SOME PROPERTIES OF THE SCHOUTEN TENSOR
AND APPLICATIONS TO CONFORMAL GEOMETRY

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Abstract. The Riemannian curvature tensor decomposes into a conformally invariant part, the Weyl tensor, and a non-conformally invariant part, the Schouten tensor. A study of the $k$th elementary symmetric function of the eigenvalues of the Schouten tensor was initiated in an earlier paper by the second author, and a natural condition to impose is that the eigenvalues of the Schouten tensor are in a certain cone, $\Gamma_k^+$. We prove that this eigenvalue condition for $k \geq n/2$ implies that the Ricci curvature is positive. We then consider some applications to the locally conformally flat case, in particular, to extremal metrics of $\sigma_k$-curvature functionals and conformal quermassintegral inequalities, using the results of the first and third authors.

1. Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 3$, and let the Ricci tensor and scalar curvature be denoted by $\text{Ric}$ and $R$, respectively. We define the Schouten tensor

$$A_g = \frac{1}{n - 2} \left( \text{Ric} - \frac{1}{2(n - 1)} R_g \right).$$

There is a decomposition formula (see [1]):

$$\text{Riem} = A_g \odot g + W_g,$$

where $W_g$ is the Weyl tensor of $g$, and $\odot$ denotes the Kulkarni-Nomizu product (see [1]). Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor. A study of $k$-th elementary symmetric functions of the Schouten tensor was initiated in [13], it was reduced to certain fully nonlinear Yamabe type equations. In order to apply the elliptic theory of fully nonlinear equations, one often restricts the Schouten tensor to be in a certain cone $\Gamma_k^+$, defined as follows (according to Gårding [5]).

Definition 1. Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Let $\sigma_k$ denote the $k$th elementary symmetric function

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

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and let
\[ \Gamma^+_k = \text{component of } \{ \sigma_k > 0 \} \text{ containing } (1, \cdots, 1). \]
Let \( \bar{\Gamma}^+_k \) denote the closure of \( \Gamma^+_k \). If \((M, g)\) is a Riemannian manifold, and \( x \in M \), we say \( g \) has positive (nonnegative, resp.) \( \Gamma_k \)-curvature at \( x \) if its Schouten tensor \( A_g \in \Gamma^+_k (\bar{\Gamma}^+_k \), resp.\) at \( x \). In this case, we also say \( g \in \Gamma^+_k (\bar{\Gamma}^+_k \), resp.\) at \( x \).

We note that positive \( \Gamma_1 \)-curvature is equivalent to positive scalar curvature, and the condition of positive \( \Gamma_k \)-curvature has some geometric and topological consequences for the manifold \( M \). For example, when \((M, g)\) is locally conformally flat with positive \( \Gamma_1 \)-curvature, then \( \pi_1(M) = 0, \forall 1 < i \leq \frac{n}{2} \), by a result of Schoen and Yau \[11\]. In this note, we will prove that positive \( \Gamma_k \)-curvature for any \( k \geq \frac{n}{2} \) implies positive Ricci curvature.

**Theorem 1.** Let \((M, g)\) be a Riemannian manifold and \( x \in M \). If \( g \) has positive (nonnegative, resp.) \( \Gamma_k \)-curvature at \( x \) for some \( k \geq \frac{n}{2} \), then its Ricci curvature is positive (nonnegative, resp.) at \( x \). Moreover, if the \( \Gamma_k \)-curvature is nonnegative for some \( k > 1 \), then
\[
\text{Ric}_g \geq \frac{2k - n}{2n(k - 1)} R_g \cdot g.
\]
In particular, if \( k \geq \frac{n}{2} \), then
\[
\text{Ric}_g \geq \frac{(2k - n)(n - 1)}{(k - 1)} \int_M \sigma_k(A_g) g. \]

**Remark.** Theorem 1 is not true for \( k = 1 \). Namely, the condition of positive scalar curvature gives no restriction on the lower bound of the Ricci curvature.

**Corollary 1.** Let \((M^n, g)\) be a compact, locally conformally flat manifold with nonnegative \( \Gamma_k \)-curvature everywhere for some \( k \geq \frac{n}{2} \). Then \((M, g)\) is conformally equivalent to either a space form or a finite quotient of a Riemannian \( \mathbb{S}^{n-1}(c) \times \mathbb{S}^1 \) for some constant \( c > 0 \) and \( k = \frac{n}{2} \). In particular, if \( g \in \Gamma^+_k \), then \((M, g)\) is conformally equivalent to a spherical space form.

When \( n = 3, 4 \), the result in Theorem 1 was already observed in \[9\] and \[2\]. Theorem 1 and Corollary 1 will be proved in the next section.

We will also consider the equation
\[
\sigma_k(A_g) = \text{constant},
\]
for conformal metrics \( \tilde{g} = e^{-2u} g \). This equation was studied in \[13\], where it was shown that when \( k \neq n/2 \), \( \tilde{\sigma}_k \) is the conformal Euler-Lagrange equation of the functional
\[
\mathcal{F}_k(g) = \text{Vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \text{dvol}(g),
\]
when \( k = 1, 2 \) or for \( k > 2 \) when \( M \) is locally conformally flat. We remark that in the even-dimensional locally conformally flat case, \( \mathcal{F}_{n/2} \) is a conformal invariant. Moreover, it is a multiple of the Euler characteristic, see \[13\].

This problem was further studied in \[7\], where the following conformal flow was considered:
\[
\frac{d}{dt} g = -(\log \sigma_k(g) - \log r_k(g)) \cdot g,
\]
\( g(0) = g_0 \).
where

$$\log r_k = \frac{1}{\text{Vol}(g)} \int_M \log \sigma_k(g) \, d\text{vol}(g).$$

Global existence with uniform $C^{1,1}$ a priori bounds of the flow was proved in [7]. It was also proved that for $k \neq n/2$ the flow is sequentially convergent in $C^{1,\alpha}$ to a $C^\infty$ solution of $\sigma_k = \text{constant}$. Also, if $k < n/2$, then $F_k$ is decreasing along the flow, and if $k > n/2$, then $F_k$ is increasing along the flow. We remark that the existence result for equation (2) has been obtained independently in [10] in the locally conformally flat case for all $k$.

In Section 3, we will consider global properties of the functional $F_k$, and give conditions for the existence of a global extremizer. We will also derive some conformal quermassintegral inequalities, which are analogous to the classical quermassintegral inequalities in convex geometry.

2. Curvature restriction

We first state a proposition which describes some important properties of the sets $\Gamma_k^+$.

**Proposition 1.** (i) Each set $\Gamma_k^+$ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions:

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+.$$  

(ii) For any $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+ \quad \text{(resp. } \tilde{\Gamma}_k^+ \text{, resp.)}, \forall 1 \leq i \leq n$, let

$$(\Lambda|i) = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n).$$

Then $(\Lambda|i) \in \Gamma_{k-1}^+ \quad \text{(resp. } \tilde{\Gamma}_{k-1}^+ \text{, resp.}).$ In particular,

$$\Gamma_{n-1}^+ \subset V_{n-1}^+ = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_i + \lambda_j > 0, i \neq j\}.$$  

The proof of this proposition is standard, following from [5].

Our main results are consequences of the following two lemmas. In this note, we assume that $k > 1$.

**Lemma 1.** Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n) \in \mathbb{R}^n$, and define

$$A_\Lambda = \Lambda - \frac{1}{2(n-1)} \sum_{i=1}^n \lambda_i (1,1, \cdots, 1).$$

If $A_\Lambda \in \tilde{\Gamma}_k^+$, then

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)}{2n(k-1)} \sum_{i=1}^n \lambda_i.$$  

In particular, if $k \geq \frac{n}{2}$, then

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)} \left(\frac{n}{k}\right)^{-\frac{1}{2}} \sigma_k^+(A_\Lambda).$$

**Proof:** We first note that, for any nonzero vector $A = (a_1, \cdots, a_n) \in \tilde{\Gamma}_2^+$ we have $\sigma_1(A) > 0$. This can be proved as follows. Since $A \in \Gamma_2^+$, $\sigma_1(A) \geq 0$. If $\sigma_1(A) = 0$, there must be an $a_i > 0$ for some $i$, since $A$ is a nonzero vector. We may assume $a_n > 0$. Let $(A|n) = (a_1, \cdots, a_{n-1})$; we have $\sigma_1(A|n) \geq 0$ by Proposition 1. This would give $\sigma_1(A) = \sigma_1(A|n) + a_n > 0$, a contradiction.
Now without loss of generality, we may assume that \( \Lambda \) is not a zero vector. By the assumption \( A_\Lambda \in \bar{\Gamma}_n^+ \) for \( k \geq 2 \), so we have \( \sum_{i=1}^{n} \lambda_i > 0 \).

Define
\[
\Lambda_0 = (1, 1, \cdots, 1, \delta_k) \in \mathbb{R}^{n-1} \times \mathbb{R};
\]
then we have \( A_{\Lambda_0} = (a, \cdots, a, b) \), where
\[
\sigma_k = \frac{(2k - n)(n - 1)}{2nk - 2k - n}, \quad a = \frac{n - 1 + \delta_k}{2(n - 1)}, \quad b = \frac{n - 1 + \delta_k}{2(n - 1)},
\]
so that
\[
(5) \quad \sigma_k(A_{\Lambda_0}) = 0 \quad \text{and} \quad \sigma_j(A_{\Lambda_0}) > 0 \quad \text{for} \quad j < k - 1.
\]

It is clear that \( \delta_k < 1 \), and so \( a > b \). Since \( \Gamma_k \) is invariant under the transformation from \( \Lambda \) to \( s\Lambda \) for \( s > 0 \), we may assume that \( \sum_{i=1}^{n} \lambda_i = \text{tr}(\Lambda_0) = n - 1 + \delta_k \) and \( \lambda_n = \min_{i=1, \cdots, n} \lambda_i \). We write
\[
A_\Lambda = (a_1, \cdots, a_n).
\]

We claim that
\[
(6) \quad \lambda_n \geq \delta_k.
\]

This is equivalent to showing
\[
(7) \quad a_n \geq b.
\]

Assume for a contradiction that \( a_n < b \). We consider \( \Lambda_t = t\Lambda_0 + (1 - t)\Lambda \) and
\[
A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1 - t)A_\Lambda = ((1 - t)a + ta_1, \cdots, (1 - t)a + ta_{n-1}, (1 - t)b + ta_n).
\]

By the convexity of the cone \( \bar{\Gamma}_n^+ \) (see Proposition 1), we know that
\[
A_t \in \bar{\Gamma}_n^+, \quad \text{for any} \ t \in (0, 1].
\]

In particular, \( f(t) := \sigma_k(A_t) \geq 0 \) for any \( t \in [0, 1] \). By the definition of \( \delta_k \), \( f(0) = 0 \).

For any \( i \) and any vector \( V = (v_1, \cdots, v_n) \), we denote by
\[
(V|i) = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)
\]
the vector with the \( i \)-th component removed. Now we compute the derivative of \( f \) at \( 0 \):
\[
f'(0) = \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(A_0|i) + (a_n - b)\sigma_{k-1}(A_0|n).
\]

Since \( \langle A_0|i \rangle = \langle A_0|1 \rangle \) for \( i \leq n - 1 \) and \( \sum_{i=1}^{n} a_i = (n - 1)a + b \), we have
\[
f'(0) = (a_n - b)(\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1)) < 0,
\]
for \( \sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1) > 0 \). (Recall that \( b < a \).) This is a contradiction; hence \( \lambda_n \geq \delta_k \). It follows that
\[
\min_{i=1, \cdots, n} \lambda_i \geq \delta_k = \frac{2k - n}{2n(k - 1)} \sum_{i=1}^{n} \lambda_i.
\]

Finally, the last inequality in the lemma follows from the Newton-MacLaurin inequality. \( \square \)
Remark. It is clear from the above proof that the constant in Lemma 1 is optimal.

We next consider the case $A_{\Lambda} \in \bar{\Gamma}_{k}^{+}$. 

**Lemma 2.** Let $k = n/2$ and $\Lambda = (\lambda_{1}, \cdots, \lambda_{n}) \in \mathbb{R}^{n}$ with $A_{\Lambda} \in \bar{\Gamma}_{k}^{+}$. Then either $\lambda_{i} > 0$ for any $i$, or

$$\Lambda = (\lambda, \lambda, \cdots, \lambda, 0)$$

up to a permutation. If the second case is true, then we must have $\sigma_{\frac{n}{2}}(A_{\Lambda}) = 0$.

**Proof.** By Lemma 1 to prove the Lemma we only need to check that for $\Lambda = (\lambda_{1}, \cdots, \lambda_{n-1}, 0)$ with $A_{\Lambda} \in \bar{\Gamma}_{k}^{+}$,

$$\lambda_{i} = \lambda_{j}, \quad \forall i, j = 1, 2, \cdots, 2k - 1.$$ 

We use the same idea as in the proof of the previous Lemma. Without loss of generality, we may assume that $\Lambda$ is not a zero vector. By the assumption $A_{\Lambda} \in \bar{\Gamma}_{k}^{+}$ for $k \geq 2$, we have

$$\sum_{i=1}^{n-1} \lambda_{i} > 0.$$ 

Hence we may assume that $\sum_{i=1}^{n-1} \lambda_{i} = n - 1$. Define

$$A_{0} = (1, 1, \cdots, 1, 0) \in \mathbb{R}^{n}.$$ 

It is easy to check that

(8) 

$$A_{\Lambda_{0}} \in \bar{\Gamma}_{k-1}^{+} \quad \text{and} \quad \sigma_{k}(A_{\Lambda_{0}}) = 0.$$ 

That is, $A_{\Lambda_{0}} \in \bar{\Gamma}_{k}^{+}$. If the $\lambda$’s are not all the same, we have

$$\sum_{i=1}^{n-1} (\lambda_{i} - 1) = 0$$

and

$$\sum_{i=1}^{n-1} (\lambda_{i} - 1)^{2} > 0.$$ 

Now consider $\Lambda_{t} = t\Lambda_{0} + (1 - t)\Lambda$ and

$$A_{t} := A_{\Lambda_{t}} = tA_{\Lambda_{0}} + (1 - t)A_{\Lambda} = \left(\frac{1}{2} + t(\lambda_{1} - 1), \cdots, \frac{1}{2} + t(\lambda_{n-1} - 1), \frac{1}{2}\right).$$

From the assumption that $A \in \bar{\Gamma}_{k}^{+}$, (8), and the convexity of $\bar{\Gamma}_{k}^{+}$, we have

(9) 

$$A_{t} \in \bar{\Gamma}_{k}^{+} \quad \text{for} \quad t > 0.$$ 

For any $i \neq j$ and any vector $A$, we denote by $(A|ij)$ the vector with the $i$-th and $j$-th components removed. Let $\tilde{\Lambda} = \left(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}\right)$ be an $(n - 1)$-vector, and $\Lambda^{*} = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ an $(n - 2)$-vector. It is clear that $\forall i, j \leq n - 1,$

$$\sigma_{k-1}(A_{0}|i) = \sigma_{k-1}(\tilde{\Lambda}) > 0,$$

$$\sigma_{k-2}(A_{0}|ij) = \sigma_{k-2}(\Lambda^{*}) > 0.$$ 

Now we expand $f(t) = \sigma_{k}(A_{t})$ at $t = 0$. By (8), $f(0) = 0$. We compute

$$f'(0) = \sum_{i=1}^{n-1} (\lambda_{i} - 1)\sigma_{k-1}(A_{0}|i)$$

$$= \sigma_{k-1}(\tilde{\Lambda}) \sum_{i=1}^{n-1} (\lambda_{i} - 1) = 0.$$
and
\[ f''(0) = \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1)\sigma_{k-2}(A_0|ij) \]
\[ = \sigma_{k-2}(\Lambda^*) \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1) \]
\[ = -\sigma_{k-2}(\Lambda^*) \sum_{i=1}^{n-1} (\lambda_i - 1)^2 < 0, \]
for \( \sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0 \) for any \( i \neq j \) and \( \sum_{i \neq j} (\lambda_j - 1) = (1 - \lambda_i) \). Hence \( f(t) < 0 \) for small \( t > 0 \), which contradicts (9). □

Remark. From the proof of Lemma 2, there is a constant \( C > 0 \), depending only on \( n \) and \( \sigma_{2nn^2}(A\Lambda) \), such that
\[ \min_i \lambda_i \geq C\sigma_{2nn^2}(A\Lambda). \]

Proof of Theorem 1. Theorem 1 follows directly from Lemmas 1 and 2. □

Corollary 2. Let \( (M, g) \) be an \( n \)-dimensional Riemannian manifold and \( k \geq n/2 \), and let \( N = M \times S^1 \) be the product manifold. Then \( N \) does not have positive \( \Gamma_k \)-curvature. If \( N \) has nonnegative \( \Gamma_k \)-curvature, then \( (M, g) \) is an Einstein manifold, and there are two cases: either \( k = n/2 \), or \( k > n/2 \) and \( (M, g) \) is a torus.

Proof. This follows from Lemmas 1 and 2. □

Proof of Corollary 1. From Theorem 1 we know that the Ricci curvature \( \text{Ric}_g \) is nonnegative. Now we deform it by the Yamabe flow considered by Hamilton, Ye [14] and Chow [4] to obtain a conformal metric \( \tilde{g} \) of constant scalar curvature. The Ricci curvature \( \text{Ric}_{\tilde{g}} \) is nonnegative, for the Yamabe flow preserves the nonnegativity of the Ricci curvature, see [4]. Now, by a classification result given in [12] for \( S^{n-1} \times S^1 \) for some constant \( c > 0 \). In the latter case, it is clear that \( k = n/2 \), since otherwise it cannot have nonnegative \( \Gamma_k \)-curvature. □

Next, we will prove that if \( M \) is locally conformally flat with positive \( \Gamma_{n-1} \)-curvature, then \( g \) has positive sectional curvature. If \( M \) is locally conformally flat, then by (1) we may decompose the full curvature tensor as
\[ \text{Riem} = A_g \odot g, \]

Proposition 2. Assume that \( n = 3 \), or that \( M \) is locally conformally flat. Then the Schouten tensor \( A_g \in V_{n-1}^+ \) if and only if \( g \) has positive sectional curvature.

Proof. Let \( \pi \) be any 2-plane in \( T_p(N) \), and let \( X, Y \) be an orthonormal basis of \( \pi \). We have
\[ K(\sigma) = \text{Riem}(X, Y, X, Y) = A_g \odot g(X, Y, X, Y) \]
\[ = A_g(X, X)g(Y, Y) - A_g(Y, X)g(X, Y) \]
\[ + A_g(Y, Y)g(X, X) - A_g(X, Y)g(Y, X) \]
\[ = A_g(X, X) + A_g(Y, Y). \]
From this it follows that
\[
\min_{\sigma \in T_pN} K(\sigma) = \lambda_1 + \lambda_2,
\]
where \(\lambda_1\) and \(\lambda_2\) are the smallest eigenvalues of \(A_g\) at \(p\).

**Corollary 3.** If \((M,g)\) is locally conformally flat with positive \(\Gamma_{n-1}\)-curvature, then \(g\) has positive sectional curvature.

**Proof.** This follows easily from Propositions 1 and 2. \(\square\)

### 3. Extremal metrics of \(\sigma_k\)-curvature functionals

We next consider some properties of the functionals \(F_k\) associated to \(\sigma_k\). These functionals were introduced and discussed in [13], see also [7]. Further variational properties in connection to 3-dimensional geometry were studied in [9].

We recall that \(F_k\) is defined by
\[
F_k(g) = \text{Vol}(g)^{-\frac{n-k}{n}} \int_M \sigma_k(g) \, d\text{vol}(g).
\]
We denote \(C_k = \{g \in [g_0]|g \in \Gamma_k^+\}\), where \([g_0]\) is the conformal class of \(g_0\).

We now apply our results to show that if \(g_0 \in \Gamma_{n/2}^+\), then there is an extremal metric \(g_e\) which minimizes \(F_m\) for \(m < n/2\), and if \(m > n/2\), there is an extremal metric \(g_e\) which maximizes \(F_m\).

**Proposition 3.** Suppose \((M,g_0)\) is locally conformally flat and \(g_0 \in \Gamma_k^+\) for some \(k \geq \frac{n}{2}\). Then \(\forall m < \frac{n}{2}\), there is an extremal metric \(g_m^e \in [g_0]\) such that
\[
\inf_{g \in C_m} F_m(g) = F_m(g_m^e),
\]
and \(\forall m > \frac{n}{2}\), there is extremal metric \(g_m^e \in [g_0]\) such that
\[
\sup_{g \in C_m} F_k(g) = F_k(g_m^e).
\]
In fact, any solution to \(\sigma_m(g) = \text{constant}\) is an extremal metric.

**Proof.** First, by Corollary 1, \((M,g_0)\) is conformal to a spherical space form. For any \(g \in C_m\), from [7] we know there is a conformal metric \(\tilde{g}\) in \(C_m\) such that \(\sigma_m(\tilde{g})\) is constant and
\[
\begin{align*}
(a) & \text{ if } m > n/2, \text{ then } F_m(g) \leq F_m(\tilde{g}), \\
(b) & \text{ if } m < n/2, \text{ then } F_m(g) \geq F_m(\tilde{g}).
\end{align*}
\]
A classification result of [13] and [14], which is analogous to a result of Obata for the scalar curvature, shows that \(\tilde{g}\) has constant sectional curvature. Therefore \(\tilde{g}\) is the unique critical metric unless \(M\) is conformally equivalent to \(S^n\), in which case any critical metric is the image of the standard metric under a conformal diffeomorphism. This clearly implies the conclusion of the Proposition. \(\square\)

Next we consider the case \(k < n/2\). We have

**Proposition 4.** Suppose \((M,g_0)\) is locally conformally flat and \(g_0 \in \Gamma_k^+\) for some \(k < \frac{n}{2}\). Suppose furthermore that for any fixed \(C > 0\), the space of solutions to the equation \(\sigma_k = C\) is compact, with a bound independent of the constant \(C\). Then there is an extremal metric \(g_k^e \in [g_0]\) such that
\[
\inf_{g \in C_k} F_k(g) = F_k(g_k^e).
\]
Proof. From the compactness assumption, there exists a critical metric \( g^k_e \) which has least energy. If the functional assumed a value strictly lower than \( F_k(g^k_e) \), then by [7], the flow would decrease to another solution of \( \sigma_k = \text{constant} \), which is a contradiction since \( g^k_e \) has minimal energy. \( \square \)

We conclude with conformal quermassintegral inequalities, which were conjectured in [7], and verified there for some special cases when \( (M,g) \) is locally conformally flat and \( g \in \Gamma^+_k \) or \( g \in \Gamma^+_k+1 \) using the flow method. In the case of \( k = 2, n = 4 \), the inequality was proved in [8] without the locally conformally flat assumption.

Proposition 5. Suppose \( (M,g_0) \) is locally conformally flat and \( g_0 \in \Gamma^+_k \) for some \( k \geq \frac{n}{2} \). Then for any \( 1 \leq l < \frac{n}{2} \leq k \leq n \) there is a constant \( C(k,l,n) > 0 \) such that for any \( g \in [g_0] \) and \( g \in \Gamma^+_k \),

\[
(\mathcal{F}_k(g))^{1/k} \leq C(k,l,n)(\mathcal{F}_l(g))^{1/l},
\]

with equality if and only if \( (M,g) \) is a spherical space form.

Proof. By Proposition 3, we have a conformal metric \( g_e \) of constant sectional curvature such that

\[
\inf_{g \in C_l} \mathcal{F}_l(g) = \mathcal{F}_l(g_e)
\]

and

\[
\sup_{g \in C_k} \mathcal{F}_k(g) = \mathcal{F}_k(g_e).
\]

Hence, for any \( g \in \Gamma^+_k \) we have

\[
\frac{(\mathcal{F}_k(g))^{1/k}}{(\mathcal{F}_l(g))^{1/l}} \leq \frac{(\mathcal{F}_k(g_e))^{1/k}}{(\mathcal{F}_l(g_e))^{1/l}} = \frac{(l!(n-l)!)^{1/l}}{(k!(n-k)!)^{1/k}}
\]

When the equality holds, \( g \) is an extremal of \( \mathcal{F}_l \), hence a metric of constant sectional curvature by [13]. \( \square \)

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