SOME PROPERTIES OF THE SCHOUTEN TENSOR AND APPLICATIONS TO CONFORMAL GEOMETRY

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Abstract. The Riemannian curvature tensor decomposes into a conformally invariant part, the Weyl tensor, and a non-conformally invariant part, the Schouten tensor. A study of the $k$th elementary symmetric function of the eigenvalues of the Schouten tensor was initiated in an earlier paper by the second author, and a natural condition to impose is that the eigenvalues of the Schouten tensor are in a certain cone, $\Gamma_k^+$. We prove that this eigenvalue condition for $k \geq n/2$ implies that the Ricci curvature is positive. We then consider some applications to the locally conformally flat case, in particular, to extremal metrics of $\sigma_k$-curvature functionals and conformal quermassintegral inequalities, using the results of the first and third authors.

1. Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 3$, and let the Ricci tensor and scalar curvature be denoted by $Ric$ and $R$, respectively. We define the Schouten tensor

$$A_g = \frac{1}{n-2} \left( Ric - \frac{1}{2(n-1)} R g \right).$$

There is a decomposition formula (see [1]):

(1) 

$$\text{Riem} = A_g \odot g + \mathcal{W}_g,$$

where $\mathcal{W}_g$ is the Weyl tensor of $g$, and $\odot$ denotes the Kulkarni-Nomizu product (see [1]). Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor. A study of $k$-th elementary symmetric functions of the Schouten tensor was initiated in [13], it was reduced to certain fully nonlinear Yamabe type equations. In order to apply the elliptic theory of fully nonlinear equations, one often restricts the Schouten tensor to be in a certain cone $\Gamma_k^+$, defined as follows (according to Gårding [5]).

Definition 1. Let $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$. Let $\sigma_k$ denote the $k$th elementary symmetric function

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

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and let
\[ \Gamma_k^+ = \text{component of } \{ \sigma_k > 0 \} \text{ containing } (1, \ldots, 1). \]
Let \( \bar{\Gamma}_k^+ \) denote the closure of \( \Gamma_k^+ \). If \((M, g)\) is a Riemannian manifold, and \( x \in M \), we say \( g \) has positive (nonnegative, resp.) \( \Gamma_k \)-curvature at \( x \) if its Schouten tensor \( A_g \in \bar{\Gamma}_k^+ \) (\( \bar{\Gamma}_k^+ \), resp.) at \( x \). In this case, we also say \( g \in \bar{\Gamma}_k^+ \) (\( \bar{\Gamma}_k^+ \), resp.) at \( x \).

We note that positive \( \Gamma_1 \)-curvature is equivalent to positive scalar curvature, and the condition of positive \( \Gamma_k \)-curvature has some geometric and topological consequences for the manifold \( M \). For example, when \((M, g)\) is locally conformally flat with positive \( \Gamma_1 \)-curvature, then \( \pi_i(M) = 0, \forall 1 < i \leq \frac{n}{2} \), by a result of Schoen and Yau [11]. In this note, we will prove that positive \( \Gamma_k \)-curvature for any \( k \geq \frac{n}{2} \) implies positive Ricci curvature.

**Theorem 1.** Let \((M, g)\) be a Riemannian manifold and \( x \in M \). If \( g \) has positive (nonnegative, resp.) \( \Gamma_k \)-curvature at \( x \) for some \( k \geq n/2 \), then its Ricci curvature is positive (nonnegative, resp.) at \( x \). Moreover, if the \( \Gamma_k \)-curvature is nonnegative for some \( k > 1 \), then
\[
\text{Ric}_g \geq \frac{2k-n}{2n(k-1)} R_g \cdot g.
\]
In particular, if \( k \geq \frac{n}{2} \), then
\[
\text{Ric}_g \geq \frac{(2k-n)(n-1)}{(k-1)} \left( \frac{n}{k} \right)^{-\frac{1}{k}} \sigma_k^+ (A_g) \cdot g.
\]

**Remark.** Theorem 1 is not true for \( k = 1 \). Namely, the condition of positive scalar curvature gives no restriction on the lower bound of the Ricci curvature.

**Corollary 1.** Let \((M^n, g)\) be a compact, locally conformally flat manifold with nonnegative \( \Gamma_k \)-curvature everywhere for some \( k \geq n/2 \). Then \((M, g)\) is conformally equivalent to either a space form or a finite quotient of a Riemannian \( S^{n-1}(c) \times S^1 \) for some constant \( c > 0 \) and \( k = n/2 \). In particular, if \( g \in \Gamma_1^+ \), then \((M, g)\) is conformally equivalent to a spherical space form.

When \( n = 3, 4 \), the result in Theorem 1 was already observed in [9] and [2]. Theorem 1 and Corollary 1 will be proved in the next section.

We will also consider the equation
\[
\sigma_k(A_g) = \text{constant},
\]
for conformal metrics \( \bar{g} = e^{-2u} g \). This equation was studied in [13], where it was shown that when \( k \neq n/2 \), [2] is the conformal Euler-Lagrange equation of the functional
\[
\mathcal{F}_k(g) = \text{Vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, d\text{vol}(g),
\]
when \( k = 1, 2 \) or for \( k > 2 \) when \( M \) is locally conformally flat. We remark that in the even-dimensional locally conformally flat case, \( \mathcal{F}_{n/2} \) is a conformal invariant. Moreover, it is a multiple of the Euler characteristic, see [13].

This problem was further studied in [7], where the following conformal flow was considered:
\[
\frac{d}{dt} g = -(\log \sigma_k(g) - \log r_k(g)) \cdot g, \\
g(0) = g_0.
\]
where
\[ \log r_k = \frac{1}{\text{Vol}(g)} \int_M \log \sigma_k(g) \, d\text{vol}(g). \]

Global existence with uniform $C^{1,1}$ a priori bounds of the flow was proved in [7]. It was also proved that for $k \neq n/2$ the flow is sequentially convergent in $C^{1,\alpha}$ to a $C^\infty$ solution of $\sigma_k = \text{constant}$. Also, if $k < n/2$, then $F_k$ is decreasing along the flow, and if $k > n/2$, then $F_k$ is increasing along the flow. We remark that the existence result for equation (2) has been obtained independently in [10] in the locally conformally flat case for all $k$.

In Section 3, we will consider global properties of the functional $F_k$, and give conditions for the existence of a global extremizer. We will also derive some conformal quermassintegral inequalities, which are analogous to the classical quermassintegral inequalities in convex geometry.

2. CURVATURE RESTRICTION

We first state a proposition which describes some important properties of the sets $\Gamma_k^+$. 

**Proposition 1.** (i) Each set $\Gamma_k^+$ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions:

\[ \Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+ . \]

(ii) For any $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+ (\bar{\Gamma}_k^+, \text{resp.})$, $\forall 1 \leq i \leq n$, let

\[ (\Lambda|i) = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n). \]

Then $(\Lambda|i) \in \Gamma_{k-1}^+ (\bar{\Gamma}_{k-1}^+, \text{resp.})$. In particular,

\[ \Gamma_{n-1}^+ \subset V_{n-1}^+ = \{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_i + \lambda_j > 0, i \neq j \}. \]

The proof of this proposition is standard, following from [5].

Our main results are consequences of the following two lemmas. In this note, we assume that $k > 1$.

**Lemma 1.** Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n) \in \mathbb{R}^n$, and define

\[ A_\Lambda = \Lambda - \sum_{i=1}^{n} \lambda_i (1, 1, \cdots, 1) . \]

If $A_\Lambda \in \bar{\Gamma}_k^+$, then

\[ \min_{i=1, \cdots, n} \lambda_i \geq \frac{(2k-n)}{2(n-1)(k-1)} . \]

(4)

In particular, if $k \geq \frac{n}{2}$, then

\[ \min_{i=1, \cdots, n} \lambda_i \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)} \left( \frac{n}{k} \right)^{-\frac{1}{k}} . \]

**Proof.** We first note that, for any nonzero vector $A = (a_1, \cdots, a_n) \in \bar{\Gamma}_2^+$ we have $\sigma_1(A) > 0$. This can be proved as follows. Since $A \in \bar{\Gamma}_2^+$, $\sigma_1(A) \geq 0$. If $\sigma_1(A) = 0$, there must be an $a_i > 0$ for some $i$, since $A$ is a nonzero vector. We may assume $a_n > 0$. Let $(A|n) = (a_1, \cdots, a_{n-1})$; we have $\sigma_1(A|n) \geq 0$ by Proposition [4] This would give $\sigma_1(A) = \sigma_1(A|n) + a_n > 0$, a contradiction.
Now without loss of generality, we may assume that $\Lambda$ is not a zero vector. By the assumption $A_{\Lambda} \in \bar{\Gamma}^+_k$ for $k \geq 2$, so we have $\sum_{i=1}^{n} \lambda_i > 0$.

Define

$$\Lambda_0 = (1, 1, \cdots, 1, \delta_k) \in \mathbb{R}^{n-1} \times \mathbb{R};$$

then we have $A_{\Lambda_0} = (a, \cdots, a, b)$, where

$$\delta_k = \frac{(2k - n)(n - 1)}{2nk - 2k - n},$$

$$a = 1 - \frac{n - 1 + \delta_k}{2(n - 1)}, \quad b = \delta_k - \frac{n - 1 + \delta_k}{2(n - 1)},$$

so that

$$\sigma_k(A_{\Lambda_0}) = 0 \quad \text{and} \quad \sigma_j(A_{\Lambda_0}) > 0 \quad \text{for} \quad j \leq k - 1.$$

It is clear that $\delta_k < 1$, and so $a > b$. Since (4) is invariant under the transformation from $\Lambda$ to $s\Lambda$ for $s > 0$, we may assume that $\sum_{i=1}^{n} \lambda_i = \text{tr}(\Lambda_0) = n - 1 + \delta_k$ and $\lambda_n = \min_{i=1, \ldots, n} \lambda_i$. We write

$$A_{\Lambda} = (a_1, \cdots, a_n).$$

We claim that

$$\lambda_n \geq \delta_k.$$  \hspace{1cm} (6)

This is equivalent to showing

$$a_n \geq b.$$  \hspace{1cm} (7)

Assume for a contradiction that $a_n < b$. We consider $\Lambda_t = t\Lambda_0 + (1 - t)\Lambda$ and

$$A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1 - t)A_{\Lambda} = ((1 - t)a + ta_1, \cdots, (1 - t)a + ta_{n-1}, (1 - t)b + ta_n).$$

By the convexity of the cone $\Gamma^+_k$ (see Proposition 1), we know that

$$A_t \in \bar{\Gamma}^+_k, \quad \text{for any} \quad t \in (0, 1].$$

In particular, $f(t) := \sigma_k(A_t) \geq 0$ for any $t \in (0, 1]$. By the definition of $\delta_k$, $f(0) = 0$.

For any $i$ and any vector $V = (v_1, \cdots, v_n)$, we denote by

$$(V|i) = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$$

the vector with the $i$-th component removed. Now we compute the derivative of $f$ at $0$:

$$f'(0) = \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(A_0|i) + (a_n - b)\sigma_{k-1}(A_0|n).$$

Since $(A_0|i) = (A_0|1)$ for $i \leq n - 1$ and $\sum_{i=1}^{n} a_i = (n - 1)a + b$, we have

$$f'(0) = (a_n - b)(\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1)) < 0,$$

for $\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1) > 0$. (Recall that $b < a$.) This is a contradiction; hence $\lambda_n \geq \delta_k$. It follows that

$$\min_{i=1, \ldots, n} \lambda_i \geq \delta_k = \frac{2k - n}{2n(k - 1)} \sum_{i=1}^{n} \lambda_i.$$

Finally, the last inequality in the lemma follows from the Newton-MacLaurin inequality. \hfill $\square$
Remark. It is clear from the above proof that the constant in Lemma 1 is optimal.

We next consider the case $A_\Lambda \in \hat{\Gamma}^+_{k\over 2}$.

**Lemma 2.** Let $k = n/2$ and $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $A_\Lambda \in \hat{\Gamma}^+_{k\over 2}$. Then either $\lambda_i > 0$ for any $i$, or

$$\Lambda = (\lambda, \lambda, \cdots, \lambda, 0)$$

up to a permutation. If the second case is true, then we must have $\sigma_{k\over 2}(A_\Lambda) = 0$.

**Proof.** By Lemma 1 to prove the Lemma we only need to check that for $\Lambda = (\lambda_1, \cdots, \lambda_{n-1}, 0)$ with $A_\Lambda \in \hat{\Gamma}^+_{k\over 2}$,

$$\lambda_i = \lambda_j, \quad \forall i, j = 1, 2, \cdots, 2k-1.$$  

We use the same idea as in the proof of the previous Lemma. Without loss of generality, we may assume that $\Lambda$ is not a zero vector. By the assumption $A_\Lambda \in \hat{\Gamma}^+_{k\over 2}$ for $k \geq 2$, we have $\sum_{i=1}^{n-1} \lambda_i > 0$. Hence we may assume that $\sum_{i=1}^{n-1} \lambda_i = n - 1$. Define

$$A_\Lambda = (1, 1, \cdots, 1, 0) \in \mathbb{R}^n.$$  

It is easy to check that

$$A_{\Lambda_0} \in \hat{\Gamma}^+_{k\over 2-1} \quad \text{and} \quad \sigma_k(A_{\Lambda_0}) = 0.$$  

That is, $A_{\Lambda_0} \in \hat{\Gamma}^+_{k\over 2}$. If the $\lambda$'s are not all the same, we have

$$\sum_{i=1}^{n-1} (\lambda_i - 1) = 0$$

and

$$\sum_{i=1}^{n-1} (\lambda_i - 1)^2 > 0.$$  

Now consider $A_t = tA_0 + (1 - t)\Lambda$ and

$$A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1 - t)A_\Lambda = \left(\frac{1}{2} + t(\lambda_1 - 1), \cdots, \frac{1}{2} + t(\lambda_n - 1), -\frac{1}{2}\right).$$

From the assumption that $A \in \hat{\Gamma}^+_{k\over 2}$, (8), and the convexity of $\hat{\Gamma}^+_{k\over 2}$, we have

(9)  

$$A_t \in \hat{\Gamma}^+_{k\over 2} \quad \text{for} \ t > 0.$$  

For any $i \neq j$ and any vector $A$, we denote by $(A|ij)$ the vector with the $i$-th and $j$-th components removed. Let $\tilde{\Lambda} = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be an $(n - 1)$-vector, and $\Lambda^* = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ an $(n - 2)$-vector. It is clear that $\forall i \neq j, \ i, j \leq n - 1$,

$$\sigma_{k-1}(A_0|i) = \sigma_{k-1}(\tilde{\Lambda}) > 0,$$

$$\sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0.$$  

Now we expand $f(t) = \sigma_k(A_t)$ at $t = 0$. By (8), $f(0) = 0$. We compute

$$f'(0) = \sum_{i=1}^{n-1} (\lambda_i - 1)\sigma_{k-1}(A_0|i)$$

$$= \sigma_{k-1}(\tilde{\Lambda})\sum_{i=1}^{n-1} (\lambda_i - 1) = 0$$
and

\[ f''(0) = \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1)\sigma_{k-2}(A_0|ij) \]
\[ = \sigma_{k-2}(\Lambda^*) \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1) \]
\[ = -\sigma_{k-2}(\Lambda^*) \sum_{i=1}^{n-1} (\lambda_i - 1)^2 < 0, \]

for \( \sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0 \) for any \( i \neq j \) and \( \sum_{i \neq j} (\lambda_j - 1) = (1 - \lambda_i) \). Hence

\[ f(t) < 0 \text{ for small } t > 0, \]

which contradicts (9).

\[ \Box \]

Remark. From the proof of Lemma 2, there is a constant \( C > 0 \), depending only on \( n \) and \( \sigma_{nn}^2 \), such that

\[ \min_i \lambda_i \geq C\sigma_{nn}^2(A\Lambda). \]

Proof of Theorem 1. Theorem 1 follows directly from Lemmas 1 and 2.

\[ \Box \]

Corollary 2. Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold and \( k \geq n/2 \), and let \( N = M \times S^1 \) be the product manifold. Then \( N \) does not have positive \( \Gamma_k \)-curvature. If \( N \) has nonnegative \( \Gamma_k \)-curvature, then \((M, g)\) is an Einstein manifold, and there are two cases: either \( k = n/2 \), or \( k > n/2 \) and \((M, g)\) is a torus.

Proof. This follows from Lemmas 1 and 2.

Proof of Corollary 1. From Theorem 1 we know that the Ricci curvature \( \text{Ric}_g \) is nonnegative. Now we deform it by the Yamabe flow considered by Hamilton, Ye [15] and Chow [4] to obtain a conformal metric \( \tilde{g} \) of constant scalar curvature. The Ricci curvature \( \text{Ric}_{\tilde{g}} \) is nonnegative, for the Yamabe flow preserves the nonnegativity of the Ricci curvature, see [3]. Now, by a classification result given in [12, 3], we know that \((M, \tilde{g})\) is isometric to either a space form or a finite quotient of a Riemannian \( S^{n-1}(c) \times S^1 \) for some constant \( c > 0 \). In the latter case, it is clear that \( k = n/2 \), since otherwise it cannot have nonnegative \( \Gamma_k \)-curvature.

Next, we will prove that if \( M \) is locally conformally flat with positive \( \Gamma_{n-1} \)-curvature, then \( g \) has positive sectional curvature. If \( M \) is locally conformally flat, then by (1) we may decompose the full curvature tensor as

\[ \text{Riem} = A_g \odot g, \]

Proposition 2. Assume that \( n = 3 \), or that \( M \) is locally conformally flat. Then the Schouten tensor \( A_g \in V_{n-1}^+ \) if and only if \( g \) has positive sectional curvature.

Proof. Let \( \pi \) be any 2-plane in \( T_p(N) \), and let \( X, Y \) be an orthonormal basis of \( \pi \). We have

\[ K(\sigma) = \text{Riem}(X, Y, X, Y) = A_g \odot g(X, Y, X, Y) \]
\[ = A_g(X, X)g(Y, Y) - A_g(Y, X)g(X, Y) \]
\[ + A_g(Y, Y)g(X, X) - A_g(X, Y)g(Y, X) \]
\[ = A_g(X, X) + A_g(Y, Y). \]
From this it follows that
\[ \min_{\sigma \in T_p N} K(\sigma) = \lambda_1 + \lambda_2, \]
where \( \lambda_1 \) and \( \lambda_2 \) are the smallest eigenvalues of \( A_g \) at \( p \).

**Corollary 3.** If \((M, g)\) is locally conformally flat with positive \( \Gamma_{n-1} \)-curvature, then \( g \) has positive sectional curvature.

**Proof.** This follows easily from Propositions 1 and 2. \( \square \)

### 3. Extremal metrics of \( \sigma_k \)-curvature functionals

We next consider some properties of the functionals \( F_k \) associated to \( \sigma_k \). These functionals were introduced and discussed in [13], see also [7]. Further variational properties in connection to 3-dimensional geometry were studied in [9].

We recall that \( F_k \) is defined by
\[ F_k(g) = \text{Vol}(g) - n-2 \int_M \sigma_k(g) \, d\text{vol}(g). \]

We denote \( C_k = \{ g \in [g_0] | g \in \Gamma_k^+ \} \), where \([g_0]\) is the conformal class of \( g_0 \).

We now apply our results to show that if \( g_0 \in \Gamma_k^+ \) for some \( k \geq \frac{n}{2} \), then there is an extremal metric \( g^e \) which minimizes \( F_m \) for \( m < n/2 \), and if \( m > n/2 \), there is an extremal metric \( g^e \) which maximizes \( F_m \).

**Proposition 3.** Suppose \((M, g_0)\) is locally conformally flat and \( g_0 \in \Gamma_k^+ \) for some \( k \geq \frac{n}{2} \). Then \( \forall m < \frac{n}{2} \), there is an extremal metric \( g^m_0 \in [g_0] \) such that
\[ \inf_{g \in C_m} F_m(g) = F_m(g^m_0), \]
and \( \forall m > \frac{n}{2} \), there is extremal metric \( g^m_0 \in [g_0] \) such that
\[ \sup_{g \in C_m} F_k(g) = F_k(g^m_0). \]

In fact, any solution to \( \sigma_m(g) = \text{constant} \) is an extremal metric.

**Proof.** First, by Corollary 1, \((M, g_0)\) is conformal to a spherical space form. For any \( g \in C_m \), from [7] we know there is a conformal metric \( \tilde{g} \) in \( C_m \) such that \( \sigma_m(\tilde{g}) \) is constant and
\[ \begin{align*}
(\text{a}) & \quad \text{if } m > n/2, \text{ then } F_m(g) \leq F_m(\tilde{g}), \\
(\text{b}) & \quad \text{if } m < n/2, \text{ then } F_m(g) \geq F_m(\tilde{g}).
\end{align*} \]

A classification result of [13] and [14], which is analogous to a result of Obata for the scalar curvature, shows that \( \tilde{g} \) has constant sectional curvature. Therefore \( \tilde{g} \) is the unique critical metric unless \( M \) is conformally equivalent to \( S^n \), in which case any critical metric is the image of the standard metric under a conformal diffeomorphism. This clearly implies the conclusion of the Proposition. \( \square \)

Next we consider the case \( k < n/2 \). We have

**Proposition 4.** Suppose \((M, g_0)\) is locally conformally flat and \( g_0 \in \Gamma_k^+ \) for some \( k < \frac{n}{2} \). Suppose furthermore that for any fixed \( C > 0 \), the space of solutions to the equation \( \sigma_k = C \) is compact, with a bound independent of the constant \( C \). Then there is an extremal metric \( g^k_0 \in [g_0] \) such that
\[ \inf_{g \in C_k} F_k(g) = F_k(g^k_0). \]
Proof. From the compactness assumption, there exists a critical metric $g_k^e$ which has least energy. If the functional assumed a value strictly lower than $F_k(g_k^e)$, then by [7], the flow would decrease to another solution of $\sigma_k = \text{constant}$, which is a contradiction since $g_k^e$ has minimal energy. □

We conclude with conformal quermassintegral inequalities, which were conjectured in [7], and verified there for some special cases when $(M, g)$ is locally conformally flat and $g \in \Gamma^+_k$ or $g \in \Gamma^+_k + 1$ using the flow method. In the case of $k = 2, n = 4$, the inequality was proved in [8] without the locally conformally flat assumption.

**Proposition 5.** Suppose $(M, g_0)$ is locally conformally flat and $g_0 \in \Gamma^+_k$ for some $k \geq \frac{n}{2}$. Then for any $1 \leq l < \frac{n}{2} \leq k \leq n$ there is a constant $C(k, l, n) > 0$ such that for any $g \in [g_0]$ and $g \in \Gamma^+_k$,

$$\left(\frac{F_k(g)}{F_l(g)}\right)^{1/k} \leq C(k, l, n) \left(\frac{F_l(g)}{F_k(g)}\right)^{1/l},$$

with equality if and only if $(M, g)$ is a spherical space form.

**Proof.** By Proposition 3, we have a conformal metric $g_e$ of constant sectional curvature such that

$$\inf_{g \in C_k} F_l(g) = F_l(g_e)$$

and

$$\sup_{g \in C_k} F_k(g) = F_k(g_e).$$

Hence, for any $g \in \Gamma^+_k$ we have

$$\frac{(F_k(g))^{1/k}}{(F_l(g))^{1/l}} \leq \frac{(F_k(g_e))^{1/k}}{(F_l(g_e))^{1/l}} = \frac{(l!(n-l)!)^{1/l}}{(k!(n-k)!)^{1/k}}.$$  

When the equality holds, $g$ is an extremal of $F_l$, hence a metric of constant sectional curvature by [13]. □

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