

MAXIMAL FUNCTIONS WITH POLYNOMIAL DENSITIES IN LACUNARY DIRECTIONS

KATHRYN HARE AND FULVIO RICCI

ABSTRACT. Given a real polynomial $p(t)$ in one variable such that $p(0) = 0$, we consider the maximal operator in \mathbb{R}^2 ,

$$M_p f(x_1, x_2) = \sup_{h>0, i, j \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt .$$

We prove that M_p is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$ with bounds that only depend on the degree of p .

1. INTRODUCTION

Maximal operators on the real line of the form

$$(1.1) \quad f(x) \mapsto \sup_{h>0} \frac{1}{h} \int_0^h |f(x - p(t))| dt ,$$

where p is a real polynomial with $p(0) = 0$, were considered in [CRW1], and it was shown that they satisfy weak-type 1-1 estimates that are uniform over all polynomials of fixed degree. Natural extensions of these operators to higher dimensions are discussed in [CRW2], in connection with \mathbb{R}^n -valued polynomials defined on \mathbb{R}^m .

We consider here a different kind of multi-dimensional analogue of (1.1), which is modelled on the maximal function in lacunary directions introduced in [NSW]. For simplicity, we restrict ourselves to two dimensions and to dyadic lacunary directions, i.e., determined by the vectors $v_k = (1, 2^k)$ with $k \in \mathbb{Z}$. In addition, we allow dyadic scaling along each of these directions.

To be precise, given a real polynomial $p(t)$ in one variable such that $p(0) = 0$, we define

$$(1.2) \quad \begin{aligned} M_p f(x_1, x_2) &= \sup_{h>0, i, k \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x - 2^i p(t)v_k)| dt \\ &= \sup_{h>0, i, j \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt . \end{aligned}$$

We prove the following result.

Theorem 1. *M_p is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$ with bounds that only depend on the degree of p .*

Received by the editors May 27, 2002.

2000 *Mathematics Subject Classification.* Primary 42B25.

The research of the first author was supported in part by NSERC and the Swedish Natural Sciences Research Council.

It is easy to check that M_p cannot satisfy a weak-type 1-1 estimate.

The proof of Theorem 1 is based on the analysis of a general class of two-parameter maximal operators in the plane defined by compactly supported measures, subject to a decay assumption on their Fourier transforms. This result is in the spirit of [DR] and [RS], but here we consider the possibility that the Fourier transform of the measure has no decay within an angle that does not contain the coordinate axes.

Theorem 2. *For a probability measure μ supported on the unit square, let $\mu_{i,j}$ be the measure such that*

$$\int f d\mu_{i,j} = \int f(2^i x_1, 2^j x_2) d\mu(x_1, x_2) .$$

Assume that

(i) *there are constants $C, \delta > 0$ and $s > 1$ such that*

$$(1.3) \quad |\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\delta}$$

away from the set where $s^{-1} < \frac{|\xi_1|}{|\xi_2|} < s$;

(ii) *the one-parameter maximal operator*

$$(1.4) \quad M_\mu^0 f(x) = \sup_{i \in \mathbb{Z}} |f * \mu_{i,i}(x)|$$

is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$.

Then also, the two-parameter maximal operator,

$$(1.5) \quad M_\mu f(x) = \sup_{i,j \in \mathbb{Z}} |f * \mu_{i,j}(x)| ,$$

is bounded on $L^q(\mathbb{R}^2)$ for $q > 1$, with bounds that only depend on s , the constants C, δ in (1.3) and the norm of M_μ^0 .

We start with the proof of Theorem 2, which combines methods from [NSW], [C] and [RS]. This is done in Section 2. Theorem 1 is proved in Section 3.

We thank both W. Beckner and the referee for useful comments on a first version of this paper.

2. PROOF OF THEOREM 2

Let σ_1 and σ_2 be the measures on the line defined by

$$\int_{\mathbb{R}} f(t) d\sigma_j(t) = \int_{\mathbb{R}^2} f(x_j) d\mu(x) .$$

Then $\hat{\sigma}_1(\tau) = \hat{\mu}(\tau, 0)$ and $\hat{\sigma}_2(\tau) = \hat{\mu}(0, \tau)$, so that

$$(2.1) \quad |\hat{\sigma}_j(\tau)| \leq C(1 + |\tau|)^{-\delta} .$$

Let φ be a nonnegative smooth function on the line, supported on $[-1, 1]$ and with integral equal to 1. Define

$$\nu = \mu - \sigma_1 \otimes \varphi - \varphi \otimes \sigma_2 + \varphi \otimes \varphi .$$

Clearly, $\hat{\nu}$ satisfies (1.3), is supported on the unit square and

$$(2.2) \quad \hat{\nu}(\xi_1, 0) = \hat{\nu}(0, \xi_2) = 0 .$$

Since

$$M_\mu f \leq M_\nu f + M_{\sigma_1 \otimes \varphi} f + M_{\varphi \otimes \sigma_2} f + M_{\varphi \otimes \varphi} ,$$

we can discuss each of the maximal functions on the right-hand side separately.

The last term is controlled by the two-parameter strong maximal operator of Jessen, Marcinkiewicz and Zygmund. The L^q -boundedness of the two intermediate terms follows from Theorem 3.2 in [RS], once we observe that, by (2.1),

$$|\widehat{\sigma_1 \otimes \varphi}(\xi)| \leq C'(1 + |\xi|)^{-\delta} ,$$

and similarly for $\varphi \otimes \sigma_2$. (Alternatively, one can argue that $M_{\sigma_1 \otimes \varphi}$ is controlled by the composition of the Hardy-Littlewood maximal operator in the x_2 -variable with the one-parameter operator M_{σ_1} in the x_1 -variable; to this operator one can apply Theorem A in [DR].)

Thus it remains to estimate $M_\nu f$. Due to the cancellations of ν that are implicit in (2.2), it is convenient to introduce appropriate square functions. Given a measure σ , we shall need two types of such functions:

$$(2.3.a) \quad S_\sigma f(x) = \left(\sum_{i,j \in \mathbb{Z}} |f * \sigma_{i,j}(x)|^2 \right)^{\frac{1}{2}} ,$$

$$(2.3.b) \quad \tilde{S}_\sigma f(x) = \left(\sum_{k \in \mathbb{Z}} \left(\sup_{i \in \mathbb{Z}} |f * \sigma_{i,i+k}(x)| \right)^2 \right)^{\frac{1}{2}} .$$

Clearly, $M_\sigma f \leq \tilde{S}_\sigma f \leq S_\sigma f$. We shall also assume that q is finite, because there is nothing to prove for $q = \infty$.

Let $\eta_\ell(x) = 2^{2\ell} \eta(2^\ell x)$, $\ell \geq 0$, be a smooth approximate identity in \mathbb{R}^2 , with η supported on the unit disk. We set $\psi_0 = \eta_0$, and $\psi_\ell = \eta_\ell - \eta_{\ell-1}$ for $\ell \geq 1$. Then

$$\nu = \sum_{\ell=0}^{\infty} \nu * \psi_\ell$$

and

$$S_\nu f \leq \sum_{\ell=0}^{\infty} S_{\nu * \psi_\ell} f .$$

Lemma 2.1. *For every $\varepsilon > 0$ and $1 < q < \infty$, $\|S_{\nu * \psi_\ell} f\|_q \leq A2^{2\ell\varepsilon} \|f\|_q$, where the constant A depends only on ε and q .*

Proof. By the standard randomization argument, we can estimate the L^q -operator norm of the singular integral operators

$$f \mapsto \sum_{i,j} \pm (\nu * \psi_\ell)_{i,j} * f .$$

We apply Lemma 2.3 in [RS]. Thus, it is necessary to prove that

$$\sup_{0 < |h_2| < 2} |h_2|^{-\varepsilon} \int \left(\sup_{0 < |h_1| < 2} |h_1|^{-\varepsilon} \int |\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell)(x)| dx_1 \right) dx_2 \leq C2^{2\ell\varepsilon} ,$$

where

$$\begin{aligned} \Delta_{h_1}^1 f(x_1, x_2) &= f(x_1 + h_1, x_2) - f(x_1, x_2) , \\ \Delta_{h_2}^2 f(x_1, x_2) &= f(x_1, x_2 + h_2) - f(x_1, x_2) . \end{aligned}$$

We observe that

$$\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell) = \nu * (\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell)$$

and that $\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell(x)$ is smaller than a constant times $2^{(2+2\varepsilon)\ell} |h_1|^\varepsilon |h_2|^\varepsilon$, and it is supported, for each x, h_1, h_2 , on a set that is the union of four disks of radius $2^{-\ell}$. Therefore,

$$\begin{aligned} \int |\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell)(x)| dx_1 &\leq \int_{\mathbb{R}^2} \left(\int |\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell(x-y)| dx_1 \right) d|\nu|(y) \\ &\leq C 2^{(1+2\varepsilon)\ell} |h_1|^\varepsilon |h_2|^\varepsilon \int_{\mathbb{R}^2} \chi_{y, h_2}(x_2) d|\nu|(y) , \end{aligned}$$

where χ_{y, h_2} is the characteristic function of a set of measure $2^{-\ell}$ depending on y and h_2 .

This concludes the proof. □

In order to obtain better estimates, we introduce a spectral decomposition of ν . Let $\Phi(\xi)$ be homogeneous of degree 0, smooth away from the origin, identically equal to 1 inside the angle $\Gamma_1 = \{ \xi : s^{-1} < |\xi_1|/|\xi_2| < s \}$, and identically equal to 0 outside of the angle $\Gamma_2 = \{ \xi : (2s)^{-1} < |\xi_1|/|\xi_2| < 2s \}$.

We then define the “bad part” ν_b of ν as the distribution such that

$$\hat{\nu}_b(\xi) = \hat{\nu}(\xi)\Phi(\xi) ,$$

and the “good part” ν_g as $\nu_g = \nu - \nu_b$.

The square functions $S_{\nu_b} f, S_{\nu_b * \psi_\ell} f$, etc. are defined as in (2.3.a) and (2.3.b) for Schwartz functions f .

We show first that each part of ν shares the good properties of ν given by Lemma 2.1.

Lemma 2.2. *The conclusion of Lemma 2.1 remains valid if we replace ν by ν_b or ν_g .*

Proof. For $k \in \mathbb{Z}$, let $P_k f = \mathcal{F}^{-1}(\Phi(\xi_1, 2^{-k}\xi_2)\hat{f}(\xi))$. Because of the finite overlapping of the supports of the multipliers $\Phi(\xi_1, 2^{-k}\xi_2)$, we have the Littlewood-Paley estimate

$$(2.4) \quad \left\| \left(\sum_{k \in \mathbb{Z}} |P_k f|^2 \right)^{\frac{1}{2}} \right\|_q \sim \|f\|_q ,$$

for $1 < q < \infty$. Also, observe that

$$(\nu_b)_{i,j} * f = \nu_{i,j} * (P_{i-j} f) , \quad (\nu_b * \psi_\ell)_{i,j} * f = (\nu * \psi_\ell)_{i,j} * (P_{i-j} f) .$$

Therefore,

$$\begin{aligned} S_{\nu_b * \psi_\ell} f &= \left(\sum_{i,j \in \mathbb{Z}} |(\nu * \psi_\ell)_{i,j} * (P_{i-j} f)(x)|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i,j,k \in \mathbb{Z}} |(\nu * \psi_\ell)_{i,j} * (P_k f)(x)|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

The last quantity equals the L^2 -norm on $[0, 1]^3$ of the function

$$(t, u, v) \longmapsto \sum_{i,j,k \in \mathbb{Z}} (\nu * \psi_\ell)_{i,j} * (P_k f)(x) r_i(t) r_j(u) r_k(v) ,$$

where r_n is the n th Rademacher function. By the properties of Rademacher functions, the L^2 -norm is equivalent to the L^q -norm. Therefore,

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C \int_{\mathbb{R}^2} \int_{[0,1]^3} \left| \sum_{i,j,k \in \mathbb{Z}} (\nu * \psi_\ell)_{i,j} * (P_k f)(x) r_i(t) r_j(u) r_k(v) \right|^q dt du dv dx .$$

We denote

$$K_{t,u} = \sum_{i,j} r_i(t) r_j(u) (\nu * \psi_\ell)_{i,j} , \quad f_v = \sum_k r_k(v) P_k f .$$

Changing the order of integration, we have

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C \int_{[0,1]^3} \|K_{t,u} * f_v\|_q^q dt du dv .$$

The proof of Lemma 2.1 shows that the L^q -operator norms of the $K_{t,u}$ are uniformly bounded by a constant times $2^{2\ell\varepsilon}$. Hence,

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C 2^{2\ell\varepsilon} \int_{[0,1]} \|f_v\|_q^q dv .$$

Changing the order of integration again, replacing the L^q -norm on $[0, 1]$ with the L^2 -norm, and using (2.4), we obtain the conclusion for ν_b .

For ν_g it is sufficient to observe that $S_{\nu_g * \psi_\ell} f \leq S_{\nu * \psi_\ell} f + S_{\nu_b * \psi_\ell} f$. □

We shall now improve the estimate on $S_{\nu_g * \psi_\ell}$, using the uniform decay of $\hat{\nu}_g(\xi)$ as ξ goes to infinity. In fact, as we already observed, $\hat{\nu}$ satisfies (1.3); hence,

$$(2.5) \quad |\hat{\nu}_g(\xi)| \leq C(1 + |\xi|)^{-\delta} .$$

We shall assume, w.l.o.g., that $\delta < 1$.

Lemma 2.3. $\|S_{\nu_g * \psi_\ell} f\|_2 \leq A 2^{-\ell\delta/4} \|f\|_2$, with A depending only on δ and C .

Proof. By the Plancherel formula, we have to prove that

$$(2.6) \quad \sum_{i,j \in \mathbb{Z}} |\hat{\nu}_g(2^i \xi_1, 2^j \xi_2)|^2 |\hat{\psi}_\ell(2^i \xi_1, 2^j \xi_2)|^2 \leq A 2^{-\ell\delta/2} .$$

By (2.2),

$$\hat{\nu}(\xi) = \int (e^{-ix_1 \xi_1} - 1)(e^{-ix_2 \xi_2} - 1) d\nu(\xi) .$$

Since ν is supported on the unit square,

$$|\hat{\nu}(\xi)| \leq C |\xi_1| |\xi_2| .$$

Combining this with (2.5), we obtain that, if $0 < \varepsilon < 1$,

$$|\hat{\nu}_g(\xi)| \leq C \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)}} .$$

If $\ell \geq 1$, then

$$|\hat{\psi}_\ell(\xi)| = |\hat{\psi}_1(2^{-(\ell-1)} \xi)| \leq C 2^{-\ell\varepsilon} |\xi|^\varepsilon ,$$

because $\hat{\psi}_1(0) = 0$. Hence,

$$|\hat{\nu}_g(\xi) \hat{\psi}_\ell(\xi)| \leq C 2^{-\ell\varepsilon} \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)-\varepsilon}} .$$

We can assume that $|\xi_1| \sim |\xi_2| \sim 1$ in (2.6). Then we simply have to observe that, taking $\varepsilon = \delta/4$, the exponent in the denominator is bigger than $\delta/2 = 2\varepsilon$, and that the series

$$\sum_{i,j \in \mathbb{Z}} \frac{2^{2\varepsilon i} 2^{2\varepsilon j}}{(1 + 2^i + 2^j)^{2\alpha}}$$

is convergent for $\alpha > 2\varepsilon$. □

Interpolating between the L^2 -estimate in Lemma 2.3 and the L^q -estimate in Lemma 2.2 for $S_{\nu_g * \psi_\ell}$, we obtain that for every $q \in (1, \infty)$ there is an $\varepsilon_q > 0$ such that $\|S_{\nu_g * \psi_\ell} f\|_q \leq A 2^{-\ell \varepsilon_q} \|f\|_q$. Therefore,

Proposition 2.4. *S_{ν_g} is bounded on L^q for $1 < q \leq 2$.*

In order to complete the proof of Theorem 2, we may just observe that we are in the hypotheses of Theorem B in [C] (attributed to M. Christ). We give, however, an independent proof, based on the extrapolation argument in [NSW], adapted to \tilde{S}_{ν_b} .

End of the proof of Theorem 2. The starting point is that \tilde{S}_{ν_b} is bounded on L^2 . In fact, assumption (ii) implies that $M_{\nu_{0,k}}^0$ is uniformly bounded on L^q independently of k . Therefore,

$$\begin{aligned} \int \tilde{S}_{\nu_b} f(x)^2 dx &= \sum_{k \in \mathbb{Z}} \int \sup_{i \in \mathbb{Z}} |\nu_{i,i+k} * P_k f(x)|^2 dx \\ &= \sum_{k \in \mathbb{Z}} \int (M_{\nu_{0,k}}^0 P_k f(x))^2 dx \\ &\leq C \sum_{k \in \mathbb{Z}} \int (P_k f(x))^2 dx \\ &= C \|f\|_2^2. \end{aligned}$$

In general, the boundedness of \tilde{S}_{ν_b} on some L^q implies, by Proposition 2.4, the boundedness of M_ν on the same L^q , and hence that of M_μ .

Assume now that M_μ is bounded on some L^q , and consider the inequality

$$(2.7) \quad \left\| \left(\sum_{k \in \mathbb{Z}} M_{\mu_{0,k}}^0 f_k(x)^r \right)^{1/r} \right\|_s \leq C \left\| \left(\sum_{k \in \mathbb{Z}} |f_k(x)|^r \right)^{1/r} \right\|_s.$$

This is equivalent to saying that the linear operator

$$T : \{f_k\} \mapsto \{\mu_{i,i+k} * f_k\}$$

is bounded from $L^s(\ell^r)$ to $L^s(\ell^r(\ell^\infty))$.

Since μ is a positive measure and we are assuming that M_μ is bounded on L^q , (2.7) is verified for $r = \infty$ and $s = q$. In addition, it is verified for $r = s > 1$ by the uniform boundedness of $M_{\mu_{0,k}}^0$. Hence, T is bounded from $L^q(\ell^\infty)$ to $L^q(\ell^\infty(\ell^\infty))$ and from $L^r(\ell^r)$ to $L^r(\ell^r(\ell^\infty))$ for $r > 1$. By interpolation, (2.7) holds for $r = 2$ and $\frac{1}{q} < \frac{1}{s} < \frac{1}{2}(1 + \frac{1}{q})$.

The same inequality holds with μ replaced by $\sigma \otimes \varphi$, $\varphi \otimes \sigma$ and $\varphi \otimes \varphi$, and hence with μ replaced by ν .

Taking $f_k = P_k f$, this implies that \tilde{S}_{ν_b} is bounded on the same spaces L^s . Since each $q \in (1, 2)$ can be reached by iteration in a finite number of steps, we conclude that M_μ is bounded on L^q for every $q > 1$. \square

3. PROOF OF THEOREM 1

The starting point for the proof of Theorem 1 is Lemma 2.5 in [CRW1]. We give a slightly different (and less complete) formulation of it.

Lemma 3.1. *For every n there are constants $A(n) \geq 1$ and $B = B(n)$ with the following property: if $p(t)$ is a monic real polynomial of degree n such that $p(0) = 0$, $A \geq A(n)$, and $m \in \mathbb{Z}$ is such that no complex zero of p lies in the strip*

$$\{z : A^{m-1} \leq |z| \leq A^{m+2}\},$$

then the following properties hold:

- (i) p has constant sign and is strictly monotonic on $I_m = [A^m, A^{m+1}]$;
- (ii) $|p(t)| \leq Bt|p'(t)|$ for $t \in I_m$;
- (iii) $\max_{t \in I_m} |p(t)| \leq B \min_{t \in I_m} |p(t)|$.

Observe that we are allowed to replace the polynomial $p(t)$ in (1.2), when convenient, by $\tilde{p}(t) = bp(at)$, with $a, b > 0$. In fact, the identity

$$M_{\tilde{p}}f(x) = M_p f_b\left(\frac{x}{b}\right),$$

where $f_b(x) = f(bx)$, implies that M_p and $M_{\tilde{p}}$ have the same operator norm. In particular, we can assume that p is monic.

Also, the maximal function M_p can be replaced by

$$\tilde{M}_p f(x_1, x_2) = \sup_{m \in \mathbb{Z}} \tilde{M}_{p,m} f(x_1, x_2)$$

where

$$(3.1) \quad \tilde{M}_{p,m} f(x_1, x_2) = \sup_{i,j \in \mathbb{Z}} A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt.$$

Let I_m be one of the “good” dyadic intervals satisfying properties (i)–(iii) in Lemma 3.1. Making the change of variable $u = p(t)$, we have

$$\begin{aligned} A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt &\leq A \int_{A^m}^{A^{m+1}} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| \frac{dt}{t} \\ &\leq AB \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| \frac{|p'(t)|}{|p(t)|} dt \\ &= AB \int_{p(I_m)} |f(x_1 - 2^i u, x_2 - 2^j u)| \frac{du}{|u|}. \end{aligned}$$

By (i) and (iii), the interval $p(I_m)$ is contained in an interval of the form $\pm[\alpha_m, B\alpha_m]$, with $\alpha_m > 0$. Therefore, assuming w.l.o.g. that p is positive on I_m ,

$$\begin{aligned}
 A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt &\leq AB \int_{\alpha_m}^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| \frac{du}{u} \\
 &\leq \frac{AB}{\alpha_m} \int_{\alpha_m}^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| du \\
 &\leq \frac{AB^2}{B\alpha_m} \int_0^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| du .
 \end{aligned}$$

This shows that the contribution to $\tilde{M}_p f$ given by the “good” intervals is controlled by the maximal function in lacunary directions

$$\mathcal{M}f(x_1, x_2) = \sup_{h>0, k \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - t, x_2 - 2^k t)| dt$$

of [NSW]. Since \mathcal{M} is bounded on L^q for $q > 1$ [NSW], it remains to consider the contribution from the “bad” intervals. Since there are at most $3n$ of these intervals, it is enough to prove that $\tilde{M}_{p,m}$ acts on L^q for $q > 1$, with operator norm bounded independently of the polynomial p and integer m .

We claim it suffices to show that there exists a constant $C_{q,n}$ such that

$$\sup_{m \in \mathbb{Z}} \|\tilde{M}_{p,m} f\|_q \leq C_{q,n} \|f\|_q$$

for every $f \in L^q$ and monic polynomial p of degree n satisfying $p(0) = 0$ and

$$(3.2) \quad A^{-n} \leq \max_{t \in I_m} |p(t)| \leq 1.$$

To see this, suppose p is an arbitrary monic polynomial with $p(0) = 0$, and choose $k \in \mathbb{Z}$ such that

$$A^{-n} \leq \max_{t \in I_m} A^{-kn} |p(t)| \leq 1.$$

Let $\tilde{p}(t) = A^{-kn} p(A^k t)$. Since

$$A^{-n} \leq \max_{t \in I_{m-k}} |\tilde{p}(t)| \leq 1,$$

the (L^q, L^q) operator norm of $\tilde{M}_{\tilde{p}, m-k}$ is at most $C_{q,n}$. Since

$$\tilde{M}_{p,m} f(x) = \tilde{M}_{\tilde{p}, m-k} f_{A^k}(A^{-k} x),$$

$\tilde{M}_{p,m}$ also acts on L^q with bounds that are independent of m and p .

Consequently, we need to investigate the measure μ given by

$$\int f d\mu = A^{-m} \int_{I_m} f(p(t), p(t)) dt$$

where p satisfies (3.2). This measure is supported on the segment $\{(u, u) : -1 \leq u \leq 1\}$ and, up to a factor depending on A , is a probability measure.

The proof of Theorem 1 will be complete once we show that the operator M_μ is bounded on L^q for $q > 1$ with bounds that depend only on n and q . We apply Theorem 2.

The Fourier transform of μ is

$$(3.3) \quad \hat{\mu}(\xi_1, \xi_2) = A^{-m} \int_{I_m} e^{-i(\xi_1 + \xi_2)p(t)} dt .$$

Lemma 3.2. *There is an integer $k \in \{1, 2, \dots, n\}$ such that, if A is large enough (depending on n), then*

$$|\hat{\mu}(\xi_1, \xi_2)| \leq CA^n(1 + |\xi_1 + \xi_2|)^{-1/k} ,$$

with C independent of p and m .

Proof. Let $t_1 = 0, t_2, \dots, t_n$ be the zeroes of p , ordered so that $0 \leq |t_2| \leq \dots \leq |t_n|$. Let m' be the smallest integer greater than m such that $I_{m'}$ does not contain any of the $|t_j|$. Then $m' \leq m + n$, so that $A^{m'}$ is comparable with A^m . Also let k be such that $|t_j| < A^{m'}$ for $j \leq k$ and $|t_j| > A^{m'+1}$ for $j > k$.

The k th derivative of p equals

$$p^{(k)}(t) = \prod_{j=k+1}^n (t - t_j) + r(t) ,$$

where $r(t)$ is a sum where each term is a product of $n - k$ factors $t - t_j$, with at least one of the j less than or equal to k .

If $t \in I_m$, $|t - t_j| < 2A^{m'}$ for $j \leq k$, and $|t - t_j| > (1 - A^{-1})|t_j| > (A - 1)A^{m'}$ for $j > k$. Therefore, if A is large enough,

$$|p^{(k)}(t)| \geq C \prod_{j=k+1}^n |t_j| ,$$

for $t \in I_m$.

By van der Corput's lemma,

$$A^{-m} \left| \int_{I_m} e^{-i\lambda p(t)} dt \right| \leq CA^{-m} \left(\prod_{j=k+1}^n |t_j| \right)^{-1/k} |\lambda|^{-1/k} .$$

If $\bar{t} \in I_m$ is such that $|p(\bar{t})| \geq A^{-n}$, we have

$$A^{-n} \leq |p(\bar{t})| \leq 2^n A^{km'} \prod_{j=k+1}^n |t_j| .$$

Therefore, $\prod_{j=k+1}^n |t_j| \geq CA^{-n} A^{-km'}$, so that

$$A^{-m} \left| \int_{I_m} e^{-i\lambda p(t)} dt \right| \leq C|\lambda|^{-1/k} A^n ,$$

with C independent of p and m . Since the left-hand side is trivially bounded by 1, this concludes the proof. \square

Thus, $\hat{\mu}$ clearly satisfies hypothesis (i) of Theorem 2. It remains to prove that the one-parameter maximal operator M_μ^0 in (1.4) is bounded on L^q for $q > 1$ with bounds that only depend on n and q . This follows from a transference argument: because μ is supported on a line, it is sufficient to consider the maximal operator on \mathbb{R} ,

$$M_{\tilde{\mu}}g(x) = \sup_{i \in \mathbb{Z}} |g * \tilde{\mu}_i(x)| ,$$

where

$$\int_{\mathbb{R}} g d\tilde{\mu} = A^{-m} \int_{I_m} g(p(t)) dt .$$

By Lemma 3.2, $|\hat{\mu}(\eta)| \leq C(1 + |\eta|)^{-1/k}$, with $1 \leq k \leq n$ and C depending only on n . The conclusion follows from Theorem A in [DR].

Remark. In [CRW1] the authors show that the “supermaximal function” on the real line

$$f(x) \longmapsto \sup_{\substack{h>0 \\ \deg p \leq k, p(0)=0}} \frac{1}{h} \int_0^h |f(x-p(t))| dt$$

is of restricted weak type $k-k$ and hence of strong type $q-q$ for $q > k$.

The proof can be adapted to show that the operator

$$\begin{aligned} \mathcal{M}_k f(x) &= \sup_{\deg p \leq k, p(0)=0} M_p f(x) \\ &= \sup_{\substack{h>0, j \in \mathbb{Z} \\ \deg p \leq k, p(0)=0}} \frac{1}{h} \int_0^h |f(x_1 - p(t), x_2 - 2^j p(t))| dt \end{aligned}$$

is bounded on $L^q(\mathbb{R}^2)$ for $q > k$.

In fact, if f is the characteristic function of a measurable set in the plane, the same proof as in [CRW1] gives the pointwise domination

$$\mathcal{M}_k f(x) \leq C(M^* f^k(x))^{1/k},$$

where M^* is the maximal function in dyadic direction of [NSW]. This implies that \mathcal{M}_k is of restricted weak type $q-q$ for $q > k$, and hence of strong type.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA, N2L 3G1

E-mail address: kehare@math.uwaterloo.ca

SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI 7, 56126 PISA, ITALY

E-mail address: fricci@sns.it

URL: <http://www.math.sns.it/HomePages/Ricci/>