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SOME PROPERTIES OF THE SCHOUTEN TENSOR AND APPLICATIONS TO CONFORMAL GEOMETRY

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ABSTRACT. The Riemannian curvature tensor decomposes into a conformally invariant part, the Weyl tensor, and a non-conformally invariant part, the Schouten tensor. A study of the kth elementary symmetric function of the eigenvalues of the Schouten tensor was initiated in an earlier paper by the second author, and a natural condition to impose is that the eigenvalues of the Schouten tensor are in a certain cone, Γ_k^+. We prove that this eigenvalue condition for $k \geq n/2$ implies that the Ricci curvature is positive. We then consider some applications to the locally conformally flat case, in particular, to extremal metrics of σ_k-curvature functionals and conformal quermassintegral inequalities, using the results of the first and third authors.

1. INTRODUCTION

Let (M^n, g) be an n-dimensional Riemannian manifold, $n \geq 3$, and let the Ricci tensor and scalar curvature be denoted by Ric and R, respectively. We define the Schouten tensor

$$A_g = \frac{1}{n-2}\left(Ric - \frac{1}{2(n-1)}Rg\right).$$

There is a decomposition formula (see [1]):

$$\text{Riem} = A_g \odot g + \mathcal{W}_g, \tag{1}$$

where $\mathcal{W}_g$ is the Weyl tensor of g, and $\odot$ denotes the Kulkarni-Nomizu product (see [1]). Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor. A study of k-th elementary symmetric functions of the Schouten tensor was initiated in [13], it was reduced to certain fully nonlinear Yamabe type equations. In order to apply the elliptic theory of fully nonlinear equations, one often restricts the Schouten tensor to be in a certain cone Γ_k^+, defined as follows (according to Gårding [5]).

Definition 1. Let $(\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n$. Let σ_k denote the kth elementary symmetric function

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

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and let

$$\Gamma_k^+ = \text{component of } \{\sigma_k > 0\} \text{ containing } \quad (1, \cdots, 1).$$

Let $\bar{\Gamma}_k^+$ denote the closure of Γ_k^+. If (M, g) is a Riemannian manifold, and $x \in M$, we say g has positive (nonnegative, resp.) Γ_k-curvature at x if its Schouten tensor $A_g \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$, resp.) at x. In this case, we also say $g \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$, resp.) at x.

We note that positive Γ_1-curvature is equivalent to positive scalar curvature, and the condition of positive Γ_k-curvature has some geometric and topological consequences for the manifold M. For example, when (M, g) is locally conformally flat with positive Γ_1-curvature, then $\pi_i(M) = 0, \forall 1 < i \leq \frac{n}{2}$, by a result of Schoen and Yau [11]. In this note, we will prove that positive Γ_k-curvature for any $k \geq \frac{n}{2}$ implies positive Ricci curvature.

Theorem 1. *Let (M, g) be a Riemannian manifold and $x \in M$. If g has positive (nonnegative, resp.) Γ_k-curvature at x for some $k \geq n/2$, then its Ricci curvature is positive (nonnegative, resp.) at x. Moreover, if the Γ_k-curvature is nonnegative for some $k > 1$, then*

$$Ric_g \geq \frac{2k - n}{2n(k-1)} R_g \cdot g.$$

In particular, if $k \geq \frac{n}{2}$, then

$$Ric_g \geq \frac{(2k-n)(n-1)}{(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_g) \cdot g.$$

Remark. Theorem 1 is not true for $k = 1$. Namely, the condition of positive scalar curvature gives no restriction on the lower bound of the Ricci curvature.

Corollary 1. *Let (M^n, g) be a compact, locally conformally flat manifold with nonnegative Γ_k-curvature everywhere for some $k \geq n/2$. Then (M, g) is conformally equivalent to either a space form or a finite quotient of a Riemannian $\mathbf{S}^{n-1}(c) \times \mathbf{S}^1$ for some constant $c > 0$ and $k = n/2$. In particular, if $g \in \Gamma_k^+$, then (M, g) is conformally equivalent to a spherical space form.*

When $n = 3, 4$, the result in Theorem 1 was already observed in [9] and [2]. Theorem 1 and Corollary 1 will be proved in the next section.

We will also consider the equation

$$\sigma_k(A_{\tilde{g}}) = constant, \tag{2}$$

for conformal metrics $\tilde{g} = e^{-2u} g$. This equation was studied in [13], where it was shown that when $k \neq n/2$, (2) is the conformal Euler-Lagrange equation of the functional

$$\mathcal{F}_k(g) = \mathrm{Vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, d\mathrm{vol}(g), \tag{3}$$

when $k = 1, 2$ or for $k > 2$ when M is locally conformally flat. We remark that in the even-dimensional locally conformally flat case, $\mathcal{F}_{n/2}$ is a conformal invariant. Moreover, it is a multiple of the Euler characteristic, see [13].

This problem was further studied in [7], where the following conformal flow was considered:

$$\frac{d}{dt} g = -(\log \sigma_k(g) - \log r_k(g)) \cdot g,$$
$$g(0) = g_0,$$

where

$$\log r_k = \frac{1}{\mathrm{Vol}(g)} \int_M \log \sigma_k(g) d\mathrm{vol}(g).$$

Global existence with uniform $C^{1,1}$ a priori bounds of the flow was proved in [7]. It was also proved that for $k \neq n/2$ the flow is sequentially convergent in $C^{1,\alpha}$ to a C^∞ solution of $\sigma_k = constant$. Also, if $k < n/2$, then $\mathcal{F}_k$ is decreasing along the flow, and if $k > n/2$, then $\mathcal{F}_k$ is increasing along the flow. We remark that the existence result for equation (2) has been obtained independently in [10] in the locally conformally flat case for all k.

In Section 3, we will consider global properties of the functional $\mathcal{F}_k$, and give conditions for the existence of a global extremizer. We will also derive some conformal quermassintegral inequalities, which are analogous to the classical quermassintegral inequalities in convex geometry.

2. Curvature restriction

We first state a proposition which describes some important properties of the sets Γ_k^+.

Proposition 1. *(i) Each set Γ_k^+ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions:*

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+.$$

(ii) For any $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$, resp.), $\forall 1 \leq i \leq n$, let

$$(\Lambda|i) = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n).$$

Then $(\Lambda|i) \in \Gamma_{k-1}^+$ ($\bar{\Gamma}_{k-1}^+$, resp.). In particular,

$$\Gamma_{n-1}^+ \subset V_{n-1}^+ = \{(\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n : \lambda_i + \lambda_j > 0, i \neq j\}.$$

The proof of this proposition is standard, following from [5].

Our main results are consequences of the following two lemmas. In this note, we assume that $k > 1$.

Lemma 1. *Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n) \in \mathbb{R}^n$, and define*

$$A_\Lambda = \Lambda - \frac{\sum_{i=1}^n \lambda_i}{2(n-1)}(1, 1, \cdots, 1).$$

If $A_\Lambda \in \bar{\Gamma}_k^+$, then

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)}{2n(k-1)} \sum_{i=1}^n \lambda_i. \tag{4}$$

In particular, if $k \geq \frac{n}{2}$, then

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_\Lambda).$$

Proof. We first note that, for any nonzero vector $A = (a_1, \cdots, a_n) \in \bar{\Gamma}_2^+$ we have $\sigma_1(A) > 0$. This can be proved as follows. Since $A \in \bar{\Gamma}_2^+$, $\sigma_1(A) \geq 0$. If $\sigma_1(A) = 0$, there must be an $a_i > 0$ for some i, since A is a nonzero vector. We may assume $a_n > 0$. Let $(A|n) = (a_1, \cdots, a_{n-1})$; we have $\sigma_1(A|n) \geq 0$ by Proposition 1. This would give $\sigma_1(A) = \sigma_1(A|n) + a_n > 0$, a contradiction.

Now without loss of generality, we may assume that Λ is not a zero vector. By the assumption $A_\Lambda \in \bar{\Gamma}_k^+$ for $k \geq 2$, so we have $\sum_{i=1}^n \lambda_i > 0$.

Define

$$\Lambda_0 = (1, 1, \cdots, 1, \delta_k) \in \mathbb{R}^{n-1} \times \mathbb{R};$$

then we have $A_{\Lambda_0} = (a, \cdots, a, b)$, where

$$\delta_k = \frac{(2k-n)(n-1)}{2nk - 2k - n},$$

$$a = 1 - \frac{n-1+\delta_k}{2(n-1)}, \quad b = \delta_k - \frac{n-1+\delta_k}{2(n-1)},$$

so that

$$\sigma_k(A_{\Lambda_0}) = 0 \quad \text{ and } \sigma_j(A_{\Lambda_0}) > 0 \text{ for } j \leq k-1. \tag{5}$$

It is clear that $\delta_k < 1$, and so $a > b$. Since (4) is invariant under the transformation from Λ to $s\Lambda$ for $s > 0$, we may assume that $\sum_{i=1}^n \lambda_i = \mathrm{tr}(\Lambda_0) = n - 1 + \delta_k$ and $\lambda_n = \min_{i=1,\cdots,n} \lambda_i$. We write

$$A_\Lambda = (a_1, \cdots, a_n).$$

We claim that

$$\lambda_n \geq \delta_k. \tag{6}$$

This is equivalent to showing

$$a_n \geq b. \tag{7}$$

Assume for a contradiction that $a_n < b$. We consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$\begin{aligned} A_t := A_{\Lambda_t} &= tA_{\Lambda_0} + (1-t)A_\Lambda \\ &= ((1-t)a + ta_1, \cdots, (1-t)a + ta_{n-1}, (1-t)b + ta_n). \end{aligned}$$

By the convexity of the cone Γ_k^+ (see Proposition 1), we know that

$$A_t \in \bar{\Gamma}_k^+, \qquad \text{for any } t \in (0, 1].$$

In particular, $f(t) := \sigma_k(A_t) \geq 0$ for any $t \in [0, 1]$. By the definition of δ_k, $f(0) = 0$.

For any i and any vector $V = (v_1, \cdots, v_n)$, we denote by

$$(V|i) = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$$

the vector with the i-th component removed. Now we compute the derivative of f at 0:

$$f'(0) = \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(A_0|i) + (a_n - b)\sigma_{k-1}(A_0|n).$$

Since $(A_0|i) = (A_0|1)$ for $i \leq n-1$ and $\sum_{i=1}^n a_i = (n-1)a + b$, we have

$$f'(0) = (a_n - b)(\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1)) < 0,$$

for $\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1) > 0$. (Recall that $b < a$.) This is a contradiction; hence $\lambda_n \geq \delta_k$. It follows that

$$\min_{i=1,\cdots,n} \lambda_i \geq \delta_k = \frac{2k-n}{2n(k-1)} \sum_{i=1}^n \lambda_i.$$

Finally, the last inequality in the lemma follows from the Newton-MacLaurin inequality. $\square$

Remark. It is clear from the above proof that the constant in Lemma 1 is optimal.

We next consider the case $A_\Lambda \in \bar{\Gamma}^+_{\frac{n}{2}}$.

Lemma 2. *Let $k = n/2$ and $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $A_\Lambda \in \bar{\Gamma}^+_k$. Then either $\lambda_i > 0$ for any i, or*

$$\Lambda = (\lambda, \lambda, \cdots, \lambda, 0)$$

up to a permutation. If the second case is true, then we must have $\sigma_{\frac{n}{2}}(A_\Lambda) = 0$.

Proof. By Lemma 1, to prove the Lemma we only need to check that for $\Lambda = (\lambda_1, \cdots, \lambda_{n-1}, 0)$ with $A_\Lambda \in \bar{\Gamma}^+_k$,

$$\lambda_i = \lambda_j, \quad \forall i, j = 1, 2, \cdots, 2k-1.$$

We use the same idea as in the proof of the previous Lemma. Without loss of generality, we may assume that Λ is not a zero vector. By the assumption $A_\Lambda \in \bar{\Gamma}^+_k$ for $k \geq 2$, we have $\sum_{i=1}^{n-1} \lambda_i > 0$. Hence we may assume that $\sum_{i=1}^{n-1} \lambda_i = n-1$. Define

$$\Lambda_0 = (1, 1, \cdots, 1, 0) \in \mathbb{R}^n.$$

It is easy to check that

$$A_{\Lambda_0} \in \Gamma^+_{k-1} \quad \text{and} \quad \sigma_k(A_{\Lambda_0}) = 0. \tag{8}$$

That is, $A_{\Lambda_0} \in \bar{\Gamma}^+_k$. If the λ's are not all the same, we have

$$\sum_{i=1}^{n-1} (\lambda_i - 1) = 0$$

and

$$\sum_{i=1}^{n-1} (\lambda_i - 1)^2 > 0.$$

Now consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1-t)A_\Lambda = (\frac{1}{2} + t(\lambda_1 - 1), \cdots, \frac{1}{2} + t(\lambda_{n-1} - 1), -\frac{1}{2}).$$

From the assumption that $A \in \bar{\Gamma}^+_k$, (8), and the convexity of $\bar{\Gamma}^+_k$, we have

$$A_t \in \bar{\Gamma}^+_k \quad \text{for } t > 0. \tag{9}$$

For any $i \neq j$ and any vector A, we denote by $(A|ij)$ the vector with the i-th and j-th components removed. Let $\widetilde{\Lambda} = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be an $(n-1)$-vector, and $\Lambda^* = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ an $(n-2)$-vector. It is clear that $\forall i \neq j, \quad i, j \leq n-1$,

$$\sigma_{k-1}(A_0|i) = \sigma_{k-1}(\widetilde{\Lambda}) > 0,$$

$$\sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0.$$

Now we expand $f(t) = \sigma_k(A_t)$ at $t = 0$. By (8), $f(0) = 0$. We compute

$$\begin{aligned} f'(0) &= \sum_{i=1}^{n-1} (\lambda_i - 1)\sigma_{k-1}(A_0|i) \\ &= \sigma_{k-1}(\widetilde{\Lambda}) \sum_{i=1}^{n-1} (\lambda_i - 1) = 0 \end{aligned}$$

and

$$\begin{aligned} f''(0) &= \sum_{i\neq j}(\lambda_i - 1)(\lambda_j - 1)\sigma_{k-2}(A_0|ij) \\ &= \sigma_{k-2}(\Lambda^*)\sum_{i\neq j}(\lambda_i - 1)(\lambda_j - 1) \\ &= -\sigma_{k-2}(\Lambda^*)\sum_{i=1}^{n-1}(\lambda_i - 1)^2 < 0, \end{aligned}$$

for $\sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0$ for any $i \neq j$ and $\sum_{i\neq j}(\lambda_j - 1) = (1 - \lambda_i)$. Hence $f(t) < 0$ for small $t > 0$, which contradicts (9). $\square$

Remark. From the proof of Lemma 2, there is a constant $C > 0$, depending only on n and $\frac{\sigma_{\frac{n}{2}}^{\frac{2}{n}}(A_\Lambda)}{\sigma_1(A_\Lambda)}$, such that

$$\min_i \lambda_i \geq C\sigma_{\frac{n}{2}}^{\frac{2}{n}}(A_\Lambda).$$

Proof of Theorem 1. Theorem 1 follows directly from Lemmas 1 and 2. $\square$

Corollary 2. *Let (M, g) be an n-dimensional Riemannian manifold and $k \geq n/2$, and let $N = M \times \mathbf{S}^1$ be the product manifold. Then N does not have positive Γ_k-curvature. If N has nonnegative Γ_k-curvature, then (M, g) is an Einstein manifold, and there are two cases: either $k = n/2$, or $k > n/2$ and (M, g) is a torus.*

Proof. This follows from Lemmas 1 and 2. $\square$

Proof of Corollary 1. From Theorem 1, we know that the Ricci curvature Ric_g is nonnegative. Now we deform it by the Yamabe flow considered by Hamilton, Ye [15] and Chow [4] to obtain a conformal metric $\widetilde{g}$ of constant scalar curvature. The Ricci curvature $Ric_{\widetilde{g}}$ is nonnegative, for the Yamabe flow preserves the nonnegativity of the Ricci curvature, see [4]. Now, by a classification result given in [12, 3], we know that $(M, \widetilde{g})$ is isometric to either a space form or a finite quotient of a Riemannian $\mathbf{S}^{n-1}(c) \times \mathbf{S}^1$ for some constant $c > 0$. In the latter case, it is clear that $k = n/2$, since otherwise it cannot have nonnegative Γ_k-curvature. $\square$

Next, we will prove that if M is locally conformally flat with positive Γ_{n-1}-curvature, then g has positive sectional curvature. If M is locally conformally flat, then by (1) we may decompose the full curvature tensor as

$$\mathrm{Riem} = A_g \odot g,$$

Proposition 2. *Assume that $n = 3$, or that M is locally conformally flat. Then the Schouten tensor $A_g \in V_{n-1}^+$ if and only if g has positive sectional curvature.*

Proof. Let π be any 2-plane in $T_p(N)$, and let X,Y be an orthonormal basis of π. We have

$$\begin{aligned} K(\sigma) = \mathrm{Riem}(X, Y, X, Y) &= A_g \odot g(X, Y, X, Y) \\ &= A_g(X, X)g(Y, Y) - A_g(Y, X)g(X, Y) \\ &\quad + A_g(Y, Y)g(X, X) - A_g(X, Y)g(Y, X) \\ &= A_g(X, X) + A_g(Y, Y). \end{aligned}$$

From this it follows that

$$\min_{\sigma \in T_pN} K(\sigma) = \lambda_1 + \lambda_2,$$

where λ_1 and λ_2 are the smallest eigenvalues of A_g at p. □

Corollary 3. *If (M, g) is locally conformally flat with positive Γ_{n-1}-curvature, then g has positive sectional curvature.*

Proof. This follows easily from Propositions 1 and 2. □

3. Extremal metrics of σ_k-curvature functionals

We next consider some properties of the functionals $\mathcal{F}_k$ associated to σ_k. These functionals were introduced and discussed in [13], see also [7]. Further variational properties in connection to 3-dimensional geometry were studied in [9].

We recall that $\mathcal{F}_k$ is defined by

$$\mathcal{F}_k(g) = \mathrm{Vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, d\mathrm{vol}(g).$$

We denote $\mathcal{C}_k = \{g \in [g_0] | g \in \Gamma_k^+\}$, where $[g_0]$ is the conformal class of g_0.

We now apply our results to show that if $g_0 \in \Gamma^+_{\frac{n}{2}}$, then there is an extremal metric g_e which minimizes $\mathcal{F}_m$ for $m < n/2$, and if $m > n/2$, there is an extremal metric g_e which maximizes $\mathcal{F}_m$.

Proposition 3. *Suppose (M, g_0) is locally conformally flat and $g_0 \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$. Then $\forall m < \frac{n}{2}$, there is an extremal metric $g_e^m \in [g_0]$ such that*

$$\inf_{g \in \mathcal{C}_m} \mathcal{F}_m(g) = \mathcal{F}_m(g_e^m), \tag{10}$$

and $\forall m > \frac{n}{2}$, there is extremal metric $g_e^m \in [g_0]$ such that

$$\sup_{g \in \mathcal{C}_m} \mathcal{F}_k(g) = \mathcal{F}_k(g_e^m). \tag{11}$$

In fact, any solution to $\sigma_m(g) = constant$ is an extremal metric.

Proof. First, by Corollary 1, (M, g_0) is conformal to a spherical space form. For any $g \in \mathcal{C}_m$, from [7] we know there is a conformal metric $\widetilde{g}$ in $\mathcal{C}_m$ such that $\sigma_m(\widetilde{g})$ is constant and

(a) if $m > n/2$, then $\mathcal{F}_m(g) \leq \mathcal{F}_m(\widetilde{g})$,
(b) if $m < n/2$, then $\mathcal{F}_m(g) \geq \mathcal{F}_m(\widetilde{g})$.

A classification result of [13] and [14], which is analogous to a result of Obata for the scalar curvature, shows that $\widetilde{g}$ has constant sectional curvature. Therefore $\widetilde{g}$ is the unique critical metric unless M is conformally equivalent to $\mathbf{S}^n$, in which case any critical metric is the image of the standard metric under a conformal diffeomorphism. This clearly implies the conclusion of the Proposition. □

Next we consider the case $k < n/2$. We have

Proposition 4. *Suppose (M, g_0) is locally conformally flat and $g_0 \in \Gamma_k^+$ for some $k < \frac{n}{2}$. Suppose furthermore that for any fixed $C > 0$, the space of solutions to the equation $\sigma_k = C$ is compact, with a bound independent of the constant C. Then there is an extremal metric $g_e^k \in [g_0]$ such that*

$$\inf_{g \in \mathcal{C}_k} \mathcal{F}_k(g) = \mathcal{F}_k(g_e^k).$$

Proof. From the compactness assumption, there exists a critical metric g_e^k which has least energy. If the functional assumed a value strictly lower than $\mathcal{F}_k(g_e^k)$, then by [7], the flow would decrease to another solution of $\sigma_k = constant$, which is a contradiction since g_e^k has minimal energy. $\square$

We conclude with conformal quermassintegral inequalities, which were conjectured in [7], and verified there for some special cases when (M, g) is locally conformally flat and $g \in \Gamma^+_{\frac{n}{2}-1}$ or $g \in \Gamma^+_{\frac{n}{2}+1}$ using the flow method. In the case of $k = 2, n = 4$, the inequality was proved in [8] without the locally conformally flat assumption.

Proposition 5. *Suppose (M, g_0) is locally conformally flat and $g_0 \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$. Then for any $1 \leq l < \frac{n}{2} \leq k \leq n$ there is a constant $C(k, l, n) > 0$ such that for any $g \in [g_0]$ and $g \in \Gamma_k^+$,*

$$(\mathcal{F}_k(g))^{1/k} \leq C(k, l, n)(\mathcal{F}_l(g))^{1/l}, \tag{12}$$

with equality if and only if (M, g) is a spherical space form.

Proof. By Proposition 3, we have a conformal metric g_e of constant sectional curvature such that

$$\inf_{g \in \mathcal{C}_l} \mathcal{F}_l(g) = \mathcal{F}_l(g_e)$$

and

$$\sup_{g \in \mathcal{C}_k} \mathcal{F}_k(g) = \mathcal{F}_k(g_e).$$

Hence, for any $g \in \Gamma_k^+$ we have

$$\begin{aligned} \frac{(\mathcal{F}_k(g))^{1/k}}{(\mathcal{F}_l(g))^{1/l}} &\leq \frac{(\mathcal{F}_k(g_e))^{1/k}}{(\mathcal{F}_l(g_e))^{1/l}} \\ &= \frac{(l!(n-l)!)^{1/l}}{(k!(n-k)!)^{1/k}}. \end{aligned}$$

When the equality holds, g is an extremal of $\mathcal{F}_l$, hence a metric of constant sectional curvature by [13]. $\square$

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