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# LINEAR PARABOLIC EQUATIONS WITH STRONG SINGULAR POTENTIALS

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ABSTRACT. Using an extension of a recent method of Cabré and Martel (1999), we extend the blow-up and existence result in the paper of Baras and Goldstein (1984) to parabolic equations with variable leading coefficients under almost optimal conditions on the singular potentials. This problem has been left open in Baras and Goldstein. These potentials lie at a borderline case where standard theories such as the strong maximum principle and boundedness of weak solutions fail. Even in the special case when the leading operator is the Laplacian, we extend a recent result in Cabré and Martel from bounded smooth domains to unbounded nonsmooth domains.

## 1. INTRODUCTION

The purpose of the paper is to study the existence and nonexistence of positive solutions to the linear parabolic Cauchy problem

$$\begin{cases} \nabla(A\nabla u(x,t)) + V(x)u(x,t) - \partial_t u(x,t) = 0, \ x \in D \subset \mathbf{R}^n, t > 0, \\ u(x,t) = 0, x \in \partial D, t > 0; u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here $A = A(x)$ is a symmetric, uniformly elliptic matrix with bounded measurable coefficients. $D$ is a domain (bounded or unbounded) or $\mathbf{R}^n$, $n \geq 2$. Of course such a problem has been well understood if $V$ belongs to the standard $L^p(\mathbf{R}^n)$ $(p > n/2)$, or the slightly more singular Kato class. Our potentials belong to a larger class, which includes but is not restricted to potentials of the form $V = a(x)/|x|^2$ with $a \in L^\infty(D)$. The novelty and difficulty of (1.1) lie in the special nature of the potential. For instance, if $a > 0$ is a positive constant, it is well known that singularities of the type $a/|x|^2$ belong to a borderline case where standard theories such as the strong maximum principle and Gaussian bounds in [A] fail. Such $V$ also lie outside the Kato class potentials, which have been studied extensively. The investigation on these kinds of potentials also has applications to nonlinear problems (see [BC]).

Despite the long history, even such fundamental questions as whether the simple-looking (1.1) has positive solutions is not well understood. To illustrate an interesting phenomenon generated by such a singular potential, let us recall an early result in a special case when $A = I$, $V = a/|x|^2$ and $a$ is a constant. Define $H_a = -\Delta - \frac{a}{|x|^2}$ on $C_0^\infty(\mathbf{R}^n - \{0\})$. By Hardy's inequality, the symmetric operator $H_a$ is nonnegative in $L^2(\mathbf{R}^n)$ iff $a \leq ((n-2)/2)^2$. Let $A_a$ be the Friedrichs extension of $H_a$ when

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$a \leq ((n-2)/2)^2$ or any selfadjoint extension when $a > ((n-2)/2)^2$. The Cauchy problem

$$\frac{du}{dt} + A_a u = 0, \quad u(0) = f \in L^2(\mathbf{R}^n)$$

is well-posed in $L^2(\mathbf{R}^n)$ for $a \leq ((n-2)/2)^2$; it is not well-posed for $a > ((n-2)/2)^2$, but it has many global solutions as can be shown using the spectral theorem. The question as to the existence of nontrivial nonnegative solutions is more subtle and depends on delicate arguments based on the maximum principle. This question was settled by Baras and Goldstein ([BG1]).

In that paper they discovered a critical behavior of the Cauchy problem (1.1) with $A = I$ and $V = a/|x|^2$. Here $a$ is a constant and $D = \mathbf{R}^n$. They found that if $a > ((n-2)/2)^2$, then the above problem has no nonnegative solutions except $u \equiv 0$ and if $a \leq ((n-2)/2)^2$, positive weak solutions do exist. One notes that the critical value of $a$ is, in fact, the best constant in Hardy's inequality. The result in [BG1] stimulated several interesting results in the study of heat equations with singular potentials. Some recent developments can be found in the papers [GP], [CM], [VZ], [AP], [Du] and [MS].

The proof in [BG1] relies on Hardy's inequality and scaling properties of the heat equation in $\mathbf{R}^n$ and, in general, the result fails for parabolic equations with variable coefficients in the principal part. The question of to what extent the result in [BG1] can be generalized to equations with variable coefficients in the principal part has been left open since 1984. The goal of the current paper is to answer this question for uniformly elliptic equations in divergence form. Moreover, our condition on the singular potential allows new nonradial singularities such as $a(x)/|x|^2$ with $a = a(x)$ being an unbounded function and potentials with nonpointwise singularities such as $c/d(x, \partial D)^2$. We emphasize that these potentials are not covered in [BG1], even for the heat equation. This problem was introduced in [CM]. In fact, our approach, which was greatly influenced both by the ideas of [CM] and the Harnack chain arguments of [GZ], produces near optimal results for blow-up and existence of positive solutions not only for (1.1) but also for some degenerate equations (Corollary 3.1).

It turns out that the existence and nonexistence of positive solutions to (1.1) is largely determined by the size of the infimum of the spectrum of the symmetric operator $S = -\nabla(A\nabla) - V$. Define

$$\sigma_{\inf} = \sigma_{\inf}(V; D) = \inf_{0 \neq \phi \in Q} \frac{\int_D (A\nabla\phi\nabla\phi - V\phi^2)dx}{\int_D \phi^2 dx}.$$

Here $Q$ is a core of $S$. In dimension $n \geq 3$, we take $Q = C_0^\infty(D)$. But for $0 \in D \subset \mathbf{R}^n$ and $n \leq 2$, it is convenient to take $Q = C_0^\infty(D - \{0\})$ when considering $V$ of the form $a(x)/|x|^2$.

The fact that $\sigma_{\inf} = -\infty$ is related to blow-up was first observed in [BG2] in the case $V = a/|x|^2$ where $a$ is a constant. It was clarified and extended in the interesting paper [CM] for the equation (1.1) with $A = I$ on bounded smooth domains with general potentials. The proof in [CM] involves nice and delicate computations using the special structure of $\Delta$ and the smoothness of the boundary. At first glance it does not seem to apply immediately to our case of operators with discontinuous coefficients. Nonetheless, we will show that the approach in [CM] can be refined to obtain sharp results in virtually all settings of equations and spaces

as long as the strong maximum principle and an embedding as (2.5) hold. The smoothness of the boundary is also irrelevant. Moreover, the method also covers the case when $D$ is an unbounded domain, a case that was left open in [CM]. We wish to record our thanks to Cabré and Martel for writing the fine paper [CM], which contains clever and novel ideas which turned out to be generalizable.

Another noteworthy progress is that we do not even need to assume that $u(.,t)$ is in some weighted $L^1(D)$ space. This assumption was used in [CM]. Instead, the blow-up result presented below works for all possible positive solutions for the equation in (1.1), with or without boundary values. These solutions only need to be locally integrable in $D$ if $A$ is $C^1$. Throughout the paper we will use the standard notion of weak solution as in [L], p. 99.

Definition (a). In case $A$ is just $L^\infty$, we say that $u(.,t) \in W^{1,2}_{\text{loc}}(D)$ is a solution to the equation in (1.1) if $V(.)u(.,t) \in L^1_{\text{loc}}(D)$ and

$$\int_D (u\phi)|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_D u\phi_t dxdt + \int_{t_1}^{t_2}\int_D A\nabla u \nabla\phi dxdt - \int_{t_1}^{t_2}\int_D Vu\phi dxdt = 0$$

for all compactly supported $C^{1,1}$ functions $\phi$ and $t_2 > t_1$.

Definition (b). If $A$ is $C^1$, then $u$ is a solution of the equation in (1.1) if the above equality with the third integral replaced by $-\int_{t_1}^{t_2}\int_D u\nabla(A\nabla\phi)dxdt$ holds.

The first definition is slightly different from the one used in [BG1], due to the presence of nonsmooth variable coefficients in the leading term. For convenience we will assume that the domain $D$ is Lipschitz. Our argument applies to any bounded domain for which weak solutions are well-defined and the embedding (2.5) holds.

If $D$ is an unbounded domain, we make the following assumptions. (i) $D$ is uniformly Lipschitz in the following sense: there exists $r_0 > 0$ such that for any $x \in \partial D$, there exists a one-to-one Lipschitz map $f$ that sends $\partial D \cap B(x, r_0)$ onto $\{x \mid x = (x_1, ..., x_n) \in B(0,1), x_1 \geq 0\}$. Moreover, $\nabla f$ is uniformly bounded a.e. (ii) Any $x \in D$ can be connected to a fixed point $x_0$ by a curve $l$ in $D$ whose length is bounded from above by $c_1|x - x_0|$. Moreover, $\text{dist}(l, \partial D) \geq c_2 \min\{1, \text{dist}(x, \partial D)\}$.

The following theorem is the main result of the paper.

**Theorem 1.1.** *Let $A$ be a uniformly elliptic matrix with bounded measurable coefficients. Let $D$ be either any bounded Lipschitz domain, or $\mathbf{R}^n$, or any unbounded domain satisfying (i) and (ii) above. Assume $0 \leq V \in L^1_{\text{loc}}(D)$.*

(i) *Suppose $\sigma_{\inf}((1-\epsilon)V; D) = -\infty$ for some $\epsilon > 0$. Then problem (1.1) has no nonnegative solutions except $u \equiv 0$. Moreover, all positive solutions blow up completely and instantaneously. This means that $\lim_{k\to\infty} u_k(x,t) = \infty$ for all $x \in D$ and $t > 0$. Here $u_k$ is the unique positive solution to (1.1) when $V$ is replaced by $V_k = \max\{V, k\}$.*

(ii) *Suppose $\sigma_{\inf}((1+\epsilon)V; D) > -\infty$ for some $\epsilon > 0$. Then (1.1) has a unique nonnegative solution for any $0 \leq u_0 \in L^2(D)$.*

(iii) *Suppose $\sigma_{\inf}(V; D) > -\infty$. Then (1.1) has a unique nonnegative solution for any $0 \leq u_0 \in L^2(D)$ if, in addition, $A$ is $C^1$.*

These hypotheses reduce to those of [CM] when $A$ is the identity matrix. Even though the condition in Theorem 1.1 is almost sharp, we still need some more verifiable conditions to illustrate its range.

**Condition 1.1.** There exist $\lambda > 1$ and $R > 0$ and an infinite sequence of integers $\{k_j\}$ such that

$$\int_{B(0,R)} \frac{a(x/k_j)}{1+|x|^2}\phi_{k_j}^2\,dx \geq \lambda \int_{B(0,R)} |\nabla \phi_{k_j}|^2 dx \tag{1.2}$$

for some nonzero $\phi_{k_j} \in C_0^\infty(B(0,R))$.

As to be shown in the proof, this condition implies that, for some $\epsilon > 0$, the bottom of the spectrum $\sigma_{\inf}((1-\epsilon)V; D)$ of the operator $-\nabla(A\nabla) - \frac{(1-\epsilon)a(x)}{|x|^2}$ is $-\infty$. This condition also includes the original blow-up condition in [BG1] (see Remark 1.2 below). Let us mention that Condition 1.1 is somewhat less general than $\sigma_{\inf} = -\infty$. However, it is easier to verify.

**Corollary 1.1.** *Suppose $0 \in D$, $V = a(x)/|x|^2$ satisfies Condition 1.1, $a(x) \in L^\infty(D)$, $A(0) = I$ and $A$ is continuous at $0$. Then problem (1.1) has no nonnegative solutions except $u \equiv 0$. Moreover, all positive solutions blow up completely and instantaneously.*

*Remark* 1.1. As in [BG1], for $a(x) \equiv a_0 \leq ((n-2)/2)^2$ and $A = I$, the Cauchy problem (using the Friedrichs extension) for (1.1) with initial data $u_0 \in L^2(\mathbf{R}^n)$ is well-posed, and $u(x,t) \geq 0$ holds for all $x, t$ provided $u_0 \geq 0$. There will be additional positive solutions for certain $u_0 \in L^p(\mathbf{R}^n)\backslash L^2(\mathbf{R}^n)$ with $p \neq 2$, but we will not characterize all of them here.

*Remark* 1.2. Here we give some examples of $V$ satisfying Condition 1.1. The proofs are given at the end of Section 3.

(i). Given $a > ((n-2)/2)^2$, if $V(x) \geq a/|x|^2$ in any ball $B(0,r)$, $r > 0$, then $V$ satisfies Condition (1.1).

(ii). Let us write $x = (x_1, x_n)$ where $x_1 \in \mathbf{R}^{n-1}$. Choose $V = a(x)/|x|^2$ such that $a(x) = ((n-2)/2)^2$ if $x_n < 0$ and $a(x) \equiv a_0 > ((n-2)/2)^2$ if $x_n \geq 0$. Then $V$ satisfies Condition 1.1.

## 2. Proof of Theorem 1.1

*Proof of the blow-up part.* As mentioned in the Introduction, we will provide a short and self-contained proof that builds on and refines the argument given in [CM].

It turns out that the special structure of $\Delta$ and the smoothness of $D$ are all irrelevant for the blow-up of positive solutions.

Given any $T > 0$, let $u$ be a solution to (1.1) in $D \times (0,T)$ with $u_0 \geq 0$ but not identically zero. Our goal is to show that $u(x,t) = \infty$ for any $t > 0$ and $x \in D$. Without loss of generality, we assume that $u$ is the monotone limit of a sequence $u_k$ where $u_k$ is the solution of (1.1) with $V$ replaced by $V_k = \min\{V, k\}$. This is so because $u$ dominates $\lim_{k\to\infty} u_k$ in case they are not equal.

Our starting point is the same as that in [CM]. We need to prove that for all $\phi \in C_0^\infty(D)$, and $0 < t_1 < t_2 < T$,

$$\int_D V(x)\phi^2(x)dx - \int_D A\nabla\phi\nabla\phi(x)dx \leq \frac{1}{t_2 - t_1}\int_D \log(u(x,t_2)/u(x,t_1))\phi^2(x)dx. \tag{2.1}$$

If $V$ is so singular at 0 that $\int_{|x|\leq\epsilon} |V(x)|dx = \infty$, we replace $\phi \in C_0^\infty(D)$ with $\phi \in C_0^\infty(D - \{0\})$. What we need is that $\int_D V(x)\phi^2(x)dx$ makes sense for all such test functions $\phi$.

Here is a proof of (2.1). Since $u_k$ satisfies (1.1) with $V$ replaced by $V_k$, we can multiply (1.1) by the test function $\phi^2/u_k$ to obtain

$$\int_D V_k(x)\phi^2(x)dx = \int_D (\partial_t u_k)\frac{\phi^2}{u_k}dx + \int_D A\nabla u_k \nabla(\frac{\phi^2}{u_k})dx.$$

This implies

$$\begin{aligned}\int_D V_k(x)\phi^2(x)dx &= \partial_t \int_D (\log u_k)\phi^2 dx + 2\int_D (A\nabla u_k \nabla\phi)\frac{\phi}{u_k} - \int_D (A\nabla u_k \nabla u_k)\frac{\phi^2}{u_k^2}dx \\ &\le \partial_t \int_D (\log u_k)\phi^2 dx + \int_D A\nabla\phi\nabla\phi dx.\end{aligned}$$

Inequality (2.1) is proven by integrating the above from $t_1$ to $t_2$ and taking the limit $k \to \infty$.

From here on our proof differs from that in [CM]. For clarity we divide the proof into two cases.

*Case* 1. $D$ is bounded.

Let $\rho \in C_0(D)$ be any strictly positive function in $D$ such that $\log\rho \in L^p(D)$ for any $p > 1$. We claim that there exists at most one point $t_1 \in (0,T)$ such that

$$u(.,t_1)\rho(.) \in L^1(D).$$

Suppose the contrary holds. Then there exist $t_1, t_2 \in (0,T)$ such that $t_2 > t_1$ and for $i = 1, 2$,

$$u(.,t_i)\rho(.) \in L^1(D).$$

For a small $\delta > 0$, let $D_\delta \equiv \{x \in D \mid \operatorname{dist}(x,D) \ge \delta\}$. Note $u(x,t_i)\rho(x) \ge c > 0$ when $x \in D_\delta$ for some $c$ depending on $t_1, t_2$ and $\delta$. By Jensen's inequality, for any $p > 1$,

$$\int_{D_\delta} |\log(u(x,t_i)\rho(x))|^p dx < C. \tag{2.2}$$

Observe that, for $m > 1$,

$$D - D_\delta = S_1 \cup S_2 \equiv \{x \in D_\delta^c \mid u(x,t_1)\rho(x) \ge m\} \cup \{x \in D_\delta^c \mid u(x,t_1)\rho(x) < m\}.$$

Note that $\log^p s$ is a concave function of $s$, for $s \ge m$ with $m$ sufficiently large. Applying Jensen's inequality on $S_1$, we see that, for any $p > 1$,

$$\int_{S_1} |\log(u(x,t_i)\rho(x))|^p dx < C. \tag{2.3}$$

Since, for $x \in S_2$,

$$m > u(x,t_1)\rho(x) \ge \int_D G_0(x,t_1;y,0)u_0(y)dy\ \rho(x) \equiv h(x,t_1)\rho(x),$$

we also know that $|\log(u(x,t_1)\rho(x))| \le C + |\log(h(x,t_1)\rho(x))|$.

When $x \in D - D_\delta$, by the boundary Harnack inequality in [FGS] (Theorem 1.6), there exists $c' > 0$ depending on $t_1, t_2$ and $\delta$ such that

$$0 < c' \le h(x,t_1)/\phi(x) \le 1/c'.$$

Here $\phi(x)$ is a normalized ground state of the operator $\nabla(A\nabla)$ with Dirichlet boundary condition. Since $\partial D$ is Lipschitz, it is well known that $\phi(x) \ge cd(x,\partial D)^a$ for

some $a \geq 1$. We mention that, using the interior Harnack inequality, one can also show that, for $x$ close to the boundary,

$$h(x,t) \geq c(t)d(x)^{a(t)}$$

for some $a(t)$ and $c(t) > 0$. The proof, which actually works for some unbounded domains too, will be given in Case 2. This method has the advantage that it works for domains for which the boundary Harnack principle may not hold.

Therefore,

$$\int_{S_2} |\log(u(x,t_1)\rho(x))|^p dx < C. \tag{2.4}$$

Repeating this process for $u(.,t_2)$ and using (2.2)-(2.4), we conclude that

$$\log \frac{u(.,t_2)}{u(.,t_1)} = \log(u(.,t_2)\rho(.)) - \log(u(.,t_1)\rho(.)) \in L^p(D)$$

for any $p > 1$. In particular, $\log \frac{u(.,t_2)}{u(.,t_1)} \in L^n(D)$. By a standard embedding theorem (see Proposition 2.1 below for an independent proof), for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\frac{1}{t_2 - t_1} \int_D (\log \frac{u(x,t_2)}{u(x,t_1)})\phi^2 dx \leq \epsilon \int_D A\nabla\phi\nabla\phi(x)dx + C(\epsilon) \int_D \phi^2(x)dx \tag{2.5}$$

for any $\phi \in C_0^1(D)$. Substituting the above into (2.1), we obtain

$$\int_D V(x)\phi^2(x)dx - \int_D A\nabla\phi\nabla\phi(x)dx \leq \epsilon \int_D A\nabla\phi\nabla\phi(x)dx + C(\epsilon) \int_D \phi^2(x)dx.$$

It follows, for another $\epsilon > 0$, that

$$\inf_{\phi \in C_0^\infty(D), \phi \neq 0} \frac{\int_D (A\nabla\phi\nabla\phi - (1-\epsilon)V\phi^2)dx}{\int_D \phi^2 dx} > -\infty.$$

This contradicts our assumption of Theorem 1.1. The claim is proven.

Given $x \in D$ and $t \in (0,T)$, we take $\rho = \rho(y) = G_0(x,t/2;y,0)$. This is admissible since $\log\rho \in L^p(D)$ as was the case for $h$. By the strong maximum principle, $\rho(y) > 0$ for any $y \in D$. If there is no $s \in (0,t/2]$ such that $\rho(.)u(.,s) \in L^1(D)$, then

$$u(x,t) \geq \int_D G_0(x,t/2;y,0)u(y,t/2)dy = \int_D \rho(y)u(y,t/2)dy = \infty.$$

In case $s \in (0,t/2]$ is the only point such that $\rho(.)u(.,s) \in L^1(D)$, then

$$u(x,(t+s)/2) \geq \int_D G_0(x,t/2;y,0)u(y,s/2)dy = \int_D \rho(y)u(y,s/2)dy = \infty.$$

Recall that $G_0$ is the heat kernel of $\nabla(A\nabla u) - u_t = 0$. So for any small $\delta > 0$, there exist $c, r > 0$ such that $G_0(z,(t+\delta)/2;y,0) \geq cG_0(x,t/2;y,0)$ when $z \in B(x,r) \subset D$. We emphasize that $c$, $r$ and $\delta$ are independent of $y$. This is due to the Harnack inequality (see [L] e.g.). Therefore, for $z \in B(x,r) \subset D$,

$$\begin{aligned} u\left(z, \frac{t+s+\delta}{2}\right) &\geq \int_D G_0\left(z, \frac{t+\delta}{2}; y, 0\right) u(y,s/2)dy \\ &\geq c\int_D G_0\left(x, \frac{t}{2}; y, 0\right) u(y,s/2)dy = \infty. \end{aligned}$$

By the reproducing formula, we know that

$$\begin{aligned} u(x,t) &\geq \int_D G_0(x,(t-s-\delta)/2;z,0)u(z,(t+s+\delta)/2)dz \\ &\geq \int_{|x-z|\leq r} G_0(x,(t-s-\delta)/2;z,0)u(z,(t+s+\delta)/2)dz = \infty. \end{aligned}$$

Since $(x,t)$ is arbitrary, this proves the blow-up part when $D$ is bounded.

Let us mention that the above argument can be shortened further for any domains such that the boundary Harnack principle holds. In this case, we can just choose $\rho(x) = cd(x)^a$ for a sufficiently large $a > 0$ and using the fact that $G_0(x,t;y,0) \geq c(x,t)d(y)^a$. We are presenting the longer proof here since it is expandable in case the boundary Harnack principle fails. In this event, we just use the inequality before (2.4), proven by an interior Harnack chain argument only. This approach is used in case $D$ is unbounded.

We also remark that the above arguments are, in fact, rigorous since we could have replaced $u$ by $u_k$ and proven that $\lim_{k\to\infty} u_k(x,t) = \infty$ in exactly the same manner.

*Case* 2. $D$ is unbounded.

In this case, we will use $L^1_w(D)$ to denote the weighted $L^1$ space with the weight $w \equiv e^{-|x|}$.

Let $\rho \in C_0(D)$ be any strictly positive function in $D$ such that $\rho$ converges to zero at $\infty$ and $\log\rho \in L^p_w(D)$ for any $p > 1$. We claim that there exists at most one point $t_1 \in (0,T)$ such that $u(.,t_1)\rho(.) \in L^1_w(D)$.

Suppose the contrary holds. Then there exist $t_1, t_2 \in (0,T)$ such that $t_2 > t_1$ and for $i = 1,2$,

$$u(.,t_i)\rho(.) \in L^1_w(D).$$

We wish to prove that

$$\log(u(.,t_i)\rho(.)) \in L^p_w(D)$$

for any $p \geq 1$. To this end we write

$$D = S_1 \cap S_2 \equiv \{x \in D \mid u(x,t_1)\rho(x) \geq 1\} \cup \{x \in D \mid u(x,t_1)\rho(x) < 1\}.$$

Since $\log^p s$ is a concave function of $s$ for $s \geq m$, $m$ large, by Jensen's inequality,

$$\int_{S_1} |\log(u(x,t_1)\rho(x))|^p w(x)dx < \infty \tag{2.6}$$

for any $p \geq 1$.

When $x \in S_2$,

$$m > u(x,t_1)\rho(x) \geq \int_D G_0(x,t_1;y,0)u_0(y)dy\ \rho(x) \equiv h(x,t_1)\rho(x).$$

Hence we know that

$$|\log(u(x,t_1)\rho(x))| \leq |\log(h(x,t_1)\rho(x))|. \tag{2.7}$$

Here $G_0$ is the heat kernel of $\nabla(A\nabla)$.

Next we show that

$$\log h(x,t_1) \in L^p_w(D) \tag{2.8}$$

for any $p > 1$. This is an immediate consequence of the following bounds for $h$: there exist positive constants $c_1, ..., c_4$ and $c_1(t), c_2(t)$ and $a > 0$ such that, for all $x$, $y$ and $t > 0$, it follows that

$$c_1(t)\min\{d(x)^a, 1\}e^{-c_1|x|}e^{-c_2\frac{|x|^2}{t}} \le h(x,t) \le c_2(t)e^{c_3|x|}e^{c_4\frac{|x|^2}{t}}.$$

Here $d(x) = \mathrm{dist}(x, \partial D)$. These bounds, which are crude but sufficient for our purpose, can be obtained by the following Harnack chain argument. Note that, if we assume $u_0 \le c$, then $h(x,t) \le c_2(t)$ immediately. But we want to minimize the number of assumptions.

Without loss of generality, we assume that $0 \in D$. According to the strong maximum principle, there is, for any $t > 0$, a positive constant $c(t)$ such that

$$h(0, t/2) \ge c(t).$$

If $d(x) \ge 1$, by assumption, one can connect $x$ and 0 by a curve $l = l(s)$ in $D$, whose length is less than $c|x|$ and $\mathrm{dist}(\partial D, l) \ge \delta > 0$. Let $l$ be parametrized by length, $x = l(0)$, $x_j = l(j)$. Using the standard Harnack inequality with a time gap of $\frac{t}{k|x|}$ for a suitable $k > 0$, we see that

$$h(x_j, t_j) \ge ce^{-c_1k|x|/t}h(x_{j-1}, t_{j-1})$$

where $t_j = (t/2) + j\frac{t}{k|x|}$ with $j = 0, 1, ..., k|x|/2$. Hence,

$$h(x,t) \ge e^{-c_1|x|}e^{-c_2\frac{|x|^2}{t}}$$

if $d(x) \ge 1$. In case $d(x) < 1$, we can find a point $x_1$ such that $d(x_1) = 1$ and $|x - x_1| \le 1$. Using the local comparison argument for the function $h$ on the bounded domain $D \cap B(y, 3)$, we see that

$$h(x,t) \ge d(x)^a h(x, 3t/4) \ge c(t)\min\{d(x)^a, 1\}e^{-c_1|x|}e^{-c_2\frac{|x|^2}{t}}.$$

This is the lower bound. The upper bound is obtained similarly. Therefore,

$$|\log h(x,t)| \le c_3(t) + c_4 d(x) + c_5|x| + c_2\frac{|x|^2}{t}.$$

This proves (2.8).

Combining (2.7) and (2.8), we obtain

$$\int_{S_2} |\log(u(x,t_1)\rho(x))|^p w(x)dx < C.$$

Repeating this process for $u(., t_2)$, we conclude that

$$\log\frac{u(., t_2)}{u(., t_1)} = \log(u(., t_2)\rho(.)) - \log(u(., t_1)\rho(.)) \in L^p_w(D)$$

for any $p > 1$. Writing

$$h \equiv w(x)\frac{1}{t_2 - t_1}\left(\log\frac{u(x,t_2)}{u(x,t_1)}\right),$$

we know that $h \in L^p(D)$ for any $p > 1$. We will show that, for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\int_D h\phi^2 dx \le \epsilon\int_D A\nabla\phi\nabla\phi(x)dx + C(\epsilon)\int_D \phi^2(x)dx \tag{2.9}$$

for any $\phi \in W_0^{1,2}(D)$. Here is a proof. Given $\epsilon > 0$, we choose $R$ sufficiently large so that $\int_{B(x,R)^c \cap D} h^{n/2}(x)dx \le \epsilon^{n/2}$. As in case 1 (by Proposition 2.1 below), there exists $c(\epsilon)$ such that

$$\int_{B(x,R)\cap D} h(x)\phi^2(x)dx \le \frac{\epsilon}{2}\int_D A\nabla\phi\nabla\phi(x)dx + c(\epsilon)\int_D \phi^2(x)dx.$$

Combining the above and using Sobolev embedding, we obtain

$$\begin{aligned}\int_D h\phi^2 dx &= \int_{B(x,R)^c\cap D} h(x)\phi^2(x)dx + \int_{B(x,R)\cap D} h(x)\phi^2(x)dx \\ &\le \left(\int_{B(x,R)^c\cap D} h^{n/2}(x)dx\right)^{2/n}\left(\int_D \phi^{2n/(n-2)}(x)dx\right)^{(n-2)/n} \\ &\quad + \int_{B(x,R)\cap D} h(x)\phi^2(x)dx \\ &\le c_n\epsilon\int_{B(x,R)^c\cap D}|\nabla\phi(x)|^2dx + \frac{\epsilon}{2}\int_D A\nabla\phi\nabla\phi(x)dx + c(\epsilon)\int_D \phi^2(x)dx.\end{aligned}$$

Adjusting $\epsilon$ if necessary, we have proven (2.9).

Substituting the above into (2.1), we obtain

$$\int_D V(x)\phi^2(x)dx - \int_D A\nabla\phi\nabla\phi(x)dx \le \epsilon\int_D A\nabla\phi\nabla\phi(x)dx + C(\epsilon)\int_D \phi^2(x)dx.$$

It follows, for another $\epsilon > 0$, that

$$\inf_{\phi\in C_0^\infty(D),\phi\neq 0}\frac{\int_D(A\nabla\phi\nabla\phi - (1-\epsilon)V\phi^2)dx}{\int_D \phi^2 dx} > -\infty.$$

This contradicts our assumption of Theorem 1.1. The claim is proven.

Given $x \in D$ and $t \in (0,T)$, we take $\rho = \rho(y) = G_0(x,t/2;y,0)$. By the strong maximum principle, $\rho(y) > 0$ for any $y \in D$. If there is no $s \in (0,t/2]$ such that $\rho(.)u(.,s)w(.) \in L^1(D)$, then

$$u(x,t) \ge \int_D G_0(x,t/2;y,0)u(y,t/2)w(y)dy = \int_D \rho(y)u(y,t/2)w(y)dy = \infty.$$

In case $s \in (0,t/2]$ is the only point such that $\rho(.)u(.,s)w(y) \in L^1(D)$, then

$$u(x,(t+s)/2) \ge \int_D G_0(x,\tfrac{t}{2};y,0)u(y,\tfrac{s}{2})w(y)dy = \int_D \rho(y)u(y,\tfrac{s}{2})w(y)dy = \infty.$$

As in Case 1, for any small $\delta > 0$, there exist $c, r > 0$ such that $G_0(z,(t+\delta)/2;y,0) \ge cG_0(x,t/2;y,0)$ when $z \in B(x,r) \subset D$. Therefore, for $z \in B(x,r) \subset D$,

$$\begin{aligned}u\left(z,\frac{t+s+\delta}{2}\right) &\ge \int_D G_0\left(z,\frac{t+\delta}{2};y,0\right)u(y,s/2)dy \\ &\ge c\int_D G_0(x,t/2;y,0)u(y,s/2)dy = \infty.\end{aligned}$$

By the reproducing formula again, we know that

$$\begin{aligned}u(x,t) &\ge \int_D G_0(x,(t-s-\delta)/2;z,0)u(z,(t+s+\delta)/2)dz \\ &\ge \int_{|x-z|\le r} G_0(x,(t-s-\delta)/2;z,0)u(z,(t+s+\delta)/2)dz = \infty.\end{aligned}$$

Since $(x,t)$ is arbitrary, this proves the blow-up part when $D$ is unbounded.

□

*Proof of the existence part.* We will use a modified version of the original idea in the paper [BG1]. Since the argument is similar, we will be brief and we will only consider the case when $D = \mathbf{R}^n$. The other cases follow from the full space case by a standard comparison method.

Let $u_k$ be the solution to (1.1) with $V$ replaced by the truncated potential $V_k$.

*Step* 1. We show that the $u_k$ converge pointwise to a locally integrable function.

Let $J(t) \equiv \int_D u_k^2(x,t)dx$. Then

$$J'(t) = 2\int_D [-A\nabla u_k \nabla u_k + V_k u_k^2]dx.$$

By our assumption on $V$,

$$\frac{\int_D (A\nabla u_k \nabla u_k - V_k u_k^2)dx}{\int_D u_k^2 dx} \geq \frac{\int_D (A\nabla u_k \nabla u_k - V u_k^2)dx}{\int_D u_k^2 dx} \geq -c > -\infty.$$

Therefore,

$$J'(t) \leq 2cJ(t),$$

which implies

$$\int_D u_k^2(x,t)dx \leq \int_D u_0^2(x)dx\ e^{2ct}.$$

Therefore, if $u_0 \in L^2(D)$, we conclude that $u_k(x,t)$ increases to a finite positive limit $u(x,t)$ as $k \to \infty$, for all $t$ and for a.e. $x$.

*Step* 2. We show that the above $u$ is a solution to (1.1).

Pick a point $(x_0, t_0)$ such that $u(x_0, t_0)$ is finite. Then for any $k$, a small $\delta > 0$, and a compact subdomain $D' \subset D$,

$$u_k(x_0,t_0) \geq \int_\delta^{t_0} \int_{D'} G_0(x_0,t_0;y,s)V_k(y,s)u_k(y,s)dyds.$$

By the strong maximum principle, there exists $c_0 > 0$ such that $G_0(x_0,t_0;y,s) \geq c_0$ in $D' \times (\delta, t_0)$. This shows

$$\int_\delta^{t_0} \int_{D'} V_k(y,s)u_k(y,s)dyds \leq c_0^{-1}u_k(x_0,t_0) \leq c_0^{-1}u(x_0,t_0). \tag{2.10}$$

Since $u_k$ is a solution to (1.1) with $V$ replaced by $V_k$, for any $\psi \in C_0^\infty(D' \times (0,t_0))$, we have

$$\int_{D'} (u_k\psi)|_{t_1}^{t_2}dx - \int_{t_1}^{t_2}\int_{D'} u_k\psi_t dxdt + \int_{t_1}^{t_2}\int_{D'} A\nabla u_k \nabla\psi dxdt - \int_{t_1}^{t_2}\int_{D'} V_k u_k \psi dxdt = 0 \tag{2.11}$$

for all $t_1, t_2 \in (\delta, t_0)$.

If $A$ is $C^1$, then

$$\int_{D'} (u_k\psi)|_{t_1}^{t_2}dx - \int_{t_1}^{t_2}\int_{D'} u_k\psi_t dxdt - \int_{t_1}^{t_2}\int_{D'} u_k\nabla(A\nabla\psi)dxdt - \int_{t_1}^{t_2}\int_{D'} V_k u_k \psi dxdt = 0.$$

Taking $k \to \infty$ and using (2.10), we obtain

$$\int_{D'} (u\psi)|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_{D'} u\psi_t dxdt - \int_{t_1}^{t_2} \int_{D'} u\nabla(A\nabla\psi) dxdt - \int_{t_1}^{t_2} \int_{D'} Vu\psi dxdt = 0.$$

This shows that $u$ is a solution to (1.1) (Definition (b)).

Next we only assume that $A$ is $L^\infty$. By the assumption that $\sigma_{\inf}((1+\epsilon)V; D) > \infty$, we know that

$$\frac{\int_D (A\nabla u_k \nabla u_k - (1+\epsilon)V_k u_k^2) dx}{\int_D u_k^2 dx} \geq -c > -\infty.$$

Hence,

$$\int_D V_k u_k^2 dx \leq \frac{1}{1+\epsilon} \int_D [A\nabla u_k \nabla u_k + c u_k^2] dx.$$

Using the first inequality at the beginning of step 1 and the above, we obtain

$$\int_0^t \int_D |\nabla u_k|^2 \leq CJ(t) \leq Ce^{ct}.$$

Since $\{\nabla u_k\}$ is weakly compact in $L^2(D \times (\delta, t_0))$, we can take a sequential limit in (2.11) to reach

$$\int_{D'} (u\psi)|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_{D'} u\psi_t dxdt + \int_{t_1}^{t_2} \int_{D'} A\nabla u \nabla\psi dxdt - \int_{t_1}^{t_2} \int_{D'} Vu\psi dxdt = 0.$$

This shows that $u$ is a solution to (1.1) (Definition (a)).

□

*Remark* 2.1. In addition to the self-contained proof above, we can give another using the machinery of Dirichlet forms. The operator $H$, defined by the form

$$h(u,v) = \int_D A\nabla u \nabla v dx - \int_D Vuv dx$$

on $H_0^1(D) \times H_0^1(D)$, is selfadjoint and bounded from below on $L^2(D)$, since $\sigma_{\inf} > -\infty$. Consequently, for some $w \geq 0$, $H + wI$ generates a positive contraction semigroup on $L^p(D)$ for $1 \leq p < \infty$, and the semigroup is analytic for $1 < p < \infty$. Then (1.1) has a global positive solution, corresponding to nonnegative $u_0$ in $\bigcup\{L^p(D) \ : \ 1 \leq p < \infty\}$. This follows from the work of Liskevich and Semenov [LS].

We close the section by stating and proving the embedding result used in Section 2. The proof is modelled after Lemma 1.1 in [EE], Chapter VII.

**Proposition 2.1.** *Let $V \in L^n(D)$ where $D$ is a bounded Lipschitz domain. Then for every $u \in W^{1,2}(D)$, and any $\epsilon > 0$, there exists $C > 0$ depending only on $D$, $\epsilon$ such that*

$$|\int_D Vu^2| \leq \epsilon ||\nabla u||_{L^2}^2 + \left[C\epsilon^{-1}||V||_{L^n}^2 + |\bar{V}_D|\right] \ ||u||_{L^2}^2,$$

*where $\bar{V}_D$ is the average of $V$ over $D$. In the above and below, all norms are over $D$.*

*Proof.* To simplify notation we will use $\bar{V}$ to denote $\bar{V}_D$ in the proof. Since

$$|\int_D Vu^2| \le \int_D |V-\bar{V}|\ u^2 + |\bar{V}| \int_D u^2,$$

it is enough to prove the Proposition for $V$ such that $\bar{V}=0$.

Now that we assume $\bar{V}=0$, we have for $u \in W^{1,1}(D)$,

$$\int_D Vu^2 = \int_D V(y)[u^2(y) - \bar{u^2}]dy \le ||V||_{L^n}||u^2 - \bar{u^2}||_{L^{n/(n-1)}}.$$

Recall the Sobolev-Poincaré inequality, for a $C = C(D)$,

$$||u^2 - \bar{u^2}||_{L^{n/(n-1)}} \le C||\nabla(u^2)||_{L^1} \le C||u|\nabla u|||_{L^1} \le C||u||_{L^2}||\nabla u||_{L^2}. \tag{2.12}$$

Here we remark that (2.12) holds for all bounded Lipschitz domains. In fact, it also holds for all bounded extension domains (see Corollary 4.2.3 in [Zi] and p. 64 [Zi] e.g.) Thus we conclude

$$|\int_D Vu^2| \le C||V||_{L^n}||u||_{L^2}||\nabla u||_{L^2}.$$

This implies

$$|\int_D Vu^2| \le \epsilon||\nabla u||^2_{L^2} + C\epsilon^{-1}||V||^2_{L^n}\ ||u||^2_{L^2}$$

for all $\epsilon > 0$. □

## 3. Proof of Corollary 1.1 and extensions

In this section, we will prove Corollary 1.1 and also generalize Theorem 1.1 to broader settings including heat equations on some Lie groups and noncompact Riemannian manifolds.

*Proof of Corollary 1.1.* Since $V$ satisfies Condition 1.1, there exist $\lambda > 1$ and $R > 0$ and an infinite sequence of integers $\{k_j\}$ such that

$$\int_{B(0,R)} \frac{(1-\epsilon)a(x/k_j)}{1+|x|^2}\phi^2_{k_j}dx \ge \lambda \int_{B(0,R)} |\nabla\phi_{k_j}|^2 dx \tag{3.1}$$

for some $\phi_{k_j} \in C_0^\infty(B(0,R))$ and $\epsilon > 0$.

By the assumption that $A(0) = 1$ and $A = A(x)$ is continuous at 0, the above implies

$$-\int_{B(0,R/k_j)} |A\nabla\phi_{k_j}|^2 dx + \int_{B(0,R/k_j)} \frac{(1-\epsilon)a(x)}{(1/k_l^2)+|x|^2}\phi_k^2 dx \ge (\lambda - 1 - o(R/k_j)) \int_{B(0,R/k_j)} |\nabla\phi_{k_j}|^2 dx.$$

Using the Poincaré inequality, we obtain, for a $c > 0$,

$$-\int_{B(0,R/k_j)} |A\nabla\phi_{k_j}|^2 dx + \int_{B(0,R/k_j)} \frac{(1-\epsilon)a(x)}{(1/k_l^2)+|x|^2}\phi^2_{k_j}dx \ge c(\tfrac{k_j}{R})^2 \int_{B(0,R/k_j)} \phi^2_{k_j}dx.$$

Note that the ground state energy $\lambda_{k_j}$ of the operator $L_j = -\nabla(A\nabla) - (1-\epsilon)V_{k_j}$ in $D$ is given by

$$\inf_{\phi\in C_0^\infty(D)} \frac{\int_{\mathbf{R}^n}(|A\nabla\phi|^2 dx - \frac{(1-\epsilon)a(x)}{(1/k_l^2)+|x|^2}\phi^2)dx}{\int_{\mathbf{R}^n}\phi^2(x)dx}.$$

When $k_j$ is sufficiently large, we have $B(0, R/k_j) \subset D$. Hence the above shows that

$$\lambda_{k_j} \leq -c_0 k_j^2.$$

By Hardy's inequality and scaling, it is easy to see that $\lambda_{k_j} \geq -c_1 k_j^2$, where $c_1 > 0$ is independent of $j$. This shows that $\sigma_{\inf}((1-\epsilon)V; D) = -\infty$. □

Next we prove that the examples given in Remark 1.2 indeed satisfy Condition 1.1.

(i). If $V(x) \geq a/|x|^2$ in any ball $B(0,r)$, $r > 0$, where $a > ((n-2)/2)^2$ is a constant, then $V$ satisfies Condition 1.1. Here is a proof.

Since the potential $a/|x|^2$ violates Hardy's inequality, there exists a nonnegative $\phi \in C_0^\infty(\mathbf{R}^n)$ and $\lambda > 1$ such that

$$\int_{\mathbf{R^n}} \frac{a}{|x|^2}\phi^2 dx \geq \lambda \int_{\mathbf{R^n}} |\nabla\phi|^2 dx.$$

By a scaling $x = kx'$, we know that the support of the above $\phi$ can be chosen as the ball $B(0,r)$. Therefore, for any small $\delta \in (0, \lambda - 1)$, there exists an $r_1 > 0$ and $k_1 > 0$ such that

$$\int_{B(0,r)} \frac{a}{(1/k_1^2) + |x|^2}\phi^2 dx \geq (\lambda - \delta) \int_{B(0,r)} |\nabla\phi|^2 dx.$$

Writing $\phi_{k_1}(y) = \phi(y/k_1)$ we have, after a scaling,

$$\int_{B(0,k_1 r)} \frac{a}{1+|y|^2}\phi_{k_1}^2 dy \geq (\lambda - \delta) \int_{B(0,k_1 r)} |\nabla\phi_{k_1}|^2 dy.$$

Note that the support of $\phi_{k_1}$ is a bounded domain. Hence $V$ satisfies Condition 1.1 by choosing $\phi_{k_j} = \phi_{k_1}$, $j = 1, 2, \ldots$.

(ii) Let us write $x = (x_1, x_n)$ where $x_1 \in \mathbf{R}^{n-1}$. Suppose, for $x \in D$, $V = a(x)/|x|^2$ such that $a(x) = ((n-2)/2)^2$ if $x_n < 0$ and $a(x) = a > ((n-2)/2)^2$ if $x_n \geq 0$. Then $V$ satisfies Condition 1.1. Here is a proof.

By direct computation (see [BG1] or [D]), there exists a radial function $\phi = \phi(x)$ such that, for some $\epsilon > 0$,

$$\int_{\mathbf{R}^n} \frac{b}{|x|^2}\phi^2 > (1+\epsilon) \int_{\mathbf{R}^n} |\nabla\phi|^2,$$

when $b > a_0 = ((n-2)/2)^2$. Hence, taking $b = a_0 + \frac{a-a_0}{2}$, we have

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{a(x)}{|x|^2}\phi^2 &= \int_{\mathbf{R}^n} \frac{a_0}{|x|^2}\phi^2 + \int_{x_n \geq 0} \frac{a-a_0}{|x|^2}\phi^2 = \int_{\mathbf{R}^n} \frac{a_0}{|x|^2}\phi^2 + \frac{1}{2}\int_{\mathbf{R}^n} \frac{a-a_0}{|x|^2}\phi^2 \\ &\geq (1+\epsilon) \int_{\mathbf{R}^n} |\nabla\phi|^2.\end{aligned}$$

Approximating $\phi$ by a compactly supported function $\phi_1 \in C^\infty(B(0,r))$ for a large $r$, we can find $\lambda > 1$ such that

$$\int_{B(0,r)} \frac{a(x)}{|x|^2}\phi_1^2 \geq \lambda \int_{B(0,r)} |\nabla\phi_1|^2.$$

Now the proof is reduced to repeating example (i).

The following recovers the blow-up result of a recent paper [GZ] on degenerate equations. Even though the current method does not need involved tools such as the Harnack chain and is much shorter, the method in [GZ] provides a pointwise estimate on the blow-up rate of positive solutions.

Let $\Delta$ be the subelliptic Laplacian on the Heisenberg group $\mathbf{H}^n$ of degree $n \in \mathbf{N}$. Here we just mention that $\Delta$ is a degenerate elliptic operator and $\mathbf{H}^n$ is the space $\mathbf{R}^{2n+1}$ endowed with a group structure. For a detailed description on those objects, we refer the reader to [GL] and [GZ]. □

**Corollary 3.1.** (i) *Suppose $a > a^* = ((q-2)/2)^2 = n^2$. Then the problem*

(3.2)
$$\begin{cases} \Delta u(z,l,t) + \frac{a|z|^2}{|z|^4+l^2} u(z,l,t) - u_t(z,l,t) = 0, \\ u(z,l,0) = u_0(z,l), (z,l) = (x,y,l) \in \mathbf{H}^n,\ x,y \in \mathbf{R}^n,\ l \in \mathbf{R}, \quad t \in (0,T],\ T > 0 \end{cases}$$

*has no nonnegative solutions except $u \equiv 0$.*

(ii) *Suppose $a \le a^*$. Then the above problem has a positive solution for any $u_0 > 0$ in $L^2(\mathbf{H}^n)$.*

*Proof.* The proof is almost identical to that in Section 2. Indeed, let us write $V = \frac{a|z|^2}{|z|^4+l^2}$. By [GZ], in case (i), $\sigma_{\inf}((1-\epsilon)V; D) = -\infty$ and in case (ii) $\sigma_{\inf}(V; D) > -\infty$. Here $D$ is any ball containing 0 and $\epsilon$ is sufficiently small. It is known that the embedding (2.6) still holds when $n/(n-1)$ is replaced by a suitable $p > 1$. The Harnack inequality also holds (see [MS-C] or [GN] and the references there e.g.). Case (ii) is proven in [GZ]. The proof is complete. □

**Corollary 3.2.** *Let $\mathbf{M}$ be an $n(\ge 2)$-dimensional Riemannian manifold and $\Delta$ be the Laplace-Beltrami operator. Let $D$ be a bounded Lipschitz domain of $\mathbf{M}$.*

*Suppose $\sigma_{\inf}((1-\epsilon)V; D) = -\infty$ for some $\epsilon > 0$. Then the problem*

$$\begin{cases} \Delta u(x,t) + V(x)u(x,t) - \partial_t u(x,t) = 0,\ x \in D, t > 0, \\ u(x,t) = 0, x \in \partial D, t > 0; u(x,0) = u_0(x) \end{cases} \tag{3.3}$$

*has no nonnegative solutions except $u \equiv 0$. Moreover, all positive solutions blow up completely and instantaneously.*

(ii) *Suppose $\sigma_{\inf}(V; D) > -\infty$. Then (3.3) has a positive solution for some $u_0 > 0$.*

*Proof.* The proof is similar to that of Theorem 1.1. We only need to replace the Sobolev embedding and the Poincaré inequality in the Euclidean case by their counterparts on $D$ so that the embedding (2.5) still holds. Since $D$ is a bounded domain, this is always possible (see [MS-C] or [SC]). □

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