TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 1, Pages 1–36
S 0002-9947(02)03092-1
Article electronically published on August 28, 2002

ON THE MINIMAL FREE RESOLUTION OF $n+1$ GENERAL FORMS

J. MIGLIORE AND R. M. MIRÓ-ROIG

ABSTRACT. Let $R = k[x_1, \dots, x_n]$ and let I be the ideal of $n+1$ generically chosen forms of degrees $d_1 \le \cdots \le d_{n+1}$. We give the precise graded Betti numbers of R/I in the following cases:

- $n = 3$;
- $n = 4$ and $\sum_{i=1}^{5} d_i$ is even;
- $n = 4$, $\sum_{i=1}^{5} d_i$ is odd and $d_2 + d_3 + d_4 < d_1 + d_5 + 4$;
- n is even and all generators have the same degree, a, which is even;
- $(\sum_{i=1}^{n+1} d_i) - n$ is even and $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$;
- $(\sum_{i=1}^{n+1} d_i) - n$ is odd, $n \ge 6$ is even, $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$ and $d_1 + \cdots + d_n - d_{n+1} - n \gg 0$.

We give very good bounds on the graded Betti numbers in many other cases. We also extend a result of M. Boij by giving the graded Betti numbers for a generic compressed Gorenstein algebra (i.e., one for which the Hilbert function is maximal, given n and the socle degree) when n is even and the socle degree is large. A recurring theme is to examine when and why the minimal free resolution may be forced to have redundant summands. We conjecture that if the forms all have the same degree, then there are *no* redundant summands, and we present some evidence for this conjecture.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be a homogeneous polynomial ring over some field k, and let $I = (G_1, \dots, G_d)$ be an ideal of forms of fixed degrees (not necessarily equal) chosen sufficiently generally. A very long-standing problem in commutative algebra is to determine the Hilbert function of R/I. Then a much more subtle question is to understand all of the syzygies, i.e., to find the minimal free resolution of R/I. If $d \le n$, then I is a complete intersection, and its minimal free resolution is given by the Koszul resolution. So we assume $d > n$, which in particular means that R/I is Artinian. A. Iarrobino and R. Fröberg have made conjectures about the Hilbert function, and Iarrobino has made a conjecture for the minimal free resolution in this case. One of the consequences of our work is to give a counterexample to the latter conjecture.

Several contributions to this very difficult problem have been made. We first discuss the Hilbert function. If $d = n+1$, then the Hilbert function is well known, coming from a result of R. Stanley [28] and of J. Watanabe [29], which implies

Received by the editors October 1, 2001 and, in revised form, March 10, 2002.
2000 *Mathematics Subject Classification.* Primary 13D02, 13D40; Secondary 13P10, 13C40, 13H10.
The first author was partially supported by the University of Barcelona.
The second author was partially supported by BFM2001-3584.

that a general Artinian complete intersection has the Strong Lefschetz property (cf. Definition 2.2). We will discuss this shortly, but for now we do not yet assume $d = n + 1$. The case $n = 2$ was solved by R. Fröberg [13]. The case $n = 3$ was solved by D. Anick [1]. M. Hochster and D. Laksov [19] showed that a generically chosen set of forms of the same degree span as much as possible in the next degree. Note that this gives the value of the Hilbert function in the next degree, and it also gives the number of linear syzygies of the forms. This was extended by M. Aubry [2]. Also, R. Fröberg and J. Hollman [14] determined the Hilbert function for s general forms of degree 2 if $n \leq 11$, and for forms of degree 3 if $n \leq 8$.

Apart from the work of M. Hochster and D. Laksov, nothing seems to be published about the problem of finding the minimal free resolution for general forms. This is the central problem which we address in this paper. We remark that very different approaches to this subject are being carried out by Ben Richert and Keith Pardue [26] and by Karen Chandler [8].

An analogous problem is the Minimal Resolution Conjecture [23]. A generic set of points in projective space has so-called *generic Hilbert function*, which depends only on the number of points. (Here by "a generic set of points" we mean corresponding to a dense Zariski-open subset of the corresponding product of projective spaces.) The Minimal Resolution Conjecture asked whether the entire resolution similarly is the "expected" one, in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function. In particular, it requires that there not be any "ghost" terms in the resolution, i.e., that consecutive terms in the resolution never have a summand in common. (Ghost terms cannot be detected from the Hilbert function alone.) A very interesting result, due to D. Eisenbud and S. Popescu [12], is that this conjecture is not true! The first counterexample is the case of a generic set of 11 points in $\mathbb{P}^6$, discovered in computational experiments by F. Schreyer, where there is a summand $R(-5)$ in both the third and fourth syzygies which does not split off.

A generic set of eleven points in $\mathbb{P}^6$ lies on 17 independent quadrics. A natural question is then whether 17 general quadrics in $k[x_1, \dots, x_6]$ also pick up a ghost term, or whether, in fact, it has the expected resolution. One can check that, in fact, it does have the expected resolution (cf. [22], page 197). So the Artinian reduction of a generic set of 11 points in $\mathbb{P}^6$ is not "general" enough, as an Artinian algebra, to have the minimum possible Betti numbers consistent with the Hilbert function. When one considers furthermore that the ideal of a generic set of points is generated in either one degree or two consecutive degrees, while our ideals of general forms have any fixed generator degrees, it is clear that these problems are rather different despite the apparent similarities. Nevertheless, a surprising connection is developed in Section 3 of this paper, using the work of A. Hirschowitz and C. Simpson [18] on the Minimal Resolution Conjecture. Furthermore, the case of general forms all of the same degree is an important special case that is closer to the problem of generic points. In this paper we also consider this case (e.g., Corollary 4.4 and Theorem 5.4).

This question of whether ghost terms exist in the minimal free resolution was of central interest to us in writing this paper. It is clear that they cannot be entirely avoided. For instance, if our chosen degrees include two forms of degree 4 and one of degree 8, then we naturally expect a Koszul syzygy of degree 8. So there is a summand $R(-8)$ in the first syzygy module which does not split off with the summand $R(-8)$ corresponding to the generator.

A natural conjecture, due to A. Iarrobino [21], is that the ghost terms arising as a result of Koszul syzygies should be the only kind of exception. Called the Thin Resolution Conjecture, it says that "the minimal free resolution ... is the minimum one that is consistent with their (expected) Hilbert function; that is, the Koszul resolution up to the smallest degree where" $(R/I)_i = 0$ (cf. [22], page 197).

One result of our work is a clearer understanding of the fact that other ghost terms do in fact arise! For instance, we show in Example 4.3 that when $n = 3$ and for general forms of degree 4, 4, 4 and 8 respectively, the minimal free resolution is

$$0 \to \begin{pmatrix} R(-10) \\ \oplus \\ R(-11)^2 \end{pmatrix} \to \begin{pmatrix} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{pmatrix} \to \begin{pmatrix} R(-4)^3 \\ \oplus \\ R(-8) \end{pmatrix} \to R \to R/I \to 0.$$

We see that the $R(-8)$ does not split off, as predicted above, but that furthermore there is a summand $R(-10)$ shared by the second and third modules, which also does not split off. Notice that the Hilbert function of R/I is

$$1 \ \ 3 \ \ 6 \ \ 10 \ \ 12 \ \ 12 \ \ 10 \ \ 6 \ \ 2,$$

and that the summand $R(-10)$ does not correspond to a Koszul syzygy.

A. Iarrobino informs us that the above example is a counterexample to his Thin Resolution Conjecture, and furthermore that it is a counterexample to his published statement [21] that the Thin Resolution Conjecture had been shown to be equivalent to Fröberg's Conjecture on the Hilbert function. Other examples of ghost terms that arise can be found in Example 3.11, Example 4.3 and Example 5.7, but one can produce more from the theorems.

This paper concerns solely the case of $n + 1$ general forms in $k[x_1, \dots, x_n]$. By this we mean the following.

Definition 1.1. Let $I = (G_1, \dots, G_{n+1})$, where $\deg G_i = d_i$ and $d_1 \le \cdots \le d_{n+1}$. We always assume that $(G_1, \dots, G_n)$ is a complete intersection, and that $d_{n+1} \le (\sum_{i=1}^n d_i) - n$. (This latter condition is assumed because otherwise G_{n+1} is in the ideal generated by the first n generators, and so I is a complete intersection.) When we say that $G_1, \dots, G_{n+1}$ are *general forms* in R, we mean that they belong to a suitable dense open subset of $R_{d_1} \times \cdots \times R_{d_{n+1}}$. Note that I is an Artinian almost complete intersection.

Our first observation (which is not new) is that such an ideal I can be linked to a Gorenstein ideal G via the complete intersection J defined by the first n generators of I. In Lemma 2.6, we give some facts about the Hilbert function of R/G. One is to note that the Hilbert function of R/G is strictly increasing until it attains its maximum value, which occurs either once ("one peak") or twice ("two peaks"), and then is strictly decreasing. In particular, it is unimodal. Furthermore, we describe exactly when R/G is a *compressed* Gorenstein algebra, i.e., the Hilbert function of R/G is maximal, given the socle degree and the embedding dimension n.

It is worth noting that the Hilbert function has one peak if and only if the socle degree $\left(\sum_{i=1}^{n+1} d_i\right) - n$ is even. The technical condition for R/G to be compressed is $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$.

As a consequence of Lemma 2.6, we show in Corollary 2.7 that $A := R/G$ has the Strong Lefschetz property. This is central especially for Section 5, where we have our strongest results, because it allows us to compute the Hilbert function of A/LA for a general linear form L.

We observe in Section 3 that a free resolution for R/I can be given in terms of one for R/G (again this is not new), and that we can control to a large degree the possible splitting off. So the problem is reduced to finding a minimal free resolution for R/G. Sections 3, 4 and 5 give different approaches to this, for different situations.

In Section 3, we first use a result (Corollary 3.5) of the first author and U. Nagel [25], which gives the precise minimal free resolution for R/G when it satisfies a condition that is slightly weaker than being compressed with Hilbert function having only one peak. However, we only need it for the case when it is compressed with only one peak. The referee has pointed out to us that in this latter situation, the result also follows from [15], Proposition 16(iii), or [11], Theorem A1, since these results show that the resolution is pure, the minimum one consistent with the Hilbert function. We then determine exactly what splitting can occur for the linked ideal, giving the minimal free resolution for R/I (Corollary 3.10).

The more difficult situation (still assuming that R/G is compressed) is when the Hilbert function of R/G has two peaks, (i.e., the socle degree is odd). Here the results of [25] do not give sharp bounds on the graded Betti numbers for R/G. However, a result of M. Boij [5] on generic compressed Gorenstein algebras is helpful here when $n = 4$. We generalize Boij's result, giving the minimal free resolution of a generic compressed Gorenstein algebra when n is even and the socle degree is large (Proposition 3.13). In the case where n is odd and the socle degree is large, we give bounds for the graded Betti numbers in Remark 3.14.

In section 4 we give a complete answer to the resolution problem for $n = 3$. Our method is to apply the work of S. Diesel [10] to find the minimal free resolution for the generic Gorenstein algebra with the known Hilbert function (coming only from the choice of $d_1, \dots, d_4$), and then apply our methods to determine all the splitting that occurs. The main result here is Theorem 4.2. Note that when $n = 3$, Diesel has shown that the family of graded Artinian Gorenstein algebras with given Hilbert function is irreducible. So it makes sense to talk about the generic Gorenstein algebra with given Hilbert function. Diesel's work in turn relied heavily on the Buchsbaum-Eisenbud structure theorem [7].

In Section 5 we use a different result of the first author and U. Nagel [25] to make a more subtle study of the minimal free resolution of the Gorenstein algebra $A = R/G$ when $n > 3$. Note that in general we can no longer assume that A is compressed, which greatly complicates the problem. The procedure is the following. First determine the Hilbert function of A/LA for a general linear form L, which is known from the Weak Lefschetz property. Then determine the graded Betti numbers of A/LA over $R/(L)$. This information, together with the result from [25], allows us to establish very good bounds for the graded Betti numbers of A. A careful analysis then shows that these bounds are actually sharp! Finally, the link to I is studied, and it is determined exactly what splitting occurs, resulting in the minimal free resolution of the almost complete intersection R/I. This program gives the following (Theorems 5.4 and 5.6):

- Assume all $n+1$ generators of I have the same degree, a. Let $s(n,a) = (n-1)a - n$. Then
  - If n is odd, we give upper bounds on the graded Betti numbers of R/I that may allow some ghost terms.
  - If n is even and $s(n,a)$ is odd, we give upper bounds on the graded Betti numbers of R/I that may allow some ghost terms.
  - If n is even and $s(n,a)$ is even, then we give the precise minimal free resolution for R/I.
- Assume that $n = 4$ and that $\sum_{i=1}^{5} d_i$ is even. Then we give the precise minimal free resolution for R/I.

We note that the approach of this section can be applied in other situations, when the generators do not all have the same degree, but that the notation quickly becomes overwhelming.

Most of this work was done while the first author was a guest of the University of Barcelona, and he would like to thank the professors and students of the Departament d'Algebra i Geometria for their warm hospitality. The authors also thank Karen Chandler, Tony Geramita and Tony Iarrobino for helpful references and comments. Finally, the authors sincerely thank the referee for the careful reading and helpful comments.

2. Hilbert function calculations

Let $R = k[x_1, \dots, x_n]$, where k is an algebraically closed field (although we remark that this hypothesis is needed only for §3 beginning with Proposition 3.13). For any homogeneous ideal $I \subset R$, we denote the Hilbert function of R/I by $h_{R/I}(t)$. If R/I is Gorenstein, we sometimes refer to I itself as being Gorenstein. In this paper, for a numerical function f, we denote by Δf the first difference function $\Delta f(t) = f(t) - f(t-1)$ for all $t \in \mathbb{Z}$. From now on in this paper, an ideal I shall denote an ideal with $n+1$ minimal generators, cutting out an Artinian quotient (hence I is an almost complete intersection), and the generators of I will be supposed to be general in the sense of Definition 1.1, i.e., our stated results about an ideal of general forms hold on a suitable dense open set.

Definition 2.1. Let $\underline{h} = (h_0, \dots, h_s)$ be a sequence of positive integers. $\underline{h}$ is called a *Gorenstein sequence* if it is the Hilbert function of some Gorenstein Artinian k-algebra. $\underline{h}$ is *unimodal* if $h_0 \le h_1 \le \cdots \le h_j \ge h_{j+1} \ge \cdots \ge h_s$ for some j. $\underline{h}$ is called an *SI-sequence* (for Stanley-Iarrobino) if it satisfies the following two conditions:

(i) $\underline{h}$ is symmetric, i.e., $h_{s-i} = h_i$ for all $i = 0, \dots, \lfloor \frac{s}{2} \rfloor$;
(ii) $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_j - h_{j-1})$ is an O-sequence, where $j = \lfloor \frac{s}{2} \rfloor$; i.e., the "first half" of $\underline{h}$ is a *differentiable* O-sequence.

Definition 2.2. A standard graded Artinian k-algebra $A = \bigoplus_{i \ge 0} A_i$ has the *Weak Lefschetz property* (sometimes called the *Weak Stanley property*) if for each i, the multiplication $A_i \to A_{i+1}$ induced by a general linear form L has maximal rank. A has the *Strong Lefschetz property* if for each i and each $d \ge 1$, the multiplication $A_i \to A_{i+d}$ induced by an L^d, for a general linear form L, has maximal rank.

It was shown by T. Harima [16] that a given finite sequence of integers $\underline{h}$ is the Hilbert function of some graded Artinian Gorenstein k-algebra with the Weak Lefschetz property if and only if $\underline{h}$ is an SI-sequence. We also remark that if we do not restrict to the Gorenstein case, the set of all possible Hilbert functions for Artinian k-algebras with the Weak Lefschetz property was described in [17], and this set *coincides with* the analogous set for the Strong Lefschetz property. It is not known if these two sets coincide in the Gorenstein case.

Notation 2.3. Let $I = (G_1, \dots, G_n, G_{n+1}) \subset R$ be an ideal, where $\deg G_i = d_i$ for $1 \le i \le n+1$, and $G_1, \dots, G_{n+1}$ are general forms in the sense of Definition 1.1. We will call I a *general Artinian almost complete intersection of type* $(d_1, \dots, d_n, d_{n+1})$.

Remark 2.4. Let I be a general Artinian almost complete intersection of type $(d_1, \dots, d_n, d_{n+1})$. Without loss of generality, assume that $d_1 \le \cdots \le d_{n+1}$. Because the forms are chosen generically, we may assume that $J := (G_1, \dots, G_n)$ forms a regular sequence. If $d_{n+1} > (\sum_{i=1}^n d_i) - n$, then $G_{n+1} \in J$ (since it is Artinian) and hence $I = J$, and the Hilbert function and minimal free resolution of I are hence known (from the Koszul resolution). So without loss of generality, *from now on we assume that* $d_{n+1} \le (\sum_{i=1}^n d_i) - n$.

The Hilbert function of R/I is well known, and we remind the reader of the main idea. R. Stanley [28] and J. Watanabe [29] independently showed that a general Artinian complete intersection J has the Strong Lefschetz property. Hence we have $h_{R/I}(t) = \max\{h_{R/J}(t) - h_{R/J}(t - d_{n+1}), 0\}$, and the values of $h_{R/J}(t)$ are known thanks to the Koszul resolution. In particular, we have

Lemma 2.5. *The maximal socle degree of R/I (i.e., the degree of the last nonzero component of R/I) is*

$$\left\lfloor \frac{1}{2} \left(\left(\sum_{i=1}^{n+1} d_i \right) - n - 1 \right) \right\rfloor ,$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

Proof. We have assumed that $d_1 \le \cdots \le d_n \le d_{n+1}$. Furthermore, we know that the Hilbert function of R/J is symmetric and ends in degree $\sum_{i=1}^n d_i - n$. Hence the "flat part" of the Hilbert function has length less than d_{n+1}. Because of the Strong Lefschetz property of R/J, the degree we are looking for is the greatest j for which $\dim(R/J)_j > \dim(R/J)_{j-d_{n+1}}$. This is a slightly tedious but easy computation, checking different parity cases. It requires only the facts mentioned, and not the precise values of these dimensions. ☐

The ideal J links I to an ideal $G := [J : I]$, which is easily seen to be arithmetically Gorenstein, using the standard mapping cone construction (cf. [27], [24]). Since the Hilbert functions of R/J and of R/I are known, we can compute the Hilbert function of R/G (cf. [9], [24]). More precisely, we have

(2.1)

$$
\begin{aligned}
h_{R/G}(t) &= h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - t \right) - h_{R/I}\left((\sum_{i=1}^{n} d_i) - n - t \right) \\
&= h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - t \right) \\
&\qquad - \left[h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - t \right) - h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - t - d_{n+1} \right) \right]_+ ,
\end{aligned}
$$

where we denote by $[x]_+$ the maximum of x and zero. Since R/G is Gorenstein, this Hilbert function is symmetric; so we only have to compute half of it.

For our purposes below we need to know precisely the number of degrees t for which the Hilbert function $h_{R/G}(t)$ achieves its maximum value. We will say that the Hilbert function *has r peaks* if there are r such degrees, and we will note that these must occur consecutively (i.e., the Hilbert function is unimodal). We also will need to know under what conditions the Hilbert function is compressed: that is, R/G coincides with the polynomial ring up to the degree where $h_{R/G}$ attains its maximum. Recall that we assume that I is a generally chosen almost complete intersection throughout this section.

Lemma 2.6. *Let G be the arithmetically Gorenstein ideal linked to I by the complete intersection J as above.*

(a) *The socle degree of R/G is $(\sum_{i=1}^{n} d_i) - d_{n+1} - n$.*
(b) *The Hilbert function $h_{R/G}$ is unimodal, with one peak if $(\sum_{i=1}^{n+1} d_i) - n$ is even, and two peaks if it is odd.*
(c) *For all integers $t \le \frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n}{2}$, we have $h_{R/G}(t) = h_{R/J}(t)$. By symmetry of $h_{R/G}$, this completely determines $h_{R/G}$.*
(d) *We have*

$$h_{R/G}(t) = \binom{t+n-1}{n-1} \text{ for all integers } 0 \le t \le \frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n}{2}$$

if and only if $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$.

Proof. Clearly R/I and R/J first differ in degree d_{n+1}. Since R/J ends in degree $(\sum_{i=1}^{n} d_i) - n$, (a) follows immediately from a Hilbert function calculation as indicated in (2.1).

For (b), note first that the parity does not change if we replace $\sum_{i=1}^{n+1} d_i$ by $\sum_{i=1}^{n} d_i - d_{n+1}$. Because of (a) and the symmetry of the Hilbert function of R/G, we know that the number of peaks of R/G will be odd if $(\sum_{i=1}^{n+1} d_i) - n$ is even, and even if $(\sum_{i=1}^{n+1} d_i) - n$ is odd (once we have shown unimodality).

First assume that $(\sum_{i=1}^{n+1} d_i) - n$ is even. We want to show that $h_{R/G}$ is unimodal with one peak, which by (a) and symmetry would have to occur in degree $\frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n}{2}$. Using Lemma 2.5 and the formula (2.1), one quickly can check that for any $t \geq 0$ we have

$$h_{R/G}\left(\frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n}{2} - t\right) = h_{R/J}\left(\frac{(\sum_{i=1}^{n+1} d_i) - n}{2} + t\right).$$

Because of our hypothesis that $d_{n+1} \leq (\sum_{i=1}^{n} d_i) - n$, one can check that $d_n \leq d_{n+1} \leq \frac{(\sum_{i=1}^{n+1} d_i)-n}{2}$. Hence for $t \geq 0$ the right-hand side of the above equation is strictly decreasing. This proves (b) for $(\sum_{i=1}^{n} d_i) - d_{n+1} - n$ even. Furthermore, (c) is easy to check using the symmetry of $h_{R/J}$.

If $(\sum_{i=1}^{n+1} d_i) - n$ is odd, note that the fraction in the statement of (c) is not an integer. The proof of (b) and (c) is identical to that of the previous case, simply replacing $\frac{(\sum_{i=1}^{n} d_i)-d_{n+1}-n}{2}$ by $\frac{(\sum_{i=1}^{n} d_i)-d_{n+1}-n-1}{2}$.

The proof of (d) follows immediately from (c) and a calculation, by setting $d_1 > \frac{\sum_{i=1}^{n} d_i - d_{n+1} - n}{2}$ and simplifying. □

Corollary 2.7. *R/G has the Strong Lefschetz property.*

Proof. We have $J = (G_1, \ldots, G_n)$, $I = J + (G_{n+1})$ and $G = [J : I]$ in the polynomial ring $R = K[x_1, \ldots, x_n]$. Thanks to the result of R. Stanley [28] and of J. Watanabe [29], R/J has the Strong Lefschetz property. Now consider the exact sequence (not graded)

$$\begin{array}{ccccccc}
0 \to [0 :_{R/J} G_{n+1}] \to & R/J & \xrightarrow{G_{n+1}} & R/J & \to R/I \to 0 \\
& & \searrow \quad \nearrow & & \\
& & I/J & & \\
& & \nearrow \quad \searrow & & \\
& & 0 \qquad\quad 0 & &
\end{array}$$

By Theorem 4.14 of [22], if A is a graded algebra with the Strong Lefschetz property, then so is $A/[0 : F]$, where F is a general element of A. Therefore, the cokernel I/J has the Strong Lefschetz property. But J is a complete intersection linking I to G. So we have an isomorphism

$$I/J \cong K_G(n - d),$$

where K_G is the canonical module of R/G and $d = \sum_{i=1}^{n} d_i$. Since R/G is Gorenstein, K_G is isomorphic to R/G up to twist. It follows that R/G has the Strong Lefschetz property, as claimed. □

3. First bounds for the graded Betti numbers of an almost complete intersection

In this section we are interested in describing the minimal free resolution for a general almost complete intersection in $R = k[x_1, \ldots, x_n]$.

First we consider the case where one of the generators has degree 1. Our approach is the same as that of Lemma 2.6 and Corollary 2.7 of [17], although the result we obtain in (b) is not explicit, as it is in [17] for the case of three variables. We give the precise graded Betti numbers if d_{n+1} is large enough, and we show how to reduce it to the same problem in a smaller polynomial ring otherwise.

Proposition 3.1. *Let $I = (L, G_2, \ldots, G_{n+1})$ be a general almost complete intersection in $R = k[x_1, \ldots, x_n]$, with $d_1 = \deg L = 1$, $d_i = \deg G_i$ for $i \geq 2$, and $d_2 \leq \cdots \leq d_n \leq d_{n+1}$. Let $J = (G_2, \ldots, G_{n+1})$ and $\bar{R} = R/(L)$, and let $\bar{J} \subset \bar{R}$ be* *the reduction of J modulo L. Note that $\bar{J}$ is a general almost complete intersection in $\bar{R}$ of type $(d_2, \dots, d_{n+1})$.*

(a) *If $d_{n+1} > (\sum_{i=1}^{n} d_i) - n$, then $G_{n+1} \in (L, G_2, \dots, G_n)$; so $I = (L, G_2, \dots, G_n) \subset R$ is an Artinian complete intersection, and its minimal free resolution over R is given by the Koszul resolution.*

(b) *If $d_{n+1} \leq (\sum_{i=1}^{n} d_i) - n$, let $J' = R\bar{J} \subset R$. Then J' is the saturated ideal of an almost complete intersection ideal in R with* $\operatorname{depth} R/J' = 1$, *and the minimal free resolution of R/I is given by the tensor product of R/J' and $R/(L)$. In particular,*

$$[\operatorname{tor}_i^R(R/I, k)]_j = [\operatorname{tor}_i^{\bar{R}}(\bar{R}/\bar{J}, k)]_j + [\operatorname{tor}_{i-1}^{\bar{R}}(\bar{R}/\bar{J}, k)]_{j+1}$$

for $1 \leq i \leq n$.

Proof. Part (a) is immediate from Remark 2.4. For part (b), note that $I = J + (L) = J' + (L)$. The fact that J' is Cohen-Macaulay of depth one follows since $\bar{J}$ is Artinian in $\bar{R}$. So L is not a zero divisor for R/J'. So the graded Betti numbers of J' over R are the same as those of $\bar{J}$ over $\bar{R}$. Note that $\operatorname{Tor}_i^{\bar{R}}(\bar{R}/\bar{J}, k) = \operatorname{Tor}_i^R(R/J', k) = 0$ for $i \geq n$. □

For the remainder of this section, we will use in an essential way some results from [25]. We first collect some notation.

Notation 3.2. Let $R = k[x_1, \dots, x_n]$ and let $I \subset R$ be a general Artinian almost complete intersection of type $(d_1, \dots, d_n, d_{n+1})$. As before, let J be the complete intersection given by the first n generators of I, and let G be the linked Gorenstein ideal. Let $A = R/G$.

For a graded R-module M, set $[\operatorname{tor}_i^R(M, k)]_j := \dim_k [\operatorname{Tor}_i^R(M, k)]_j$.

We set

$$s = \sum_{i=1}^{n} d_i - d_{n+1} - n,$$

$$\alpha = \left\lfloor \frac{s+1}{2} \right\rfloor.$$

Note that s is the socle degree of A (Lemma 2.6) and α is the degree in which the last peak (of which there are either one or two) occurs in the Hilbert function of A. Since A has the Weak Lefschetz property (even the Strong Lefschetz property – see Corollary 2.7), then we also have $\alpha = \operatorname{in}[0 :_A L]$ for a general linear form L, where for a homogeneous ideal I, $\operatorname{in}(I)$ is its initial degree.

We will need the following results, which we specialize slightly for our purposes.

Notation 3.3. Let $\underline{h} = (1, h_1, \dots, h_s)$ be the h-vector of an Artinian k-algebra. Let $c \geq h_1$ be an integer. Then there is a uniquely determined lex-segment ideal $I \subset k[z_1, \dots, z_c] =: T$ such that T/I has $\underline{h}$ as its Hilbert function. We define

$$\beta_{i,j}(\underline{h}, c) := \left[\operatorname{tor}_i^T(T/I, k)\right]_j.$$

If $c = h_1$, we simply write $\beta_{i,j}(\underline{h})$ instead of $\beta_{i,j}(\underline{h}, h_1)$.

Using the main result of [4] and [20], the following was obtained:

Theorem 3.4 ([25], Theorem 8.13)**.** *Let $\underline{h} = (1, h_1, h_2, \dots, h_\ell, \dots, h_s)$ be an SI-sequence where $h_{\ell-1} < h_\ell = \cdots = h_{s-\ell} > h_{s-\ell+1}$. Put $\underline{g} = (1, h_1 - 1, h_2 - h_1, \dots, h_\ell - h_{\ell-1})$. Then we have:*

(a) *If $A = R/G$ is a Gorenstein k-algebra with $c = \operatorname{codim} G$ and having $\underline{h}$ as h-vector and an Artinian reduction which has the Weak Lefschetz property, then*

$$\left[\operatorname{tor}_i^R(A,k)\right]_j \leq \begin{cases} \beta_{i,j}(\underline{g}, c-1) & \text{if } j \leq s-\ell+i-1, \\ \beta_{i,j}(\underline{g}, c-1) + \beta_{c-i,s+c-j}(\underline{g}, c-1) & \text{if } s-\ell+i \leq j \leq \ell+i, \\ \beta_{c-i,s+c-j}(\underline{g}, c-1) & \text{if } j \geq \ell+i+1. \end{cases}$$

(b) *Suppose the field k has "sufficiently many elements" (e.g., infinitely many). Then there is a reduced, nondegenerate arithmetically Gorenstein subscheme $X \subset \mathbb{P}^n = \operatorname{Proj}(R)$ of codimension c whose Artinian reduction has the Weak Lefschetz property and h-vector $\underline{h}$, and with equality holding for all $i, j \in \mathbb{Z}$ in* (a)*. This subscheme can be constructed explicitly.*

In the following result, the case of most interest for this paper is when $s = 2\ell$. In this case, the Gorenstein algebras considered are also known as "extremal Gorenstein algebras" or "extremely compressed Gorenstein algebras" of even socle degree and maximum Hilbert function given the socle degree. As mentioned previously, the case $s = 2\ell$ also follows from [15], Proposition 16(iii), or [11], Theorem A1.

Corollary 3.5 ([25], Corollary 8.14)**.** *Let s, ℓ be positive integers, where either $s = 2\ell$ or $s \geq 2\ell + 2$. Define $\underline{h} = (h_0, \dots, h_s)$ by*

$$h_i = \begin{cases} \binom{n-1+i}{n-1} & \text{if } 0 \leq i \leq \ell, \\ \binom{n-1+\ell}{n-1} & \text{if } \ell \leq i \leq s-\ell, \\ \binom{s-i+n-1}{n-1} & \text{if } s-\ell \leq i \leq s. \end{cases}$$

Let $G \subset R$ be an Artinian Gorenstein ideal such that R/G has the Weak Lefschetz property and Hilbert function $\underline{h}$. Then the minimal free R-resolution of R/G has the shape

$$0 \to R(-s-n) \to F_{n-1} \to \cdots \to F_1 \to R \to R/G \to 0,$$

where

$$F_i = R(-\ell-i)^{\alpha_i} \oplus R(\ell-s-i)^{\gamma_i} \quad \text{and} \quad \alpha_i = \binom{n+\ell-1}{i+\ell}\binom{\ell-1+i}{\ell} = \gamma_{n-i}.$$

Remark 3.6. a. In the language of this paper, ℓ refers to the first degree where the peak is achieved, s is the last degree at which the Hilbert function is non-zero, and $s - \ell$ is the last degree where there is a peak. The condition $s = 2\ell$ means that there is one peak, while the case $s \geq 2\ell + 2$ means three or more peaks (which does not occur for us). In our situation,

$$s = (\sum_{i=1}^{n} d_i) - d_{n+1} - n,$$

$$\ell = \begin{cases} \frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n}{2} & \text{if } (\sum_{i=1}^{n+1} d_i) - n \text{ is even,} \\ \\ \frac{(\sum_{i=1}^{n} d_i) - d_{n+1} - n - 1}{2} & \text{if } (\sum_{i=1}^{n+1} d_i) - n \text{ is odd.} \end{cases}$$

b. The Hilbert function described in this corollary is the same as that of Lemma 2.6 (d). That is, it grows maximally up to degree ℓ, is constant in the middle degrees, and then decreases symmetrically. Hence $\underline{h}$ is the maximum Gorenstein sequence of embedding dimension n, socle degree s and bounded above by $\binom{n-1+\ell}{n-1}$.

c. The hypothesis given in Corollary 3.5 that R/G has the Weak Lefschetz property is not needed in the case $s = 2\ell$, since the growth described is precisely that of R up to degree ℓ: this means that multiplication by a general linear form is injective in this range, and the surjectivity in the other degrees comes from the self-duality of R/G. In any case, we have seen that our G even has the Strong Lefschetz property, hence in particular the Weak Lefschetz property.

We now make the basic construction of this section, and then we will draw consequences. We have a general almost complete intersection $I = (G_1, \dots, G_n, G_{n+1})$ with $d_1 \le \cdots \le d_n \le d_{n+1} \le (\sum_{i=1}^{n} d_i) - n$, the complete intersection $J = (G_1, \dots, G_n) \subset I$, and the Gorenstein ideal $G = [J : I]$. Let $d = d_1 + \cdots + d_n$. Consider the minimal free R-resolution of R/J given by the Koszul resolution:

$$0 \to K_n \to K_{n-1} \to \cdots \to K_2 \to K_1 \to R \to R/J \to 0,$$

where

$$K_i = \bigwedge^{i} \left(\bigoplus_{i=1}^{n} R(-d_i) \right).$$

In particular, $K_1 \cong \bigoplus_{i=1}^{n} R(-d_i)$, $K_{n-1} \cong \bigoplus_{j=1}^{n} R(d_j - d)$ and $K_n \cong R(-d)$. Consider the minimal free resolution of R/G:

$$0 \to R(-e) \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R/G \to 0,$$

where $e = (\sum_{i=1}^{n} d_i) - d_{n+1}$ thanks to Lemma 2.6 (a).

Applying the mapping cone construction to the diagram

$$\begin{array}{ccccccccccccccccc} 0 & \to & R(-d) & \to & K_{n-1} & \to & \cdots & \to & K_2 & \to & K_1 & \to & R & \to & R/J & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R(-e) & \to & F_{n-1} & \to & \cdots & \to & F_2 & \to & F_1 & \to & R & \to & R/G & \to & 0 \end{array}$$

gives a free R-resolution for R/I:

(3.1)

$$0 \to F_1^\vee(-d) \to \begin{array}{c} K_1^\vee(-d) \\ \oplus \\ F_2^\vee(-d) \end{array} \to \begin{array}{c} K_2^\vee(-d) \\ \oplus \\ F_3^\vee(-d) \end{array} \to \cdots \to \begin{array}{c} K_{n-1}^\vee(-d) \\ \oplus \\ R(e-d) \end{array} \to R \to R/I \to 0.$$

A simple calculation gives that I has $n+1$ generators, in degrees $d_1, \dots, d_n, d_{n+1}$, as expected. The challenge is to determine how much splitting can occur and to try to narrow down as much as possible what the free modules F_i can be. In any case, we immediately obtain

Proposition 3.7. *With the above notation, we have*

$$[\mathrm{tor}_i^R(R/I,k)]_j \le [\mathrm{tor}_{n-i}^R(R/J,k)]_{d-j} + [\mathrm{tor}_{n-i+1}^R(R/G,k)]_{d-j}.$$

The first term on the right-hand side of the inequality in Proposition 3.7 is obtained from the Koszul resolution and is completely determined. The second term can be bounded as follows:

Corollary 3.8. *With the above notation we have*

$$\mathrm{tor}_i^R(R/I,k)_j \le \mathrm{tor}_{n-i}^R(R/J,k)_{d-j} + B_{n-i+1,d-j},$$

where $B_{a,b}$ is the bound for $[\mathrm{tor}_a^R(R/G,k)]_b$ obtained in Theorem 3.4.

Corollary 3.8 is slightly too general for our purposes (usually). It assumes that the "first half" of R/G is as bad as possible, while in our case it agrees with a complete intersection. The "obvious" first place to look for splitting in Proposition 3.7 is to see if any of the generators G_i of J are minimal generators of G. If this occurs, each such generator leads to a splitting off of a rank one free summand at the end of the resolution of R/I. This need not happen, however, as illustrated by the following example.

Example 3.9. Let $n=4$, $d_1 = \cdots = d_4 = 5$, $d_5 = 10$. By Lemma 2.6 (b), this has one peak, which, as one can compute, occurs in degree $\ell = 3$. By Lemma 2.6 (d), we have that the Hilbert function of R/G is

$$1 \quad 4 \quad 10 \quad 20 \quad 10 \quad 4 \quad 1.$$

Then Corollary 3.5 gives the *minimal* free resolution for R/G, and (3.1) gives the following R-resolution for R/I:

$$0 \to R(-16)^{25} \to R(-15)^{52} \to \begin{array}{c} R(-10)^6 \\ \oplus \\ R(-14)^{25} \end{array} \to \begin{array}{c} R(-5)^4 \\ \oplus \\ R(-10) \end{array} \to R \to R/I \to 0.$$

So the type of splitting mentioned above does not occur. The only possible splitting off would be the summand $R(-10)$ at the beginning of the resolution. However, since we have assumed that $d_{n+1} \le (\sum_{i=1}^n d_i) - n$ (Remark 2.4) and the G_i are chosen generically, G_5 is a minimal generator of I and so this splitting off does not occur either, and the above is the minimal free R-resolution of R/I.

In the next section, and in Example 3.11, we will see examples where overlaps arise in other parts of the resolution and still cannot be split off.

The idea behind Example 3.9 leads to one situation where we can give a minimal free resolution of a general almost complete intersection in any number of variables. If one of the generators has degree 1, then we may pass to the analogous problem in a ring of one fewer variable. So we will assume that our ideal is nondegenerate, i.e., that $d_1 \ge 2$.

Corollary 3.10. *Let $I = (G_1, \dots, G_{n+1})$ be a general almost complete intersection in $R = k[x_1, \dots, x_n]$, with $d_i = \deg G_i$ and $2 \le d_1 \le d_2 \le \cdots \le d_n \le d_{n+1} \le$*

$(\sum_{i=1}^{n} d_i) - n$. *Let* $d = d_1 + \cdots + d_n$. *Assume that* $(\sum_{i=1}^{n+1} d_i) - n$ *is even and that* $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$. *Then* R/I *has a free* R*-resolution of the form*

$$0 \to F_1^\vee(-d) \to \begin{matrix} K_1^\vee(-d) \\ \oplus \\ F_2^\vee(-d) \end{matrix} \to \begin{matrix} K_2^\vee(-d) \\ \oplus \\ F_3^\vee(-d) \end{matrix} \to \cdots \to \begin{matrix} K_{n-2}^\vee(-d) \\ \oplus \\ F_{n-1}^\vee(-d) \end{matrix}$$

$$\to \bigoplus_{i=1}^{n+1} R(-d_i) \to R \to R/I \to 0,$$

where K_i *is the* i*-th free module in the Koszul resolution of* $R/(G_1, \ldots, G_n)$,

$$F_i = R(-\ell - i)^{\alpha_i + \gamma_i} \quad \textit{with} \quad \alpha_i = \binom{n+\ell-1}{i+\ell}\binom{\ell-1+i}{\ell} = \gamma_{n-i},$$

and ℓ *is as defined in Remark 3.6. If* $\ell + 1 = d_i$ *for any* $1 \le i \le n$, *then for each such occurrence, there is a corresponding splitting off of a free summand from* $F_1^\vee(-d)$ *and* $K_1^\vee(-d)$. *Except for such splitting, this resolution is minimal.*

Proof. As we noted in Remark 3.6, Lemma 2.6 shows that the hypotheses on the d_i are precisely what we need in order to apply Corollary 3.5, and we obtain $s = 2\ell$ and generators of degrees d_i, $1 \le i \le n+1$. The resolution was already observed in (3.1).

The splitting off of summands of $F_1^\vee(-d)$ when $\ell + 1 = d_i$ occurs since $\ell + 1$ is the initial degree of G and $J \subset G$. So any such polynomial can be taken to be a minimal generator of G. Hence it splits off, thanks to a usual mapping cone argument.

Now we show that other overlapping terms do not split off. Because of the way that the homomorphisms in the mapping cone are formed, there is no splitting off between summands of $K_i^\vee(-d)$ and $F_{i+2}^\vee(-d)$, since there is no map between these modules. (See Example 3.11.) There is no splitting off of summands of $K_i^\vee(-d)$ and $K_{i+1}^\vee(-d)$, since the K_i come from a minimal free resolution, and similarly there is no splitting off of summands of $F_i^\vee(-d)$ and $F_{i+1}^\vee(-d)$. We now check that there is no possible overlap between summands of $F_i^\vee(-d)$ and $K_i^\vee(-d)$ for $i \ge 2$ (we have already accounted for overlaps when $i = 1$).

Clearly it is enough to compare summands of F_i and K_i. Each summand of F_i is of the form $R(-\ell - i)$, while the summands of K_i are of the form $R(-d_{r_1} - \cdots - d_{r_i})$. So we have to show that it is impossible to have an equality of the form

$$\frac{d - d_{n+1} - n}{2} + i = d_{r_1} + \cdots + d_{r_i}$$

for $i \ge 2$, i.e., it is impossible to have $d - d_{n+1} - n + 2i = 2d_{r_1} + \cdots + 2d_{r_i}$. Let $A > 0$ be the integer such that $d_2 + \cdots + d_n = d_1 + d_{n+1} + n - A$. Then we have

$$\begin{aligned} d - d_{n+1} - n + 2i &= (d_1 + d_{n+1} + n - A) + d_1 - d_{n+1} - n + 2i \\ &= 2d_1 + 2i - A \\ &= 2d_{r_1} + \cdots + 2d_{r_i} + (2d_1 + 2i - A - 2d_{r_1} - \cdots - 2d_{r_i}). \end{aligned}$$

So the suggested equality holds if and only if

$$\begin{aligned} A &= 2d_1 + 2i - 2d_{r_1} - \cdots - 2d_{r_i} \\ &= 2[(d_1 - d_{r_1}) + (i - d_{r_2} - \cdots - d_{r_i})]. \end{aligned}$$

If $i = 1$, this could be strictly positive (if and only if $d_1 = d_{r_1}$), as we have seen. If $i = 2$, this could be nonnegative (if and only if $i = 2$ and $d_1 = d_{r_1} = d_{r_2} = 2$) but not strictly positive. If $i \geq 3$, this is strictly negative, and we have our contradiction. □

Example 3.11. Note that there can be "ghost" terms that do not split off, coming from overlaps between $K_i^\vee(-d)$ and $F_{i+2}^\vee(-d)$. For instance, let $n = 4$, $d_1 = 3$, $d_2 = d_3 = d_4 = 5$, $d_5 = 10$. Then one computes $\ell = 2$ and $d = 18$, and (3.1) predicts a free R-resolution

$$0 \to R(-15)^{16} \to \begin{array}{c} \left[\begin{array}{c} R(-15) \\ \oplus \\ R(-13)^3 \end{array}\right] \\ \oplus \\ R(-14)^{30} \end{array} \to \begin{array}{c} \left[\begin{array}{c} R(-10)^3 \\ \oplus \\ R(-8)^3 \end{array}\right] \\ \oplus \\ R(-13)^{16} \end{array} \to \begin{array}{c} \left[\begin{array}{c} R(-3) \\ \oplus \\ R(-5)^3 \end{array}\right] \\ \oplus \\ R(-10) \end{array}$$
$$\to R \to R/I \to 0.$$

Our observation above allows a splitting off of a summand $R(-15)$, and Corollary 3.10 guarantees that the rest of the resolution is minimal, despite the overlap of three copies of $R(-13)$ in the second and third free modules in the resolution and the overlap of one copy of $R(-10)$ in the first and second. (We have already seen in Example 3.9 that the summand $R(-10)$ in the first module does not split off.)

The hypotheses of Corollary 3.10 guarantee that the Hilbert function of R/G has one peak and R/G is compressed, respectively (Lemma 2.6 (b) and (d)). These are enough to completely determine the minimal free R-resolution. We would like to weaken these hypotheses as much as possible. One way is Corollary 3.8, where these hypotheses are removed completely but a bound for the Betti numbers of R/G is used, which greatly exceeds those appearing in our "general" situation.

We will make a slight improvement, allowing two peaks (by Lemma 2.6 no more peaks can occur) but keeping the fact that the R/G is compressed. Note first that in this situation the minimal free resolution is not uniquely determined just from the Hilbert function.

Example 3.12. Consider the case $n = 4$, $d_1 = d_2 = d_3 = d_4 = 4$, $d_5 = 5$. The corresponding Gorenstein Hilbert function is

$$1 \;\; 4 \;\; 10 \;\; 20 \;\; 20 \;\; 10 \;\; 4 \;\; 1.$$

Thanks to [25] the maximal possible graded Betti numbers are

$$0 \to R(-11) \to \begin{array}{c} R(-6)^{10} \\ \oplus \\ R(-7)^{15} \end{array} \to \begin{array}{c} R(-5)^{24} \\ \oplus \\ R(-6)^{24} \end{array} \to \begin{array}{c} R(-4)^{15} \\ \oplus \\ R(-5)^{10} \end{array} \to R \to R/G \to 0,$$

while thanks to [5] the general such graded Betti numbers are

$$0 \to R(-11) \to R(-7)^{15} \to \begin{array}{c} R(-5)^{14} \\ \oplus \\ R(-6)^{14} \end{array} \to R(-4)^{15} \to R \to R/G \to 0.$$

So, we first need to determine the generic Betti numbers of certain Gorenstein Artinian graded algebras. In [5], Corollary 3.10, M. Boij established the existence of generic Betti numbers of compressed level algebras, and in Conjecture 3.13 he guessed that the generic Betti numbers are as small as we can hope for. This conjecture was well-known for even socle degree, where it follows from the almost purity of the minimal free resolution of R/G in that case.

Using the fact that the Minimal Resolution Conjecture (MRC) has been proved for large numbers of points in any $\mathbb{P}^r$ by A. Hirschowitz and C. Simpson ([18]) and that the canonical module of the homogeneous coordinate ring of points can be identified with an ideal of the ring itself, we will prove (resp. partially prove) that Boij's Conjecture ([5], Conjecture 3.13) holds for compressed Gorenstein Artinian algebras of even (resp. odd) embedded dimension n, initial degree $t \gg 0$ and socle degree $2t-1$.

Proposition 3.13. *Let A be a generic compressed Gorenstein Artinian graded algebra of even embedding dimension $n = 2p$ with initial degree t and socle degree $2t-1$. Assume $n = 4$, or $n > 4$ and $t \gg 0$. Then A has a minimal free R-resolution of the following type:*

$$\begin{aligned} &0 \to R(-2t-n+1) \to R(-t-n+1)^{\alpha_1} \to ... \to R(-t-p-1)^{\alpha_{p-1}} \\ &\to \begin{pmatrix} R(-t-p)^{\alpha_p} \\ \oplus \\ R(-t-p+1)^{\alpha_p} \end{pmatrix} \to R(-t-p+2)^{\alpha_{p-1}} \to .... \to R(-t-1)^{\alpha_2} \\ &\to R(-t)^{\alpha_1} \to R \to A \to 0, \end{aligned}$$

where

$$\alpha_i = \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{t+n-1}{t+n-i}\binom{t+n-i-1}{n-i}$$

for $i = 1, ..., p$.

Proof. For $n = 4$, see [5], Proposition 3.24. Assume $n = 2p > 4$ and $t \gg 0$. According to A. Hirschowitz and C. Simpson [18], there exists an integer $\pi(n-1)$ ($\pi(n-1) \sim 6^{(n-1)^3 \log(n-1)}$) such that the MRC holds for a generic set of $s > \pi(n-1)$ points in $\mathbb{P}^{n-1}$. Let $X \subset \mathbb{P}^{n-1}$ be a generic set of

$$\rho(t,n) := \left\lceil \frac{t(t+1)...(t+p-1)(t+p+1)...(t+n-1)}{(n-1)!} \right\rceil$$

points. (For any $x \in \mathbb{R}$, we set $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}$.) Since $t \gg 0$, we have $\rho(t,n) > \pi(n-1)$.

We first observe that

$$\begin{aligned} \binom{t+n-2}{n-1} \leq \rho(t,n) &= \left\lceil \frac{t(t+1)...(t+p-1)(t+p+1)...(t+n-1)}{(n-1)!} \right\rceil \\ &< \binom{t+n-1}{n-1}. \end{aligned}$$

In particular, $R/I(X)$ has generic Hilbert function with initial degree t. As a result, $I(X)$ has a minimal free $R = k[x_0, ..., x_{n-1}]$-resolution of the following type:

$$0 \to \begin{pmatrix} R(-t-n+1)^{b_{n-1}} \\ \oplus \\ R(-t-n+2)^{a_{n-1}} \end{pmatrix} \to \begin{pmatrix} R(-t-n+2)^{b_{n-2}} \\ \oplus \\ R(-t-n+3)^{a_{n-2}} \end{pmatrix}$$
$$\to .... \to \begin{pmatrix} R(-t-2)^{b_2} \\ \oplus \\ R(-t-1)^{a_2} \end{pmatrix} \to \begin{pmatrix} R(-t-1)^{b_1} \\ \oplus \\ R(-t)^{a_1} \end{pmatrix}$$
$$\to R \to R/I(X) \to 0,$$

where

$$a_i = \max\{0, h^0(\mathbb{P}^{n-1}, \Omega^{i-1}_{\mathbb{P}^{n-1}}(t+i-1)) - \mathrm{rk}(\Omega^{i-1}_{\mathbb{P}^{n-1}})\rho(t,n)\}$$

and

$$b_i = \max\{0, \mathrm{rk}(\Omega^{i}_{\mathbb{P}^{n-1}})\rho(t,n) - h^0(\mathbb{P}^{n-1}, \Omega^{i}_{\mathbb{P}^{n-1}}(t+i))\}$$

for any $i = 1, ..., n-1$ (see [18], pg. 2). Note that we set $\Omega^0_{\mathbb{P}^{n-1}} = \mathcal{O}_{\mathbb{P}^{n-1}}$.

An intricate calculation shows that for any i we have

$$\begin{aligned} a_i &= \max\{0, h^0(\mathbb{P}^{n-1}, \Omega^{i-1}_{\mathbb{P}^{n-1}}(t+i-1)) - \mathrm{rk}(\Omega^{i-1}_{\mathbb{P}^{n-1}})\rho(t,n)\} \\ &= \max\left\{0, \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{n-1}{i-1}\rho(t,n)\right\} \end{aligned}$$

and

$$\begin{aligned} b_i &= \max\{0, \mathrm{rk}(\Omega^{i}_{\mathbb{P}^{n-1}})\rho(t,n) - h^0(\mathbb{P}^{n-1}, \Omega^{i}_{\mathbb{P}^{n-1}}(t+i))\} \\ &= \max\left\{0, \binom{n-1}{i}\rho(t,n) - \binom{t+n-1}{t+i}\binom{t+i-1}{i}\right\}. \end{aligned}$$

Furthermore,

- if $p+1 \le i \le n-1$, then $a_i = 0$;
- if $1 \le i \le p-1$, then $b_i = 0$.

Hence, the minimal free R-resolution of $R/I(X)$ has the form

$$0 \to R(-t-n+1)^{b_{n-1}} \to ..... \to R(-t-p-1)^{b_{p+1}} \to \begin{matrix} R(-t-p)^{b_p} \\ \oplus \\ R(-t-p+1)^{a_p} \end{matrix}$$
$$\to R(-t-p+2)^{a_{p-1}} \to ..... \to R(-t-1)^{a_2} \to R(-t)^{a_1} \to R \to R/I(X) \to 0,$$

where

$$a_i = \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{n-1}{i-1}\rho(t,n)$$

and

$$b_i = \binom{n-1}{i}\rho(t,n) - \binom{t+n-1}{t+i}\binom{t+i-1}{i}.$$

Let $A(X) = R/I(X)$ be the homogeneous coordinate ring of X. The canonical module ω_X of $A(X)$ can be embedded as an ideal $\omega_X \subset A(X)$ of initial degree t, and we have a short exact sequence

$$0 \to \omega_X \to A(X) \to A \to 0,$$

where A is a Gorenstein Artinian algebra of embedding dimension n with initial degree t and socle degree $2t-1$ (cf. [6]). A straightforward calculation shows that

$$\begin{aligned} a_i + b_{n-i} &= h^0(\mathbb{P}^{n-1}, \Omega^{i-1}_{\mathbb{P}^{n-1}}(t+i-1)) - h^0(\mathbb{P}^{n-1}, \Omega^{i}_{\mathbb{P}^{n-1}}(t+i)) \\ &= \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{t+n-1}{t+n-i}\binom{t+n-i-1}{n-i}. \end{aligned}$$

Hence, applying the mapping cone construction to the diagram

$$\begin{array}{ccccccccc} & & & & & & & 0 & \\ & & & & & & & \downarrow & \\ 0\to & R(-2t-n+1) & \to & R(-t-n+1)^{a_1} & \to \; ... \; \to & R(-t)^{b_{n-1}} & \to & \omega_X & \to 0 \\ & \downarrow & & \downarrow & & & & \downarrow & \downarrow \\ 0\to & R(-t-n+1)^{b_{n-1}} & \to & R(-t-n+2)^{b_{n-2}} & \to \; ... \; \to & R & \to & R/I(X) & \to 0 \\ & & & & & & & \downarrow & \\ & & & & & & & A & \\ & & & & & & & \downarrow & \\ & & & & & & & 0 & \end{array}$$

we get the minimal free R-resolution of A:

$$\begin{aligned} &0 \to R(-2t-n+1) \to R(-t-n+1)^{\alpha_1} \to ... \to R(-t-p-1)^{\alpha_{p-1}} \\ &\to \begin{pmatrix} R(-t-p)^{\alpha_p} \\ \oplus \\ R(-t-p+1)^{\alpha_p} \end{pmatrix} \to R(-t-p+2)^{\alpha_{p-1}} \to .... \to R(-t-1)^{\alpha_2} \\ &\to R(-t)^{\alpha_1} \to R \to A \to 0 \end{aligned}$$

with

$$\alpha_i = \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{t+n-1}{t+n-i}\binom{t+n-i-1}{n-i}$$

for $i = 1, ..., p$. □

Remark 3.14. Arguing as above, we can prove that the generic minimal free R-resolution of a Gorenstein Artinian graded algebra A of odd embedding dimension $n = 2p+1 > 3$ with initial degree $t \gg 0$ and socle degree $2t-1$ has the form

$$\begin{aligned} &0 \to R(-2t-n+1) \to R(-t-n+1)^{\alpha_1} \\ &\to ... \to R(-t-p-1)^{\alpha_p} \oplus R(-t-p)^{\leq b_p} \\ &\to R(-t-p+1)^{\alpha_p} \oplus R(-t-p)^{\leq b_p} \to \cdots \to R(-t-1)^{\alpha_2} \\ &\to R(-t)^{\alpha_1} \to R \to A \to 0 \end{aligned}$$

with

$$\alpha_i = \binom{t+n-1}{t+i-1}\binom{t+i-2}{i-1} - \binom{t+n-1}{t+n-i}\binom{t+n-i-1}{n-i}$$

for $i = 1, ..., p$ and

$$b_p = \binom{n-1}{p}\rho(t,n) - \binom{t+n-1}{t+p}\binom{t+p-1}{p},$$

but we do not know if the overlap appearing in the middle of the resolution actually occurs.

We have already found the minimal free resolution of a general almost complete intersection when the related Gorenstein algebra is compressed and has even socle degree (Corollary 3.10). So without loss of generality, we will assume two peaks (i.e., the socle degree $\sum_{i=1}^{n+1} d_i - n$ is odd). The case $n = 3$ will be treated in the next section. We now use Proposition 3.13 to give an analogous result when $n = 4$, or when n is even and the socle degree $2\ell+1$ of the related Gorenstein algebra R/G is odd. We have

Proposition 3.15. *Let $I = (G_1, \dots, G_{n+1})$ be a general almost complete intersection in $R = k[x_1, \dots, x_n]$, with $d_i = \deg G_i$, $2 \le d_1 \le d_2 \cdots \le d_n \le d_{n+1} \le (\sum_{i=1}^{n} d_i) - n$, $\sum_{i=1}^{n+1} d_i - n$ odd and $n = 2p$. Let*

$$\begin{aligned} d &= d_1 + \cdots + d_n, \\ \ell &= \frac{d - d_{n+1} - n - 1}{2}. \end{aligned}$$

Assume either that $n = 4$ and $d_2+d_3+d_4 < d_1+d_5+4$, or that $n > 4$, $d_2+\cdots+d_n < d_1 + d_{n+1} + n$ and $\ell \gg 0$. Then R/I has a free R-resolution of the form

$$0 \to F_1^\vee(-d) \to \begin{array}{c} K_1^\vee(-d) \\ \oplus \\ F_2^\vee(-d) \end{array} \to \begin{array}{c} K_2^\vee(-d) \\ \oplus \\ F_3^\vee(-d) \end{array} \to \cdots \to \begin{array}{c} K_{n-2}^\vee(-d) \\ \oplus \\ F_{n-1}^\vee(-d) \end{array}$$

$$\to \bigoplus_{i=1}^{n+1} R(-d_i) \ \to R \to R/I \to 0,$$

where K_i is the i-th free module in the Koszul resolution of $R/(G_1, \dots, G_n)$ and

$$F_i = \begin{cases} R(-\ell - i)^{\alpha_i} & \text{if } 1 \le i \le p-1, \\ R(-\ell - p)^{\alpha_p} \oplus R(-\ell - p - 1)^{\alpha_p} & \text{if } i = p, \\ R(-\ell - i - 1)^{\alpha_{n-i}} & \text{if } p+1 \le i \le n-1, \end{cases}$$

where

$$\alpha_i = \binom{\ell+n}{\ell+i}\binom{\ell+i-1}{i-1} - \binom{\ell+n}{\ell+1+n-i}\binom{\ell+n-i}{n-i}$$

for $i = 1, ..., p$.

a. *If $\ell + 1 = d_i$ for any $1 \le i \le n$, then for each such occurrence there is a corresponding splitting off of a free summand at the end of the resolution.*
b. *If $n = 4$, $d_1 = d_2 = 2$ and $d_3 + d_4 = d_5 + 3$, then there is a splitting off of one term of the form $R(\ell + 3 - d)$ with $R(-d_3 - d_4)$.*

Except for such splitting off, this resolution is minimal.

Proof. The calculations are almost identical with those of Corollary 3.10, but there is a difference in the reasoning. Here we will first prove the existence of an almost complete intersection with the claimed resolution. Note that a priori this resolution is not necessarily minimal, and the almost complete intersection is not necessarily general in the sense of Definition 1.1. We then describe all the splittings off that could conceivably occur for numerical reasons, and show that these in fact do occur. After such splitting, by semicontinuity, the result must be the minimal free R-resolution of a general almost complete intersection.

Letting t in Proposition 3.13 be $\ell+1$, and since $\ell \gg 0$, we have that for a general compressed Gorenstein Artinian quotient of $R = k[x_1, \dots, x_n]$ $(n = 2p)$ of socle degree $2\ell+1$, the minimal free resolution is

$$0 \to R(-2\ell-n-1) \to R(-\ell-n)^{\alpha_1} \to ... \to R(-\ell-p-2)^{\alpha_{p-1}}$$
$$\to R(-\ell-p-1)^{\alpha_p} \oplus R(-\ell-p)^{\alpha_p} \to R(-\ell-p+1)^{\alpha_{p-1}} \to ... \to R(-\ell-2)^{\alpha_2}$$

$$\to R(-\ell-1)^{\alpha_1} \to R \to R/G \to 0,$$

where

$$\alpha_i = \binom{\ell+n}{\ell+i}\binom{\ell+i-1}{i-1} - \binom{\ell+n}{\ell+n+1-i}\binom{\ell+n-i}{n-i}$$

for $i = 1, ..., p$.

Now, notice that such a general Gorenstein Artinian quotient has all its generators in degree $\ell+1$, and a simple calculation (using the hypotheses in the statement of the proposition) shows that $\ell+1 \leq d_1$. Hence a complete intersection $J \subset G$ exists with generators of degrees $d_1, ..., d_n$. By a standard mapping cone argument, the residual $I = [J : G]$ is an almost complete intersection (Cohen-Macaulay of height n with $n+1$ minimal generators). The discussion above shows that I has a free resolution of the form claimed.

Now we consider splitting. The type of splitting mentioned in part a has already been discussed. As we have already observed, the only possible splitting off comes (in the resolution (3.1)) between a summand of $F_i^\vee(-d)$ and one of $K_i^\vee(-d)$. So let us study the possible overlaps between summands of $F_i^\vee(-d)$ and of $K_i^\vee(-d)$ for $i \geq 2$. (We have already accounted in part a for overlaps when $i = 1$.) First we make a numerical calculation.

Claim. *Under the hypotheses of part b there is exactly one summand in common between $K_2^\vee(-d)$ and $F_2^\vee(-d)$. If these hypotheses are not met, then there is no other summand in common.*

Arguing as in Corollary 3.10, we check that if $n = 2p$ and $2 \leq i \leq p$, then there is no overlapping between summands of $R(\ell+i-d)^{\alpha_i}$ and summands of

$$K_i^\vee(-d) = \bigoplus_{d_1 \leq d_{r_1} < \dots < d_{r_i} \leq d_n} R(d_{r_1} + ... + d_{r_i} - d).$$

Assume now that $n = 2p$ and $p \leq i \leq n-1$. There is a summand in common between $R(\ell+i+1-d)^{\alpha_{n-i}}$ and

$$K_i^\vee(-d) = \bigoplus_{d_1 \leq d_{r_1} < \dots < d_{r_i} \leq d_n} R(d_{r_1} + ... + d_{r_i} - d)$$

if and only if $\ell+i+1 = d_{r_1} + \cdots + d_{r_i}$. Let $A > 0$ be the integer such that $d_2 + \cdots + d_n = d_1 + d_{n+1} + n - A$. We have

$$\begin{aligned} A &= 2d_1 + 2i - 2d_{r_1} - \cdots - 2d_{r_i} \\ &= 2(d_1 - d_{r_1}) + 2(i - d_{r_2} - \cdots - d_{r_i}) + 1. \end{aligned}$$

So, necessarily, we have $i = 2$ (and hence $n = 4$), and $d_{r_1} = d_{r_2} = 2$. Since we have assumed $d_1 > 1$, this forces $n = 4$, $d_1 = d_2 = 2$ and $d_3 + d_4 = d_5 + 3$, as desired.

So after the splitting off of part a we have a minimal free resolution *unless* the numerical conditions of part b hold. In this case we have to prove one more splitting off.

Assume that the conditions of part b hold. Then one checks that $\ell = 1$. So the Hilbert function of R/G is $(1\ \ 4\ \ 4\ \ 1)$, and Proposition 3.13 gives that G is generated in degree 2 and has 5 syzygies in degree 4. In fact, the minimal free resolution of R/G is

$$0 \to R(-7) \to R(-5)^6 \to R(-4)^5 \oplus R(-3)^5 \to R(-2)^6 \to R \to R/G \to 0. \tag{3.2}$$

One can check that the generators of I are of degrees 2, 2, d_3, d_4, d_5, where $d_3 > 2$. Then J also has exactly two generators of degree 2, and hence exactly one first syzygy of degree 4. It is the corresponding summand of $K_2^\vee(-d)$ that we would like to split off with a summand of $F_2^\vee(-d)$. As before, our strategy will be to construct a specific G and link which meet our needs, and then the same will hold for the general case.

Let $Z \subset \mathbb{P}^3$ be a generic set of 5 points. Then Z is arithmetically Gorenstein with Hilbert function

$$1\ \ 4\ \ 5\ \ 5\ \ \to$$

and minimal free resolution

$$0 \to R(-5) \to R(-3)^5 \to R(-2)^5 \to R \to R/I_Z \to 0.$$

Let Q be a generally chosen form of degree 2. Then $G := I_Z + (Q)$ is the saturated ideal of a height 4 Gorenstein ideal with Hilbert function $(1\ \ 4\ \ 4\ \ 1)$ and minimal free resolution (3.2) (obtained by a tensor product). Note that G has five linear syzygies and five quadratic syzygies, and that I_Z has five linear syzygies. Hence the five linear syzygies of G are precisely the five linear syzygies of I_Z, and the five quadratic syzygies of G are the Koszul syzygies of Q with the five generators of I_Z. Now choose J to consist of Q, one quadric generator of I_Z, and then general forms in G of suitable degrees satisfying the numerical conditions (e.g., $d_3 = d_4 = 4$, $d_5 = 5$). This produces an I as desired, and clearly the Koszul syzygy of degree 4 for J is the one from G.

Since no further numerical overlap exists, the remaining resolution is minimal, and hence is the minimal free R-resolution of the general almost complete intersection of type $(d_1, ..., d_{n+1})$. ☐

Example 3.16. Suppose $n = 4$. Let $I = (G_1, \dots, G_5)$ be a general almost complete intersection in $R = k[x_1, \dots, x_4]$, with $d_i = \deg G_i$, $2 \le d_1 \le d_2 \le d_3 \le d_4 \le d_5 \le (\sum_{i=1}^4 d_i) - 4$ and $\sum_{i=1}^5 d_i - 4$ odd. Let $d = d_1 + d_2 + d_3 + d_4$ and $\ell = \frac{d-d_5-5}{2}$. Assume that $d_2 + d_3 + d_4 < d_1 + d_5 + 4$. Then R/I has a free R-resolution of the form

$$0 \to R(\ell+1-d)^a \to \begin{pmatrix} \bigoplus_{i=1}^4 R(d_i - d) \\ \oplus \\ R(\ell+2-d)^b \\ \oplus \\ R(\ell+3-d)^b \end{pmatrix} \to \begin{pmatrix} \bigoplus_{1\le i<j\le 4} R(-d_i-d_j) \\ \oplus \\ R(\ell+4-d)^a \end{pmatrix}$$
$$\to \bigoplus_{i=1}^5 R(-d_i) \to R \to R/I \to 0,$$

where we have $a = \binom{\ell+3}{2}$ and $b = \binom{\ell+3}{2} - 1$.

a. If $\ell+1=d_i$ for any $1\le i\le 4$, then for each such occurrence there is a corresponding splitting off of a free summand at the end of the resolution.
b. If $d_1=d_2=2$ and $d_3+d_4=d_5+3$, then there is a splitting off of one term of the form $R(\ell+3-d)$ with $R(-d_3-d_4)$.

Except for such splitting off, this resolution is minimal.

Remark 3.17. Let $I=(G_1,\dots,G_{n+1})$ be a general almost complete intersection in $R=k[x_1,\dots,x_n]$, with $d_i=\deg G_i$, $2\le d_1\le d_2...\le d_n\le d_{n+1}\le(\sum_{i=1}^n d_i)-n$, $\sum_{i=1}^{n+1} d_i-n$ odd and $n=2p+1>3$. Let

$$\begin{aligned} d &= d_1+...+d_n,\\ \ell &= \frac{d-d_{n+1}-n-1}{2}. \end{aligned}$$

Assume that $d_2+...+d_n<d_1+d_{n+1}+n$ and $\ell\gg 0$. Then R/I has a free R-resolution of the form

$$0\to F_1^\vee(-d)\to \begin{matrix} K_1^\vee(-d)\\ \oplus\\ F_2^\vee(-d)\end{matrix} \to \begin{matrix} K_2^\vee(-d)\\ \oplus\\ F_3^\vee(-d)\end{matrix} \to\cdots\to \begin{matrix} K_{n-2}^\vee(-d)\\ \oplus\\ F_{n-1}^\vee(-d)\end{matrix}$$

$$\to\bigoplus_{i=1}^{n+1} R(-d_i)\to R\to R/I\to 0,$$

where K_i is the i-th free module in the Koszul resolution of $R/(G_1,\dots,G_n)$ and

$$F_i=\begin{cases} R(-\ell-i)^{\alpha_i} & \text{if } 1\le i\le p-1,\\ R(-\ell-p)^{\alpha_p}\oplus R(-\ell-p-1)^{\le b_p} & \text{if } i=p,\\ R(-\ell-p-2)^{\alpha_p}\oplus R(-\ell-p-1)^{\le b_p} & \text{if } i=p+1,\\ R(-\ell-i-1)^{\alpha_{n-i}} & \text{if } p+2\le i\le n-1,\end{cases}$$

where

$$\alpha_i=\binom{\ell+n}{\ell+i}\binom{\ell+i-1}{i-1}-\binom{\ell+n}{\ell+1+n-i}\binom{\ell+n-i}{n-i}$$

for $i=1,...,p$ and

$$b_p=\binom{n-1}{p}\rho(t,n)-\binom{t+n-1}{t+p}\binom{t+p-1}{p}.$$

Moreover, if $\ell+1=d_i$ for any $1\le i\le n$, then for each such occurrence there is a corresponding splitting off of a free summand at the end of the resolution.

4. The case $n=3$

We now specialize to the case $n=3$, and for simplicity of notation we will write $R=k[x,y,z]$. The goal of this section is to find the minimal free resolution for any general almost complete intersection $I=(G_1,G_2,G_3,G_4)$ in R, with generators of arbitrary degrees $d_i=\deg G_i$. As before we assume that $d_1\le d_2\le d_3\le d_4\le d_1+d_2+d_3-3$ (Remark 2.4).

We begin with the resolution (3.1), which in our context now becomes

$$(4.1)\qquad 0 \to F_1^\vee(-d) \to \begin{pmatrix} \bigoplus_{i=1}^{3} R(d_i - d) \\ \oplus \\ F_2^\vee(-d) \end{pmatrix} \to \bigoplus_{i=1}^{4} R(-d_i) \to R \to R/I \to 0,$$

where $d = d_1 + d_2 + d_3$, and the F_i come from the minimal free R-resolution of R/G:

$$0 \to R(-e) \to F_2 \to F_1 \to R \to R/G \to 0$$

with $e = d_1 + d_2 + d_3 - d_4$.

Our approach is similar to that of Proposition 3.15. We know the Hilbert function of R/G, from which we can calculate all possible minimal free resolutions using [10]. The general such can be determined, and a link gives the resolution of the general almost complete intersection.

Proposition 4.1. *Let $I = (G_1, G_2, G_3, G_4)$ be a general almost complete intersection in R of type (d_1, d_2, d_3, d_4) with $d_1 \le d_2 \le d_3 \le d_4 \le d_1 + d_2 + d_3 - 3$. Let G be the Gorenstein ideal linked to I by the complete intersection $J = (G_1, G_2, G_3)$. Let*

$$s = d_1 + d_2 + d_3 - d_4 - 3 \quad \text{and} \quad \ell = \left\lfloor \frac{s}{2} \right\rfloor.$$

a. *We have maximal Hilbert function*

$$h_{R/G}(t) = \binom{t+2}{2} \text{ for all integers } 0 \le t \le \ell$$

if and only if $d_2 + d_3 < d_1 + d_4 + 3$. In any case, the Hilbert function is described in Lemma 2.6.

b. *We have $d_2 > \ell + 1$.*

c. *For the* general *Gorenstein ideal G' with the Hilbert function described in part* a*, the minimal free resolution of G' is described as follows.*

Case I. *$d_2 + d_3 < d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ odd:*

$$0 \to R(-2\ell - 3) \to R(-\ell - 2)^{2\ell+3} \to R(-\ell - 1)^{2\ell+3} \to R \to R/G' \to 0.$$

Case II. *$d_2 + d_3 < d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ even:*

$$0 \to R(-2\ell - 4) \to \begin{matrix} R(-\ell - 3)^{\ell+2} \\ \oplus \\ R(-\ell - 2)^{\delta} \end{matrix} \to \begin{matrix} R(-\ell - 1)^{\ell+2} \\ \oplus \\ R(-\ell - 2)^{\delta} \end{matrix} \to R \to R/G' \to 0,$$

where $\delta = 1$ if ℓ is even and 0 otherwise.

Case III. *$d_2 + d_3 \ge d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ odd:*

$$0 \to R(-2\ell - 3) \to \begin{matrix} R(-2\ell - 3 + d_1) \\ \oplus \\ R(-\ell - 2)^{2d_1} \end{matrix} \to \begin{matrix} R(-d_1) \\ \oplus \\ R(-\ell - 1)^{2d_1} \end{matrix} \to R \to R/G' \to 0.$$

Case IV. *$d_2 + d_3 \geq d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ even:*

$$0 \to R(-2\ell-4) \to \begin{array}{c} R(-2\ell-4+d_1) \\ \oplus \\ R(-\ell-3)^{d_1} \\ \oplus \\ R(-\ell-2)^{\delta} \end{array} \to \begin{array}{c} R(-d_1) \\ \oplus \\ R(-\ell-1)^{d_1} \\ \oplus \\ R(-\ell-2)^{\delta} \end{array} \to R \to R/G' \to 0,$$

where $\delta = 1$ if d_1 is odd and 0 otherwise.

Proof. Note that by Lemma 2.6, the knowledge of the Hilbert function up to degree ℓ determines the full Hilbert function of R/G by symmetry. The same lemma then gives the values of $h_{R/G}(t)$, completing part a. Part b is an easy calculation.

For part c, we consider Case IV (the other three being similar but easier). We note that we are in the situation where there are two peaks, but the Hilbert function is not maximal for the entire first half.

We know that $h_{R/G'}(t) = h_{R/J}(t)$ for all $t \leq \ell$ (Lemma 2.6 (c)). Hence G' and J agree up to degree ℓ. Since $d_2 > \ell+1$, we get that the only generator is of degree d_1 in this range. (This holds for G, hence also for G' by semicontinuity.) Hence we can compute the Hilbert function of R/G' as follows (we compute only the part that is relevant to our subsequent calculation):

$$h_{R/G'}(t) = \begin{cases} \binom{t+2}{2} & \text{if } 0 \leq t < d_1, \\ \binom{t+2}{2} - \binom{t-d_1+2}{2} & \text{if } d_1 \leq t \leq \ell, \\ \text{(symmetric)} & \text{otherwise.} \end{cases}$$

It is known (cf. for instance [10], Corollary 2.6) that in each degree, the negative third difference, $-\Delta^3 h_{R/G'}(t)$, of the Hilbert function gives a lower bound for the number of minimal generators of G' in that degree. When the socle degree is even, there is a G' with precisely these minimal generators. When the socle degree s is odd, one more generator may be needed, which occurs in degree $\frac{s+3}{2}$, if the remaining number of generators is even (since the total number of minimal generators must be odd – cf. [7]). In Case IV, the socle degree is $d_1 + d_2 + d_3 - d_4 - 3 = 2\ell + 1$, which is odd. We compute

$$\Delta^3 h_{R/G'}(t) = \begin{cases} 0 & \text{if } 0 < t < d_1, \\ -1 & \text{if } t = d_1, \\ -d_1 & \text{if } t = \ell + 1, \\ \geq 0 & \text{otherwise.} \end{cases}$$

This gives the result. □

Note that Case I is really a special case of Corollary 3.10.

Theorem 4.2. *Let $I = (G_1, G_2, G_3, G_4)$ be a general almost complete intersection in R of type (d_1, d_2, d_3, d_4). If $d_4 > d_1 + d_2 + d_3 - 3$, then $G_4 \in (G_1, G_2, G_3)$ and I is a complete intersection, hence has a Koszul resolution. So without loss of generality, assume that $d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_1 + d_2 + d_3 - 3$. Choose F_1 and F_2 to be the free modules appearing in Proposition 4.1 (depending on the values of the d_i, i.e., depending on which of Cases I-IV applies). Then, using these F_i, (4.1) is a free R-resolution for I.*

Furthermore, the following represent all the splitting off that occurs.

I. *If* $d_2 + d_3 < d_1 + d_4 + 3$*,* $d_1 + d_2 + d_3 + d_4$ *is odd and* $d_1 + d_4 = d_2 + d_3 - 1$*, then one summand* $R(d_1 - d)$ *splits off at the end of (4.1).*

II. *Assume that* $d_2 + d_3 < d_1 + d_4 + 3$ *and* $d_1 + d_2 + d_3 + d_4$ *is even.*
   (i) *If* $d_1 + d_4 = d_2 + d_3 - 2$*, then one summand* $R(d_1 - d)$ *splits off at the end of the resolution.*
   (ii) *If* $d_1 = d_2$*,* $d_3 = d_4$ *and* ℓ *is even (where* ℓ *is defined in Proposition 4.1), then one summand* $R(d_2 - d)$ *splits off at the end of the resolution.*
   (iii) *If* $d_4 - d_3 = d_2 - d_1$ *and* ℓ *is even, then one summand* $R(d_1 - d)$ *splits off at the end of the resolution.*
   (iv) *If the hypothesis of (ii) holds, then clearly the hypothesis of (iii) also holds. In this case there is only one splitting off, by applying either (ii) or (iii). Other than this, no two of (i), (ii) and (iii) can happen simultaneously.*

III. *If* $d_2 + d_3 \geq d_1 + d_4 + 3$ *and* $d_1 + d_2 + d_3 + d_4$ *is odd, then one summand* $R(d_1 - d)$ *splits off at the end of the resolution.*

IV. *If* $d_2 + d_3 \geq d_1 + d_4 + 3$ *and* $d_1 + d_2 + d_3 + d_4$ *is even, then one summand* $R(d_1 - d)$ *splits off at the end of the resolution.*

Proof. The approach is similar to that of Proposition 3.15. In each of the four cases, we start with the general Gorenstein resolution described in Proposition 4.1 and link with general forms in G of degrees d_1, d_2, d_3 to produce an almost complete intersection I of type (d_1, d_2, d_3, d_4). Thanks to (4.1) we have a free resolution for I. We want to produce from it the minimal free resolution for a general almost complete intersection of the same type.

Because the forms are general, if one of the d_i is the degree of a minimal generator of G, there is a corresponding splitting off of the resolution. (We check below that the numerical conditions given in the statement of the theorem give precisely this situation.) This produces the claimed resolution in the statement of the theorem. It is possible that there are still summands that numerically are candidates for splitting off, in this free resolution.

Suppose that the general almost complete intersection of this type has further splitting off. Then linking back gives a Gorenstein ideal whose minimal free resolution is smaller than the one described in Proposition 4.1, which is a contradiction.

So we have only to check that the numerical conditions in the statement of the theorem are exactly what is required to have a generator of degree d_1 or d_2 (and that both cannot happen simultaneously).

We saw in Proposition 4.1 (b) that $d_2 \geq \ell + 2$; so in cases I and III the only possibility is a splitting off of a summand $R(d_1 - d)$. This is automatic in III, and it is easy to check that in case I, we have $d_1 = \ell + 1$ if and only if $d_1 + d_4 = d_2 + d_3 - 1$.

In case II, we can check that $d_1 = \ell + 1$ if and only if $d_1 + d_4 = d_2 + d_3 - 2$. We can also check that $d_2 = \ell + 2$ if and only if $d_2 + d_4 = d_1 + d_3$, which in turn is equivalent to $d_1 = d_2$ and $d_3 = d_4$. Clearly, then, we cannot have both $d_1 = \ell + 1$ and $d_2 = \ell + 2$. Finally, we can also have $d_1 = \ell + 2$ if ℓ is even, and this happens if and only if $d_4 - d_3 = d_2 - d_1$.

In case IV, we automatically get a splitting off of a summand $R(d_1 - d)$. We can again check that $d_2 = \ell + 2$ if and only if $d_1 = d_2$ and $d_3 = d_4$. However, this time it is incompatible with the hypothesis $d_2 + d_3 \geq d_1 + d_4 + 3$. □

Example 4.3. Combining Proposition 4.1 and Theorem 4.2, it is easy to give examples of general almost complete intersections which have "ghost" terms in the resolution, i.e., summands which occur in consecutive modules in the minimal free resolution but cannot be split off. Some examples for (d_1, d_2, d_3, d_4) are $(4,4,4,8)$, $(5,5,6,8)$ and $(3,6,6,7)$.

When there is such a ghost term, it is a summand $R(\ell+2-d)$, which one checks is $R(\frac{-d_1-d_2-d_3-d_4}{2})$.

For instance, the minimal free R-resolution for $(4,4,4,8)$ is

$$0 \to \begin{pmatrix} R(-10) \\ \oplus \\ R(-11)^2 \end{pmatrix} \to \begin{pmatrix} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{pmatrix} \to \begin{pmatrix} R(-4)^3 \\ \oplus \\ R(-8) \end{pmatrix} \to R \to R/I \to 0$$

(here $\ell = 0$). The summand $R(-8)$ in the first and second free modules is clearly not going to split off, since one represents a minimal generator and the other represents a Koszul relation between two other generators. But the summand $R(-10)$ in the second and third free modules is an illustration of the phenomenon that we are describing. To our knowledge this is the first counterexample to Iarrobino's Thin Resolution Conjecture [21].

A special case of interest is when the generators of our general almost complete intersection all have the same degree, and here we record the minimal free resolution in this case.

Corollary 4.4. *Let I be a general almost complete intersection of type (a,a,a,a). Then R/I has a minimal free R-resolution*

$$0 \to R(-2a-1)^a \to \begin{pmatrix} R(-2a)^3 \\ \oplus \\ R(-2a+1)^a \end{pmatrix} \to R(-a)^4 \to R \to R/I \to 0.$$

Proof. The argument is a simple application of the preceding work. The only thing to note is that if a is even then $\delta = 1$, and a summand $R(-2a)$ is split off, but using a summand from $K_1^\vee(-d)$ rather than from $F_2^\vee(-d)$. □

5. Consequences of a more careful analysis of R/G

In this section we will first extend Corollary 4.4 to rings of higher dimension. Our strongest result comes when this dimension n is even and a is also even, but we also have results for the more general situation. These are contained in Theorem 5.4. These resolutions are very explicit. Our methods apply more generally, but the computations and notation become very cumbersome. As a middle road, we conclude the section with a result for $n = 4$ which carefully shows how to apply the method of this section and an inductive approach to obtain the free resolution for R/I, which is minimal "half" of the time.

The following result from [25] is crucial to our work in this section:

Proposition 5.1 ([25], Proposition 8.7)**.** *Let $A = R/G$ be a graded Artinian Gorenstein k-algebra, let $L \in R$ be a general linear form, and let $\bar{R} = R/(L)$.*

Let s be the socle degree of A, and let $\alpha = \text{in}[0 :_A L]$. Then, for all $i \in \mathbb{Z}$,

$$\left[\operatorname{tor}_i^R(A,k)\right]_j = \begin{cases} \left[\operatorname{tor}_i^{\bar{R}}(A/LA,k)\right]_j & \text{if } j \leq \alpha + i - 2, \\ \leq \left[\operatorname{tor}_i^{\bar{R}}(A/LA,k)\right]_j + \left[\operatorname{tor}_{n-i}^{\bar{R}}(A/LA,k)\right]_{s+n-j} & \\ & \text{if } \alpha + i - 1 \leq j \leq s - \alpha + i + 1, \\ \left[\operatorname{tor}_{n-i}^{\bar{R}}(A/LA,k)\right]_{s+n-j} & \text{if } j \geq s - \alpha + i + 2. \end{cases}$$

This can be applied to our situation. Since the calculations are somewhat complicated, we illustrate it with two examples before we proceed to a more general statement.

Example 5.2. Let $R = k[x_1, x_2, x_3, x_4]$ and let I be a general almost complete intersection of type $(4, 4, 4, 4, 4)$. Let J be the ideal given by the first four generators and let $G = [J : I]$ be the linked Gorenstein ideal. The Hilbert function of $A := R/G$ is

$$1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 20 \quad 10 \quad 4 \quad 1 \quad 0.$$

Let L be a general linear form. We have seen (Corollary 2.7) that $A = R/G$ has the Strong Lefschetz property, hence the Weak Lefschetz property. So the Hilbert function of A/LA is

$$1 \quad 3 \quad 6 \quad 10 \quad 11 \quad 0.$$

We want to find the minimal free resolution of A/LA over $\bar{R} = R/(L) \cong k[x, y, z]$.

Note that the regularity of A/LA is 5. Hence it has a minimal free $\bar{R}$-resolution of the form

$$0 \to \begin{pmatrix} \bar{R}(-6)^{c_1} \\ \oplus \\ \bar{R}(-7)^{c_2} \end{pmatrix} \to \begin{pmatrix} \bar{R}(-5)^{b_1} \\ \oplus \\ \bar{R}(-6)^{b_2} \end{pmatrix} \to \begin{pmatrix} \bar{R}(-4)^{a_1} \\ \oplus \\ \bar{R}(-5)^{a_2} \end{pmatrix} \to \bar{R} \to A/LA \to 0.$$

Clearly $a_1 = 4$, from the Hilbert function. The four generators in degree 4 are the restriction $\bar{J}$ of J, hence can be viewed as giving a general almost complete intersection in $\bar{R}$. b_1 represents all linear syzygies of these four polynomials. From Corollary 4.4, we see that $\bar{J}$ has no linear syzygies, and so $b_1 = 0$. This means that the four generators in degree 4 span a subspace of dimension $3 \cdot 4 = 12$ in $\bar{R}_5$. So we need $21 - 12 = 9$ more generators in degree 5 and $a_2 = 9$. Once $b_1 = 0$, this forces $c_1 = 0$, since the smallest number is strictly increasing from one free module to the next. Then by exactness one computes $b_2 = 23$ and $c_2 = 11$. So the minimal free $\bar{R}$-resolution has the form

$$0 \to \bar{R}(-7)^{11} \to \bar{R}(-6)^{23} \to \begin{pmatrix} \bar{R}(-4)^4 \\ \oplus \\ \bar{R}(-5)^9 \end{pmatrix} \to \bar{R} \to A/LA \to 0.$$

Now we can use Proposition 5.1. In this case $s = 8$, and, thanks to the Weak Lefschetz property, $\alpha = 4$. We have discussed A/LA above and have computed the minimal free $\bar{R}$-resolution of A/LA. We conclude (after a calculation) that R/G

has a minimal free resolution of the form

$$0 \to R(-12) \to \begin{pmatrix} R(-7)^{\le 20} \\ \oplus \\ R(-8)^{\le 4} \end{pmatrix} \to R(-6)^{\le 46} \to \begin{pmatrix} R(-4)^{\le 4} \\ \oplus \\ R(-5)^{\le 20} \end{pmatrix} \\ \to R \to R/G \to 0.$$

A computation from the Hilbert function gives that the inequalities are in fact equalities. Therefore we have the precise minimal free resolution of R/G.

To compute the minimal free resolution of R/I, note that we have linked using four quartics, which are minimal generators of G. Therefore, using (3.1) we see that we can split off four summands $R(-12)$ from the end of the resolution of R/I. The remaining resolution is

$$0 \to R(-11)^{20} \to R(-10)^{46} \to \begin{pmatrix} R(-8)^{10} \\ \oplus \\ R(-9)^{20} \end{pmatrix} \to R(-4)^5 \to R \to R/I \to 0,$$

which is minimal.

Example 5.3. Let $R = k[x_1, x_2, x_3, x_4]$ and let I be a general almost complete intersection of type $(5, 5, 5, 5, 5)$. Let J be the ideal given by the first four generators and let $G = [J : I]$ be the linked Gorenstein ideal. The Hilbert function of $A := R/G$ is

$$1 \;\; 4 \;\; 10 \;\; 20 \;\; 35 \;\; 52 \;\; 52 \;\; 35 \;\; 20 \;\; 10 \;\; 4 \;\; 1 \;\; 0.$$

Let L be a general linear form. We have seen (Corollary 2.7) that $A = R/G$ has the Strong Lefschetz property, hence the Weak Lefschetz property. So the Hilbert function of A/LA is

$$1 \;\; 3 \;\; 6 \;\; 10 \;\; 15 \;\; 17 \;\; 0.$$

We want to find the minimal free resolution of A/LA over $\bar{R} = R/(L) \cong k[x, y, z]$. Note that the regularity of A/LA is 6. Then precisely the same reasoning as in Example 5.2 now gives that A/LA has a minimal $\bar{R}$-resolution of the form

$$0 \to \bar{R}(-8)^{17} \to \bar{R}(-7)^{36} \to \begin{pmatrix} \bar{R}(-5)^4 \\ \oplus \\ \bar{R}(-6)^{16} \end{pmatrix} \to \bar{R} \to A/LA \to 0.$$

Now we can use Proposition 5.1. In this case $s = 11$, and, thanks to the Weak Lefschetz property, $\alpha = 6$. We have discussed A/LA above and have computed the minimal free $\bar{R}$-resolution of A/LA. We conclude (after a calculation) that R/G has a minimal free R-resolution of the form

$$0 \to R(-15) \to \begin{pmatrix} R(-8)^y \\ \oplus \\ R(-9)^{\le 16} \\ \oplus \\ R(-10)^4 \end{pmatrix} \to \begin{pmatrix} R(-7)^{19+y} \\ \oplus \\ R(-8)^{19+y} \end{pmatrix} \to \begin{pmatrix} R(-5)^4 \\ \oplus \\ R(-6)^{\le 16} \\ \oplus \\ R(-7)^y \end{pmatrix} \\ \to R \to R/G \to 0,$$

where $y \le 17$. A computation from the Hilbert function gives that the inequality "≤ 16" is in fact an equality, but it does not determine the value of y. Experiments with Macaulay [3] indicate that in fact $y = 0$, but we have not been able to prove this.

To compute the minimal free resolution of R/I, note that we have linked using four quartics, which are minimal generators of G. Therefore, using (3.1) we see that we can split off four summands $R(-12)$ from the end of the resolution of R/I. The remaining resolution is

$$0 \to \begin{pmatrix} R(-14)^{16} \\ \oplus \\ R(-13)^{y} \end{pmatrix} \to \begin{pmatrix} R(-13)^{19+y} \\ \oplus \\ R(-12)^{19+y} \end{pmatrix} \to \begin{pmatrix} R(-10)^{10} \\ \oplus \\ R(-11)^{16} \\ \oplus \\ R(-12)^{y} \end{pmatrix}$$
$$\to R(-5)^5 \to R \to R/I \to 0.$$

As mentioned above, computations with Macaulay [3] indicate that $y = 0$.

This approach leads us to the following more general results, for the minimal free resolution of a general almost complete intersection. We first consider the case of general forms of the same degree, a. Our goal is to extend Corollary 4.4 to rings of higher dimension. To this end we need to introduce some extra notation. For any integers $0 < a, n \in \mathbb{Z}$, we set

$$\begin{aligned}
s(n,a) &= (n-1)a - n, \\
\ell(n,a) &= \left\lfloor \frac{s(n,a)}{2} \right\rfloor, \\
t(n,a) &= \max\{t \mid \ell(n,a) + t - 1 \geq ta\}, \\
\alpha_j(n,a) &= \sum_{i=0}^{t(n,a)} (-1)^{i+j-1} \binom{n}{i} \binom{\ell(n,a) + n - 2 + j - ia}{n-2} \\
&\quad - \sum_{r=1}^{j-1} (-1)^{r+j} \binom{n-2+j-r}{j-r} \alpha_r(n,a), \\
&\quad \text{defined inductively for } 1 \leq j \leq n-2,
\end{aligned}$$

$$\begin{aligned}
\alpha_{n-1} &= \sum_{i=2}^{n-t(n,a)-1} (-1)^i \alpha_{n-i}(n,a) \\
&\quad + \sum_{i=n-t(n,a)}^{n-1} (-1)^i (\alpha_{n-i}(n,a) + \beta_{n-i}(n)) + (-1)^n, \\
\beta_i(n) &= \binom{n}{i} \text{ for } i \geq 1, \\
\binom{a}{b} &= 0 \text{ if } a < b.
\end{aligned}$$

Theorem 5.4. *Let I be a general almost complete intersection in $R = k[x_1, \dots, x_n]$ of type $(a, \dots, a)$, $a > 1$.*

If n is odd, then R/I has a minimal free R-resolution bounded as follows:

$$0 \to \begin{pmatrix} R(\ell(n,a)+1-na)^{\alpha_1(n,a)} \\ \oplus \\ R(\ell(n,a)+2-na)^{\le \alpha_{n-1}(n,a)} \end{pmatrix} \to \cdots \to \begin{pmatrix} R(\ell(n,a)+(\frac{n-1}{2})-na)^{\le \alpha_{\frac{n-1}{2}}(n,a)} \\ \oplus \\ R(\ell(n,a)+(\frac{n+1}{2})-na)^{\le \alpha_{\frac{n+1}{2}}(n,a)} \end{pmatrix}$$
$$\to \begin{pmatrix} R(-\frac{n+1}{2}a)^{\beta_{\frac{n+1}{2}}(n)} \\ \oplus \\ R(\ell(n,a)+(\frac{n+1}{2})-na)^{\le \alpha_{\frac{n+1}{2}}(n,a)} \\ \oplus \\ R(\ell(n,a)+(\frac{n+3}{2})-na)^{\le \alpha_{\frac{n-1}{2}}(n,a)} \end{pmatrix} \to \begin{pmatrix} R(-\frac{n-1}{2}a)^{\beta_{\frac{n-1}{2}}(n+1)} \\ \oplus \\ R(\ell(n,a)+(\frac{n+3}{2})-na)^{\le \alpha_{\frac{n+3}{2}}(n,a)} \\ \oplus \\ R(\ell(n,a)+(\frac{n+5}{2})-na)^{\le \alpha_{\frac{n-3}{2}}(n,a)} \end{pmatrix}$$
$$\to \cdots \to \begin{pmatrix} R(-2a)^{\beta_2(n+1)} \\ \oplus \\ R(\ell(n,a)+n-1-na)^{\le \alpha_{n-1}(n,a)} \\ \oplus \\ R(\ell(n,a)+n-na)^{\alpha_1(n,a)} \end{pmatrix} \to R(-a)^{\beta_1(n+1)} \to R \to R/I \to 0.$$

If n is even and $s(n,a)$ is odd, then R/I has a minimal free R-resolution bounded as follows:

$$0 \to \begin{pmatrix} R(\ell(n,a)+1-na)^{\alpha_1(n,a)} \\ \oplus \\ R(\ell(n,a)+2-na)^{\le \alpha_{n-1}(n,a)} \end{pmatrix}$$
$$\to \cdots \to \begin{pmatrix} R(\ell(n,a)+(\frac{n}{2})-na)^{\le \alpha_{\frac{n}{2}}(n,a)} \\ \oplus \\ R(\ell(n,a)+(\frac{n}{2})+1-na)^{\le \alpha_{\frac{n}{2}}(n,a)} \end{pmatrix}$$
$$\to \begin{pmatrix} R(-\frac{n}{2}a)^{\beta_{\frac{n}{2}}(n+1)} \\ \oplus \\ R(\ell(n,a)+(\frac{n}{2})+1-na)^{\le \alpha_{\frac{n}{2}+1}(n,a)} \\ \oplus \\ R(\ell(n,a)+(\frac{n}{2})+2-na)^{\le \alpha_{\frac{n}{2}-1}(n,a)} \end{pmatrix}$$
$$\to \cdots \to \begin{pmatrix} R(-2a)^{\beta_2(n+1)} \\ \oplus \\ R(\ell(n,a)+n-1-na)^{\le \alpha_{n-1}(n,a)} \\ \oplus \\ R(\ell(n,a)+n-na)^{\alpha_1(n,a)} \end{pmatrix}$$
$$\to R(-a)^{\beta_1(n+1)} \to R \to R/I \to 0.$$

If n is even and $s(n,a)$ is even, then R/I has a minimal free R-resolution

$$\begin{aligned}
0 &\to R(\ell(n,a)+1-na)^{\alpha_1(n,a)+\alpha_{n-1}(n,a)} \\
&\to \cdots \to R(\ell(n,a)+(\frac{n}{2})-na)^{2\alpha_{\frac{n}{2}}(n,a)} \\
&\to \begin{pmatrix} R(-\frac{n}{2}a)^{\beta_{\frac{n}{2}}(n+1)} \\ \oplus \\ R(\ell(n,a)+(\frac{n}{2})+1-na)^{\alpha_{\frac{n}{2}+1}(n,a)+\alpha_{\frac{n}{2}-1}(n,a)} \end{pmatrix} \\
&\to \cdots \to \begin{pmatrix} R(-2a)^{\beta_2(n+1)} \\ \oplus \\ R(\ell(n,a)+n-1-na)^{\alpha_{n-1}(n,a)+\alpha_1(n,a)} \end{pmatrix} \\
&\to R(-a)^{\beta_1(n+1)} \to R \to R/I \to 0.
\end{aligned}$$

Proof. We proceed by induction on n. The case $n=3$ is covered by Corollary 4.4, and, even more, all possible splitting off does occur.

For arbitrary n, one first checks that the complete intersection J formed by taking n of the generators of I links I to a Gorenstein ideal G with socle degree $s(n,a)$. According to Lemma 2.6, if $s(n,a)$ is even, then there is only one peak, occurring in degree $\ell(n,a)$. If $s(n,a)$ is odd, then there are two peaks, the first of which is in degree $\ell(n,a)$. Let $A=R/G$ and let $L\in R_1$ be a general linear form.

The Hilbert function of R/G is

$$h_{R/G}(t)=\begin{cases} \binom{t+n-1}{n-1} & \text{if } t\le a-1,\\ \binom{t+n-1}{n-1}-n\cdot\binom{t-a+n-1}{n-1} & \text{if } a\le t\le \ell(n,a),\\ (symmetric) & \text{otherwise.}\end{cases}$$

By the Weak Lefschetz property, the Hilbert function of A/LA is

$$h_{A/LA}(t)=\begin{cases} \binom{t+n-2}{n-2} & \text{if } t\le a-1,\\ \binom{t+n-2}{n-2}-n\cdot\binom{t-a+n-2}{n-2} & \text{if } a\le t\le \ell(n,a),\\ 0 & \text{if } t>\ell(n,a).\end{cases}$$

Note that if $\bar R=R/(L)$ and $\bar G=\frac{G+(L)}{(L)}$, then $A/LA\cong \bar R/\bar G$. We will compute the minimal free $\bar R$-resolution of A/LA.

We first observe that the regularity of A/LA is $\ell(n,a)+1=\lfloor\frac{(n-1)a-n}{2}\rfloor+1$. Note also that $(A/LA)_i\cong(\bar R/\bar G)_i=(\bar R/\bar J)_i$ for all $i\le\ell(n,a)$ by Lemma 2.6. Since $\bar R/\bar J$ is an Artinian almost complete intersection in $\bar R$, by the induction hypothesis we have very good bounds on the graded Betti numbers of $\bar R/\bar J$.

We have seen that the minimal generators of G in degree $\le \ell(n,a)$ agree with those in J of degree $\le\ell(n,a)$; hence all appear in degree a. Furthermore, since $\operatorname{reg} A/LA=\ell(n,a)+1$, the minimal free $\bar R$-resolution of A/LA begins

$$\cdots\to \begin{matrix}\bar R(-a)^n\\ \oplus\\ \bar R(-\ell(n,a)-1)^{\alpha_1(n,a)}\end{matrix} \to \bar R\to A/LA\to 0.$$

Claim. (a) *If* $1 \le i \le t(n,a)$, *then*

$$[\mathrm{tor}_i^{\bar R}(A/LA,k)]_j = \begin{cases} \beta_i(n) & \text{if } j = ia, \\ \alpha_i(n,a) & \text{if } j = \ell(n,a)+i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If* $n-1 \ge i > t(n,a)$, *then*

$$[\mathrm{tor}_i^{\bar R}(A/LA,k)]_j = \begin{cases} \alpha_i(n,a) & \text{if } j = \ell(n,a)+i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the Claim. The generators of A/LA occur in degrees a and $\ell(n,a)+1$, and the generators of degree a of A/LA are precisely the generators of $\bar J$. So $(A/LA)_t \cong (\bar R/\bar J)_t$ for all $t \le \ell(n,a)$, and we deduce that $[\mathrm{tor}_i^{\bar R}(A/LA,k)]_j = [\mathrm{tor}_i^{\bar R}(\bar R/\bar J,k)]_j$ for all $j \le \ell(n,a)+i-1$. Furthermore, we have very good bounds on the graded Betti numbers of $\bar R/\bar J$ by the inductive hypothesis. In particular,

$$[\mathrm{tor}_i^{\bar R}(\bar R/\bar J,k)]_j \ne 0 \Leftrightarrow \begin{cases} j = ia \text{ and } i \le \lfloor \frac{n+1}{2} \rfloor, \\ j = -\ell(n-1,a)-(n-1)+(n-1)a+i-1, \\ j = -\ell(n-1,a)-(n-1)+(n-1)a+i-2 \text{ (possibly)} \end{cases}$$

and

$$[\mathrm{tor}_i^{\bar R}(\bar R/\bar J,k)]_{ia} = \beta_{ia}(n) \text{ if } i \le \lfloor \frac{n-1}{2} \rfloor.$$

An easy calculation shows that

$$\ell(n,a)+i-1 < -\ell(n-1,a)-(n-1)+(n-1)a+i-2$$

and

$$\ell(n,a)+i-1 < ia \Leftrightarrow i > t(n,a),$$

which together with the fact that reg $(A/LA) = \ell(n,a)+1$ tell us that the values of j given in parts (a) and (b) of the claim are the only ones where $[\mathrm{tor}_i^{\bar R}(A/LA,k)]_j \ne 0$. The precise values of $\beta_i(n)$ are given by the free $\bar R$-resolution of $\bar R/\bar J$, and the precise values of $\alpha_i(n,a)$ then follow from a calculation involving the Hilbert function of A/AL and the exactness of the $\bar R$-resolution.

It follows from the above claim that A/LA has a minimal free $\bar R$-resolution of the form

$$0 \to \bar R(-\ell(n,a)-n+1)^{\alpha_{n-1}(n,a)} \to \cdots \to \bar R(-\ell(n,a)-t(n,a)-1)^{\alpha_{t(n,a)+1}(n,a)}$$
$$\to \begin{pmatrix} \bar R(-t(n,a)a)^{\beta_{t(n,a)}(n)} \\ \oplus \\ \bar R(-\ell(n,a)-t(n,a))^{\alpha_{t(n,a)}(n,a)} \end{pmatrix} \to \cdots \to \begin{pmatrix} \bar R(-a)^{\beta_1(n)} \\ \oplus \\ \bar R(-\ell(n,a)-1)^{\alpha_1(n,a)} \end{pmatrix}$$
$$\to \bar R \to A/LA \to 0.$$

Now we take the now-known resolution for A/LA and plug the values into Proposition 5.1. When n is odd we obtain (the other two cases are similar), after a calculation, that:

If $1 \le i \le t(n,a)$, then

$$[\mathrm{tor}_i^R(A,k)]_j = \begin{cases} \beta_i(n) & \text{if } j = ia, \\ \le \alpha_i(n,a) & \text{if } j = \ell(n,a)+i, \\ \le \alpha_{n-i}(n,a) & \text{if } j = \ell(n,a)+i+1, \\ 0 & \text{otherwise;} \end{cases}$$

if $\frac{n-1}{2} \geq i > t(n,a)$, then

$$[\operatorname{tor}_i^R(A,k)]_j = \begin{cases} \leq \alpha_i(n,a) & \text{if } j = \ell(n,a)+i, \\ \leq \alpha_{n-i}(n,a) & \text{if } j = \ell(n,a)+i+1, \\ 0 & \text{otherwise;} \end{cases}$$

and by symmetry (since G is Gorenstein), we compute $[\operatorname{tor}_i^R(A,k)]_j$ for $\frac{n+1}{2} \leq i \leq n-1$.

From the Hilbert function and the above considerations we can compute that the number of minimal generators of G in degree $\ell(n,a)+1$ is precisely α_1. So we have equality for $[\operatorname{tor}_1^R(A,k)]_{\ell(n,a)+1}$. But then by symmetry (since G is Gorenstein), we get equality for $[\operatorname{tor}_{n-1}^R(A,k)]_{\ell(n,a)+n}$.

Finally, (3.1) gives the claimed free R-resolution of R/I after splitting off the summands of the left half part of the resolution, since they correspond to minimal generators of J and syzygies involving these minimal generators. When n and $s(n,a)$ are both even, this resolution is clearly minimal. □

Example 5.5. Suppose $n = 4$. Notice that the case $a = 2$ is covered by Corollary 3.10, the case $a = 3$ by Corollary 3.16, the case $a = 4$ by Example 5.2, and the case $a = 5$ by Example 5.3. So suppose that $a \geq 6$ and assume that a is even. Then

$$\begin{aligned} s = s(4,a) &= 3a-4, \\ \ell = \ell(4,a) &= \frac{3a-4}{2}, \\ t = t(n,a) &= 1, \\ \alpha_1 = \alpha_1(4,a) &= \binom{\ell+3}{2} - 4\binom{\ell+3-a}{2}, \\ \alpha_2 = \alpha_2(4,a) &= -\binom{\ell+4}{2} + 4\binom{\ell+4-a}{2} + 3\alpha_1, \\ \alpha_3 = \alpha_3(n,a) &= \alpha_2 - \alpha_1 - 3, \end{aligned}$$

and R/I has a minimal free R-resolution

$$0 \to R(\ell+1-4a)^{\alpha_1+\alpha_3} \to R(\ell+2-4a)^{2\alpha_2} \to \begin{pmatrix} R(-2a)^{10} \\ \oplus \\ R(\ell+3-4a)^{\alpha_1+\alpha_3} \end{pmatrix} \to R(-a)^5 \to R \to R/I \to 0.$$

It should be clear from the above considerations that for any values of n and of $d_1, \dots, d_{n+1}$, we can say quite a bit about the minimal free resolution using our methods, and that in some cases we can give the precise resolution. This is of course the most desirable, and interesting, situation. It is also clear that the notation gets progressively more cumbersome as n grows. As a middle road, we show that the precise minimal free resolution can be obtained when $n = 4$ and when we have only one peak, otherwise allowing the values of the d_i to be arbitrary.

Theorem 5.6. *Assume that $n = 4$. Let $I = (G_1, \dots, G_5)$ be a general almost complete intersection of type $(d_1, \dots, d_5)$ and assume that $\sum_{i=1}^5 d_i$ is even, with*

$2 \le d_1 \le \cdots \le d_5 \le \sum_{i=1}^4 d_i - 4$. *(The latter condition only assures that I is not a complete intersection.) Then the minimal free resolution of R/I can be obtained by the above procedure, and is explicitly given in the proof below.*

Proof. Let $J = (G_1, \ldots, G_4)$ and let $G = [J : I]$ as usual. We have $d = \sum_{i=1}^4 d_i$, $s = d - d_5 - 4$ and $\ell = \frac{s}{2}$. Note that we are in the case of one peak, which occurs in degree ℓ. We have seen in Lemma 2.6 that $h_{R/G}(t) = h_{R/J}(t)$ for $t \le \ell$.

Let L be a general linear form and let $A = R/G$. For a numerical function f, we again denote by Δf the first difference function $\Delta f = f(t) - f(t-1)$ for $t \in \mathbb{Z}$. By the Weak Lefschetz property for A (Corollary 2.7), we get

$$h_{A/LA}(t) = \begin{cases} \Delta h_{R/J}(t) & \text{for } t \le \ell, \\ 0 & \text{for } t > \ell. \end{cases}$$

In particular, the regularity of A/LA is $\ell + 1$.

Let $\bar{R} = R/(L)$ and $\bar{J} = \frac{J+(L)}{(L)}$. Let $f = d_1 + d_2 + d_3$. If $d_4 \le d_1 + d_2 + d_3 - 3$, then a free $\bar{R}$-resolution for $\bar{R}/\bar{J}$ is given by

$$(5.1) \qquad 0 \to F_1^\vee(-f) \to \begin{pmatrix} \bigoplus_{i=1}^3 \bar{R}(d_i - f) \\ \oplus \\ F_2^\vee(-f) \end{pmatrix} \to \bigoplus_{i=1}^4 \bar{R}(-d_i) \to \bar{R} \to \bar{R}/\bar{J} \to 0$$

(with all splitting off as described in Theorem 4.2). If $d_4 > d_1 + d_2 + d_3 - 3$, then $\bar{J}$ is a complete intersection; this case is easier, and we leave it to the reader.

Our first task is to describe the minimal free $\bar{R}$-resolution of A/LA. Since $G_t = J_t$ for $t \le \ell$, we have that $\bar{R}/\bar{J} \cong A/LA$ in degrees $\le \ell$, and $h_{A/LA}(t) = h_{\bar{R}/\bar{J}}(t)$ for $t \le \ell$. Since the regularity of A/LA is $\ell + 1$, we have

$$(5.2) \qquad [\operatorname{tor}_i^{\bar{R}}(A/LA, k)]_j = \begin{cases} [\operatorname{tor}_i^{\bar{R}}(\bar{R}/\bar{J}, k)]_j, & j \le \ell + i - 1, \\ ?, & j = \ell + i, \\ 0, & j > \ell + i. \end{cases}$$

We have seen that there may be overlap ("ghost terms") in the minimal free resolution of $\bar{R}/\bar{J}$, but these have been completely described in terms of the d_i. They are all contained in the first line of (5.2). No further overlaps can arise from the second line.

Since A/LA ends in degree ℓ, we have in particular that

$$\begin{aligned} [\operatorname{tor}_3^{\bar{R}}(A/LA, k)]_{\ell+3} &= h_{A/LA}(\ell) \\ &= \Delta h_{R/J}(\ell) := b_3. \end{aligned}$$

Using Proposition 4.1, Theorem 4.2 and the resolution (5.1), it is tedious but possible to check that

$$[\operatorname{tor}_3^{\bar{R}}(\bar{R}/\bar{J}, k)]_j = 0 \ \text{ for } j \le \ell + 2.$$

Combining the above, we obtain a precise description of $[\operatorname{tor}_3^{\bar{R}}(A/LA, k)]_j$ for all j.

Again using Proposition 4.1, Theorem 4.2 and the resolution (5.1), we get that if $j \le \ell + 1$, then there is only one possibility for $[\operatorname{tor}_2^{\bar{R}}(\bar{R}/\bar{J}, k)]_j \ne 0$:

$$\begin{aligned} [\operatorname{tor}_2^{\bar{R}}(\bar{R}/\bar{J}, k)]_j \ne 0 \quad &\Leftrightarrow \quad j = f - d_3 \le \ell + 1 \\ &\Leftrightarrow \quad d_1 + d_2 + d_5 + 2 \le d_3 + d_4. \end{aligned}$$

Notice that even if $f - d_3 = \ell + 1$, there is an overlap in the resolution which does not split off. This comes from Theorem 4.2. Such a summand corresponds to a nontrivial syzygy of generators of lower degree.

Therefore A/LA has the following minimal free $\bar{R}$-resolution:

$$0 \to \bar{R}(-\ell-3)^{b_3} \to \begin{pmatrix} \bar{R}(d_3-f)^{a_2} \\ \oplus \\ \bar{R}(-\ell-2)^{b_2} \end{pmatrix} \to \begin{pmatrix} \bigoplus_{i=1}^4 \bar{R}(-d_i)^{a_1^i} \\ \oplus \\ \bar{R}(-\ell-1)^{b_1} \end{pmatrix} \to \bar{R} \to A/LA \to 0, \tag{5.3}$$

where

$$a_2 = \begin{cases} 1 & \text{if } f-d_3 \leq \ell+1, \\ 0 & \text{otherwise,} \end{cases} \qquad a_1^i = \begin{cases} 1, & \text{if } d_i \leq \ell, \\ 0, & \text{otherwise,} \end{cases}$$

and b_2, b_1 are determined by the exactness of the above exact sequence (use the equation involving only ranks and the equation involving the first Chern classes). Note that b_3 was defined above.

Now we take the now-known $\bar{R}$-resolution of A/LA and plug the values into Proposition 5.1. Note that α is what we now are calling ℓ (because there is only one peak), and in fact $\alpha = \ell = s - \alpha$. Then, we have

$$\left[\operatorname{tor}_i^R(A,k)\right]_j$$

$$= \begin{cases} \left[\operatorname{tor}_i^{\bar{R}}(A/LA,k)\right]_j & \text{if } j \leq \ell+i-2, \\ \leq \left[\operatorname{tor}_i^{\bar{R}}(A/LA,k)\right]_j + \left[\operatorname{tor}_{4-i}^{\bar{R}}(A/LA,k)\right]_{2\ell+4-j} & \\ & \text{if } \ell+i-1 \leq j \leq \ell+i+1, \\ \left[\operatorname{tor}_{4-i}^{\bar{R}}(A/LA,k)\right]_{2\ell+4-j} & \text{if } j \geq \ell+i+2. \end{cases}$$

Now, using this fact together with the fact that A and R/J agree in degree $\leq \ell$, plus the symmetry of the resolution, we get that the minimal free R-resolution of R/G has the form

$$0 \to R(-2\ell-4) \to \begin{pmatrix} \bigoplus_{i=1}^4 R(d_i-2\ell-4)^{a_1^i} \\ \oplus \\ R(-\ell-3)^{\leq b_1+b_3} \end{pmatrix} \to \begin{pmatrix} R(d_3-f)^{a_2} \\ \oplus \\ R(-\ell-2)^{\leq 2b_2} \\ \oplus \\ R(f-d_3-2\ell-4)^{a_2} \end{pmatrix}$$

$$\to \begin{pmatrix} \bigoplus_{i=1}^4 R(-d_i)^{a_1^i} \\ \oplus \\ R(-\ell-1)^{\leq b_1+b_3} \end{pmatrix} \to R \to R/G \to 0.$$

(We are also again using the fact that if $d_3 - f = \ell+1$, then this syzygy represents a minimal syzygy of forms of lower degree, and cannot be split off.)

We now claim that the above inequalities are in fact equalities. Indeed, we know that A and R/J agree up to degree ℓ. Let us denote by $J_{\leq \ell}$ the ideal generated by the components of J in degree $\leq \ell$. Note that $J_{\leq \ell}$ is a complete intersection of height < 4; hence depth $R/J_{\leq \ell} \geq 1$. It follows that we need b_1 minimal generators in degree $\ell+1$ in order to bring $\dim(R/J_{\leq \ell})_{\ell+1}$ down to the level of $\dim A_\ell$, and then another b_3 generators to bring it down to the level of $\dim A_{\ell+1}$ ($= \dim A_{\ell-1}$). Therefore we have the first and third free modules. The equality for the second free module is exactly obtained from the ranks of the minimal free resolution (5.3). Therefore we have the precise minimal free resolution of R/G.

To compute the minimal free R-resolution of R/I, note that we have linked using G_1, G_2, G_3 and G_4, and that G_i is a minimal generator of G if $d_i \leq \ell$ or if $d_i = \ell+1$. Therefore, using (3.1), we see that for any i such that $d_i \leq \ell + 1$ we can split off a summand $R(d_i - d)$ from the end of the resolution of R/I, and the remaining resolution is minimal. The fact that no further splitting off occurs follows as in the first part of the proof of Theorem 4.2. □

Example 5.7. Let $(d_1, \ldots, d_5) = (3, 3, 4, 6, 6)$. Then the above proof tells us that a minimal free resolution for R/G is

$$0 \to R(-10) \to \begin{pmatrix} R(-6)^{17} \\ \oplus \\ R(-7)^2 \end{pmatrix} \to R(-5)^{36} \to \begin{pmatrix} R(-3)^2 \\ \oplus \\ R(-4)^{17} \end{pmatrix} \to R \to R/G \to 0$$

and a minimal free resolution for R/I is

$$0 \to R(-12)^{16} \to \begin{pmatrix} R(-10) \\ \oplus \\ R(-11)^{36} \end{pmatrix} \to \begin{pmatrix} R(-6) \\ \oplus \\ R(-7)^2 \\ \oplus \\ R(-9)^4 \\ \oplus \\ R(-10)^{18} \end{pmatrix} \to \begin{pmatrix} R(-3)^2 \\ \oplus \\ R(-4) \\ \oplus \\ R(-6)^2 \end{pmatrix}$$
$$\to R \to R/I \to 0.$$

Notice that all the summands $R(-6), R(-7)$ and $R(-9)$ are Koszul, but that there is also an $R(-10)$ that does not split off.

We observe that all of the examples where redundant ("ghost") terms occurred were cases where the degrees were not all the same. In contrast, Corollary 4.4 and Theorem 5.4 at least suggest – and prove some cases – of the following conjecture:

Conjecture 5.8. *Let $I \subset R = k[x_1, \ldots, x_n]$ be the ideal of $d > n$ generically chosen forms of the same degree. Then there is no redundant term in the minimal free resolution of R/I. Consequently, the minimal free resolution is the minimum one consistent with the Hilbert function, whose conjectured value is given by R. Fröberg.*

Note that the first syzygy case of this conjecture was proved by M. Hochster and D. Laksov [19].

References

[1] D. Anick, *Thin algebras of embedding dimension three*, J. Algebra **100** (1986), 235–259. MR **88d:**13016a

[2] M. Aubry, *Série de Hilbert d'une algèbre de polynomes quotient*, J. Algebra **176** (1995), 392–416. MR **96h:**13039

[3] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from ftp://math.harvard.edu via anonymous ftp.

[4] A. Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2317–2334. MR **94c:**13014

[5] M. Boij, *Betti numbers of compressed level algebras*, J. Pure and Applied Algebra **134** (1999), 111–131. MR **99m:**13023

[6] M. Boij, *Gorenstein Artin Algebras and Points in Projective Space*, Bull. London Math. Soc. **31** (1999), 11–16. MR **99i:**13033
[7] D. Buchsbaum and D. Eisenbud, *Algebra Structures for Finite Free Resolutions, and some Structure Theorems for Ideals of Codimension 3*, Amer. J. of Math. **99** (1977), 447–485. MR **56:**11983
[8] K. Chandler, *The Geometric Fröberg-Iarrobino Conjecture*, in preparation.
[9] E. Davis, A.V. Geramita, and F. Orecchia, *Gorenstein Algebras and the Cayley-Bacharach Theorem*, Proc. Amer. Math. Soc. **93** (1985), 593–597. MR **86k:**14034
[10] S. Diesel, *Irreducibility and Dimension Theorems for Families of Height 3 Gorenstein Algebras*, Pacific J. Math. **172** (1996), no. 2, 365–397. MR **99f:**13016
[11] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133. MR **85f:**13023
[12] D. Eisenbud and S. Popescu, *Gale Duality and Free Resolutions of Ideals of Points*, Invent. Math. **136** (1999), 419–449. MR **2000i:**13014
[13] R. Fröberg, *An inequality for Hilbert series of graded algebras*, Math. Scand. **56** (1985), 117–144. MR **87f:**13021
[14] R. Fröberg and J. Hollman, *Hilbert series for ideals generated by generic forms*, J. Symbolic Comput. **17** (1994), 149–157. MR **95h:**13015
[15] R. Fröberg and D. Laksov, *Compressed algebras*, Complete intersections (Acireale, 1983), 121–151, Lecture Notes in Math., 1092, Springer, Berlin, 1984.
[16] T. Harima, *Characterization of Hilbert functions of Gorenstein Artin algebras with the weak Stanley property*, Proc. Amer. Math. Soc. **123** (1995), 3631–3638. MR **96b:**13014
[17] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The Weak and Strong Lefschetz properties for Artinian K-algebras*, to appear in J. Algebra.
[18] A. Hirschowitz and C. Simpson, *La résolution minimale de l'idéal d'un arrangement général d'un grand nombre de points dans $\mathbb{P}^n$*, Invent. Math. **126** (1996), 467–503. MR **97i:**13015
[19] M. Hochster and D. Laksov, *The Linear Syzygies of Generic Forms*, Comm. Algebra **15** (1987), 227–239. MR **88d:**13016b
[20] H. Hulett, *Maximum Betti numbers of homogeneous ideals with a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2335–2350. MR **94c:**13015
[21] A. Iarrobino, *Inverse system of a symbolic power III. Thin algebras and fat points*, Compositio Math. **108** (1997) 319-356. MR **98k:**13017
[22] A. Iarrobino and V. Kanev, "Power Sums, Gorenstein Algebras, and Determinantal Loci," Springer LNM 1721 (1999). MR **2001d:**14056
[23] A. Lorenzini, *The Minimal Resolution Conjecture*, J. Algebra **156** (1993), 5–35. MR **94g:**13005
[24] J. Migliore, "Introduction to Liaison Theory and Deficiency Modules," Birkhäuser, Progress in Mathematics 165, 1998. MR **2000g:**14058
[25] J. Migliore and U. Nagel, *Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers*, to appear in Adv. Math.
[26] Keith Pardue and Ben Richert, *Resolutions of Generic Ideals*, in preparation.
[27] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Invent. Math. **26** (1974), 271–302. MR **51:**526
[28] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184. MR **82j:**20083
[29] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. 11, Kinokuniya Co. North Holland, Amsterdam (1987), 303–312. MR **89k:**13015

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
E-mail address: Juan.C.Migliore.1@nd.edu

Facultat de Matemàtiques, Departament d'Algebra i Geometria, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
E-mail address: miro@cerber.mat.ub.es