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# SPIN STRUCTURES AND CODIMENSION TWO EMBEDDINGS OF 3-MANIFOLDS UP TO REGULAR HOMOTOPY

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Abstract. We clarify the structure of the set of regular homotopy classes containing embeddings of a 3-manifold into 5-space inside the set of all regular homotopy classes of immersions with trivial normal bundles. As a consequence, we show that for a large class of 3-manifolds $M^3$, the following phenomenon occurs: there exists a codimension two immersion of the 3-sphere whose double points cannot be eliminated by regular homotopy, but can be eliminated after taking the connected sum with a codimension two embedding of $M^3$. This involves introducing and studying an equivalence relation on the set of spin structures on $M^3$. Their associated $\mu$-invariants also play an important role.

## 1. Introduction

Let $f\colon M^3 \looparrowright \mathbf{R}^4$ be an immersion of an oriented 3-manifold $M^3$ into oriented $\mathbf{R}^4$. Then, via the bundle isomorphism $TM^3 \oplus \varepsilon^1 \cong f^*T\mathbf{R}^4$ ($\varepsilon^1$ being the trivial line bundle), the unique spin structure on $\mathbf{R}^4$ induces a spin structure on $M^3$ that is clearly a regular homotopy invariant of $f$.

Now let $F\colon M^3 \looparrowright \mathbf{R}^5$ be an immersion with trivial normal bundle. By taking a normal framing for $F$, we obtain a spin structure on $M^3$ that usually depends on the choice of the normal framing. Then, as has been seen in [9, Section 3], we are naturally led to an equivalence relation on the set of spin structures on $M^3$, regarded as an affine space over $H^1(M^3;\mathbf{Z}_2)$. We say that two spin structures are *equivalent modulo* $\operatorname{Im}\rho$ if their difference lies in the image of the modulo two reduction map $\rho\colon H^1(M^3;\mathbf{Z}) \to H^1(M^3;\mathbf{Z}_2)$. The spin structure modulo $\operatorname{Im}\rho$ associated to $F$ is independent of the choice of the normal framing and is a regular homotopy invariant of $F$.

In [9], a geometric characterisation of regular homotopy classes of immersions of $M^3$ into $\mathbf{R}^5$ has also been given. As a consequence, we have encountered a rather interesting situation for embeddings of the 3-torus $T^3$ up to regular homotopy: *there exists an immersion $g\colon S^3 \looparrowright \mathbf{R}^5$ such that* (1) *$g$ is not regularly homotopic to an embedding $S^3 \hookrightarrow \mathbf{R}^5$, but* (2) *for any embedding $E\colon T^3 \hookrightarrow \mathbf{R}^5$, the connected sum $E\sharp g\colon T^3 \looparrowright \mathbf{R}^5$ is again regularly homotopic to an embedding $T^3 \hookrightarrow \mathbf{R}^5$.* In other words, the double points of the immersion $g$ cannot be eliminated by regular homotopy, but one can eliminate them after taking the connected sum with an embedding of $T^3$. We call such an immersion $g\colon S^3 \looparrowright \mathbf{R}^5$ a *$T^3$-pseudo-embedding* of $S^3$ (see Definition 3.1). This phenomenon is closely related to the diversity of

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spin structures of $T^3$, and, in the proof given in [9], spin structures modulo $\operatorname{Im}\rho$ associated to embeddings of $T^3$ play an important role.

In this paper, we prove the existence of $M^3$-pseudo-embeddings $S^3 \looparrowright \mathbf{R}^5$ for a large class of 3-manifolds $M^3$ (see Proposition 3.2 and Theorem 3.7). This involves a study of spin structures modulo $\operatorname{Im}\rho$ on $M^3$, and of their associated $\mu$-invariants (see Section 3). In Section 4, we study how the regular homotopy classes of embeddings are distributed in the set of all regular homotopy classes of immersions, and give a result which implies that the cases of $S^3$ and $T^3$ are typical (Theorem 4.2). We also give an example of a 3-manifold for which the classes of embeddings are distributed in a complicated manner (Proposition 4.4). In Section 5, we generalise the above study, replacing $S^3$ by an arbitrary 3-manifold $N^3$. Finally, in the last section, we introduce the notion of virtual homotopy: we say that two immersions are *virtually homotopic* if they are regularly homotopic after taking connected sums with embeddings. Virtual homotopy classes of immersions of closed connected oriented 3-manifolds into $\mathbf{R}^5$ with trivial normal bundles form a group under connected sum. We show that this group is isomorphic to $\mathbf{Z}/12\mathbf{Z}$.

Throughout the paper, manifolds are of class $C^\infty$ and $M^3$ is a closed oriented 3-manifold. We write "$A \approx B$" if there is a bijective correspondence between the sets $A$ and $B$; we use the symbol "$\cong$" for a group isomorphism. The symbol "$\sim_r$" means "regularly homotopic". We often do not distinguish between an immersion $f$ and its regular homotopy class, which we also denote by $f$.

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## 2. Preliminaries

In this section, we recall some known results. We review them only in our case — that of immersions of 3-manifolds into $\mathbf{R}^5$ — although some of them were originally stated in more general contexts.

2.1. **Immersions of the 3-sphere.** Let $\operatorname{Imm}[X, Y]$ denote the set of regular homotopy classes of immersions of a manifold $X$ into a manifold $Y$. Denote by $\operatorname{Emb}[X, Y]$ the subset of regular homotopy classes containing an embedding. Note that the set $\operatorname{Imm}[S^3, \mathbf{R}^5]$ has a group structure given by the connected sum, and that the Smale invariant

$$\Omega\colon\ \operatorname{Imm}[S^3, \mathbf{R}^5] \to \mathbf{Z}$$

gives a group isomorphism (see [6, 3]). Hughes and Melvin [4] have shown that $\operatorname{Emb}[S^3, \mathbf{R}^5]$ forms the subgroup isomorphic to $24\mathbf{Z}$. Furthermore, for an embedding $f\colon\ S^3 \hookrightarrow \mathbf{R}^5$ we have $\Omega(f) = 3\sigma(W_f^4)/2 \in 24\mathbf{Z}$, where $W_f^4$ is a Seifert surface for $f$ and $\sigma(W_f^4)$ is its signature.

This last result has been extended by Ekholm and Szűcs [1] as follows. For an immersion $f\colon\ S^3 \looparrowright \mathbf{R}^5$, consider a *singular Seifert surface* for $f$, that is, a generic map $\widetilde{f}\colon\ W_f^4 \to \mathbf{R}^5$ that is bounded by $f$ and has no singularity near the boundary $S^3 = \partial W_f^4 \subset W_f^4$. If we denote by $\#\Sigma^{1,1}(\widetilde{f})$ the algebraic number of cusp points of $\widetilde{f}$, then

$$\Omega(f) = \tfrac{1}{2}(3\sigma(W_f^4) + \#\Sigma^{1,1}(\widetilde{f})).$$

Note that for an embedding we can consider a usual nonsingular Seifert surface and obtain the result of Hughes and Melvin as a corollary.

2.2. **Immersions of 3-manifolds with trivial normal bundles.** Let $M^3$ be a closed oriented 3-manifold. Denote by $\mathrm{Imm}[M^3, \mathbf{R}^5]_0$ the subset of $\mathrm{Imm}[M^3, \mathbf{R}^5]$ consisting of the regular homotopy classes of immersions with trivial normal bundles. Note that $\mathrm{Emb}[M^3, \mathbf{R}^5] \subset \mathrm{Imm}[M^3, \mathbf{R}^5]_0$. In this subsection, we recall the results in [9], which give a geometric description of the set $\mathrm{Imm}[M^3, \mathbf{R}^5]_0$.

First, we introduce an equivalence relation on the set of spin structures on an oriented manifold.

**Definition 2.1** (Spin structures modulo $\mathrm{Im}\,\rho$)**.** Denote by $\mathrm{Spin}(M^n)$ the set of all spin structures on an oriented $n$-manifold $M^n$. Recall that $\mathrm{Spin}(M^n)$ is an affine space over $H^1(M^n; \mathbf{Z}_2)$. Let

$$\rho\colon\ H^1(M^n; \mathbf{Z}) \to H^1(M^n; \mathbf{Z}_2)$$

be the modulo two reduction.

Two spin structures $\omega$ and $\omega' \in \mathrm{Spin}(M^n)$ are said to be *equivalent modulo* $\mathrm{Im}\,\rho$ if their difference lies in $\mathrm{Im}\,\rho \subset H^1(M^n; \mathbf{Z}_2)$. We denote the equivalence class containing $\omega$ by $[\omega]$ and call it *a spin structure modulo* $\mathrm{Im}\,\rho$. Let

$$\begin{array}{rccc} [\ ]\colon & \mathrm{Spin}(M^n) & \to & \mathrm{Spin}(M^n)/\,\mathrm{Im}\,\rho \\ & \omega & \mapsto & [\omega] \end{array}$$

be the natural projection. Note that $\mathrm{Spin}(M^n)/\,\mathrm{Im}\,\rho \approx H^1(M^n; \mathbf{Z}_2)/\,\mathrm{Im}\,\rho$ (non-canonically).

As has been shown in [9], an element in the set $\mathrm{Imm}[M^3, \mathbf{R}^5]_0$ is characterised by two invariants $c$ and $i$ defined as follows.

Let $F\colon\ M^3 \looparrowright \mathbf{R}^5$ be an immersion with trivial normal bundle. If we take a normal framing for $F$, then the unique spin structure on $T\mathbf{R}^5$ induces a spin structure $\omega_F$ of $M^3$. If we take another normal framing, then we get another spin structure $\omega'_F$ of $M^3$. We have shown in [9, Section 3] that the difference of two such spin structures always lies in $\mathrm{Im}\,\rho \subset H^1(M^3; \mathbf{Z}_2)$. Thus to $F$ we can associate $[\omega_F]$ $(= [\omega'_F])$ — the spin structure modulo $\mathrm{Im}\,\rho$ — which is a regular homotopy invariant of $F$. Let

$$c\colon\ \mathrm{Imm}[M^3, \mathbf{R}^5]_0 \to \mathrm{Spin}(M^3)/\,\mathrm{Im}\,\rho$$

be this correspondence.

The second invariant $i\colon\ \mathrm{Imm}[M^3, \mathbf{R}^5]_0 \to \mathbf{Z}$ is an analogue of a geometric formula for the Smale invariant given in [1]. Denote by $\tau H^1(M^3; \mathbf{Z})$ the torsion subgroup of $H^1(M^3; \mathbf{Z})$, and let $\alpha = \alpha(M^3)$ be the dimension of $\tau H^1(M^3; \mathbf{Z}) \otimes \mathbf{Z}_2$ over $\mathbf{Z}_2$. Note that the number of elements in the set $\mathrm{Spin}(M^3)/\,\mathrm{Im}\,\rho$ is equal to $2^\alpha$. Then, for an immersion $F\colon\ M^3 \looparrowright \mathbf{R}^5$ with trivial normal bundle, we define

$$i(F) := \tfrac{1}{2}(3(\sigma(W_F^4) - \alpha(M^3)) + \#\Sigma^{1,1}(\widetilde{F})),$$

where $\widetilde{F}\colon\ W_F^4 \to \mathbf{R}^5$ is a singular Seifert surface for $F$ and $\#\Sigma^{1,1}(\widetilde{F})$ is the algebraic number of cusp points of $\widetilde{F}$. Note that $i(F)$ is always an integer.

The following results were obtained in [9]. Put $\mathrm{Imm}[M^3, \mathbf{R}^5]_0^C := c^{-1}(C)$ for $C \in \mathrm{Spin}(M^3)/\,\mathrm{Im}\,\rho$.

(a) For any $[\omega] \in \mathrm{Spin}(M^3)/\,\mathrm{Im}\,\rho$, $\mathrm{Imm}[M^3, \mathbf{R}^5]_0^{[\omega]}$ contains a class represented by an embedding.

(b) The map $(c, i)\colon \operatorname{Imm}[M^3, \mathbf{R}^5]_0 \to (\operatorname{Spin}(M^3)/\operatorname{Im}\rho) \times \mathbf{Z}$ gives a bijection. As a corollary, two embeddings in $\operatorname{Imm}[M^3, \mathbf{R}^5]_0^{[\omega]} \approx \mathbf{Z}$ ($[\omega] \in \operatorname{Spin}(M^3)/\operatorname{Im}\rho$) are regularly homotopic if and only if they have Seifert surfaces with the same signature.

(c) By taking connected sums of immersions, we can define an action of the group $\operatorname{Imm}[S^3, \mathbf{R}^5]$ on the set $\operatorname{Imm}[M^3, \mathbf{R}^5]_0$. This action is effective, and each orbit coincides with $\operatorname{Imm}[M^3, \mathbf{R}^5]_0^{[\omega]}$ for some $[\omega] \in \operatorname{Spin}(M^3)/\operatorname{Im}\rho$.

*Remark* 2.2. The above items (a) and (b) imply that each "$\mathbf{Z}$-component" of

$$\operatorname{Imm}[M^3, \mathbf{R}^5]_0 \approx (\operatorname{Spin}(M^3)/\operatorname{Im}\rho) \times \mathbf{Z} = \mathbf{Z} \amalg \cdots \amalg \mathbf{Z} \tag{2.1}$$

contains a class represented by an embedding. Furthermore, (c) implies that if we take an embedding $F$ in $\operatorname{Imm}[M^3, \mathbf{R}^5]_0^{[\omega]} \approx \mathbf{Z}$, then we can define the bijection

$$\begin{array}{rccc} \sharp_F\colon & \operatorname{Imm}[S^3, \mathbf{R}^5] & \to & \operatorname{Imm}[M^3, \mathbf{R}^5]_0^{c(F)}, \\ & g & \mapsto & F\sharp g. \end{array}$$

From now on, we always consider each $\mathbf{Z}$-component of (2.1) to be endowed with such a group structure, induced from that of $\operatorname{Imm}[S^3, \mathbf{R}^5]$. In other words, whenever we mention a $\mathbf{Z}$-component of (2.1), we fix in it an embedding $F$, which determines the group structure via the bijection $\sharp_F$ above.

*Remark* 2.3. According to Wu's computation [11] of $\operatorname{Imm}[M^3, \mathbf{R}^5]$, the set $\operatorname{Imm}[M^3, \mathbf{R}^5]_0$ corresponds bijectively to $\Gamma_2(M^3) \times \mathbf{Z}$, where $\Gamma_2(M^3)$ denotes the set of order two elements of $H^2(M^3; \mathbf{Z})$ together with the zero element. The Bockstein homomorphism $H^1(M^3; \mathbf{Z}_2) \to H^2(M^3; \mathbf{Z})$ induces a bijection between $H^1(M^3; \mathbf{Z}_2)/\operatorname{Im}\rho$ and $\Gamma_2(M^3)$ (see [9, Remark 3.6]).

## 3. $M^3$-pseudo-embeddings of the 3-sphere

As mentioned in the introduction, we have observed in [9, Section 6] a surprising situation for embeddings of $T^3$ up to regular homotopy. We first give a definition to describe this phenomenon.

**Definition 3.1.** Let $M^n$ and $N^n$ be closed oriented $n$-manifolds and $g\colon N^n \looparrowright \mathbf{R}^{n+k}$ be an immersion. We say that $g$ is an *$M^n$-pseudo-embedding* if (1) $g$ is not regularly homotopic to any embedding, and (2) for some embedding $F\colon M^n \hookrightarrow \mathbf{R}^{n+k}$, the connected sum $F\sharp g\colon M^n \sharp N^n \looparrowright \mathbf{R}^{n+k}$ is regularly homotopic to an embedding.

In this paper, we consider only the case of codimension two immersions of 3-manifolds, i.e., $n = 3$ and $k = 2$.

Under this notion of "$M^n$-pseudo-embedding", the curious phenomenon mentioned in the introduction (see [9, Section 6]) can be described as follows: there exists a $T^3$-pseudo-embedding $h\colon S^3 \looparrowright \mathbf{R}^5$. Here we show that for a large class of 3-manifolds $M^3$, there exists an $M^3$-pseudo-embedding $S^3 \looparrowright \mathbf{R}^5$.

The following is a necessary and sufficient condition on spin structures of $M^3$ for the existence of an $M^3$-pseudo-embedding $S^3 \looparrowright \mathbf{R}^5$. Let $M^3$ be a closed oriented 3-manifold. If $\omega$ is a spin structure on $M^3$, then we denote by $\mu(M^3, \omega)$ the *$\mu$-invariant* (or *Rohlin invariant*) of the spin manifold $(M^3, \omega)$, i.e., the signature modulo 16 of a spin 4-manifold that is spin bounded by $(M^3, \omega)$ (for example, see [7]).

**Proposition 3.2.** *There exists an $M^3$-pseudo-embedding $S^3 \looparrowright \mathbf{R}^5$ if and only if $M^3$ admits two spin structures $\omega$ and $\omega'$ such that*

(1) $[\omega] = [\omega']$, *i.e.,* $\omega - \omega' \in \operatorname{Im}\rho \subset H^1(M^3; \mathbf{Z}_2)$, *and*
(2) $\mu(M^3, \omega) \neq \mu(M^3, \omega')$.

*Proof.* Suppose that $M^3$ has two spin structures $\omega$ and $\omega'$ satisfying the two conditions (1) and (2). There exist compact spin 4-manifolds $W^4$ and $W'^4$ that are spin bounded by $(M^3, \omega)$ and $(M^3, \omega')$, respectively, such that each of them has a handlebody decomposition with one 0-handle and some 2-handles with even framings (see [5]). The condition $\mu(M^3, \omega) \neq \mu(M^3, \omega')$ implies that $\sigma(W^4) \not\equiv \sigma(W'^4) \pmod{16}$. Furthermore, by the same argument as in [9, Section 6], we can take embeddings $\widetilde{F}\colon W^4 \hookrightarrow \mathbf{R}^5$ and $\widetilde{G}\colon W'^4 \hookrightarrow \mathbf{R}^5$, and then their restrictions $F := \widetilde{F}|_{M^3}$ and $G := \widetilde{G}|_{M^3}$ to the boundaries have the same associated spin structure modulo $\operatorname{Im}\rho$, that is, $c(F) = [\omega] = [\omega'] = c(G)$.

Then, from the fact that

$$\sharp_F\colon\ \operatorname{Imm}[S^3, \mathbf{R}^5] \longrightarrow \operatorname{Imm}[M^3, \mathbf{R}^5]_0^{[\omega]}$$

is a bijection (see Remark 2.2), there exists an immersion $g\colon S^3 \looparrowright \mathbf{R}^5$ with $F\sharp g \sim_r G$.

Let us assume that $g$ is regularly homotopic to an embedding. Then, by the result of Hughes and Melvin [4], $g$ extends to an immersion of a compact oriented 4-manifold $V^4$ with signature $16k$ ($k \in \mathbf{Z}$). Hence,

$$i(F\sharp g) = \tfrac{3}{2}(\sigma(W^4 \natural V^4) - \alpha(M^3)) = \tfrac{3}{2}(\sigma(W^4) - \alpha(M^3)) + 24k,$$

where $W^4 \natural V^4$ stands for the boundary connected sum of $W^4$ and $V^4$. This contradicts the fact that $F\sharp g \sim_r G$, since

$$i(G) = \tfrac{3}{2}(\sigma(W'^4) - \alpha(M^3))$$

and $\sigma(W^4) \not\equiv \sigma(W'^4) \pmod{16}$.

Conversely, if there is an $M^3$-pseudo-embedding $g\colon S^3 \looparrowright \mathbf{R}^5$, then we have two embeddings $F$ and $G\colon M^3 \hookrightarrow \mathbf{R}^5$ such that $F\sharp g \sim_r G$. Denote by $\omega_F$ and $\omega_G$ the spin structures determined by $F$ and $G$ respectively. Then, $[\omega_F] = c(F) = c(G) = [\omega_G]$, since $F$ and $G$ restricted to $M^3 \smallsetminus \operatorname{Int} D^3$ are regularly homotopic (see [9, Section 3]), where $\operatorname{Int} D^3$ denotes the interior of a 3-disc embedded in $M^3$. We also see from (b) in Subsection 2.2 that $\mu(M^3, \omega_F) \neq \mu(M^3, \omega_G)$, since $i(g) = i(G) - i(F)$ and $g$ is not regularly homotopic to an embedding.

This completes the proof. □

**Example 3.3.** Let $T^3$ be the 3-torus. Since all the classes in $H^1(M^3; \mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ are modulo two reductions of integral classes, we see that $\operatorname{Spin}(M^3)/\operatorname{Im}\rho = 0$ and $\operatorname{Imm}[T^3, \mathbf{R}^5]_0 \approx \mathbf{Z}$.

The 3-torus $T^3$ has eight distinct spin structures, which all belong to the unique equivalence class in $\operatorname{Spin}(M^3)/\operatorname{Im}\rho$. It is well known that seven of them are spin bounded by a spin 4-manifold with signature 0 modulo 16 and the remaining one is spin bounded by a spin 4-manifold with signature 8 modulo 16.

Thus, $T^3$ satisfies the conditions in Proposition 3.2. Actually, $\operatorname{Emb}[T^3, \mathbf{R}^5]$ forms the subgroup isomorphic to $12\mathbf{Z}$ of $\operatorname{Imm}[T^3, \mathbf{R}^5]_0 \approx \mathbf{Z}$. Then, an immersion $g\colon S^3 \looparrowright \mathbf{R}^5$ with Smale invariant $12 + 24k$ ($k \in \mathbf{Z}$) is not regularly homotopic to any embedding, but the connected sum of $g$ and an arbitrary embedding $T^3 \hookrightarrow \mathbf{R}^5$ is regularly homotopic to an embedding (for details, see [9, Section 6]).

The following theorem, due to Kaplan [5], together with Lemma 3.5 below, provides some 3-manifolds with two spin structures that satisfy the two conditions in Proposition 3.2.

**Theorem 3.4** (Kaplan [5, Theorem 6.12])**.** *Suppose that $M^3$ is a closed connected oriented 3-manifold that bounds a spin 4-manifold of signature $k$. Suppose further that there exist elements $x_i \in H^1(M^3;\mathbf{Z})$, $i = 1, 2, 3$, such that $x_1 \smile x_2 \smile x_3$ is an odd multiple of the generator of $H^3(M^3;\mathbf{Z})$. Then $M^3$ bounds a spin 4-manifold of signature $k + 8$.* □

Actually, in the proof given in [5] of the above theorem, two spin structures of $M^3$ for which the $\mu$-invariants differ by 8 are specified. We have:

**Lemma 3.5.** *These two spin structures satisfy the condition* (1) *in Proposition* 3.2.

Before proving this lemma, we need a definition.

**Definition 3.6.** Let $L = \{K_1, K_2, \dots, K_n\}$ be a framed link in $S^3$. A sublink $L' \subset L$ is said to be *characteristic*, if $lk(L', K_i) \equiv lk(K_i, K_i) \pmod 2$ for $1 \le i \le n$.

*Proof of Lemma 3.5.* Consider $M^3 = M^3_L$ as the boundary of the 4-manifold $W^4_L$ for a framed link $L = \{K_1, K_2, \dots, K_n\}$ with even framings, that is, the boundary of the spin 4-manifold obtained by attaching a 2-handle to $D^4$ along each component of $L \subset S^3 = \partial D^4$ with the given framing.

Then, characteristic sublinks $L'$ of $L$ bijectively correspond to spin structures $\omega_{L'}$ of $M^3$. Note that each component $K_i$ of $L$ corresponds to an element $[K_i] \in H_2(W^4_L;\mathbf{Z})$, which is represented by the union of the core of the handle over $K_i$ and the cone on $K_i$ in $D^4$. Note also that the classes $[K_i]$ form a basis of $H_2(W^4_L;\mathbf{Z})$ that is free abelian. Furthermore, each characteristic sublink $L'$ represents the Poincaré dual of the relative obstruction to extending the spin structure $\omega_{L'}$ over $W^4_L$, $w_2(W^4_L, \omega_{L'}) \in H^2(W^4_L, M^3;\mathbf{Z}_2) \cong H_2(W^4_L;\mathbf{Z}_2)$, which maps to the second Stiefel-Whitney class $w_2(W^4_L)$ of $W^4_L$ under the map

$$j^*\colon\ H^2(W^4_L, M^3;\mathbf{Z}_2) \to H^2(W^4_L;\mathbf{Z}_2),$$

where $j : W^4_L \to (W^4_L, M^3)$ is the inclusion (see the diagram below). Obviously, the empty (characteristic) sublink corresponds to the spin structure that extends over $W^4_L$.

Therefore, in order to obtain Theorem 3.4, we need to find a characteristic sublink corresponding to a spin structure that provides the $\mu$-invariant $\sigma(W^4_L) + 8 \pmod{16}$. In fact, in the proof of Theorem 3.4, Kaplan [5] has found in a suitable $L$ such a characteristic sublink $\ell$ of one component with the property that $[\ell] \cdot x = 0$ for all $x \in H_2(W^4_L;\mathbf{Z})$. This property of $[\ell]$ implies that $[\ell]$ lies in the kernel of the map $j_*\colon\ H_2(W^4_L;\mathbf{Z}) \to H_2(W^4_L, M^3;\mathbf{Z})$ (see the diagram below), and hence it is the image of a class $(\ell)$ in $H_2(M^3;\mathbf{Z})$ under the homomorphism $i_*\colon\ H_2(M^3;\mathbf{Z}) \to H_2(W^4_L;\mathbf{Z})$ induced by the inclusion $i$.

Now let us consider the following diagram, in which the horizontal arrows come from the exact sequences for the pair $(W^4_L, M^3)$, each vertical arrow upstairs comes from Poincaré-Lefschetz duality, and each vertical arrow downstairs is the modulo

two reduction:

$$\begin{array}{ccccc}
(\ell) & \longmapsto & [\ell] & & \\
H_2(M^3;\mathbf{Z}) & \xrightarrow[i_*]{} & H_2(W_L^4;\mathbf{Z}) & \xrightarrow[j_*]{} & H_2(W_L^4,M^3;\mathbf{Z}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^1(M^3;\mathbf{Z}) & \longrightarrow & H^2(W_L^4,M^3;\mathbf{Z}) & \longrightarrow & H^2(W_L^4;\mathbf{Z}) \\
\downarrow\rho & & \downarrow & & \downarrow \\
H^1(M^3;\mathbf{Z}_2) & \longrightarrow & H^2(W_L^4,M^3;\mathbf{Z}_2) & \xrightarrow{j^*} & H^2(W_L^4;\mathbf{Z}_2) \\
\omega_\ell & \longmapsto & w_2(W_L^4,\omega_\ell) & \longmapsto & w_2(W_L^4)=0.
\end{array}$$

Then we see that the difference between the two spin structures corresponding to the characteristic sublink $\ell$ and to the empty sublink corresponds to $\rho((\ell)^*)$, where $(\ell)^* \in H^1(M^3;\mathbf{Z})$ is the Poincaré dual of $(\ell)$. This completes the proof. □

Thus Theorem 3.4, together with Lemma 3.5, implies that a large class of 3-manifolds exhibit the same phenomenon as the 3-torus. Namely, we have the following theorem.

**Theorem 3.7.** *If there exist elements $x_i \in H^1(M^3;\mathbf{Z})$, $i=1,2,3$, such that $x_1 \smile x_2 \smile x_3$ is an odd multiple of the generator of $H^3(M^3;\mathbf{Z})$, then there exists an $M^3$-pseudo-embedding $S^3 \looparrowright \mathbf{R}^5$.* □

We see easily that $S^1 \times F^2$ is such a 3-manifold satisfying the conditions of the theorem above, where $F^2$ is a closed connected orientable surface of positive genus.

## 4. Regular homotopy classes of embeddings

In all the cases that we have dealt with, $\mathrm{Emb}[M^3,\mathbf{R}^5]$ forms the subgroup $24\mathbf{Z}$ (for $S^3$ or $\mathbf{Z}_2$-homology 3-spheres [4, 10]) or $12\mathbf{Z}$ (for $T^3$ or $S^1 \times F^2$) of a $\mathbf{Z}$-component in $\mathrm{Imm}[M^3,\mathbf{R}^5]_0$. The following proposition implies that these are all the possibilities.

**Proposition 4.1.** *Let $\omega$ and $\omega'$ be spin structures of $M^3$ such that $[\omega]=[\omega']$, i.e., $\omega-\omega' \in \mathrm{Im}\,\rho \subset H^1(M^3;\mathbf{Z}_2)$. Then,*

$$\mu(M^3,\omega) \equiv \mu(M^3,\omega') \pmod 8.$$

*Proof.* Let $W^4$ and $W'^4$ be compact spin 4-manifolds spin bounded by $(M^3,\omega)$ and $(M^3,\omega')$, respectively, and put $V^4 := W^4 \cup -W'^4$. By the assumption, we can take an oriented surface $F \subset M^3$ with $[F] \in H_2(M^3;\mathbf{Z})$ being an integral dual to $\omega-\omega' \in H^1(M^3;\mathbf{Z}_2)$. Since $[F] \in H_2(V^4;\mathbf{Z})$ is also an integral dual to the second Stiefel-Whitney class $w_2(V^4) \in H^2(V^4;\mathbf{Z}_2)$, considering that $F$ lies in $M^3 \subset V^4$, we have

$$\sigma(V^4) \equiv [F]\cdot[F] \equiv 0 \pmod 8,$$

where " $\cdot$ " stands for the intersection number in $V^4$ (see [7, p. 25, Lemma 3.4], for example). Thus by Novikov additivity, we have

$$\mu(M^3,\omega)-\mu(M^3,\omega') \equiv \sigma(W^4)-\sigma(W'^4) \equiv \sigma(V^4) \equiv 0 \pmod 8.$$

This completes the proof. □

**Theorem 4.2.** *In each $\mathbf{Z}$-component of $\mathrm{Imm}[M^3,\mathbf{R}^5]_0 \approx \mathbf{Z}\amalg\cdots\amalg\mathbf{Z}$, $\mathrm{Emb}[M^3,\mathbf{R}^5]$ is a subgroup isomorphic either to $24\mathbf{Z}$ or to $12\mathbf{Z}$.*

*Proof.* In view of the result of Hughes and Melvin and (a), (c) in Subsection 2.2, we see that $\mathrm{Emb}[M^3, \mathbf{R}^5]$ contains the subgroup $24\mathbf{Z}$ in each $\mathbf{Z}$-component of $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \cdots \amalg \mathbf{Z}$.

Suppose that two embeddings $F$ and $G\colon M^3 \hookrightarrow \mathbf{R}^5$ belong to the same $\mathbf{Z}$-component of $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \cdots \amalg \mathbf{Z}$. Let $W_F^4$ and $W_G^4$ be Seifert surfaces for $F$ and $G$ respectively. Then, we can consider spin structures $\omega_F$ and $\omega_G$ for $M^3$ induced from the normal framings given by $M^3 \subset W_F^4$ and $M^3 \subset W_G^4$ respectively. Since $c(F) = c(G)$, we have $[\omega_F] = [\omega_G]$. Furthermore, it is clear that $\mu(M^3, \omega_F) \equiv \sigma(W_F^4) \pmod{16}$ and $\mu(M^3, \omega_G) \equiv \sigma(W_G^4) \pmod{16}$. Therefore, by Proposition 4.1, we have $\sigma(W_F^4) - \sigma(W_G^4) \in 8\mathbf{Z}$. Then, by the definitions of the invariant $i$ and the group structure on each $\mathbf{Z}$-component, we have the desired result. □

*Remark* 4.3. By the above proof, we see that all the spin structures in a class $C \in \mathrm{Spin}(M^3)/\operatorname{Im}\rho$ give the same $\mu$-invariant if and only if $\mathrm{Emb}[M^3, \mathbf{R}^5]$ forms the subgroup $24\mathbf{Z}$ in $\mathrm{Imm}[M^3, \mathbf{R}^5]_0^C \approx \mathbf{Z}$, and that a class $C \in \mathrm{Spin}(M^3)/\operatorname{Im}\rho$ contains two spin structures that give different $\mu$-invariants if and only if $\mathrm{Emb}[M^3, \mathbf{R}^5]$ forms the subgroup $12\mathbf{Z}$ in $\mathrm{Imm}[M^3, \mathbf{R}^5]_0^C \approx \mathbf{Z}$.

Thus, by Proposition 3.2, we see that there exists an $M^3$-pseudo-embedding $S^3 \looparrowright \mathbf{R}^5$ if and only if $\mathrm{Emb}[M^3, \mathbf{R}^5]$ forms the subgroup $12\mathbf{Z}$ in a $\mathbf{Z}$-component of $\mathrm{Imm}[M^3, \mathbf{R}^5]_0$.

In view of Theorem 4.2, it is natural to ask: given a 3-manifold $M^3$, does $\mathrm{Emb}[M^3, \mathbf{R}^5]$ necessarily form the subgroup $24\mathbf{Z}$ (or $12\mathbf{Z}$) simultaneously in all the $\mathbf{Z}$-components of $\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \cdots \amalg \mathbf{Z}$? We give an example that answers this question negatively, as follows.

**Proposition 4.4.** *Let $X^3$ be the (total space of the) $S^1$-bundle over $T^2 = S^1 \times S^1$ with Euler class 2. Then,* $\mathrm{Imm}[X^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \mathbf{Z}$, *and* $\mathrm{Emb}[X^3, \mathbf{R}^5]$ *forms the subset* $12\mathbf{Z} \amalg 24\mathbf{Z} \subset \mathbf{Z} \amalg \mathbf{Z}$.

*Proof.* We see that $H^2(X^3; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$ and $H^1(X^3; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$, which can be computed using the Gysin exact sequence for the $S^1$-bundle $X^3 \to T^2$, for example. Therefore, we have $H^1(X^3; \mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Since two of the three $\mathbf{Z}_2$ factors come from integral classes, we have $H^1(X^3; \mathbf{Z}_2)/\operatorname{Im}\rho \cong \mathbf{Z}_2$. Thus, we have $\mathrm{Spin}(X^3)/\operatorname{Im}\rho \approx \mathbf{Z}_2$, and hence $\mathrm{Imm}[X^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \mathbf{Z}$. We want to show that in one equivalence class in $\mathrm{Spin}(X^3)/\operatorname{Im}\rho \approx \mathbf{Z}_2$, all spin structures provide the same $\mu$-invariant and that in the other class there are spin structures that provide distinct $\mu$-invariants.

It is known (see [2, p. 198] or [8, p. 714], for example) that $X^3$ has a framed link representation given by the Borromean rings $L = \{K_1, K_2, K_3\}$ with framings $(0, 0, 2)$ (see Figure 1). This framed link has eight distinct characteristic sublinks — $\emptyset$, $K_1$, $K_2$, $K_3$, $\{K_1, K_2\}$, $\{K_2, K_3\}$, $\{K_3, K_1\}$ and $L = \{K_1, K_2, K_3\}$ — each of which corresponds to a spin structure of $X^3$.

Now our strategy is to apply, for each characteristic sublink of our link $L$, Theorem 4.2 (ii) in [5], which enables us to compute the signature of a spin 4-manifold corresponding to each characteristic sublink of $L$ without actually constructing such a 4-manifold. First, the 4-manifold $W_L^4$ represented by $L$ is nothing but a spin 4-manifold corresponding to the empty characteristic sublink. Put $k := \sigma(W_L^4)$.

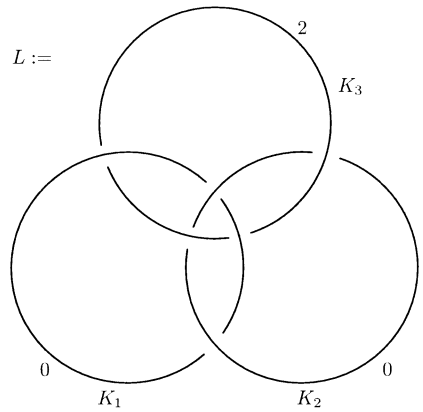


FIGURE 1. A framed link representation of $X^3$

For characteristic sublinks with one component, we can just apply [5, Theorem 4.2 (ii)]. Then the signature of the spin 4-manifold is $k$ mod 16 for $K_1$ and $K_2$, and $k-2$ mod 16 for $K_3$.

For characteristic sublinks with two components, we first follow the procedure of Case 4 in the proof of Theorem 3.1 in [5], that is, we add a push-off of one component to the other in order to obtain a new characteristic sublink with one component. This move is the so-called handle sliding and does not change the 3-manifold $X^3$ and the corresponding spin structure. Now for any two components, we get an unknotted component whose framing is 0 when the original characteristic sublink does not contain $K_3$, and is 2 when it does contain $K_3$. Therefore, applying [5, Theorem 4.2 (ii)], we see that the signature of the spin 4-manifold is $k$ mod 16 in the former case and $k-2$ mod 16 in the latter case.

When we consider $L$ itself as a characteristic sublink, we first follow the procedure of Case 4 in the proof of Theorem 3.1 in [5] as well. If we add push-offs of $K_1$ and $K_2$ to $K_3$, then the result is a trefoil knot with framing 2, whose Arf invariant is $1 \in \mathbf{Z}_2$. Therefore, again by [5, Theorem 4.2 (ii)], we see that the signature of the corresponding spin 4-manifold is $k-10$ mod 16.

Summarising the above, we see that among the eight spin structures, four spin structures provide the $\mu$-invariant $k$ mod 16, three spin structures provide the $\mu$-invariant $k-2$ mod 16, and the last one provides the $\mu$-invariant $k-10$ mod 16. By the same argument as in the last part of the proof of Lemma 3.5, we see that the first four spin structures belong to one equivalence class in $\mathrm{Spin}(X^3)/\operatorname{Im}\rho \approx \mathbf{Z}_2$ and the remaining four belong to the other. Thus, by Remark 4.3, we have the desired conclusion. This completes the proof. $\square$

*Remark* 4.5. In the case of the 3-torus $T^3$, the connected sum of a $T^3$-pseudo-embedding $g\colon S^3 \looparrowright \mathbf{R}^5$ and an *arbitrary* embedding $T^3 \hookrightarrow \mathbf{R}^5$ is regularly homotopic to an embedding. Proposition 4.4 shows, however, that this is not true in general. Namely, for the above $X^3$, the connected sum of an $X^3$-pseudo-embedding $g\colon S^3 \looparrowright \mathbf{R}^5$ and an embedding $X^3 \hookrightarrow \mathbf{R}^5$ chosen from $24\mathbf{Z} \subset 12\mathbf{Z} \amalg 24\mathbf{Z} \approx \mathrm{Emb}[X^3, \mathbf{R}^5]$ is not regularly homotopic to an embedding. This shows that we have to choose a correct embedding $F$ for a given $M^3$-pseudo-embedding in Definition 3.1 in general.

*Remark* 4.6. In Proposition 4.4, if we consider the 3-manifold $Y^3$ whose framed link representation is obtained by replacing the framings $(0,0,2)$ for the Borromean rings $L = \{K_1, K_2, K_3\}$ with the framings $(0,2,2)$, then we see that $\mathrm{Imm}[Y^3, \mathbf{R}^5]_0 \approx \mathbf{Z} \amalg \mathbf{Z} \amalg \mathbf{Z} \amalg \mathbf{Z}$ and $\mathrm{Emb}[Y^3, \mathbf{R}^5]$ forms the subset $12\mathbf{Z} \amalg 24\mathbf{Z} \amalg 24\mathbf{Z} \amalg 24\mathbf{Z} \subset \mathbf{Z} \amalg \mathbf{Z} \amalg \mathbf{Z} \amalg \mathbf{Z}$, by using a similar argument. These examples show that the distribution of $12\mathbf{Z}$ and $24\mathbf{Z}$ can be complicated in general. For other examples obtained by connected sum, see Section 5.

## 5. $M^3$-pseudo-embeddings of general 3-manifolds

In this section, we consider a question which naturally arises from the definition of $M^3$-pseudo-embeddings: *for a pair of closed oriented* 3*-manifolds* $N^3$ *and* $M^3$*, does there exist an* $M^3$*-pseudo-embedding of* $N^3$ *into* $\mathbf{R}^5$*?* In other words, when does there exist an immersion $g\colon N^3 \looparrowright \mathbf{R}^5$ such that (1) $g$ is not regularly homotopic to any embedding, and (2) for some embedding $F\colon M^3 \hookrightarrow \mathbf{R}^5$, the connected sum $F \sharp g$ is regularly homotopic to an embedding $M^3 \sharp N^3 \hookrightarrow \mathbf{R}^5$?

Concerning this question, we have the following.

**Proposition 5.1.** *Let* $M^3$ *and* $N^3$ *be closed connected oriented* 3*-manifolds. Assume that*

$$\mathrm{Imm}[M^3, \mathbf{R}^5]_0 \approx Z(1) \amalg \cdots \amalg Z(i) \amalg \cdots \amalg Z(r),$$
$$\mathrm{Imm}[N^3, \mathbf{R}^5]_0 \approx Z(1) \amalg \cdots \amalg Z(j) \amalg \cdots \amalg Z(s),$$

*where* $i \in \{1, 2, \dots, r\}$, $j \in \{1, 2, \dots, s\}$ *and each* $Z(i)$ *or* $Z(j)$ *denotes a copy of* $\mathbf{Z}$*. Furthermore, assume that*

$$\mathrm{Emb}[M^3, \mathbf{R}^5] \approx m_1\mathbf{Z} \amalg \cdots \amalg m_i\mathbf{Z} \amalg \cdots \amalg m_r\mathbf{Z},$$
$$\mathrm{Emb}[N^3, \mathbf{R}^5] \approx n_1\mathbf{Z} \amalg \cdots \amalg n_j\mathbf{Z} \amalg \cdots \amalg n_s\mathbf{Z}$$

*with respect to the above decompositions of* $\mathrm{Imm}[M^3, \mathbf{R}^5]_0$ *and* $\mathrm{Imm}[N^3, \mathbf{R}^5]_0$*, respectively, where each* $m_i$ *or* $n_j$ *is equal either to* 12 *or to* 24*. Then, we have the following.*

(a) *The set* $\mathrm{Imm}[M^3 \sharp N^3, \mathbf{R}^5]_0$ *corresponds bijectively to the disjoint union of* $rs$ *copies of* $\mathbf{Z}$*. More precisely, we have*

$$\mathrm{Imm}[M^3 \sharp N^3, \mathbf{R}^5]_0 \approx Z(1,1) \amalg \cdots \amalg Z(i,j) \amalg \cdots \amalg Z(r,s),$$

*where each* $Z(i,j)$ *denotes a copy of* $\mathbf{Z}$ *and each class in* $Z(i,j)$ *is represented by the connected sum of an immersion in* $Z(i) \subset \mathrm{Imm}[M^3, \mathbf{R}^5]_0$ *and an immersion in* $Z(j) \subset \mathrm{Imm}[N^3, \mathbf{R}^5]_0$*.*

(b) *In each* $Z(i,j)$*, the classes containing an embedding correspond to*

$$\begin{cases} 24\mathbf{Z}, & \text{if } (m_i, n_j) = (24, 24), \\ 12\mathbf{Z}, & \text{if } (m_i, n_j) = (12, 12), (12, 24), \text{ or } (24, 12). \end{cases}$$

(c) *The set* $\mathrm{Imm}[N^3, \mathbf{R}^5]_0$ *contains an* $M^3$*-pseudo-embedding* $N^3 \looparrowright \mathbf{R}^5$ *if and only if the case* $(m_i, n_j) = (12, 24)$ *occurs. More precisely, if we assume* $(m_i, n_j) = (12, 24)$*, then an immersion* $g\colon N^3 \looparrowright \mathbf{R}^5$ *in* $12 + 24\mathbf{Z} \subset Z(j) \subset \mathrm{Imm}[N^3, \mathbf{R}^5]_0$ *is not regularly homotopic to an embedding, but the connected sum of* $g$ *and any embedding* $M^3 \hookrightarrow \mathbf{R}^5$ *in* $12\mathbf{Z} \subset Z(i) \subset \mathrm{Imm}[M^3, \mathbf{R}^5]_0$ *is regularly homotopic to an embedding* $M^3 \sharp N^3 \hookrightarrow \mathbf{R}^5$ *in* $12\mathbf{Z} \subset Z(i,j) \subset \mathrm{Imm}[M^3 \sharp N^3, \mathbf{R}^5]_0$*.*

*Proof.* (a) Since a spin structure of $M^3$ and a spin structure of $N^3$ determine a spin structure of $M^3\sharp N^3$, and since $H^1(M^3\sharp N^3) \cong H^1(M^3) \oplus H^1(N^3)$ (with any coefficients), we have the bijection

$$\sharp\colon (\mathrm{Spin}(M^3)/\operatorname{Im}\rho) \times (\mathrm{Spin}(N^3)/\operatorname{Im}\rho) \to \mathrm{Spin}(M^3\sharp N^3)/\operatorname{Im}\rho.$$

Furthermore, since a normal framing of $F \in \mathrm{Imm}[M^3, \mathbf{R}^5]_0$ and a normal framing of $G \in \mathrm{Imm}[N^3, \mathbf{R}^5]_0$ determine a normal framing of $F\sharp G \in \mathrm{Imm}[M^3\sharp N^3, \mathbf{R}^5]_0$, and since the associated spin structures are coherent with respect to the connected sum, we see that

$$c(F\sharp G) = \sharp(c(F), c(G)).$$

Thus the result follows.

(b) This follows from the additivity of the $\mu$-invariant with respect to the connected sum together with the above item (a) and Remark 4.3.

(c) This follows immediately from the above items (a) and (b). □

**Example 5.2.** Let $X^3$ be the total space of the $S^1$-bundle over $T^2 = S^1 \times S^1$ with Euler class 2 as in Proposition 4.4, and $Y^3$ the 3-manifold as in Remark 4.6. Then, for $M^3 := X^3\sharp X^3$ or $X^3\sharp Y^3$ or $Y^3\sharp Y^3$, the numbers of $12\mathbf{Z}$- and $24\mathbf{Z}$-components of $\mathrm{Emb}[M^3, \mathbf{R}^5] \subset \mathrm{Imm}[M^3, \mathbf{R}^5]_0$ are as follows.

| $M^3$ | $12\mathbf{Z}$ | $24\mathbf{Z}$ |
|---|---|---|
| $X^3\sharp X^3$ | 3 | 1 |
| $X^3\sharp Y^3$ | 5 | 3 |
| $Y^3\sharp Y^3$ | 7 | 9 |

In fact, we can show that for the connected sum of $m$ copies of $X^3$ and $n$ copies of $Y^3$, the number of $24\mathbf{Z}$-components is equal to $3^n$ and that of $12\mathbf{Z}$ is equal to $2^{m+2n} - 3^n$.

## 6. Virtual homotopy classes

In this section, we introduce a new equivalence relation on the set of all immersions of 3-manifolds into 5-space with trivial normal bundles, and study the structure of the quotient set. This relation measures, in a certain sense, the difference between the set of all regular homotopy classes of immersions and those containing embeddings.

**Definition 6.1.** Let $f\colon M^n \looparrowright \mathbf{R}^{n+k}$ and $g\colon N^n \looparrowright \mathbf{R}^{n+k}$ be immersions of closed *connected* oriented $n$-manifolds with trivial normal bundles. We say that $f$ and $g$ are *virtually homotopic* if there exist embeddings $f_1\colon M_1^n \hookrightarrow \mathbf{R}^{n+k}$ and $g_1\colon N_1^n \hookrightarrow \mathbf{R}^{n+k}$ of closed connected oriented $n$-manifolds such that $M^n\sharp M_1^n$ is orientation-preservingly diffeomorphic to $N^n\sharp N_1^n$ and $f\sharp f_1$ is regularly homotopic to $g\sharp g_1$. This clearly defines an equivalence relation on the set of all immersions of closed connected oriented $n$-manifolds into $\mathbf{R}^{n+k}$ with trivial normal bundles.

Let $\overline{\mathrm{Imm}}_0(n, n+k)$ denote the set of virtual homotopy classes of immersions of closed connected oriented $n$-manifolds into $\mathbf{R}^{n+k}$ with trivial normal bundles.

**Lemma 6.2.** *The set* $\overline{\mathrm{Imm}}_0(3, 5)$ *is an additive group under connected sum.*

*Proof.* For an arbitrary dimension pair $(n, n+k)$, it is easy to see that the connected sum (considering the orientation when $k = 1$) gives rise to a well-defined

binary operation on the set $\overline{\mathrm{Imm}}_0(n, n+k)$. Then, clearly $\overline{\mathrm{Imm}}_0(n, n+k)$ becomes an additive semi-group with respect to this operation; the virtual homotopy class containing embeddings is the identity element. When $(n, n+k) = (3, 5)$, for every immersion $f\colon M^3 \looparrowright \mathbf{R}^5$ of a closed connected oriented 3-manifold $M^3$ with trivial normal bundle, there exists an immersion $f_1\colon S^3 \looparrowright \mathbf{R}^5$ such that $f \sharp f_1$ is regularly homotopic to an embedding (see [9] or Subsection 2.2 of the present paper). Thus, every virtual homotopy class of $\overline{\mathrm{Imm}}_0(3, 5)$ has an inverse. This completes the proof. □

We do not know if $\overline{\mathrm{Imm}}_0(n, n+k)$ is a group for every dimension pair $(n, n+k)$.

**Proposition 6.3.** *The group* $\overline{\mathrm{Imm}}_0(3, 5)$ *is isomorphic to* $\mathbf{Z}/12\mathbf{Z}$.

*Proof.* Let $\varphi\colon \mathrm{Imm}[S^3, \mathbf{R}^5] \to \overline{\mathrm{Imm}}_0(3, 5)$ be the natural map, which is clearly a group homomorphism. Since every immersion of a closed connected oriented 3-manifold into $\mathbf{R}^5$ with trivial normal bundle is virtually homotopic to an immersion of $S^3$ (see Subsection 2.2), $\varphi$ is an epimorphism. Furthermore, the kernel of $\varphi$ consists of the immersions $f$ of $S^3$ into $\mathbf{R}^5$ such that $f \sharp g$ is regularly homotopic to an embedding for some embedding $g$ of a closed connected oriented 3-manifold. Thus, using the results in Subsection 2.2 and Theorem 4.2 together with the existence of $T^3$-pseudo-embeddings of $S^3$ ([9]), we see that the kernel of $\varphi$ consists of those immersions of $S^3$ into $\mathbf{R}^5$ whose Smale invariants are divisible by 12. Hence the result follows. □

*Remark* 6.4. Let $f : M^3 \looparrowright \mathbf{R}^5$ be an immersion not regularly homotopic to an embedding. By the very definition of virtual homotopy, $f$ is an $N^3$-pseudo-embedding for some 3-manifold $N^3$ if and only if $f$ is virtually homotopic to an embedding, i.e., if and only if $f$ is an immersion that represents the identity element of the group $\overline{\mathrm{Imm}}_0(3, 5)$.

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