$$

If instead $p_1 = 0$, then we solve the last equation to obtain

$$\frac{v_1}{v_2} \;=\; \pm\sqrt{\frac{1}{1-r^2} + 1 - n}\,.$$

Thus for general r, there will be 2^n common tangents to the spheres with radius r and centers $\pm e_j$ for $j = 2, \ldots, n$. We investigate when these are real.

We will have p_1 real when $r^2 \;>\; 1 - 1/(n-1)$. Similarly, v_1/v_2 will be real when $1/(1-r^2) \;>\; n-1$. In particular, $1 - r^2 > 0$ and so $1 > r^2$. Using this we get

$$1 - r^2 \;<\; \frac{1}{n-1} \qquad \text{so that} \qquad r^2 \;>\; 1 - \frac{1}{n-1}\,,$$

which we previously obtained.

We conclude that there will be 2^n real common tangents to the spheres with centers $\pm e_j$ for $j = 2, \ldots, n$ and radius r when

$$\sqrt{1 - \frac{1}{n-1}} \;<\; r \;<\; 1\,.$$

This concludes the proof of Theorem 7.

5. Lines Tangent to Quadrics

Suppose that in our original question we ask for common tangents to ellipsoids, or to more general quadric hypersurfaces. Since all smooth quadric hypersurfaces are projectively equivalent, a natural setting for this question is the following:

"How many common tangents are there to $2n-2$ general quadric hypersurfaces in (complex) projective space $\mathbb{P}^n_{\mathbb{C}}$?"

Theorem 10. *There are at most*

$$2^{2n-2} \cdot \frac{1}{n}\binom{2n-2}{n-1}$$

isolated common tangent lines to $2n-2$ quadric hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$.

Proof. The space of lines in $\mathbb{P}^n_{\mathbb{C}}$ is the Grassmannian of 2-planes in $\mathbb{C}^{n+1}$. The Plücker embedding [8] realizes this as a projective subvariety of $\mathbb{P}_{\mathbb{C}}^{\binom{n+1}{2}-1}$ of degree

$$\frac{1}{n}\binom{2n-2}{n-1}\,.$$

The theorem follows from the refined Bézout theorem [6, §12.3] and from the fact that the condition for a line to be tangent to a quadric hypersurface is a homogeneous quadratic equation in the Plücker coordinates for lines [16, §5.4]. □

In Table 2, we compare the upper bound of Theorem 10 for the number of lines tangent to $2n-2$ quadrics to the number of lines tangent to $2n-2$ spheres of Theorem 2, for small values of n.

TABLE 2. Maximum number of tangents in small dimensions

n	3	4	5	6	7
# for spheres	12	24	48	96	192
# for quadrics	32	320	3580	43008	540672

The bound of 32 tangent lines to 4 quadrics in $\mathbb{P}^3_{\mathbb{C}}$ is sharp, even under the restriction to real quadrics and real tangents [18]. In a computer calculation, we found 320 lines in $\mathbb{P}^4_{\mathbb{C}}$ tangent to 6 general quadrics; thus, the upper bound of Theorem 10 is sharp also for $n=4$, and indicating that it is likely sharp for $n>4$. The question arises: what is the source of the huge discrepancy between the second and third rows of Table 2?

Consider a sphere in affine n-space

$$(x_1-c_1)^2+(x_2-c_2)^2+\cdots+(x_n-c_n)^2 \;=\; r^2\,.$$

Homogenizing this with respect to the new variable x_0, we obtain

$$(x_1-c_1x_0)^2+(x_2-c_2x_0)^2+\cdots+(x_n-c_nx_0)^2 \;=\; r^2x_0^2\,.$$

If we restrict this sphere to the hyperplane at infinity, setting $x_0=0$, we obtain

$$x_1^2+x_2^2+\cdots+x_n^2 \;=\; 0\,, \tag{5.1}$$

the equation for an imaginary quadric at infinity. We invite the reader to check that every line at infinity tangent to this quadric is tangent to the original sphere.

Thus the equations for lines in $\mathbb{P}^n_{\mathbb{C}}$ tangent to $2n-2$ spheres define the $3\cdot 2^{n-1}$ lines we computed in Theorem 2, as well as this excess component of lines at infinity tangent to the imaginary quadric (5.1). Thus, this excess component contributes some portion of the Bézout number of Theorem 10 to the total number of lines. Indeed, when $n=3$, Aluffi and Fulton [2] have given a careful argument that this excess component contributes 20, which implies that there are $32-20=12$ isolated common tangent lines to 4 spheres in 3-space, recovering the result of [9].

The geometry of that calculation is quite interesting. Given a system of equations on a space (say the Grassmannian) whose set of zeroes has a positive-dimensional excess component, one method to compute the number of isolated solutions is to first modify the underlying space by blowing up the excess component and then compute the number of solutions on this new space. In many cases, the equations on this new space have only isolated solutions. However, for this problem of lines tangent to spheres, the equations on the blown up space will *still* have an excess intersection and a further blow-up is required. This problem of lines tangent to 4 spheres in projective 3-space is by far the simplest enumerative geometric problem with an excess component of zeroes which requires two blow-ups (technically speaking, blow-ups along smooth centers) to resolve the excess zeroes.

It would be interesting to understand the geometry also when $n>3$. For example, how many blow-ups are needed to resolve the excess component?

Since all smooth quadrics are projectively equivalent, Theorem 2 has the following implication for this problem of common tangents to projective quadrics.

Theorem 11. *Given $2n-2$ quadrics in $\mathbb{P}^n_{\mathbb{C}}$ whose intersection with a fixed hyperplane is a given smooth quadric Q, but are otherwise general, there will be at most $3 \cdot 2^{n-1}$ isolated lines in $\mathbb{P}^n_{\mathbb{C}}$ tangent to each quadric.*

When the quadrics are all real, we ask: how many of these $3 \cdot 2^{n-1}$ common isolated tangents can be real? This question is only partially answered by Theorem 2. The point is that projective real quadrics are classified up to real projective transformations by the absolute value of the signature of the quadratic forms on $\mathbb{R}^{n+1}$ defining them. Theorem 2 implies that all lines can be real when the shared quadric Q has no real points (signature is $\pm n$). In [16], it is shown that when $n = 3$, each of the five additional cases concerning nonempty quadrics can have all 12 lines real.

Recently, Megyesi [11] has largely answered this question. Specifically, he showed that, for any nonzero real numbers $\lambda_3, \ldots, \lambda_n$, there are $2n-2$ quadrics of the form

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 + \sum_{j=3}^{n} \lambda_j (x_j - c_j)^2 \;=\; R$$

having all $3 \cdot 2^{n-1}$ tangents real. These all share the same quadric at infinity

$$x_1^2 + x_2^2 + \lambda_3 x_3^2 + \cdots + \lambda_n x_n^2 \;=\; 0\,,$$

and thus the upper bound of Theorem 11 is attained, when the shared quadric is this quadric.

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